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Apoloniusz Tyszka *

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*Article*

# A Short Program in *MuPAD* That Computes in the Limit a Function $f : \mathbb{N} \to \mathbb{N}$ Which Eventually Dominates Every Computable Function $g : \mathbb{N} \to \mathbb{N}$

**Apoloniusz Tyszka**

Hugo Kołłątaj University, Balicka 116B, 30-149 Kraków, Poland; rttyszka@cyf-kr.edu.pl

**Abstract**

It is known that there exists a limit-computable function $f : \mathbb{N} \to \mathbb{N}$ which is not computable. Every known proof of this fact does not lead to the existence of a short computer program that computes $f$ in the limit. For $n \in \mathbb{N}$, let $E_n = \{1 = x_k,\ x_i + x_j = x_k,\ x_i \cdot x_j = x_k :\ i, j, k \in \{0, \ldots, n\}\}$. For $n \in \mathbb{N}$, $f(n)$ denotes the smallest $b \in \mathbb{N}$ such that if a system of equations $\mathcal{S} \subseteq E_n$ has a solution in $\mathbb{N}^{n+1}$, then $\mathcal{S}$ has a solution in $\{0, \ldots, b\}^{n+1}$. The author proved earlier that the function $f : \mathbb{N} \to \mathbb{N}$ is computable in the limit and eventually dominates every computable function $g : \mathbb{N} \to \mathbb{N}$. We present a short program in *MuPAD* which for $n \in \mathbb{N}$ prints the sequence $\{f_i(n)\}_{i=0}^{\infty}$ of non-negative integers converging to $f(n)$. For $n \in \mathbb{N}$, $\beta(n)$ denotes the smallest $b \in \mathbb{N}$ such that if a system of equations $\mathcal{S} \subseteq E_n$ has a unique solution in $\mathbb{N}^{n+1}$, then this solution belongs to $\{0, \ldots, b\}^{n+1}$. The author proved earlier that the function $\beta : \mathbb{N} \to \mathbb{N}$ is computable in the limit and eventually dominates every function $\delta : \mathbb{N} \to \mathbb{N}$ with a single-fold Diophantine representation. The computability of $\beta$ is unknown. We present a short program in *MuPAD* which for $n \in \mathbb{N}$ prints the sequence $\{\beta_i(n)\}_{i=0}^{\infty}$ of non-negative integers converging to $\beta(n)$.

**Keywords:** computable function; eventual domination; limit-computable function; non-computable function; single-fold Diophantine representation

**MSC:** 03D20

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## 1. Introduction

**Definition 1.** *(cf. [9][pp. 233–235]). A computation in the limit of a function $f : \mathbb{N} \to \mathbb{N}$ is a semi-algorithm which takes as input a non-negative integer $n$ and for every $m \in \mathbb{N}$ prints a non-negative integer $\xi(n, m)$ such that $\lim_{m \to \infty} \xi(n, m) = f(n)$.*

It is known that there exists a limit-computable function $f : \mathbb{N} \to \mathbb{N}$ which is not computable, see Theorem 1. Every known proof of this fact does not lead to the existence of a short computer program that computes $f$ in the limit. In particular, this observation applies to the proof of Theorem 1 in [8], see also Observation 1.

**Observation 1.** *Let $\varphi$ be a computable bijection from $\mathbb{N}$ to the set of all Diophantine equations. For $n \in \mathbb{N}$, let*

$$\theta(n) = \begin{cases} 1, & \text{if } \varphi(n) \text{ is solvable in non-negative integers} \\ 0, & \text{otherwise} \end{cases}$$

*The function $\theta : \mathbb{N} \to \mathbb{N}$ is computable in the limit. A negative solution to Hilbert's 10th problem implies that the function $\theta$ is not computable. There is no known $\varphi$ for which there exists a short computer program that computes $\theta$ in the limit.*

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*MuPAD* is a part of the Symbolic Math Toolbox in MATLAB R2019b. In this article, we present a short program in *MuPAD* that computes in the limit a function $f : \mathbb{N} \to \mathbb{N}$ which eventually dominates every computable function $g : \mathbb{N} \to \mathbb{N}$.

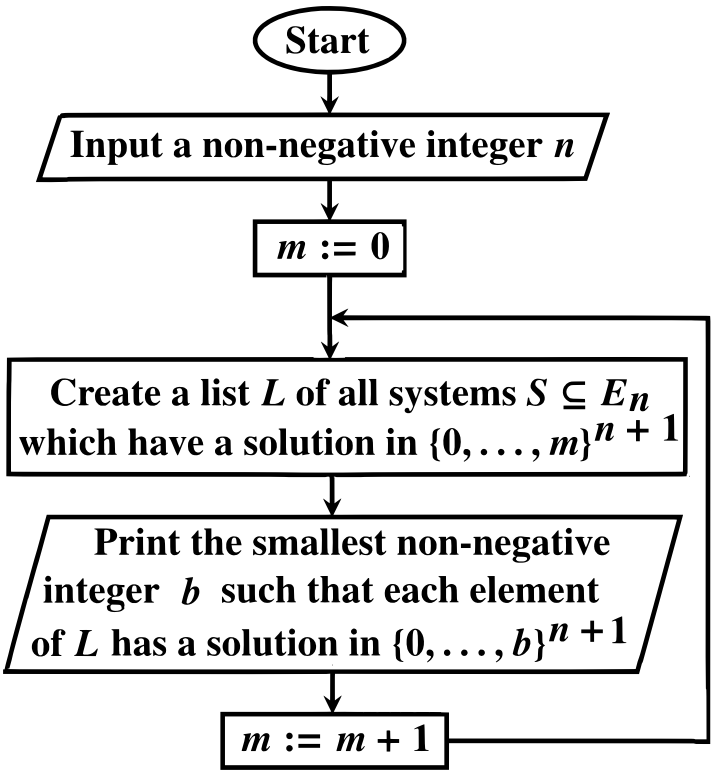
## 2. A Limit-Computable Function $f : \mathbb{N} \to \mathbb{N}$ Which Eventually Dominates Every Computable Function $g : \mathbb{N} \to \mathbb{N}$

For $n \in \mathbb{N}$, let

$$E_n = \{1 = x_k,\ x_i + x_j = x_k,\ x_i \cdot x_j = x_k :\ i, j, k \in \{0, \ldots, n\}\}$$

**Theorem 1.** *([8, p. 118]). There exists a limit-computable function $f : \mathbb{N} \to \mathbb{N}$ which eventually dominates every computable function $g : \mathbb{N} \to \mathbb{N}$.*

We present an alternative proof of Theorem 1. For $n \in \mathbb{N}$, $f(n)$ denotes the smallest $b \in \mathbb{N}$ such that if a system of equations $\mathcal{S} \subseteq E_n$ has a solution in $\mathbb{N}^{n+1}$, then $\mathcal{S}$ has a solution in $\{0, \ldots, b\}^{n+1}$. The function $f : \mathbb{N} \to \mathbb{N}$ is computable in the limit and eventually dominates every computable function $g : \mathbb{N} \to \mathbb{N}$, see [11]. The term *"dominated"* in the title of [11] means *"eventually dominated"*. Flowchart 1 shows a semi-algorithm which computes $f(n)$ in the limit, see [11].




**Flowchart 1**

A semi-algorithm which computes $f(n)$ in the limit

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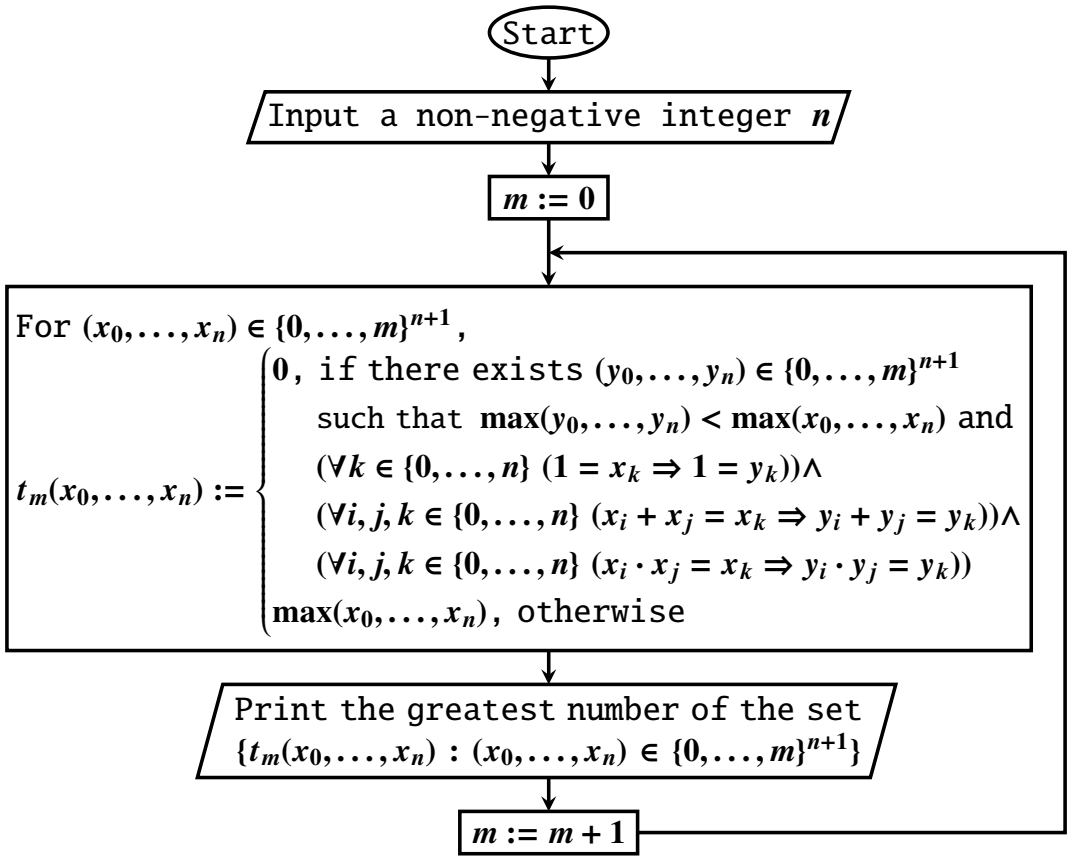
## 3. A Short Program in *MuPAD* That Computes $f$ in the Limit

Flowchart 2 shows a simpler semi-algorithm which computes $f(n)$ in the limit.




**Flowchart 2**

A simpler semi-algorithm which computes $f(n)$ in the limit

**Lemma 1.** *For every $n, m \in \mathbb{N}$, the number printed by Flowchart 2 does not exceed the number printed by Flowchart 1.*

**Proof.** For every $(a_0,\dots,a_n) \in \{0,\dots,m\}^{n+1}$,

$$E_n \supseteq \{1 = x_k :\ (k \in \{0,\dots,n\}) \wedge (1 = a_k)\} \cup$$

$$\{x_i + x_j = x_k :\ (i,j,k \in \{0,\dots,n\}) \wedge (a_i + a_j = a_k)\} \cup$$

$$\{x_i \cdot x_j = x_k :\ (i,j,k \in \{0,\dots,n\}) \wedge (a_i \cdot a_j = a_k)\}$$

□

**Lemma 2.** *For every $n, m \in \mathbb{N}$, the number printed by Flowchart 1 does not exceed the number printed by Flowchart 2.*

**Proof.** Let $n, m \in \mathbb{N}$. For every system of equations $\mathcal{S} \subseteq E_n$, if $(a_0,\dots,a_n) \in \{0,\dots,m\}^{n+1}$ and $(a_0,\dots,a_n)$ solves $\mathcal{S}$, then $(a_0,\dots,a_n)$ solves the following system of equations:

$$\{1 = x_k :\ (k \in \{0,\dots,n\}) \wedge (1 = a_k)\} \cup$$

$$\{x_i + x_j = x_k :\ (i,j,k \in \{0,\dots,n\}) \wedge (a_i + a_j = a_k)\} \cup$$

$$\{x_i \cdot x_j = x_k :\ (i,j,k \in \{0,\dots,n\}) \wedge (a_i \cdot a_j = a_k)\}$$

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□

**Theorem 2.** *For every $n, m \in \mathbb{N}$, Flowcharts 1 and 2 print the same number.*

**Proof.** It follows from Lemmas 1 and 2. □

**Definition 2.** *An approximation of a tuple $(x_0, \ldots, x_n) \in \mathbb{N}^{n+1}$ is a tuple $(y_0, \ldots, y_n) \in \mathbb{N}^{n+1}$ such that*

$$(\forall k \in \{0, \ldots, n\}\ (1 = x_k \Rightarrow 1 = y_k)) \wedge$$

$$(\forall i, j, k \in \{0, \ldots, n\}\ (x_i + x_j = x_k \Rightarrow y_i + y_j = y_k)) \wedge$$

$$(\forall i, j, k \in \{0, \ldots, n\}\ (x_i \cdot x_j = x_k \Rightarrow y_i \cdot y_j = y_k))$$

**Observation 2.** *There exists a set $\mathcal{A}(n) \subseteq \mathbb{N}^{n+1}$ such that*

$$\mathrm{card}(\mathcal{A}(n)) \leqslant 2^{\mathrm{card}(E_n)} = 2^{n+1+2\cdot(n+1)^3}$$

*and every tuple $(x_0, \ldots, x_n) \in \mathbb{N}^{n+1}$ possesses an approximation in $\mathcal{A}(n)$.*

**Observation 3.** *$f(n)$ equals the smallest $b \in \mathbb{N}$ such that every tuple $(x_0, \ldots, x_n) \in \mathbb{N}^{n+1}$ possesses an approximation in $\{0, \ldots, b\}^{n+1}$.*

**Observation 4.** *For every $n, m \in \mathbb{N}$, Flowcharts 1 and 2 print the smallest $b \in \{0, \ldots, m\}$ such that every tuple $(x_0, \ldots, x_n) \in \{0, \ldots, m\}^{n+1}$ possesses an approximation in $\{0, \ldots, b\}^{n+1}$.*

The following program in *MuPAD* implements the semi-algorithm shown in Flowchart 2.

```
input("Input a non-negative integer n",n):
m:=0:
while TRUE do
X:=combinat::cartesianProduct([s $s=0..m] $t=0..n):
Y:=[max(op(X[u])) $u=1..(m+1)^(n+1)]:
for p from 1 to (m+1)^(n+1) do
for q from 1 to (m+1)^(n+1) do
v:=1:
for k from 1 to n+1 do
if 1=X[p][k] and 1<>X[q][k] then v:=0 end_if:
for i from 1 to n+1 do
for j from i to n+1 do
if X[p][i]+X[p][j]=X[p][k] and X[q][i]+X[q][j]<>X[q][k] then v:=0 end_if:
if X[p][i]*X[p][j]=X[p][k] and X[q][i]*X[q][j]<>X[q][k] then v:=0 end_if:
end_for:
end_for:
end_for:
if max(op(X[q]))<max(op(X[p])) and v=1 then Y[p]:=0 end_if:
end_for:
end_for:
print(max(op(Y))):
m:=m+1:
end_while:
```

For $n \in \mathbb{N}$, $h(n)$ denotes the smallest $b \in \mathbb{N}$ such that if a system of equations $\mathcal{S} \subseteq \{x_i + 1 = x_k,\ x_i \cdot x_j = x_k :\ i, j, k \in \{0, \ldots, n\}\}$ has a solution in $\mathbb{N}^{n+1}$, then $\mathcal{S}$ has a solution in $\{0, \ldots, b\}^{n+1}$.

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From [11] and Lemma 3 in [10], it follows that the function $h : \mathbb{N} \to \mathbb{N}$ is computable in the limit and eventually dominates every computable function $g : \mathbb{N} \to \mathbb{N}$. A bit shorter program in *MuPAD* computes $h$ in the limit.

## 4. A Limit-Computable Function $\beta : \mathbb{N} \to \mathbb{N}$ of Unknown Computability Which Eventually Dominates Every Function $\delta : \mathbb{N} \to \mathbb{N}$ with a Single-Fold Diophantine Representation

The Davis-Putnam-Robinson-Matiyasevich theorem states that every listable set $\mathcal{M} \subseteq \mathbb{N}^n$ $(n \in \mathbb{N} \setminus \{0\})$ has a Diophantine representation, that is

$$(a_1, \ldots, a_n) \in \mathcal{M} \Longleftrightarrow \exists x_1, \ldots, x_m \in \mathbb{N} \ W(a_1, \ldots, a_n, x_1, \ldots, x_m) = 0 \qquad \text{(R)}$$

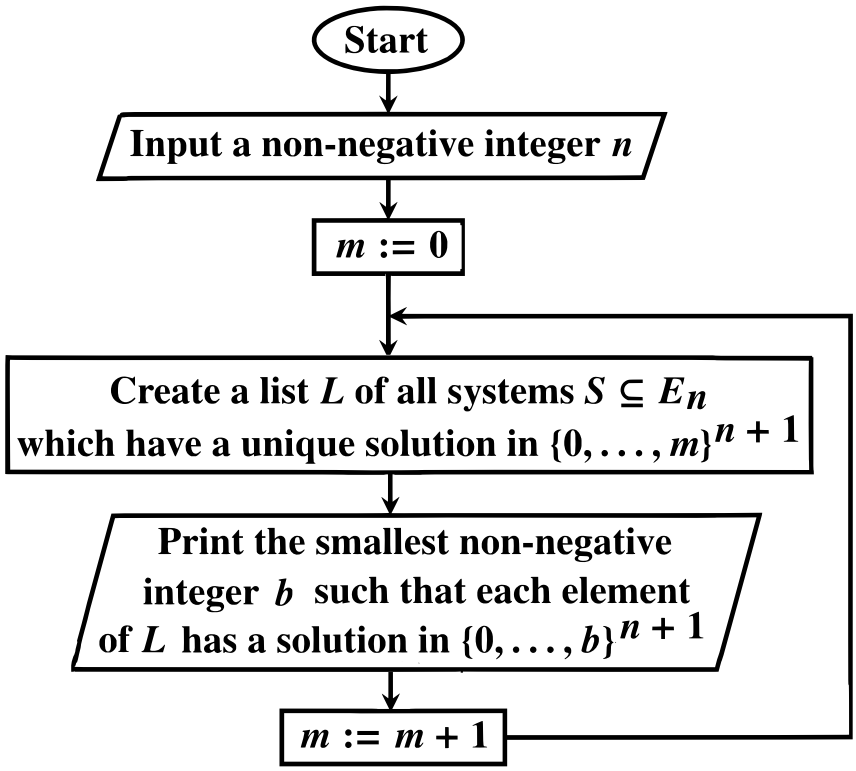
for some polynomial $W$ with integer coefficients, see [5]. The representation (R) is said to be single-fold, if for any $a_1, \ldots, a_n \in \mathbb{N}$ the equation $W(a_1, \ldots, a_n, x_1, \ldots, x_m) = 0$ has at most one solution $(x_1, \ldots, x_m) \in \mathbb{N}^m$.

**Hypothesis 1.** *([1–4][pp. 341–342], [6][p. 42], [7][p. 745]). Every listable set $\mathcal{X} \subseteq \mathbb{N}^k$ $(k \in \mathbb{N} \setminus \{0\})$ has a single-fold Diophantine representation.*

For $n \in \mathbb{N}$, $\beta(n)$ denotes the smallest $b \in \mathbb{N}$ such that if a system of equations $\mathcal{S} \subseteq E_n$ has a unique solution in $\mathbb{N}^{n+1}$, then this solution belongs to $\{0, \ldots, b\}^{n+1}$. The computability of $\beta$ is unknown.

**Theorem 3.** *The function $\beta : \mathbb{N} \to \mathbb{N}$ is computable in the limit and eventually dominates every function $\delta : \mathbb{N} \to \mathbb{N}$ with a single-fold Diophantine representation.*

**Proof.** This is proved in [11]. Flowchart 3 shows a semi-algorithm which computes $\beta(n)$ in the limit, see [11].




**Flowchart 3**

A semi-algorithm which computes $\beta(n)$ in the limit

□

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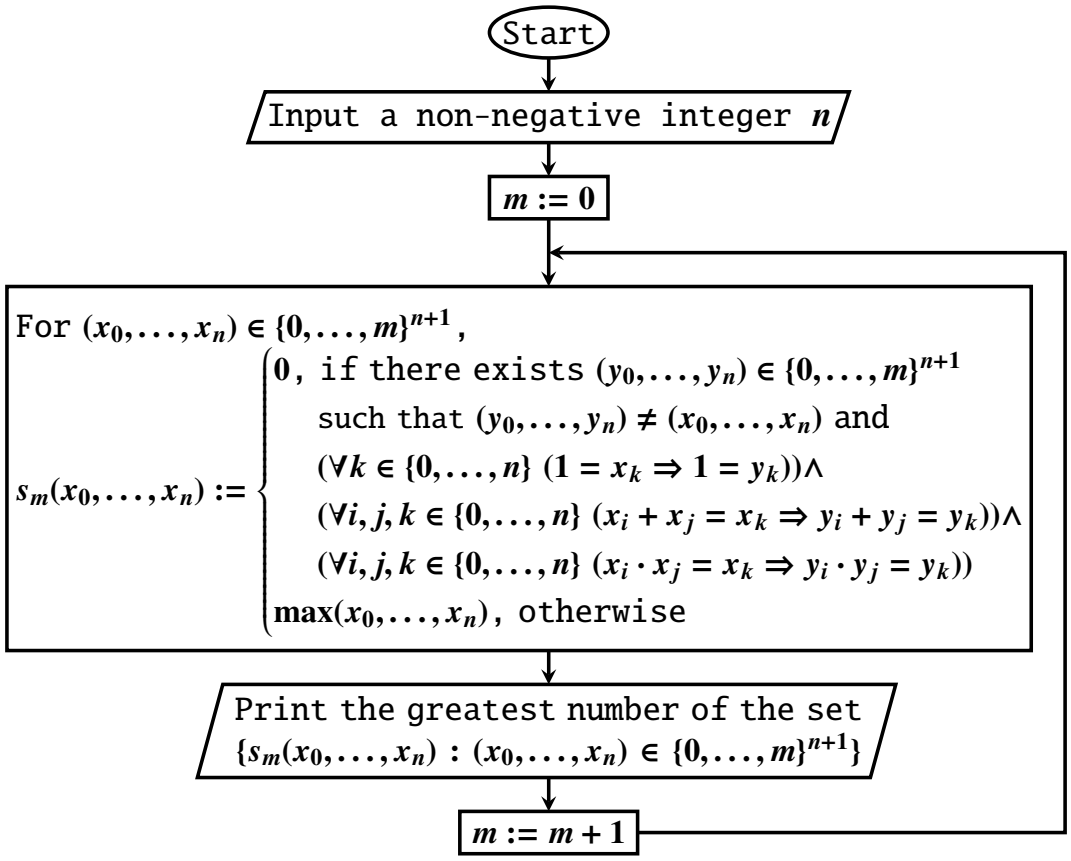
## 5. A Short Program in *MuPAD* That Computes $\beta$ in the Limit

Flowchart 4 shows a simpler semi-algorithm which computes $\beta(n)$ in the limit.




**Flowchart 4**

A simpler semi-algorithm which computes $\beta(n)$ in the limit

**Lemma 3.** *For every $n, m \in \mathbb{N}$, the number printed by Flowchart 4 does not exceed the number printed by Flowchart 3.*

**Proof.** For every $(a_0,\dots,a_n) \in \{0,\dots,m\}^{n+1}$,

$$E_n \supseteq \{1 = x_k :\ (k \in \{0,\dots,n\}) \wedge (1 = a_k)\} \cup$$

$$\{x_i + x_j = x_k :\ (i,j,k \in \{0,\dots,n\}) \wedge (a_i + a_j = a_k)\} \cup$$

$$\{x_i \cdot x_j = x_k :\ (i,j,k \in \{0,\dots,n\}) \wedge (a_i \cdot a_j = a_k)\}$$

□

**Lemma 4.** *For every $n, m \in \mathbb{N}$, the number printed by Flowchart 3 does not exceed the number printed by Flowchart 4.*

**Proof.** Let $n, m \in \mathbb{N}$. For every system of equations $\mathcal{S} \subseteq E_n$, if $(a_0,\dots,a_n) \in \{0,\dots,m\}^{n+1}$ is a unique solution of $\mathcal{S}$ in $\{0,\dots,m\}^{n+1}$, then $(a_0,\dots,a_n)$ solves the system $\widehat{\mathcal{S}}$, where

$$\widehat{\mathcal{S}} = \{1 = x_k :\ (k \in \{0,\dots,n\}) \wedge (1 = a_k)\} \cup$$

$$\{x_i + x_j = x_k :\ (i,j,k \in \{0,\dots,n\}) \wedge (a_i + a_j = a_k)\} \cup$$

$$\{x_i \cdot x_j = x_k :\ (i,j,k \in \{0,\dots,n\}) \wedge (a_i \cdot a_j = a_k)\}$$

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By this and the inclusion $\widehat{\mathcal{S}} \supseteq \mathcal{S}$, $\widehat{\mathcal{S}}$ has exactly one solution in $\{0,\dots,m\}^{n+1}$, namely $(a_0,\dots,a_n)$. □

**Theorem 4.** *For every $n, m \in \mathbb{N}$, Flowcharts 3 and 4 print the same number.*

**Proof.** It follows from Lemmas 3 and 4. □

The following program in *MuPAD* implements the semi-algorithm shown in Flowchart 4.

```
input("Input a non-negative integer n",n):
m:=0:
while TRUE do
X:=combinat::cartesianProduct([s $s=0..m] $t=0..n):
Y:=[max(op(X[u])) $u=1..(m+1)^(n+1)]:
for p from 1 to (m+1)^(n+1) do
for q from 1 to (m+1)^(n+1) do
v:=1:
for k from 1 to n+1 do
if 1=X[p][k] and 1<>X[q][k] then v:=0 end_if:
for i from 1 to n+1 do
for j from i to n+1 do
if X[p][i]+X[p][j]=X[p][k] and X[q][i]+X[q][j]<>X[q][k] then v:=0 end_if:
if X[p][i]*X[p][j]=X[p][k] and X[q][i]*X[q][j]<>X[q][k] then v:=0 end_if:
end_for:
end_for:
end_for:
if q<>p and v=1 then Y[p]:=0 end_if:
end_for:
end_for:
print(max(op(Y))):
m:=m+1:
end_while:
```

## References


1. D. Cantone, A. Casagrande, F. Fabris, E. Omodeo, *The quest for Diophantine finite-fold-ness,* Matematiche (Catania) 76 (2021), no. 1, 133–160, https://doi.org/10.4418/2021.76.1.8.
2. D. Cantone, L. Cuzziol, E. G. Omodeo, *On Diophantine singlefold specifications,* Matematiche (Catania) 79 (2024), no. 2, 585–620, https://lematematiche.dmi.unict.it/index.php/lematematiche/article/view/2703/1218.
3. D. Cantone and E. G. Omodeo, *"One equation to rule them all", revisited,* Rend. Istit. Mat. Univ. Trieste 53 (2021), Art. No. 28, 32 pp. (electronic), https://doi.org/10.13137/2464-8728/33314.
4. M. Davis, Yu. Matiyasevich, J. Robinson, *Hilbert's tenth problem, Diophantine equations: positive aspects of a negative solution;* in: Mathematical developments arising from Hilbert problems (ed. F. E. Browder), Proc. Sympos. Pure Math., vol. 28, Part 2, Amer. Math. Soc., Providence, RI, 1976, 323–378, https://doi.org/10.1090/pspum/028.2; reprinted in: The collected works of Julia Robinson (ed. S. Feferman), Amer. Math. Soc., Providence, RI, 1996, 269–324.
5. Yu. Matiyasevich, *Hilbert's tenth problem,* MIT Press, Cambridge, MA, 1993.
6. Yu. Matiyasevich, *Hilbert's tenth problem: what was done and what is to be done,* in: Proceedings of the Workshop on Hilbert's tenth problem: relations with arithmetic and algebraic geometry (Ghent, 1999), Contemp. Math. 270, Amer. Math. Soc., Providence, RI, 2000, 1–47, https://doi.org/10.1090/conm/270.
7. Yu. Matiyasevich, *Towards finite-fold Diophantine representations,* J. Math. Sci. (N. Y.) vol. 171, no. 6, 2010, 745–752, https://doi.org/10.1007%2Fs10958-010-0179-4.
8. J. S. Royer and J. Case, *Subrecursive Programming Systems: Complexity and Succinctness,* Birkhäuser, Boston, 1994.


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9. R. I. Soare, *Interactive computing and relativized computability,* in: B. J. Copeland, C. J. Posy, and O. Shagrir (eds.), Computability: Turing, Gödel, Church and beyond, MIT Press, Cambridge, MA, 2013, 203–260.
10. A. Tyszka, *A hypothetical upper bound on the heights of the solutions of a Diophantine equation with a finite number of solutions*, Open Comput. Sci. 8 (2018), no. 1, 109–114, https://doi.org/10.1515/comp-2018-0012.
11. A. Tyszka, *All functions $g : \mathbb{N} \to \mathbb{N}$ which have a single-fold Diophantine representation are dominated by a limit-computable function $f : \mathbb{N} \setminus \{0\} \to \mathbb{N}$ which is implemented in MuPAD and whose computability is an open problem*, in: Computation, cryptography, and network security (eds. N. J. Daras, M. Th. Rassias), Springer, Cham, 2015, 577–590, https://doi.org/10.1007/978-3-319-18275-9_24.

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