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Article

Generalization of Convolution

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Abstract

We generalize the concept of convolution to basis sets other than the Fourier basis. The basis set is defined by the eigenfunctions of a self-adjoint operator, which in turn defines the generalized transform. Several special cases are considered, including the scale transform and the chirplet transform, among others. We also generalize the concept of correlation between two functions. Examples are given to illustrate each result.

Keywords: convolution; transform; correlation; scale transform; chirplet transform

1. Introduction

For two functions $f(t)$ and $g(t)$ with respective Fourier transforms $F(\omega)$ and $G(\omega)$, defined by

$$F(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{-i\omega t} dt \tag{1}$$

$$G(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(t) e^{-i\omega t} dt \tag{2}$$

we multiply the two Fourier transforms,

$$H(\omega) = F(\omega)G(\omega) \tag{3}$$

and ask for the time function $h(t)$ whose Fourier transform is $H(\omega)$,

$$h(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(\omega)G(\omega) e^{i\omega t} d\omega \tag{4}$$

By substituting Equations (1) and (2) into Equation (4), one obtains that

$$h(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t-t')g(t')dt' \tag{5}$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t')g(t-t')dt' \tag{6}$$

The function $h(t)$ is called the convolution of the functions $f(t)$ and $g(t)$, often symbolized with the notation $h = f * g$.

It is remarkable that this simple idea has found applications in almost all fields of science, engineering, and mathematics [1–4]. This is the case because it naturally arises for various physical and mathematical reasons. In some sense, the fundamental reason for the importance of convolution is filtering, as one can view the situation that $H(\omega)$ is the filtered version of $F(\omega)$ when it is filtered by $G(\omega)$, and the resulting time function, the filtered time function, is the convolution. We point out that in some mathematical fields, $h(t)$, is called the convolution transform [5]. A related concept is the correlation of two functions, which we discuss in Sec. X.

Our aim in this paper is to generalize the concept of convolution to basis sets other than the Fourier basis.

2. Notation and Mathematical Preliminaries

All integrals go from $-\infty$ to ∞ unless otherwise indicated. Operators are indicated by boldface characters and operate on functions of t ; in cases where they do not, we will denote that by, for example, $\mathbf{A}(t')$ which indicates that $\mathbf{A}$ is operating on functions of t' . We shall be dealing with self-adjoint operators (called Hermitian operators in Physics and Chemistry [6–8]).

For the eigenvalue problem, for the continuous case, we write

$$\mathbf{A} u(\lambda, t) = \lambda u(\lambda, t) \quad (7)$$

where the λ 's are the eigenvalues, and $u(\lambda, t)$ are the eigenfunctions respectively. If the spectrum is discrete, we write

$$\mathbf{A} u_n(t) = \lambda_n u_n(t) \quad (8)$$

where now the eigenvalues are given by the discrete set, λ_n , and the corresponding eigenfunctions by $u_n(t)$. Self-adjoint operators have real eigenvalues, and the eigenfunctions form a complete set.

We normalize the eigenfunctions to a delta function, [8]).

$$\int u(\lambda, t) u^*(\lambda, t') d\lambda = \delta(t - t') \quad (9)$$

$$\int u(\lambda, t) u^*(\lambda', t) dt = \delta(\lambda - \lambda') \quad (10)$$

and for the discrete case the normalization is

$$\int u_n^*(t) u_m(t) dt = \delta_{nm} \quad (11)$$

$$\sum_n u_n^*(t') u_n(t) = \delta(t - t') \quad (12)$$

For the continuous case, an arbitrary function can be expanded as [8])

$$f(t) = \int F(\lambda) u(\lambda, t) d\lambda \quad (13)$$

where

$$F(\lambda) = \int f(t) u^*(\lambda, t) dt \quad (14)$$

The function $F(\lambda)$ is called the *transform* of $f(t)$ in the basis set generated by the operator $\mathbf{A}$. For the case of a discrete basis set, $u_n(t)$, indexed by n , we write

$$f(t) = \sum_n c_n u_n(t) \quad (15)$$

where $\{c_n\}$ are the expansion coefficients given by

$$c_n = \int f(t) u_n^*(t) dt \quad (16)$$

Functions of operators. In Appendix A, we prove the following statements regarding the operation of functions of operators. For a function of an operator, $Q(\mathbf{A})$, the operation on an eigenfunction is given by

$$Q(\mathbf{A}) u(\lambda, t) = Q(\lambda) u(\lambda, t) \quad (17)$$

Its action on an arbitrary function $z(t)$, is

$$Q(\mathbf{A}) z(t) = \int q(t', t) z(t') dt' \quad (18)$$

where

$$q(t', t) = \int u^*(\lambda, t') Q(\lambda) u(\lambda, t) d\lambda \quad (19)$$

For the discrete case,

$$q(t', t) = \sum_n u_n^*(t') Q(\lambda_n) u_n(t) \quad (20)$$

Consider the operator $u(\mathbf{A}(t'), t)$, meaning that for an eigenfunction, substitute the operator $\mathbf{A}(t')$ for the eigenvalue λ . Operating on a function $g(t')$, we have

$$u(\mathbf{A}(t'), t) g(t') = \int g(t'') \eta(t, t', t'') dt'' \quad (21)$$

where

$$\eta(t, t', t'') = \int u(\lambda, t) u^*(\lambda, t') u^*(\lambda, t'') d\lambda \quad (22)$$

Explicitly,

$$u(\mathbf{A}(t'), t) g(t') = \iint g(t'') u(\lambda, t) u^*(\lambda, t') u^*(\lambda, t'') d\lambda dt'' \quad (23)$$

3. Generalized Convolution

For two functions $f(t)$ and $g(t)$ with respective transforms

$$F(\lambda) = \int f(t) u^*(\lambda, t) dt \quad (24)$$

$$G(\lambda) = \int g(t) u^*(\lambda, t) dt \quad (25)$$

we form

$$H(\lambda) = F(\lambda) G(\lambda) \quad (26)$$

and ask for the function $h(t)$ whose transform is $H(\lambda)$

$$h(t) = \int H(\lambda) u(\lambda, t) d\lambda \quad (27)$$

$$= \int F(\lambda) G(\lambda) u(\lambda, t) d\lambda \quad (28)$$

We call $h(t)$ the generalized convolution.

We now give two expressions for calculating $h(t)$. These are derived in Appendix B.

Expression 1:

$$h(t) = \iint f(t') g(t'') \eta(t, t', t'') dt' dt'' \quad (29)$$

where $\eta(t, t', t'')$ is given by

$$\eta(t, t', t'') = \int u(\lambda, t) u^*(\lambda, t') u^*(\lambda, t'') d\lambda \quad (30)$$

Explicitly,

$$h(t) = \iiint f(t') g(t'') u(\lambda, t) u^*(\lambda, t') u^*(\lambda, t'') dt' dt'' d\lambda \quad (31)$$

Expression 2:

$$h(t) = F(\mathbf{A}) g(t) \quad (32)$$

The meaning of $F(\mathbf{A})$ is that in the expression for the transform, Equation (24), which we repeat here

$$F(\lambda) = \int f(t) u^*(\lambda, t) dt \quad (33)$$

we substitute the operator $\mathbf{A}$ for the eigenvalue λ . In particular

$$F(\mathbf{A}) = \int f(t') u^*(\mathbf{A}(t), t') dt' \quad (34)$$

Then

$$h(t) = F(\mathbf{A})g(t) = \int f(t') u^*(\mathbf{A}(t), t') g(t) dt' \quad (35)$$

Similarly

$$h(t) = G(\mathbf{A})f(t) = \int g(t') u^*(\mathbf{A}(t), t') f(t) dt' \quad (36)$$

4. Fourier Case

This example is the standard case, but we use the methods presented above to illustrate the general procedure. The frequency operator $\mathbf{W}$ is

$$\mathbf{W} = \frac{1}{i} \frac{d}{dt} \quad (37)$$

and the eigenvalue problem

$$\mathbf{W} u(\omega, t) = \omega u(\omega, t) \quad (38)$$

gives

$$u(\omega, t) = \frac{1}{\sqrt{2\pi}} e^{i\omega t} \quad (39)$$

for the eigenfunctions, which are normalized to a delta function. The transform and its inverse are given by Equations (1) and (2). We now calculate the convolution using the two methods described by Equations (29) and (34).

Using **Expression 1** as given by Equation (29), we first calculate $\eta(t, t', t'')$ as given by Equation (30)

$$\eta(t, t', t'') = \frac{1}{(2\pi)^{3/2}} \int e^{i\omega t} e^{-i\omega t'} e^{-i\omega t''} d\omega \quad (40)$$

$$= \frac{1}{\sqrt{2\pi}} \delta(t - t' - t'') \quad (41)$$

Substituting into Equation (29) we have

$$h(t) = \int f(t') g(t'') \eta(t, t', t'') dt' dt'' \quad (42)$$

$$= \frac{1}{\sqrt{2\pi}} \int f(t') g(t'') \delta(t - t' - t'') dt' dt'' \quad (43)$$

which gives

$$h(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t') g(t - t') dt' \quad (44)$$

which is the expected result, Equation (6).

Using **Expression 2**, Equation (32),

$$h(t) = F(\mathbf{W})g(t) \quad (45)$$

For $F(\omega)$, we have

$$F(\omega) = \frac{1}{\sqrt{2\pi}} \int f(t) e^{-i\omega t} dt \quad (46)$$

and therefore

$$F(\mathbf{W}) = \frac{1}{\sqrt{2\pi}} \int f(t) e^{-i\mathbf{W}t} dt \quad (47)$$

Using Equation (29) we have

$$h(t) = \frac{1}{\sqrt{2\pi}} \int f(t') e^{-it' \mathbf{W}} g(t) dt' \quad (48)$$

$$= \frac{1}{\sqrt{2\pi}} \int f(t') e^{-t' \frac{d}{dt}} g(t) dt' \quad (49)$$

Remembering that, $e^{-t' \frac{d}{dt}}$ translates functions, [9,10]

$$e^{-t' \frac{d}{dt}} g(t) = g(t - t') \quad (50)$$

we obtain

$$h(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t') g(t - t') dt' \quad (51)$$

which is the same as Equation (44).

5. Scale Transform

The scale or dilation operator $\mathbf{C}$ is [10–12]

$$\mathbf{C} = \frac{1}{2i} \left(t \frac{d}{dt} + \frac{d}{dt} t \right) \quad (52)$$

and has the following properties,

$$e^{i\tau \mathbf{C}} f(t) = e^{\tau/2} f(e^{\tau} t) \quad (53)$$

$$e^{i \ln \tau \mathbf{C}} f(t) = \sqrt{\tau} f(\tau t) \quad (54)$$

Also,

$$e^{-i \ln \tau \mathbf{C}} f(t) = \frac{1}{\sqrt{\tau}} f(t/\tau) \quad (55)$$

The eigenvalue problem

$$\mathbf{C} u(\lambda, t) = \lambda u(\lambda, t) \quad (56)$$

gives

$$u(\lambda, t) = \frac{1}{\sqrt{2\pi}} \frac{e^{i\lambda \ln t}}{\sqrt{t}} \quad , \quad t \geq 0 \quad (57)$$

for the eigenfunctions [11]. The scale transform, $D(\lambda)$, of a function $f(t)$ is

$$D(\lambda) = \frac{1}{\sqrt{2\pi}} \int_0^{\infty} f(t) \frac{e^{-i\lambda \ln t}}{\sqrt{t}} dt \quad (58)$$

with inverse

$$f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} D(\lambda) \frac{e^{i\lambda \ln t}}{\sqrt{t}} d\lambda \quad ; \quad t \geq 0 \quad (59)$$

We now obtain the convolution theorem for scale. Using **Expression 1**, we first calculate $\eta(t, t', t'')$ by way of Equation (30)

$$\eta(t, t', t'') = \frac{1}{(\sqrt{2\pi})^3} \int_{-\infty}^{\infty} \frac{1}{\sqrt{t t' t''}} e^{i\lambda (\ln t - \ln t' - \ln t'')} d\lambda \quad (60)$$

$$= \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{t t' t''}} \delta(\ln t - \ln t' - \ln t'') \quad (61)$$

and therefore Equation (29) yields

$$h(t) = \int_{-\infty}^{\infty} f(t') g(t'') \eta(t, t', t'') dt' dt'' \quad (62)$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{1}{\sqrt{tt't''}} f(t') g(t'') \delta(\ln t - \ln t' - \ln t'') dt' dt'' \quad (63)$$

Making the transformation

$$\tau = \ln t'' \quad (64)$$

and subsequently integrating over τ results in

$$h(t) = \frac{1}{\sqrt{2\pi}} \int_0^{\infty} \frac{1}{t'} f(t') g(t/t') dt' \quad (65)$$

which is also equal to

$$h(t) = \frac{1}{\sqrt{2\pi}} \int_0^{\infty} \frac{1}{t'} f(t/t') g(t') dt' \quad (66)$$

This is the convolution theorem for the scale transform.

Using **Expression 2**, Equation (35), we have

$$h(t) = D(\mathbf{C})g(t) = \int f(t') u^*(\mathbf{C}(t), t') dt' g(t) \quad (67)$$

Substituting for $u^*(\mathbf{C}(t), t')$ as per in Equation (57) we obtain

$$h(t) = D(\mathbf{C})g(t) = \frac{1}{\sqrt{2\pi}} \int_0^{\infty} f(t') \frac{e^{-i\mathbf{C} \ln t'}}{\sqrt{t'}} g(t) dt' \quad (68)$$

and using Equation (55) we obtain

$$h(t) = \frac{1}{\sqrt{2\pi}} \int_0^{\infty} \frac{1}{t'} f(t') g(t/t') dt' \quad (69)$$

which is Equation (65),

6. Chirplet Transform

The operator

$$\mathbf{A} = \alpha t - i\beta \frac{d}{dt} \quad (70)$$

is self adjoint for real α and β . This operator and its associated eigenfunctions are related to coherent states in quantum mechanics [13–15] and to the chirplet transform in signal processing [16–18]. Solving the eigenvalue problem

$$\left(\alpha t - i\beta \frac{d}{dt} \right) = \lambda u(\lambda, t) \quad (71)$$

gives

$$u(\lambda, t) = \frac{1}{\sqrt{2\pi\beta}} e^{i(\lambda t - \alpha t^2/2)/\beta} \quad (72)$$

where we have normalized to a delta function. Hence, we have the following transform pairs

$$F(\lambda) = \frac{1}{\sqrt{2\pi\beta}} \int f(t) e^{-i(\lambda t - \alpha t^2/2)/\beta} dt \quad (73)$$

$$f(t) = \frac{1}{\sqrt{2\pi\beta}} \int F(\lambda) e^{i(\lambda t - \alpha t^2/2)/\beta} d\lambda \quad (74)$$

Expression 1, Calculating $\eta(t, t', t'')$ as per Equation (30) we have

$$\eta(t, t', t'') = \frac{1}{(\sqrt{2\pi\beta})^3} \int e^{i\lambda(t-t'-t'')/\beta} e^{-i\alpha(t^2-t'^2-t''^2)/(2\beta)} d\lambda \tag{75}$$

which simplifies to

$$\eta(t, t', t'') = \frac{1}{\sqrt{2\pi\beta}} \delta(t - t' - t'') e^{i\alpha t'(t'-t)/\beta} \tag{76}$$

Calculating $h(t)$ as per Equation (29) we have

$$h(t) = \frac{1}{\sqrt{2\pi\beta}} \int f(t') g(t'') \delta(t - t' - t'') e^{i\alpha(t'^2-tt')/\beta} dt' dt'' \tag{77}$$

resulting in

$$h(t) = \frac{1}{\sqrt{2\pi\beta}} \int f(t') g(t - t') e^{i\alpha t'(t'-t)/\beta} dt' \tag{78}$$

Expression 2. Using Equation (32), we have

$$h(t) = F(\mathbf{A})g(t) \tag{79}$$

From Equation (74) we have that

$$F(\mathbf{A}) = \frac{1}{\sqrt{2\pi\beta}} \int f(t') e^{-i(\mathbf{A}t' - \alpha t'^2/2)/\beta} dt' \tag{80}$$

and substituting into Equation (35) we obtain

$$F(\mathbf{A}) g(t) = \frac{1}{\sqrt{2\pi\beta}} \int f(t') e^{-i[(\alpha t - i\beta \frac{d}{dt})t' - \alpha t'^2/2]/\beta} g(t) dt' \tag{81}$$

Using the Baker–Campbell–Hausdorff theorem , we have [9]

$$e^{-i(\alpha t - i\beta \frac{d}{dt})t'/\beta} = e^{-i\alpha t t'/\beta - \frac{d}{dt}t'} = e^{-i\alpha t t'/\beta} e^{i\alpha t'^2/2\beta} e^{-t' \frac{d}{dt}} \tag{82}$$

Therefore

$$e^{-i(\alpha t - i\beta \frac{d}{dt})t'/\beta} g(t) = e^{-i\alpha t t'/\beta} e^{-i\alpha t'^2/2\beta} g(t - t') \tag{83}$$

Calculating $h(t)$ by way of Equation (35) we have

$$h(t) = \frac{1}{\sqrt{2\pi\beta}} \int f(t') e^{i[\alpha t'^2/2]/\beta} e^{-i\alpha t t'/\beta} e^{i\alpha t'^2/2\beta} g(t - t') dt' \tag{84}$$

$$= \frac{1}{\sqrt{2\pi\beta}} \int f(t') g(t - t') e^{i\alpha t'(t'-t)/\beta} dt' \tag{85}$$

which is the same as Equation (78)

7. The operator $\mathbf{A} = \alpha t^2 - \frac{i}{c} \frac{d}{dt}$

The operator

$$\mathbf{A} = \alpha t^2 - \frac{i}{c} \frac{d}{dt} \tag{86}$$

is self adjoint for real α and c . We take c to be positive. This operator is connected with the Airy function. The eigenvalue problem

$$\left(\alpha t^2 - \frac{i}{c} \frac{d}{dt} \right) u(\lambda, t) = \lambda u(\lambda, t) \tag{87}$$

gives eigenfunctions,

$$u(\lambda, t) = \sqrt{\frac{c}{2\pi}} e^{ic(\lambda t - \alpha t^3/3)} \quad (88)$$

The transform is

$$F(\lambda) = \sqrt{\frac{c}{2\pi}} \int_{-\infty}^{\infty} f(t) e^{-ic(\lambda t - \alpha t^3/3)} dt \quad (89)$$

where

$$f(t) = \sqrt{\frac{c}{2\pi}} \int_{-\infty}^{\infty} F(\lambda) e^{ic(\lambda t - \alpha t^3/3)} d\lambda \quad (90)$$

To obtain the generalized convolution, we first use **Expression 1**, and we first calculate

$$\eta(t, t', t'') = \int u(\lambda, t) u^*(\lambda, t') u^*(\lambda, t'') d\lambda \quad (91)$$

Substituting Equation (88) into Equation (91) results in

$$\eta(t, t', t'') = \sqrt{\frac{c}{2\pi}} e^{ic\alpha t t' (t-t')} \delta(t - t' - t'') \quad (92)$$

Therefore,

$$h(t) = \int f(t') g(t'') \eta(t, t', t'') dt' dt''$$

which evaluates to

$$h(t) = \sqrt{\frac{c}{2\pi}} \int f(t') g(t'') e^{-ic\alpha t t' (t-t')} \delta(t - t' - t'') dt' dt'' \quad (93)$$

and hence

$$h(t) = \sqrt{\frac{c}{2\pi}} \int f(t') g(t - t') e^{-ic\alpha t t' (t-t')} dt' \quad (94)$$

To use **Expression 2**, we have

$$F(\lambda) = \sqrt{\frac{c}{2\pi}} \int_{-\infty}^{\infty} f(t) e^{-ic(\lambda t - \alpha t^3/3)} dt \quad (95)$$

and therefore

$$F(\mathbf{A}) = \sqrt{\frac{c}{2\pi}} \int_{-\infty}^{\infty} f(t') e^{-ic(\mathbf{A}t' - \alpha t'^3/3)} dt' \quad (96)$$

$$= \sqrt{\frac{c}{2\pi}} \int_{-\infty}^{\infty} f(t') e^{+ic\alpha t'^3/3} e^{-ict'(\alpha t'^2 - \frac{i}{c} \frac{d}{dt})} dt' \quad (97)$$

the convolution is hence

$$h(t) = F(\mathbf{A}) g(t) = \sqrt{\frac{c}{2\pi}} \int_{-\infty}^{\infty} f(t') e^{+ic\alpha t'^3/3} e^{-ict'(\alpha t'^2 - \frac{i}{c} \frac{d}{dt})} g(t) dt' \quad (98)$$

A variety of methods may be used to establish that

$$e^{-ict'(\alpha t'^2 - \frac{i}{c} \frac{d}{dt})} g(t) = \exp\left[-ic\alpha\left(t^2 t' - t t'^2 + \frac{t'^3}{3}\right)\right] g(t - t') \quad (99)$$

Upon substitution into Equation (98) we have

$$h(t) = F(\mathbf{A}) g(t) = \sqrt{\frac{c}{2\pi}} \int_{-\infty}^{\infty} f(t') e^{+ic\alpha t'^3/3} \exp\left[-ic\alpha\left(t^2 t' - t t'^2 + \frac{t'^3}{3}\right)\right] g(t - t') dt' \quad (100)$$

giving

$$= \sqrt{\frac{c}{2\pi}} \int_{-\infty}^{\infty} f(t') \exp[-ic\alpha t'(t-t')] g(t-t') dt' \quad (101)$$

which is the same as Equation (94)

8. Generalized Correlation

For the Fourier case, correlation is where one takes [1–4]

$$H_*(\omega) = F^*(\omega)G(\omega) \quad (102)$$

and ask for the function $h_+(t)$ such that

$$h_*(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F^*(\omega)G(\omega)e^{i\omega t} d\omega \quad (103)$$

This arises in a number of signal processing applications, particularly in detection problems [19]. By substituting Equations (1) and (2), into Equation (103) one obtains that

$$h_*(t) = \frac{1}{\sqrt{2\pi}} \int f^*(t') g(t'+t) dt' \quad (104)$$

This is called the correlation between f and g .

We now generalize and form,

$$H_*(\omega) = F^*(\lambda)G(\lambda) \quad (105)$$

and ask for the function $h_*(t)$ that corresponds to $H_*(\lambda)$,

$$h_*(t) = \int H_*(\lambda) u(\lambda, t) d\lambda \quad (106)$$

$$= \int F^*(\lambda)G(\lambda) u(\lambda, t) d\lambda \quad (107)$$

The same type of derivation used for convolution gives

$$h_*(t) = \int \int f^*(t') g(t'') u(\lambda, t) u(\lambda, t') u^*(\lambda, t'') d\lambda dt' dt'' \quad (108)$$

which we write as

$$h_*(t) = \int f^*(t') g(t'') \eta_*(t, t', t'') dt' dt'' \quad (109)$$

where

$$\eta_*(t, t', t'') = \int u(\lambda, t) u(\lambda, t') u^*(\lambda, t'') d\lambda \quad (110)$$

8.1. Example: Fourier case

Calculating $\eta_*(t, t', t'')$ as per Equation (110) and using $\lambda = \omega$, we have

$$\eta_*(t, t', t'') = \int u(\omega, t) u(\omega, t') u^*(\omega, t'') d\omega \quad (111)$$

$$= \frac{1}{\sqrt{2\pi}} \delta(t + t' - t'') \quad (112)$$

Substituting into Equation (109), we obtain that

$$h_*(t) = \int f^*(t') g(t'') \eta(t, t', t'') dt' dt'' \quad (113)$$

$$= \frac{1}{\sqrt{2\pi}} \int f^*(t') g(t'') \delta(t + t' - t'') dt' dt'' \quad (114)$$

which gives, as expected, that

$$h_*(t) = \frac{1}{\sqrt{2\pi}} \int f^*(t' - t) g(t') dt' \quad (115)$$

8.2. Example: Scale

Using Equation (110) with Equation (57) we have

$$\eta_*(t, t', t'') = \frac{1}{(\sqrt{2\pi})^3} \int \frac{1}{\sqrt{tt't''}} e^{i\lambda(\ln t + \ln t' - \ln t'')} d\lambda \quad (116)$$

$$= \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{tt't''}} \delta(\ln t + \ln t' - \ln t'') \quad (117)$$

Substituting into Equation (109) we obtain

$$h_*(t) = \frac{1}{\sqrt{2\pi}} \int f^*(t') g(t'') \frac{1}{\sqrt{tt't''}} \delta(\ln t + \ln t' - \ln t'') dt' dt'' \quad (118)$$

$$= \frac{1}{\sqrt{2\pi}} \int f^*(t') g(t'') \frac{1}{\sqrt{tt't''}} \delta(\ln tt' / t'') dt' dt'' \quad (119)$$

which evaluates to

$$h_*(t) = \frac{1}{\sqrt{2\pi}} \int f^*(t') g(tt') dt' \quad (120)$$

Appendix A

We prove Equations (18)–(21) of the text. Starting with the eigenvalue equation, for the continuous case,

$$\mathbf{A} u(\lambda, t) = \lambda u(\lambda, t) \quad (\text{A1})$$

where the λ 's are the eigenvalues, and $u(\lambda, t)$ are the eigenfunctions, respectively. For a function of an operator, $Q(\mathbf{A})$, the operation on an eigenfunction is given by

$$Q(\mathbf{A}(t)) u(\lambda, t) = Q(\lambda) u(\lambda, t) \quad (\text{A2})$$

For the operation on an arbitrary function $z(t)$, we expand

$$z(t) = \int Z(\lambda) u(\lambda, t) d\lambda \quad (\text{A3})$$

where

$$Z(\lambda) = \int z(t) u^*(\lambda, t) dt \quad (\text{A4})$$

Operating on Equation (A3) with $Q(\mathbf{A})$ and remembering that $\mathbf{A}$ operates on functions of t we obtain

$$Q(\mathbf{A}) z(t) = \int Z(\lambda) Q(\mathbf{A}) u(\lambda, t) d\lambda \quad (\text{A5})$$

Substituting for $Z(\lambda)$ as given by Equation (A4) we have

$$Q(\mathbf{A}) z(t) = \iint z(t') u^*(\lambda, t') Q(\mathbf{A}) u(\lambda, t) dt' d\lambda \quad (\text{A6})$$

which we write as

$$Q(\mathbf{A}) z(t) = \int q(t', t) z(t') dt' \quad (\text{A7})$$

with

$$q(t', t) = \int u^*(\lambda, t') Q(\mathbf{A}) u(\lambda, t) dt' \quad (\text{A8})$$

$$= \int u^*(\lambda, t') Q(\lambda) u(\lambda, t) dt' \quad (\text{A9})$$

which is Equation (29) of the text.

For the discrete case where

$$\mathbf{A} u_n(t) = \lambda_n u_n(t) \quad (\text{A10})$$

we have

$$q(t', t) = \sum_n u_n^*(t') Q(\lambda_n) u_n(t) \quad (\text{A11})$$

To derive Equation (21) and (22) consider the operator $u(\mathbf{A}(t'), t)$, which means that for an eigenfunction we substitute the operator $\mathbf{A}(t')$ for the eigenvalue λ . We use

$$g(t) = \int G(\lambda) u(\lambda, t) d\lambda \quad (\text{A12})$$

where

$$G(\lambda) = \int g(t) u^*(\lambda, t) dt \quad (\text{A13})$$

Operating with $u(\mathbf{A}(t'), t)$ on a function $g(t')$

$$u(\mathbf{A}(t'), t) g(t') = u(\mathbf{A}(t'), t) \int G(\lambda) u(\lambda, t') d\lambda \quad (\text{A14})$$

$$= \int G(\lambda) u(\mathbf{A}(t'), t) u(\lambda, t') d\lambda \quad (\text{A15})$$

$$= \int G(\lambda) u(\lambda, t) u(\lambda, t') d\lambda \quad (\text{A16})$$

Substituting for $G(\lambda)$, as per (A13) we obtain

$$u(\mathbf{A}(t'), t) g(t') = \int g(t'') u(\lambda, t) u^*(\lambda, t') u^*(\lambda, t'') dt'' d\lambda \quad (\text{A17})$$

which we write as

$$u(\mathbf{A}(t'), t) g(t') = \int g(t'') \eta(t, t', t'') dt'' \quad (\text{A18})$$

where

$$\eta(t, t', t'') = \int u(\lambda, t) u^*(\lambda, t') u^*(\lambda, t'') d\lambda \quad (\text{A19})$$

which is Equation (22).

Appendix B

We show Equation (29) and (29) of the text. Starting with

$$h(t) = \int F(\lambda) G(\lambda) u(\lambda, t) d\lambda \quad (\text{A20})$$

and using

$$F(\lambda) = \int f(t') u^*(\lambda, t') dt' \quad (\text{A21})$$

$$G(\lambda) = \int g(t'') u^*(\lambda, t'') dt'' \quad (\text{A22})$$

and substituting into Equation (A20), we have

$$h(t) = \iiint f(t') g(t'') u(\lambda, t) u^*(\lambda, t') u^*(\lambda, t'') dt' dt'' d\lambda \quad (\text{A23})$$

which is Equation (31) of the text.

To derive Equation (32), consider

$$h(t) = \int F(\lambda) G(\lambda) u(\lambda, t) d\lambda \quad (\text{A24})$$

$$= F(\mathbf{A}) \int G(\lambda) u(\lambda, t) d\lambda \quad (\text{A25})$$

giving

$$h(t) = F(\mathbf{A}) g(t) \quad (\text{A26})$$

which is Equation (32). In going from Equation (A24) to Equation (A25), we have used that

$$F(\lambda) u(\lambda, t) = F(\mathbf{A}) u(\lambda, t) \quad (\text{A27})$$

References

1. R. J. Marks II, *Handbook of Fourier Analysis & Its Applications*. Oxford Univ. Press, 2009.
2. A. Papoulis, *Signal Analysis*, McGraw-Hill, 1977.
3. L. Franks, *Signal Theory*, Prentice-Hall, Inc. Englewood Cliffs, NJ, 1969.
4. A. D. Poularikas, *Transforms; Applications Handbook* - 2nd Edition, CRC press, Boca Raton, FL, 2010.
5. I. I. Hirschman and D. V. Widder, *The convolution transform*, Dover, 2005.
6. D. Bohm, *Quantum Theory*, Prentice-Hall, New York, 1951.
7. D. Merzbacher, *Quantum Mechanics*, John Wiley & Sons, Inc., 1998.
8. G. B. Arfken, H. J. Weber, and F. E. Harris, *Mathematical Methods for Physicists: A Comprehensive Guide* (7th ed.). Academic Press., 2012
9. R. M. Wilcox, Exponential Operators and Parameter Differentiation in Quantum Physics, *J. Math. Phys.* 8, 962, 1967.
10. L. Cohen, *The Weyl Operator and its Generalization*, Birkhäuser Basel, New York. Springer, 2013.
11. L. Cohen, The scale representation, *IEEE Trans. on Signal Processing*, 41, 3275–3293, 1993.
12. J. Bertrand and P. Bertrand, Time-Frequency Representations of Broad Band Signals in: *The Physics of Phase Space* edited by Y.S. Kim and W.W. Zachary, Springer Verlag, pp. 250-252, 1987.
13. W.P. Schleich, *Quantum Optics in Phase Space*, Wiley, 2001.
14. M. O. Scully and M. S. Zubairy, *Quantum Optics*, Cambridge University Press, 1997.
15. J. R. Klauder and B. S. Skagerstam, *Coherent States*, World Scientific, 1985.
16. R.G. Baraniuk and D.L. Jones, New Signal-Space Orthonormal Bases Via the Metaplectic Transform. In *Proceedings of the IEEE-SP International Symposium on Time-Frequency and Time-Scale Analysis*, Victoria, BC, Canada, 4–6 October 1992.
17. R.G. Baraniuk and D. L. Jones Wigner-Based Formulation of the Chirplet Transform. *IEEE Trans. Signal Process.* 44, 3129–3135, 1996
18. S. Mann and S. Haykin, The chirplet transform: A generalization of Gabor's logon transform, *IEEE Transactions on Signal Processing* 43, 295–300, 1995.
19. D. Middleton, *An Introduction to Statistical Communication Theory*, McGraw-Hill, 1960.

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