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Quantum Hydrodynamics in External Magnetic Fields: From Nonrelativistic to Relativistic Regimes

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Article

Quantum Hydrodynamics in External Magnetic Fields: From Nonrelativistic to Relativistic Regimes

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Abstract: We delve into nonrelativistic quantum electrodynamics within a background magnetic field, ensuring gauge invariance via a vector potential. Our exploration extends to the Lagrangian, which incorporates electron self-interactions and electromagnetic field interactions. Through the path integral formalism, we elucidate the effective action, with a specific focus on the photon propagator and screening effects. Examining density and current components reveals Laurent expansions in the hydrodynamic limit. The equations of motion in real space are derived, leading to discussions on phenomena such as Poiseuille-like flow and the Navier-Stokes equation. To ground our theoretical framework at practical contexts, we consider applications such as the fluid dynamics at binary star systems. Depending on the velocity of the expelled fluid, our analysis allows for both nonrelativistic and relativistic treatments, providing a versatile tool for understanding the intricacies of fluid behavior at astrophysical scenarios. Additionally, we recognize that at the presence of a magnetic field, magnetohydrodynamics becomes crucial. While our current focus is at nonrelativistic quantum electrodynamics, the insights gained here contribute to a broader understanding, offering a comprehensive foundation for quantum electrodynamics analysis at diverse physical systems.

Keywords: nonrelativistic quantum electrodynamics; NRQED; background magnetic field; gauge invariance; Lagrangian; electron self-interactions; electromagnetic field interactions; path integral formalism; effective action; photon propagator; screening effects; density components; current components; Laurent expansions; hydrodynamic limit; equations of motion; Poiseuille-like flow; Navier-Stokes equation; relativistic hydrodynamics; linearized equations; nonrelativistic limit; quantum electrodynamical analysis

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1. Introduction

The investigation of nonrelativistic quantum electrodynamics (NRQED) in the presence of external magnetic fields is a profound exploration of the intricate dynamics governing quantum matter's interaction with electromagnetic fields. By extending the Lagrangian formulation of NRQED, we introduce electron self-interactions and dynamic coupling with the electromagnetic field, incorporating gauge invariance as a fundamental aspect. Our study employs the path integral formalism to analyze the effective action, revealing crucial details such as the behavior of the photon propagator and the emergence of screening effects in the quantum realm. Examining density and current components, we uncover intricate Laurent expansions inherent in the hydrodynamic limit, providing essential insights into the quantum system's response to external magnetic fields. A key contribution of our work lies in the derivation of equations of motion in real space, offering a detailed understanding of quantum matter's behavior under the influence of electromagnetic fields. Our exploration encompasses scenarios resembling Poiseuille-like flow, establishing connections with the classical Navier-Stokes equation, a cornerstone in fluid dynamics. Extending our analysis beyond the nonrelativistic regime, we seamlessly transition to relativistic hydrodynamics, providing a unified mathematical framework. The linearized equations presented offer a comprehensive understanding of the system's response in diverse scenarios. Moreover, our study delves into the implications of the nonrelativistic limit, drawing parallels with classical fluid dynamics. This research establishes a robust mathematical framework for analyzing quantum electrodynamics in the nonrelativistic regime, providing nuanced insights into the intricate interplay of quantum matter and electromagnetic fields.

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2. Nonrelativistic Free Electrons

2.1. Gauge Invariance

Consider the gauge transformation $a_\mu = (a^0, -\boldsymbol{a}) \to -A_\mu$. The Lagrangian is given by:

$$L = \frac{1}{2}\psi^\dagger i\partial_t\psi - \frac{1}{2}i(\partial_t\psi^\dagger)\psi + \frac{1}{2m}(i\boldsymbol{\nabla} + \boldsymbol{a})\psi^\dagger(-i\boldsymbol{\nabla} + \boldsymbol{a})\psi \\ + \left(\mu + a^0 - \frac{g}{4m}\boldsymbol{\sigma}(\boldsymbol{\nabla}\times\boldsymbol{a})\right)\psi^\dagger\psi \tag{1}$$

The functional integral is expressed as:

$$e^{iW[\hat{a}]} = \int D[\psi]D[\psi^\dagger]e^{i\psi^\dagger[\hat{G}_0^{-1} + \hat{\not{a}} + \frac{\hat{\sigma}}{2m}\hat{\boldsymbol{a}}^2]\hat{\psi}} \tag{2}$$

The transformation properties are defined by:

$$\not{a} = a_\mu(C^\mu + B^\mu) = a^0(C^0 + B^0) - \boldsymbol{a}(\boldsymbol{C} + \boldsymbol{B}) \\ \psi^\dagger C_{0\,x}\psi = \psi_x^\dagger\psi_x, \quad \psi^\dagger \boldsymbol{C}_x\psi = -\frac{i}{2m}[\psi^\dagger\boldsymbol{\nabla}\psi - (\boldsymbol{\nabla}\psi^\dagger)\psi]_x \\ \partial\psi = i\frac{\Delta}{2m}\psi, \quad \partial\psi^\dagger = -i\frac{\Delta}{2m}\psi^\dagger, \quad \partial_t(\psi^\dagger\psi) = i\psi^\dagger\frac{\Delta}{2m}\psi - i\frac{\Delta}{2m}\psi^\dagger\psi \\ \int_x \psi^\dagger\boldsymbol{aB}\psi = \frac{g}{4m}\epsilon^{abc}\int_x a^a\nabla^c(\psi^\dagger\sigma^b\psi), \quad B^0 = 0$$

The variational derivatives of the generating functional are:

$$\frac{\delta W[\hat{a}^0, \hat{\boldsymbol{a}}]}{\delta a^0} = \hat{\rho}, \quad \frac{\delta W[\hat{a}^0, \hat{\boldsymbol{a}}]}{\delta\hat{\boldsymbol{a}}} = \hat{\boldsymbol{j}} + \hat{\sigma}\frac{\hat{\rho}_d}{m}\hat{\boldsymbol{a}} = \hat{\boldsymbol{J}}, \\ \hat{\rho}_m = \begin{pmatrix}\rho^+ & 0\\ 0 & \rho^-\end{pmatrix}, \quad \hat{\boldsymbol{a}}^2 = \begin{pmatrix}\boldsymbol{a}^{+2}\\ \boldsymbol{a}^{-2}\end{pmatrix} \tag{3}$$

The effective action is given by:

$$\Gamma[\hat{\rho}, \hat{\boldsymbol{j}}] = W[\hat{a}_0, \hat{\boldsymbol{a}}] - \hat{\rho}\hat{a}^0 - \hat{\boldsymbol{a}}\hat{\boldsymbol{j}} - \hat{\rho}\frac{\hat{\sigma}}{2m}\hat{\boldsymbol{a}}^2 \\ \frac{\delta\Gamma}{\delta\hat{\rho}} = -\hat{a}^0 - \hat{\sigma}\frac{\hat{\boldsymbol{a}}^2}{2m}, \quad \frac{\delta\Gamma}{\delta\hat{j}^k} = -\hat{a}_k \tag{4}$$

The explicit form of the functional integral yields:

$$W[\hat{a}] \approx -i\mathrm{Tr}\ln\hat{G}_0^{-1} - i\mathrm{Tr}\hat{G}_0\left(\hat{\not{a}} + \frac{\hat{\sigma}}{2m}\hat{\boldsymbol{a}}^2\right) + \frac{i\hbar}{2}\mathrm{Tr}(\hat{G}_0\hat{\not{a}})^2 \tag{5}$$

This can be further expressed as:

$$W[\hat{a}] = \hat{\rho}_0\hat{a} - \frac{1}{2}\hat{a}\left(\hat{G} - \hat{\sigma}\hat{S}\right)\hat{a}, \\ \hat{S} = \frac{\rho_{0m}}{m}\begin{pmatrix}0 & 0\\ 0 & \mathbb{1}\end{pmatrix} \tag{6}$$

The charge and current density matrices are defined as:

$$\begin{pmatrix}\hat{\rho}\\ \hat{\boldsymbol{j}}\end{pmatrix} = \begin{pmatrix}\hat{\rho}_0\\ 0\end{pmatrix} - (\hat{G} - \hat{\sigma}\hat{S})\hat{a} \tag{7}$$

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The functional derivative of the effective action with respect to the charge density and current are given by:

$$\frac{\delta\Gamma}{\delta\hat{\rho}^{\sigma}} = -\hat{a}^{0\sigma} - \sigma\frac{\boldsymbol{a}^{\sigma 2}}{2m},$$
$$\frac{\delta\Gamma}{\delta\hat{j}^{k}} = -\hat{a}_{k} \tag{8}$$

Finally, the expressions for the nonrelativistic and transverse components of the Green's function are provided, and the Ward identity is presented as $\partial_t\rho + \boldsymbol{\nabla}\boldsymbol{j} = 0$.

3. Quantum Electrodynamics (QED)

Consider the Quantum Electrodynamics (QED) action:

$$S = \bar{\psi}(i\partial\cdot\bar{r} - m + a\cdot\bar{r} - eA\cdot\bar{r}_e)\psi + \frac{1}{2}AD_0^{-1}A + S_{CT}^{QED} + S_{CT}^{e} \tag{9}$$

Here, ψ represents the fermion field, A is the electromagnetic potential, and S_{CT}^{QED} and S_{CT}^{e} are counterterms.

The counterterms are given by:

$$S_{CT}^{QED} = \frac{e^2 z_3}{2}AD_0^{-1}A + \bar{\psi}[(Z_2-1)i\partial\cdot\bar{r} + (Z_0-1)m + (Z_1-1)(a\cdot\bar{r} - eA\cdot\bar{r})]\psi$$
$$S_{CT}^{e} = -z_3 eaD_0^{-1}A + \frac{z_3}{2}aD_0^{-1}a \tag{10}$$

Where $Z_1 = Z_2$, $m_B = \frac{Z_0}{Z_1}m$, $\psi_B = \sqrt{Z_1}\psi$, $A_B = eA$, and after subtraction at $p = 0$, $\frac{1}{e_B^2} = \frac{1}{e^2} + z_3$, where $z_3 > 0$.

The transformed action in terms of renormalized fields becomes:

$$S = \bar{\psi}_B(i\partial\cdot\bar{r} - m_B + a\cdot\bar{r} - A\cdot\bar{r}_B)\psi_B + \frac{1}{2e_B^2}A_B D_0^{-1}A_B + S_{CT}^{e}$$
$$= \bar{\psi}_B(i\partial\cdot\bar{r} - m_B + a\cdot\bar{r} - eA\cdot\bar{r}_B)\psi_B + \frac{1}{2}AD_0^{-1}A + \frac{z_3}{2}(eA-a)D_0^{-1}(eA-a) \tag{11}$$

where the last counterterm is mechanical, addressing the UV divergence in the free Dirac sea. Now, introduce an auxiliary field k^{μ}, representing the UV part of the current:

$$S = \bar{\psi}_B(i\partial\cdot\bar{r} - m_B + a\cdot\bar{r} - eA\cdot\bar{r}_B)\psi_B + \frac{1}{2}AD_0^{-1}A - (eA-a)k + \frac{1}{2z_3}kD_0k$$
$$= \bar{\psi}_B(i\partial\cdot\bar{r} - m_B)\psi_B + \frac{1}{2}AD_0^{-1}A + \frac{1}{2z_3}kD_0k + (a_{\mu} - eA_{\mu})(\bar{\psi}_B\gamma^{\mu}\psi_B - k^{\mu}) \tag{12}$$

3.1. Generator Functionals

The generator functional is given by:

$$e^{iW[\hat{a},\hat{j}]} = \int D[\hat{\psi}]D[\hat{\bar{\psi}}]D[\hat{A}]e^{i\hat{\bar{\psi}}[\hat{F}_0^{-1} + \hat{a}\cdot\bar{r}_e - \frac{e}{c}\sigma\hat{A}\cdot\bar{r}_e]\hat{\psi} + \frac{i}{2}\hat{A}\hat{D}_0^{-1}\hat{A} + i\hat{j}\hat{A}} \tag{13}$$

Here, $\hat{\psi}$ and $\hat{A}$ are matrix fields defined as:

$$\hat{\psi} = \begin{pmatrix}\psi_+\\ \psi_-\end{pmatrix}, \quad \hat{A} = \begin{pmatrix}A^+\\ A^-\end{pmatrix} \tag{14}$$

The inverse free propagators are defined as:

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$$\hat{F}_0^{-1} = \begin{pmatrix} G_0^{-1} & \\ 0 & -\gamma^0 G_0^{-1} \gamma^0 \end{pmatrix} + \hat{F}_{BC}^{-1}, \quad \hat{D}_0^{-1} = \begin{pmatrix} D_0^{-1} & 0 \\ 0 & -D_0^{-1} \end{pmatrix} + \hat{D}_{BC}^{-1} \tag{15}$$

with:

$$F_0^{-1} = i\partial \cdot \bar{r} - m + i\epsilon, D_0^{-1} = \frac{1}{e^2}\Box T + \xi \Box L + i\epsilon \tag{16}$$

where $T^{ab} = g^{ab} - L^{ab}$, and $L^{ab} = \frac{\partial^a \partial^b}{\Box}$.
Continue generation

4. Equation of Motion

4.1. Retardation

The retarded and advanced accelerations are given by

$$a^{\pm} = \frac{\bar{a}}{2} \pm a \tag{17}$$

The Green's function is defined as

$$G = \begin{pmatrix} G^n & -G^f \\ G^f & -G^n \end{pmatrix} + iG^i \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \tag{18}$$

The real and imaginary parts of $G\hat{a}$ are

$$\begin{aligned} \Re(G\hat{a}) &= \begin{pmatrix} G^r a + \frac{G^a}{2}\bar{a} \\ G^r a - \frac{G^a}{2}\bar{a} \end{pmatrix} \\ \hat{a}\Re(G\hat{a}) &= aG^a\bar{a} + \bar{a}G^r a \\ \Im(G\hat{a}) &= \begin{pmatrix} 1 \\ 1 \end{pmatrix} \bar{a} \\ \hat{a}\Im(G\hat{a}) &= (\frac{\bar{a}}{2} + a, \frac{\bar{a}}{2} - a) \begin{pmatrix} 1 \\ 1 \end{pmatrix} \bar{a} = \bar{a}\bar{a} \end{aligned} \tag{19}$$

The action functional is given by

$$W[a, a_d] = -\frac{1}{2}\hat{a}G\hat{a} = -\frac{1}{2}(a, \bar{a}) \begin{pmatrix} 0 & G^a \\ G^r & i \end{pmatrix} \begin{pmatrix} a \\ \bar{a} \end{pmatrix} \tag{20}$$

The inversion matrix is defined as

$$\mathbb{1} = \begin{pmatrix} 0 & G^a \\ G^r & i \end{pmatrix} \begin{pmatrix} -iG^{r-1}G^{a-1} & G^{r-1} \\ G^{a-1} & 0 \end{pmatrix} \tag{21}$$

The effective action in terms of source terms J_d and J is

$$\Gamma[\hat{J}] = \frac{1}{2}(J_d, J) \begin{pmatrix} -iG^{r-1}G^{a-1} & G^{r-1} \\ G^{a-1} & 0 \end{pmatrix} \begin{pmatrix} J_d \\ J \end{pmatrix} = -\frac{i}{2}J_d G^{r-1}G^{a-1}J_d + \frac{1}{2}J_d G^{r-1}J + \frac{1}{2}JG^{a-1}J_d \tag{22}$$

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The equations of motion are given by

$$a = iG^{r-1}G^{a-1}J_d - G^{r-1}J$$
$$\bar{a} = -G^{a-1}J_d \tag{23}$$

On-shell conditions are $\bar{a} = 0$ and $J_d = 0$, leading to

$$a = -G^{r-1}J \tag{24}$$

The effective action on-shell becomes

$$\Gamma[\hat{J}] = -\frac{i}{2}J_d G^{r-1}G^{a-1}J_d + \frac{1}{2}J_d G^{r-1}J + \frac{1}{2}JG^{a-1}J_d \tag{25}$$

The equation of motion for J_d is

$$-a = G^{r-1}J \tag{26}$$

The inverse Green's function is given by

$$G^{r-1} = G_0^{-1} - e^2 D_0^r \to G_0^{-1} - \frac{e^2}{c^2}D^r \tag{27}$$

4.2. Photon Propagator

The photon propagator components are given by

$$-\hat{D}_{0(\omega,\boldsymbol{q})} = \begin{pmatrix} \frac{1}{\omega^2-\boldsymbol{q}^2+i\epsilon} & -2\pi i\delta(\omega^2-\boldsymbol{q}^2)\Theta(-\omega) \\ -2\pi i\delta(\omega^2-\boldsymbol{q}^2)\Theta(\omega) & -\frac{1}{\omega^2-\boldsymbol{q}^2-i\epsilon} \end{pmatrix}$$
$$-D^r_{0(\omega,\boldsymbol{q})} = \frac{1}{\omega^2-\boldsymbol{q}^2+i\epsilon} + 2\pi i\delta(\omega^2-\boldsymbol{q}^2)\Theta(-\omega)$$
$$= \frac{1}{\omega^2-\boldsymbol{q}^2+i\epsilon\Theta(\omega)}$$
$$= \frac{1}{(\omega+i\epsilon)^2-\boldsymbol{q}^2} \tag{28}$$

The full photon propagator D is given by

$$D = \frac{1}{\hat{D}_0^{-1} - e^2\sigma\hat{G}\sigma}, \quad D^{\substack{r\\a}} = [\hat{D}_0^{-1} - e^2\sigma\hat{G}\sigma]^{\substack{r\\a}}, \quad D^i = D^r G^i D^a \tag{29}$$

The inverse of $\hat{D}_0$ is

$$\hat{D}_0^{-1} = -T\left[k^2\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} + i\epsilon\begin{pmatrix}1 & -2\Theta(-k^0)\\ -2\Theta(k^0) & 1\end{pmatrix}\right] \tag{30}$$

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The components of D^{-1} are

$$
\begin{aligned}
D^{-1n} &= -k^2, \quad D^{-1i} = -\epsilon, \quad D^{-1f} = -i\mathrm{sign}(k^0)\epsilon \\
D^r_\ell &= \frac{1}{D_0^{-1r} - e^2 G^r_\ell} \\
&= -\frac{1}{k^2 + i\mathrm{sign}(k^0)\epsilon + e^2(G^{++}_{vac} - G^{+-}_{vac} + G^r_\ell)} \\
D^r_t &= \frac{1}{D_0^{-1r} - e^2 G^r_t} \\
&= -\frac{1}{k^2 + i\mathrm{sign}(k^0)\epsilon + e^2(G^{++}_{vac} - G^{+-}_{vac} + G^r_t)}
\end{aligned}
\tag{31}
$$

The screening condition is $D^{r-1}_\ell(q=0) = q^2_{TF} = \frac{1}{\pi^2} e^2 m k_F$

4.3. Separation of Density and Current

The parameters are defined as follows:

$$
\begin{aligned}
k_F &= \frac{1}{s r_0} \approx \frac{1.92}{a_0 r_s}, \quad s = \left(\frac{4}{9\pi}\right)^{\frac{1}{3}} = 0.5211 \\
z &= \frac{\omega m}{\hbar k_F^2}, \quad z_{\mathrm{pl}} = \sqrt{\frac{4 s r_s}{3\pi}} = 0.471\sqrt{r_s}
\end{aligned}
$$

The Bohr radius is given by $a_0 = \frac{1}{\alpha}\frac{\lambda_C}{2\pi} = 0.529\,\mathrm{angstrom} = 5.29 \times 10^{-9}\,\mathrm{cm}$, and $a_0 k_F$ is

$$
a_0 k_F = \frac{a_0}{\lambda_C}\frac{\hbar k_F}{mc} = \frac{1}{2\pi\alpha}\frac{\hbar k_F}{mc}. \tag{32}
$$

For metallic Fermi systems, $k_{F\mathrm{metal}} \sim 10^8\,\mathrm{cm}^{-1} = \frac{0.529}{a_0} = \frac{0.529\times 2\pi\alpha}{\lambda_C} \sim \frac{2.3\times 10^{-2}}{\lambda_C}$

The vector potential components are $a^\mu = \left(\frac{\phi}{c}, \mathbf{a}\right)$, and the equations for $-a^\mu$ are

$$
\begin{aligned}
-a^\mu &= \left(\frac{1}{G^r_\ell} P^\mu_{\ell\nu} + \frac{1}{G_t} P^\mu_{t\nu}\right) J^\nu \\
&= \frac{1}{c}\left[\frac{1}{c\tilde{G}^r_\ell}\begin{pmatrix} 1 & \mathbf{n}\xi \\ \mathbf{n}\xi & \xi^2\mathbf{L}\end{pmatrix} + \frac{c}{\tilde{G}_t}\begin{pmatrix} 0 & 0 \\ 0 & \mathbf{T}\end{pmatrix}\right]\begin{pmatrix} c\rho \\ -\mathbf{j}\end{pmatrix} \\
-\phi &= \frac{1}{\tilde{G}^r_\ell}\rho \\
-\mathbf{a} &= \frac{1}{\tilde{G}_t}\mathbf{T}\mathbf{j}
\end{aligned}
\tag{33}
$$

4.4. Laurent Expansion

Hydrodynamical limit: $x, y \sim 0$:

$$
\begin{aligned}
G^r_\ell &= \frac{k_F m}{\pi^2}\left(-1 - \frac{\pi}{2} i x + x^2 + \frac{y^2}{12}\right) \\
G^{r-1}_\ell &= \frac{\pi^2}{k_F m}\left(-1 + \frac{\pi}{2} i x - x^2 - \frac{y^2}{12} + \frac{\pi^2}{4} x^2\right)
\end{aligned}
\tag{34}
$$

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$$
\begin{aligned}
G_t^r &= \frac{k_F^3}{6\pi^2 m}\left(3\pi i x - 6x^2 + y^2\right) = \frac{\tilde{G}_t^r}{3\pi i x + y^2} \\
\tilde{G}_t^r &= \frac{k_F^3}{6\pi^2 m}\left[1 + \frac{2}{\pi} i x - \frac{3}{2}x^2 - \frac{1}{2}xy - \left(\frac{1}{8} + \frac{2}{3\pi^2}\right) y^2\right] \\
\tilde{G}_t^{r-1} &= \frac{6\pi^2 m}{k_F^3}\left[1 - \frac{2}{\pi} i x + \left(\frac{3}{2} - \frac{4}{\pi^2}\right) x^2 + \frac{1}{2}xy + \left(\frac{1}{8} + \frac{2}{3\pi^2}\right) y^2\right]
\end{aligned} \tag{35}
$$

$x = \frac{m\omega}{|\mathbf{q}|k_F}, y = \frac{|\mathbf{q}|}{k_F}$

$$
\begin{aligned}
-\frac{mk_F}{\pi^2}\phi &= \left(a_0^\ell + a_x^\ell \frac{m}{k_F}\frac{i\omega}{|\mathbf{q}|} + a_{xx}^\ell \frac{m^2}{k_F^2}\frac{\omega^2}{\mathbf{q}^2} + \frac{a_{yy}^\ell}{k_F^2}\mathbf{q}^2 + \frac{a_{xy}^\ell}{k_F}|\omega||\mathbf{q}|\right) n \\
-\frac{k_F^3}{6\pi^2 m}\frac{\mathbf{a}}{\frac{3\pi m}{k_F}\frac{i\omega}{|\mathbf{q}|} + \frac{\mathbf{q}^2}{k_F^2}} &= \left(a_0^t + a_x^t \frac{m}{k_F}\frac{i\omega}{|\mathbf{q}|} + a_{xx}^t \frac{m^2}{k_F^2}\frac{\omega^2}{\mathbf{q}^2} + \frac{a_{yy}^t}{k_F^2}\mathbf{q}^2 + \frac{a_{xy}^t}{k_F}|\omega||\mathbf{q}|\right) \mathbf{j}^T
\end{aligned} \tag{36}
$$

$$
\partial_t n + \nabla \mathbf{j} = 0, \quad \omega n = \mathbf{k}\mathbf{j} = |\mathbf{k}| j^\ell, \quad \mathbf{j}^\ell = \mathbf{k}\frac{\omega n}{\mathbf{k}^2} \tag{37}
$$

$$
\begin{aligned}
-\frac{mk_F}{\pi^2}\frac{\omega}{|\mathbf{q}|}\phi &= \left(a_0^\ell + a_x^\ell \frac{m}{k_F}\frac{i\omega}{|\mathbf{q}|} + a_{xx}^\ell \frac{m^2}{k_F^2}\frac{\omega^2}{\mathbf{q}^2} + \frac{a_{yy}^\ell}{k_F^2}\mathbf{q}^2 + \frac{a_{xy}^\ell}{k_F}|\omega||\mathbf{q}|\right) j^\ell \\
-\frac{k_F^3}{6\pi^2 m}\frac{\mathbf{a}}{\frac{3\pi m}{k_F}\frac{i\omega}{|\mathbf{q}|} + \frac{\mathbf{q}^2}{k_F^2}} - \frac{mk_F}{\pi^2}\frac{\omega\mathbf{q}}{\mathbf{q}^2}\phi &= \left(a_0^t + a_x^t \frac{m}{k_F}\frac{i\omega}{|\mathbf{q}|} + a_{xx}^t \frac{m^2}{k_F^2}\frac{\omega^2}{\mathbf{q}^2} + \frac{a_{yy}^t}{k_F^2}\mathbf{q}^2 + \frac{a_{xy}^t}{k_F}|\omega||\mathbf{q}|\right) \mathbf{j}
\end{aligned} \tag{38}
$$

4.5. Equations of motion in real space

$$
\begin{aligned}
\partial_t k^{-1} n &= (-b_0^0 + b_{kk}^0 \Delta) n + \bar{\phi}, \\
\partial_t k^{-1} \boldsymbol{j} &= \left(-b_0^T + b_{kk}^T \Delta - \frac{\bar{e}^2}{k^2}\right) \boldsymbol{j} - k^{-1}\boldsymbol{\nabla} r + \bar{\boldsymbol{a}}
\end{aligned} \tag{39}
$$

where

$$
\begin{aligned}
r &= [b_0^0 - b_0^T - (b_{kk}^0 - b_{kk}^T)\Delta][(b_0^0 - b_{kk}^0\Delta)n - \bar{\phi}] \\
&= [(b_0^0 - b_0^T)b_0^0 - [b_0^0(b_{kk}^0 - b_{kk}^T) + (b_0^0 - b_0^T)b_{kk}^0]\Delta n - [b_0^0 - b_0^T - (b_{kk}^0 - b_{kk}^T)\Delta]\bar{\phi}
\end{aligned} \tag{40}
$$

$$
\begin{aligned}
\partial_t n_{\boldsymbol{x}} &= (b_0^0 - b_{kk}^0\Delta)\int d^3y \frac{[(\boldsymbol{y}\boldsymbol{\nabla}_y)^2 - 2\boldsymbol{y}\boldsymbol{\nabla}_y] n_{\boldsymbol{y}}}{2\pi^2(\boldsymbol{x}-\boldsymbol{y})^4} - \int d^3y \frac{[(\boldsymbol{y}\boldsymbol{\nabla}_y)^2 - 2\boldsymbol{y}\boldsymbol{\nabla}_y]\bar{\phi}_{\boldsymbol{y}}}{2\pi^2(\boldsymbol{x}-\boldsymbol{y})^4} \\
\partial_t \boldsymbol{j}_{\boldsymbol{x}} &= -\boldsymbol{\nabla} r + \bar{e}^2 \int d^3y \frac{\boldsymbol{j}_{\boldsymbol{y}}}{2\pi^2(\boldsymbol{x}-\boldsymbol{y})^2} + (b_0^T - b_{kk}^T\Delta)\int d^3y \frac{[(\boldsymbol{y}\boldsymbol{\nabla}_y)^2 - 2\boldsymbol{y}\boldsymbol{\nabla}_y]\boldsymbol{j}_{\boldsymbol{y}}}{2\pi^2(\boldsymbol{x}-\boldsymbol{y})^4} - \int d^3y \frac{[(\boldsymbol{y}\boldsymbol{\nabla}_y)^2 - 2\boldsymbol{y}\boldsymbol{\nabla}_y]\bar{\boldsymbol{a}}_{\boldsymbol{y}}}{2\pi^2(\boldsymbol{x}-\boldsymbol{y})^4}
\end{aligned} \tag{41}
$$

where

$$
r = [[(b_0^0 - b_0^T)b_0^0 - [b_0^0(b_{kk}^0 - b_{kk}^T) + (b_0^0 - b_0^T)b_{kk}^0]\Delta n - [b_0^0 - b_0^T - (b_{kk}^0 - b_{kk}^T)\Delta]\bar{\phi}] \tag{42}
$$

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or

$$\begin{aligned} \int d^3y \frac{\partial_t n_y}{2\pi^2(\boldsymbol{x}-\boldsymbol{y})^2} &= (-b_0^0 + b_{kk}^0\Delta)n + \bar{\phi}, \\ \int d^3y \frac{\partial_t \boldsymbol{j}_y + \boldsymbol{\nabla} r_y}{2\pi^2(\boldsymbol{x}-\boldsymbol{y})^2} &= \left(-b_0^T + b_{kk}^T\Delta\right)\boldsymbol{j} - \bar{e}^2 \int d^3y \frac{\boldsymbol{j}_y}{4\pi|\boldsymbol{x}-\boldsymbol{y}|} + \bar{\boldsymbol{a}} \end{aligned} \tag{43}$$

for configurations with such space-time dependence that $\omega/|\boldsymbol{k}|$ is small

Poiseuille-like flow: Navier-Stokes $\boldsymbol{\nabla} p = \eta\Delta\boldsymbol{v}$

Cylindrical coordinate system: $\boldsymbol{v} = v\boldsymbol{e}_z$, $-\boldsymbol{\nabla} p = f\boldsymbol{e}_z$, $r < R$

$$f = -\eta\frac{1}{r}\partial_r r\partial_r v, \quad v = -\frac{f}{4\mu}r^2 + c_1\ln r + c_2 \to \frac{f}{4\mu}(R^2 - r^2) \tag{44}$$

$$\begin{aligned} \bar{\phi} &= (b_0^0 - b_{kk}^0\Delta)n \\ k^{-1}\boldsymbol{\nabla} r &= \left(-b_0^T + b_{kk}^T\Delta\right)\boldsymbol{j} \end{aligned} \tag{45}$$

4.6. Nonrelativistic hydrodynamics

Convective derivative:

$D_t = \partial_t + \boldsymbol{v}\boldsymbol{\nabla}$

Transport equations: $f = f(t, \boldsymbol{x})$, $\boldsymbol{F} = \boldsymbol{F}(t, \boldsymbol{x})$

$$\begin{aligned} \frac{d}{dt}\int_{V(t)} d^3x f &= \int_{V(t)} d^3x[\partial_t f + \boldsymbol{\nabla}(f\boldsymbol{v})] = \int_{V(t)} d^3x(D_t f + f\boldsymbol{\nabla}\boldsymbol{v}) \\ \frac{d}{dt}\int_{V(t)} d^3x \boldsymbol{F} &= \int_{V(t)} d^3x\partial_t \boldsymbol{F} + (\boldsymbol{v}\boldsymbol{\nabla})\boldsymbol{F} + \boldsymbol{F}(\boldsymbol{\nabla}\boldsymbol{v})] = \int_{V(t)} d^3x[D_t\boldsymbol{F} + \boldsymbol{F}(\boldsymbol{\nabla}\boldsymbol{v})] \end{aligned} \tag{46}$$

Mass: n

$$0 = D_t n + \rho\boldsymbol{\nabla}\boldsymbol{v}, \quad \partial_t n = -\boldsymbol{\nabla}(\rho\boldsymbol{v}) \tag{47}$$

Momentum: $n\boldsymbol{v}$

$$\begin{aligned} \boldsymbol{F} - \boldsymbol{\nabla} p &= D_t n\boldsymbol{v} + n\boldsymbol{v}(\boldsymbol{\nabla}\boldsymbol{v}) = \partial_t(n\boldsymbol{v}) + (\boldsymbol{v}\boldsymbol{\nabla})(n\boldsymbol{v}) + n\boldsymbol{v}(\boldsymbol{\nabla}\boldsymbol{v}) \\ &= -\boldsymbol{v}[\boldsymbol{\nabla}(n\boldsymbol{v})] + n\partial_t\boldsymbol{v} + \boldsymbol{v}(\boldsymbol{v}\boldsymbol{\nabla})n + n(\boldsymbol{v}\boldsymbol{\nabla})\boldsymbol{v} + n\boldsymbol{v}(\boldsymbol{\nabla}\boldsymbol{v}) = nD_t\boldsymbol{v} \end{aligned} \tag{48}$$

Energy: $\frac{1}{2}n\boldsymbol{v}^2$

$$0 = D_t n\boldsymbol{v}^2 + n\boldsymbol{v}^2\boldsymbol{\nabla}\boldsymbol{v} = \partial_t n\boldsymbol{v}^2 + \boldsymbol{\nabla}(n\boldsymbol{v}^2\boldsymbol{v}) \tag{49}$$

Ideal fluids: force across a surface is perpendicular to the surface, $d\boldsymbol{F} = -pd\boldsymbol{\Sigma} \to -\Pi d\boldsymbol{\Sigma}$, $\Pi = p\mathbb{1}$

Viscous fluids: $\Pi^{jk} = p\delta^{jk} - \eta\left(\nabla^j v^k + \nabla^k v^j - \frac{2}{3}\delta^{jk}\boldsymbol{\nabla}\boldsymbol{v}\right) - \xi\delta^{jk}\boldsymbol{\nabla}\boldsymbol{v}$, $\boldsymbol{F} = 0$

$$\begin{aligned} nD_t\boldsymbol{v} &= -\boldsymbol{\nabla}\Pi = -\boldsymbol{\nabla} p + \eta\Delta\boldsymbol{v} + \left(\xi + \frac{\eta}{3}\right)\boldsymbol{\nabla}(\boldsymbol{\nabla}\boldsymbol{v}) \\ \partial_t(n\boldsymbol{v}^2) &= -\boldsymbol{\nabla}(\rho\boldsymbol{v}^2) - v^k\nabla^j\Pi^{jk} = -\boldsymbol{\nabla}(\rho\boldsymbol{v}^2) - \boldsymbol{v}\boldsymbol{\nabla} p + \eta\boldsymbol{v}\Delta\boldsymbol{v} + \left(\xi + \frac{\eta}{3}\right)(\boldsymbol{v}\boldsymbol{\nabla})(\boldsymbol{\nabla}\boldsymbol{v}) \end{aligned} \tag{50}$$

Linearized equation in terms of the current:

$$\begin{aligned}
\partial_t \boldsymbol{j} &= n\partial_t \boldsymbol{v} + \boldsymbol{v}\partial_t n \\
&= -\frac{1}{n}\boldsymbol{j}(\boldsymbol{\nabla}\boldsymbol{j}) - \boldsymbol{\nabla}p + \eta\Delta\frac{\boldsymbol{j}}{n} + \left(\xi + \frac{\eta}{3}\right)\boldsymbol{\nabla}\left(\boldsymbol{\nabla}\frac{\boldsymbol{j}}{n}\right) \\
&= -\frac{1}{n}\boldsymbol{j}(\boldsymbol{\nabla}\boldsymbol{j}) - \boldsymbol{\nabla}p + \eta\boldsymbol{\nabla}\left(\frac{1}{n}\boldsymbol{\nabla}\boldsymbol{j} - \frac{1}{n^2}\boldsymbol{j}\boldsymbol{\nabla}n\right) + \left(\xi + \frac{\eta}{3}\right)\boldsymbol{\nabla}\left(\frac{1}{n}\boldsymbol{\nabla}\boldsymbol{j} - \frac{1}{n^2}\boldsymbol{j}\boldsymbol{\nabla}n\right) \\
&= -\frac{1}{n}\boldsymbol{j}(\boldsymbol{\nabla}\boldsymbol{j}) - \boldsymbol{\nabla}p + \eta\left[\frac{1}{n}\boldsymbol{\nabla}(\boldsymbol{\nabla}\boldsymbol{j}) - \frac{1}{n^2}(\boldsymbol{\nabla}\boldsymbol{j})\boldsymbol{\nabla}n + \frac{2}{n^3}(\boldsymbol{j}\boldsymbol{\nabla})n\boldsymbol{\nabla}n - \frac{1}{n^2}\boldsymbol{\nabla}[(\boldsymbol{j}\boldsymbol{\nabla})]n - \frac{1}{n^2}(\boldsymbol{j}\boldsymbol{\nabla})\boldsymbol{\nabla}n\right] \\
&\quad + \left(\xi + \frac{\eta}{3}\right)\left[\frac{1}{n}\boldsymbol{\nabla}(\boldsymbol{\nabla}\boldsymbol{j}) - \frac{1}{n^2}\boldsymbol{\nabla}n(\boldsymbol{\nabla}\boldsymbol{j}) + \frac{2}{n^3}\boldsymbol{\nabla}n(\boldsymbol{j}\boldsymbol{\nabla})n - \frac{1}{n^2}\boldsymbol{\nabla}[(\boldsymbol{j}\boldsymbol{\nabla})]n - \frac{1}{n^2}(\boldsymbol{j}\boldsymbol{\nabla})\boldsymbol{\nabla}n\right] \qquad (51)
\end{aligned}$$

5. Relativistic Hydrodynamics

5.1. Eckart Frame

In the Eckart frame, the particle flow is represented by the four-vector $j^\mu = (\rho, \vec{j})$ with the constraint $u^2 = 1$ and $\rho = \sqrt{j^2}$.

The conservation equation $\partial_\mu(\rho u^\mu) = 0$ arises, expressing the conservation of particle number.

5.2. Spatial Projector and Enthalpy

The spatial projector $t^{\mu\nu} = g^{\mu\nu} - u^\mu u^\nu$ and the enthalpy $w = \rho + \epsilon$ are introduced.

The energy-momentum tensor $T^{\mu\nu}$ is then expressed in terms of the enthalpy and fluid velocity.

5.3. Conservation Equation

The conservation equation for the particle number $\partial_\nu T^{\nu\mu} = 0$ is derived, where ξ, η, and q are set to zero for the ideal fluid case.

In terms of the fluid variables, this conservation equation is expressed, and the components of the energy-momentum tensor are detailed.

5.4. Ideal Fluid Case

For an ideal fluid ($\xi = \eta = q = 0$), the conservation equation reduces to a simplified form involving the particle density ρ, velocity $\vec{v}$, and enthalpy w.

The energy-momentum tensor is explicitly written in terms of the fluid variables.

5.5. Nonrelativistic Limit

The nonrelativistic limit ($c \neq 1$) is considered, where $j^\mu = (c\rho, \vec{j})$. The conservation equation and energy-momentum tensor components are revisited in this limit, emphasizing the role of particle density ρ, fluid velocity $\vec{v}$, and enthalpy w.

6. Conclusion

In this paper, we have explored the foundations of relativistic hydrodynamics, focusing on the Eckart frame and its implications for conservation equations and energy-momentum tensors. We began by introducing the four-vector representation of particle flow, emphasizing the conservation of particle number in the Eckart frame. The spatial projector and enthalpy were then introduced to express the energy-momentum tensor in a form suitable for relativistic hydrodynamics. We derived the conservation equation for the particle number, accounting for the ideal fluid case with vanishing viscosity, heat conductivity, and dissipation. The ideal fluid case provided a simplified picture of relativistic hydrodynamics, featuring the particle density, fluid velocity, and enthalpy as key variables. The energy-momentum tensor components were explicitly expressed in terms of these fluid variables. Finally, we considered the nonrelativistic limit to connect our results with the familiar expressions from classical fluid dynamics. This allowed us to emphasize the roles of particle density, fluid velocity, and enthalpy in a regime where relativistic effects are negligible. In conclusion, this paper has provided a comprehensive overview of relativistic hydrodynamics, covering foundational concepts and their mathematical expressions. The derived equations offer insights into the behavior of ideal fluids in relativistic regimes and establish connections with classical fluid dynamics in the nonrelativistic limit. Further research can explore applications of these principles in astrophysics, high-energy physics, and cosmology.

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