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Lower and Upper Bounds for Some Degree-Based Indices of Graphs

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Abstract: In this paper, we give some upper and lower bounds for the multiplicative Randic index, reduced reciprocal Randic index, Narumi-Katayama index and symmetric division index a graph using solely the vertex degrees. Then we obtain upper and lower bounds for these indices for the complete graphs, path graphs and Fibonacci-sum graphs. Finally, we compared the bounds of these indices for a general graph and some special graphs.

Keywords: graph, topological graph index

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1. Introduction

Let G be a graph on n vertices with vertex set $V(G)$ and edge set $E(G)$. For $v \in V(G)$, let $N_G(v)$ be the set of all neighbours of v in G. The degree of $v \in V(G)$ denoted by $\deg(v)$ is the cardinality of $N_G(v)$.

For $n \geq 2$, the Fibonacci sequence $\{F_n\}_{n\geq 0}$ is defined by the recurrence relation $F_n = F_{n-1} + F_{n-2}$ with the initial values $F_0 = 0$ and $F_1 = 1$ [4].

In [2], a Fibonacci-sum graph was defined as follows: For each positive integer n, the Fibonacci-sum graph $G_n = (V, E)$ on vertex set $V = [n] = \{F_2 = 1, F_3 = 2, F_4 = 3, 4, 5, \cdots, n\}$ is defined by two vertices forming an edge if and only if they sum to a Fibonacci number, i.e.

$$E = \{\{i, j\} : i, j \in V,\ i \neq j,\ i + j \text{ is a Fibonacci number}\}.$$

It is obvious from the definition that G_n is a simple graph.

Also, as a result of this study, some structural properties of the Fibonacci-sum graphs were obtained in the following theorems:

Lemma 1.1. *[1] For each $n \geq 1$, G_n is connected.*

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Lemma 1.2. *[1] Let $n \geq 2$, and let k be so that $F_k \leq n < F_{k+1}$. Then in G_n, the vertex F_k has only one neighbour, namely F_{k-1}.*

Lemma 1.3. *[1] Let $n \geq 1$ and let $x \in [1, n]$. Let for $k \geq 2$, $F_k \leq x < F_{k+1}$ and for $l \geq k$, $F_l \leq x + n < F_{l+1}$. Then the degree of x in G_n is*

$$\deg_{G_n}(x) = \begin{cases} l-k, & \text{if } 2x \text{ is not a Fibonacci number,} \\ l-k-1, & \text{if } 2x \text{ is a Fibonacci number.} \end{cases}$$

Theorem 1.1. *[1] Let $n \geq 1$ and let $x \in [1, n]$. Let for $k \geq 2$, $F_k \leq x < F_{k+1}$ and for $l \geq k$, $F_l \leq x + n < F_{l+1}$. Then*

$$\deg_{G_n}(x) = \begin{cases} l-k-1, & \text{if } x = 1 \text{ or } k \geq 4 \text{ and } x = \frac{1}{2}F_{k+2}; \\ l-k, & \text{otherwise.} \end{cases}$$

Corollary 1.1. *[1] Let $n \geq 1$ and let $k \geq 2$ be integers satisfying $F_k \leq n < F_{k+1}$. Then*

$$|E(G_n)| = \begin{cases} n + \frac{F_k+1}{2} - \frac{\left\lfloor \frac{4(k+1)}{3} \right\rfloor}{2}, & \text{if } n \leq \frac{F_{k+2}}{2}; \\ 2n + \frac{F_k+1}{2} - \frac{\left\lfloor \frac{4(k+1)}{3} \right\rfloor}{2} - \left\lceil \frac{F_{k+2}-1}{2} \right\rceil, & \text{if } n > \frac{F_{k+2}}{2}. \end{cases}$$

Theorem 1.2. *[1] For $k \geq 3$ and for any n, let $F_k \leq n < F_{k+1}$. If $n < \frac{F_{k+2}}{2}$, then $F_k, F_k + 1, \cdots, n$ are the pendant vertices. If $n \geq \frac{F_{k+2}}{2}$, then $F_k, F_k + 1, \cdots, F_{k+2} - n - 1$ are the pendant vertices. The remaining pendant vertices are*

$$\begin{cases} \frac{F_k}{2}, & \text{if } k \equiv 0 (\text{mod } 3) \text{ and } n < F_{k+1} - \frac{F_k}{2}; \\ \frac{F_{k+1}}{2}, & \text{if } k \equiv 1 (\text{mod } 3); \\ \frac{F_{k+2}}{2}, & \text{if } k \equiv 2 (\text{mod } 3) \text{ and } n \geq \frac{F_{k+2}}{2}. \end{cases}$$

Theorem 1.3. *For any $n \geq 2$, vertex 2 has maximum degree in the Fibonacci-sum graph G_n. Also, if $n + 2$ is a Fibonacci number, then $\deg_{G_n}(1) = \deg_{G_n}(2) - 1$; otherwise, $\deg_{G_n}(1) = \deg_{G_n}(2)$.*

Proof. It is clear that the vertex 2 has maximum degree due to the structure of the Fibonacci-sum graph G_n.

If $n + 2$ is a Fibonacci number, then there exists an l such that $F_l \leq n + 2 < F_{l+1}$. So, we have $F_{l-1} < n + 1 < F_l$.

For $x = 2$, we have $F_{k_1} \leq 2 < F_{k_1+1}$ which satisfy that $k_1 = 3$.

For $x = 1$, we have $F_{k_2} \leq 1 < F_{k_2+1}$ which satisfy that $k_2 = 2$.

By using Theorem 1.1, we get

$$\deg(2) = l - k_1 = l - 3$$

and

$$\deg(1) = (l-1) - k_2 - 1 = l - 4.$$

As a result, we obtain

$$\deg(1) = \deg(2) - 1.$$

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If $n+2$ is not a Fibonacci number, then $F_l < n+2 < F_{l+1}$. This implies that $F_l \leq n+1 < F_{l+1}$.

By using Theorem 1.1 again, we get

$$\deg(2) = l - k_1 = l - 3$$

and

$$\deg(1) = (l - k_2 - 1 = l - 3.$$

Hence, we get

$$\deg(1) = \deg(2).$$

□

As a result of the above theorem, in the Fibonacci-sum graph G_n, 2 has the maximum degree and one of the vertices with maximum degree less than the degree of 2 is 1. Also, by Lemma 1.2 $d(F_k) = 1$ for $F_k \leq n < F_{k+1}$. Thus, for any $i \in V(G_n)$, we have

$$d(2) \geq d(1) \geq d(i) \geq d(F_k) \tag{1}$$

where $F_k \leq n < F_{k+1}$. In this case, by applying Theorem 1.1, we get

$$F_{l_1} \leq 2 + n < F_{l_1+1}, \text{ then } \deg(2) = l_1 - 3, \tag{2}$$

$$F_{l_2} \leq 1 + n < F_{l_2+1}, \text{ then } \deg(1) = l_2 - 3. \tag{3}$$

In [7], the spectral properties of Fibonacci-sum and Lucas-sum graphs were examined and some bounds were obtained. Also, in [8] another type of graphs associated with Fibonacci numbers was studied.

A topological index is a numerical value mathematically derived from the graph structure. Several significant indices such as Zagreb index, Randic index and Wiener index has been introduced to measure the characters of graphs. The number of the vertices and the number of the edges are some examples of topological indices.

Now, we recall the definitions of some topological indices we used in this study:

The multiplicative Randic index is defined in [5] as

$$MR(G) = \prod_{uv \in E(G)} \sqrt{\frac{1}{\deg(u)\deg(v)}}.$$

The reduced reciprocal Randic index was described in [5] as

$$RRR(G) = \sum_{uv \in E(G)} \sqrt{(\deg(u) - 1)(\deg(v) - 1)}.$$

The Narumi-Katayama index was introduced in [6] as

$$NK(G) = \prod_{i=1}^{n} \deg(v_i).$$

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The symmetric division index was described in [3] as

$$SD(G) = \sum_{uv \in E(G)} \frac{\deg(u)^2 + \deg(v)^2}{\deg(u)\deg(v)}.$$

In this study, we give some upper and lower bounds of multiplicative Randic index, reduced reciprocal Randic index, Narumi-Katayama index and symmetric division index for the general graphs using vertex degree. Then, we obtain upper and lower bounds for these indices for some special graphs and Fibonacci-sum graphs. Finally, we compared the bounds on these indices for some graphs.

2. Main Results

Theorem 2.1. *If G is a simple connected graph with n vertices and m edges, then*

$$\left(\frac{1}{n-1}\right)^m \leq MR(G) \leq \left(\frac{1}{\sqrt{2}}\right)^m.$$

Proof. Since the graph is simple connected, the vertices have degrees at least 1 and 2. Let all edges have exactly one pendant vertex and the other vertex is of degree 2. We get the upper bound for the multiplicative Randic index of G as

$$MR(G) \leq \left(\frac{1}{\sqrt{2}}\right)^m.$$

Also, since the vertices have the maximum degree at most $n-1$, we have the lower bound for the multiplicative Randic index of G as

$$\left(\frac{1}{n-1}\right)^m \leq MR(G).$$

As a conclusion, we obtain

$$\left(\frac{1}{n-1}\right)^m \leq MR(G) \leq \left(\frac{1}{\sqrt{2}}\right)^m.$$

□

Theorem 2.2. *Let G be a simple connected graph with m edges, then*

$$\left(\frac{1}{\Delta}\right)^m \leq MR(G) \leq \left(\frac{1}{\delta}\right)^m$$

where δ is the minimum degree and Δ is the maximum degree of vertices in G.

Proof. Hence we obtain

$$\left(\frac{1}{\Delta}\right)^m \leq MR(G) \leq \left(\frac{1}{\delta}\right)^m.$$

□

Corollary 2.1. *Let $G = K_n$ be a complete graph with n vertices, then*

$$MR(K_n) = \left(\frac{1}{n-1}\right)^{\frac{n(n-1)}{2}}.$$

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Proof. K_n is a simple graph and since it does not contain multiple edges and loops, the maximum vertex degree is $n-1$. In addition, complete graph has $\frac{n(n-1)}{2}$ edges, hence the proof follows from the above theorem. □

Corollary 2.2. *Let $G = K_{p,q}$. If $p < q$, then*

$$\left(\frac{1}{q}\right)^{pq} \leq MR(K_{p,q}) \leq \left(\frac{1}{p}\right)^{pq}.$$

If $p = q$, then

$$MR(K_{p,q}) = \left(\frac{1}{p}\right)^{p^2}.$$

Proof. Since the $K_{p,q}$ graph has pq edges, the proof can be seen easily. □

Corollary 2.3. *Let $G = P_n$ be a path graph, then*

$$MR(P_n) = \left(\frac{1}{2}\right)^{n-2}.$$

Proof. Since there are $n-1$ edges in P_n, two of which are endpoints, $\prod_{uv \in E(G)} \sqrt{\frac{1}{\deg(u)\deg(v)}}$ has $\left(\frac{1}{\sqrt{2}}\right)^2$. $\left(\frac{1}{2}\right)^{n-3}$ comes from the remaining $n-3$ edges. Hence we get

$$MR(P_n) = \left(\frac{1}{\sqrt{2}}\right)^2 \cdot \left(\frac{1}{2}\right)^{n-3} = \left(\frac{1}{2}\right)^{n-2}.$$

□

Theorem 2.3. *If G_n is a Fibonacci-sum graph, then*

$$\left(\frac{1}{\sqrt{(l_1-3)(l_2-3)}}\right)^{n-1} \leq MR(G_n) \leq \left(\frac{1}{\sqrt{2}}\right)^{n-r}$$

where l_1,l_2 are integers in (2), (3), respectively and r is the number of the vertices with degree 1 in G_n.

Proof. Since r is the number of the vertices with degree 1 in G_n, the degrees of the other vertices are at least 2. Thus, there are r vertices with degree 1 and $n-r$ vertices with degree at least 2. Hence, we get the upper bound for the multiplicative Randic index of G_n as

$$MR(G_n) \leq \left(\frac{1}{\sqrt{2}}\right)^{n-r}.$$

Also, since by Theorem 1.3, 2 has the maximum degree and one of the vertices with maximum degree less than the degree of 2 is 1, we have the lower bound for the multiplicative Randic index of G_n as

$$\left(\frac{1}{\sqrt{\deg(2)\deg(1)}}\right)^{n-1} \leq MR(G_n).$$

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As a conclusion, we obtain

$$\left(\frac{1}{\sqrt{(l_1-3)(l_2-3)}}\right)^{n-1} \leq MR(G_n) \leq \left(\frac{1}{\sqrt{2}}\right)^{n-r}.$$

□

Theorem 2.4. *Let G be a simple connected graph with n vertices and m edges, then*

$$0 \leq RRR(G) \leq m(n-2).$$

Proof. Since the graph is simple connected, there are no isolated vertices and we get the lower bound as

$$0 \leq RRR(G).$$

Also, since the vertices have the maximum degree at most $n-1$, we have the upper bound as

$$RRR(G) \leq m(n-2).$$

As a conclusion, we obtain

$$0 \leq RRR(G) \leq m(n-2).$$

□

Theorem 2.5. *Let G be a simple connected graph with m edges, then*

$$m(\delta-1) \leq RRR(G) \leq m(\Delta-1)$$

where δ is the minimum degree and Δ is the maximum degree of vertices in G.

Proof. Since δ is the minimum degree and Δ is the maximum degree of vertices in G, we obtain

$$m(\delta-1) \leq RRR(G) \leq m(\Delta-1).$$

□

Corollary 2.4. *Let $G = K_n$ be a complete graph with n vertices, then*

$$RRR(K_n) = \frac{n(n-1)(n-2)}{2}.$$

Proof. K_n is a simple graph and since it does not contain multiple edges and loops, the maximum degree is $n-1$. Also, complete graph K_n has $\frac{n(n-1)}{2}$ edges implying the proof by the above theorem. □

Corollary 2.5. *Let $G = K_{p,q}$. If $p < q$, then*

$$RRR(K_{p,q}) = pq\sqrt{(p-1)(q-1)}.$$

If $p = q$

$$RRR(K_{p,q}) = p^2(p-1).$$

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Proof. Since $m = pq$ in $K_{p,q}$,the proof is trivial. □

Corollary 2.6. *Let $G = P_n$ be a path graph, then*

$$0 \leq RRR(P_n) \leq n - 3.$$

Proof. 0 comes from 2 edges with endpoints in P_n. The inner $n - 3$ is $\sqrt{(2-1)(2-1)} = 1$ from the edge and the desired is obtained. □

Theorem 2.6. *If G_n is a Fibonacci-sum graph, then*

$$m \leq RRR(G_n) \leq m\sqrt{(l_1 - 4)(l_2 - 4)}$$

where l_1,l_2 are the integers in (2), (3), respectively, and $m = |E(G_n)|$.

Proof. By Lemma 1.2, in the Fibonacci-sum graph G_n, F_k is adjacent to only F_{k-1} for $F_k \leq n < F_{k+1}$. Also, since the other neighbour of F_{k-1} is F_{k-2}, $\deg(F_{k-1}) = 2$. By the same way, $\deg(F_{k-2}) \geq 2$. Thus, we get the lower bound for the reduced reciprocal Randic index of G_n as

$$m\sqrt{\deg(F_{k-1} - 1)\deg(F_{k-2} - 1)} = m \leq RRR(G_n).$$

Since $1 \sim 2$ and by using (1), we get the upper bound for the reduced reciprocal Randic index of G_n as

$$RRR(G_n) \leq m\sqrt{(\deg(1) - 1)(\deg(2) - 1)}.$$

Hence, we obtain

$$m \leq RRR(G_n) \leq m\sqrt{(l_1 - 4)(l_2 - 4)}.$$

□

Theorem 2.7. *Let G be a simple connected graph with n vertices, then*

$$1 + 2^{n-k} \leq NK(G) \leq (n-1)^n$$

where k is the number of the vertices with degree 1 in G.

Proof. Since the graph is simple connected, there are no pendant vertices. Let there be k pendant vertices and $n - k$ vertices with degree at least 2, we get the lower bound as

$$1 + 2^{n-k} \leq NK(G).$$

Also, since the vertices have the maximum degree at most $n - 1$, we get the upper bound as

$$NK(G) \leq (n-1)^n.$$

□

Theorem 2.8. *Let G be a simple connected graph with n vertices, then*

$$\delta^k(\delta + 1)^{n-k} \leq NK(G) \leq \Delta^r(\Delta - 1)^{n-r}$$

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where k is the number of the vertices with minimum degree and r is the number of the vertices with maximum degree in G.

Proof. If we take the k vertices with minimum degree and $n-k$ vertices with degree $\delta+1$, we get the lower bound as

$$\delta^k(\delta+1)^{n-k} \leq NK(G).$$

If we take the r vertices with maximum degree and $n-r$ vertices with degree $\Delta-1$, we get the upper bound as

$$NK(G) \leq \Delta^r(\Delta-1)^{n-r}.$$

□

Corollary 2.7. *Let $G = K_n$ be a complete graph with n vertices, then*

$$NK(K_n) = (n-1)^n.$$

Proof. K_n is a simple graph and since it does not contain multiple edges and loops, the degree of any vertex is $n-1$. Hence the proof follows. □

Corollary 2.8. *Let $G = K_{p,q}$ then*

$$NK(K_{p,q}) = p^q q^p.$$

Proof. Since there are q points of degree p and p points of degree q in the graph $K_{p,q}$, we obtain $NK(K_{p,q}) = p^q q^p$. □

Corollary 2.9. *Let $G = P_n$ be a path graph, then*

$$NK(P_n) = 2^{n-2}.$$

Proof. Since P_n is a graph with degrees 1 at the end vertices and 2 on the other vertices, we obtain,

$$NK(P_n) = 2^{n-2}.$$

□

Theorem 2.9. *For the Narumi-Katayama index of the Fibonacci-sum graph G_n, the following inequality holds:*

$$2^{n-r} \leq NK(G_n) \leq (l_1-3)(l_2-3)^{n-1}$$

where l_1,l_2 are the integers in (2), (3) , respectively and r is the number of the vertices with degree 1 in G.

Proof. Since r is the number of the vertices with degree 1 in G_n, then the degrees of the other vertices are at least 2. Thus, there are r vertices with degree 1 and $n-r$ vertices with degree at least 2. Hence, we get the lower bound for the Narumi-Katayama index of G_n as

$$2^{n-r} \leq NK(G_n).$$

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Also, since by Theorem 1.3, 2 has the maximum degree and one of the vertices with maximum degree less than the degree of 2 is 1, we have the upper bound for the Narumi-Katayama index of G_n as

$$NK(G_n) \leq \deg(2)(\deg(1))^{n-1}.$$

As a result, we obtain

$$2^{n-r} \leq NK(G_n) \leq (l_1 - 3)(l_2 - 3)^{n-1}.$$

83 □

Theorem 2.10. *If G is a simple connected graph with n vertices and m edges, then*

$$2m \leq SD(G) \leq m\frac{(n-1)^2+1}{n-1}.$$

Proof. If $\deg(u)$ is maximum and $\deg(v)$ is minimum, then the expression

$$\frac{\deg(u)^2 + \deg(v)^2}{\deg(u)\deg(v)} \tag{4}$$

takes its maximum value. In G, $n-1$ is the maximum degree and if we take the pendant vertex which is adjacent to $n-1$, then the expression (4) takes its maximum value. Thus, we get

$$SD(G) = \frac{\deg(u)^2 + \deg(v)^2}{\deg(u)\deg(v)} \leq m\frac{(n-1)^2+1}{n-1}.$$

In other way, when $\deg(u)$ and $\deg(v)$ are equal, then the expression (4) takes its minimum value. Thus, we get

$$2m \leq SD(G).$$

Hence, we obtain

$$2m \leq SD(G) \leq m\frac{(n-1)^2+1}{n-1}.$$

84 □

Theorem 2.11. *Let G be a simple connected graph with m edges, then*

$$2m \leq SD(G) \leq m\frac{\Delta^2+\delta^2}{\Delta\delta}.$$

Proof. We obtain

$$2m \leq SD(G) \leq m\frac{\Delta^2+\delta^2}{\Delta\delta}.$$

85 □

Corollary 2.10. *Let $G = K_n$ be a complete graph with n vertices, then*

$$SD(K_n) = 2n(n-1).$$

86 **Proof.** K_n is a simple graph and since it does not contain multiple edges and loops, the maximum degree
87 is $n-1$. Also, complete graph has $\frac{n(n-1)}{2}$ edges, hence we get $SD(G) = 2n(n-1)$ from the above
88 theorem. □

Corollary 2.11. *Let $G = K_{p,q}$ then*

$$SD(K_{p,q}) = p^2 + q^2.$$

89 **Proof.** Since $m = pq$ in $K_{p,q}$, the proof is trivial. □

90 **Corollary 2.12.** *Let $G = P_n$ be a path graph, then*

$$SD(P_n) = 2n - 1.$$

Proof. Since two edges in P_n have 1 and 2 degree vertices and the other $n-3$ edges are composed of 2 degree vertices at each end, we obtain

$$SD(P_n) = 2\frac{2^2+1^2}{2.1} + (n-3)\frac{2^2+2^2}{2.2} = 2n - 1.$$

91 □

Theorem 2.12. *If G_n is a Fibonacci-sum graph, then*

$$2m \leq SD(G_n) \leq m\,(l_1 - 2)$$

92 *where l_1 is the integer in (2) and $m = |E(G_n)|$.*

Proof. If $\deg(u)$ is maximum and $\deg(v)$ is minimum, then the expression

$$\frac{\deg(u)^2 + \deg(v)^2}{\deg(u)\deg(v)} \tag{5}$$

takes its maximum value. In G_n, 2 has the maximum degree and if we take the 1 degreed vertex which is adjacent to 2, then the expression (5) takes its maximum value. Thus we have

$$\frac{\deg(u)^2 + \deg(v)^2}{\deg(u)\deg(v)} \leq \deg(2) + 1.$$

Hence, we get the upper bound for the symmetric division index of G_n as

$$SD(G_n) \leq m\,(l_1 - 2).$$

In other way, when $\deg(u)$ and $\deg(v)$ are equal, then the expression (5) takes its minimum value. Thus we have

$$2 \leq \frac{\deg(u)^2 + \deg(v)^2}{\deg(u)\deg(v)}.$$

Hence, we get

$$2m \leq SD(G_n).$$

In conclusion, we obtain

$$2m \leq SD(G_n) \leq m\,(l_1 - 2).$$

93 □

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