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On $k-$Unitary Perfect Polynomials over $\mathbb{F}_2$

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Abstract

Let k be a positive integer. A polynomial $A \in \mathbb{F}_2[x]$ is called *k-unitary perfect* if the sum of the k-th powers of its distinct unitary divisors equals A^k. In this paper, we focus on the case $k = 2^n$ and prove that every 2^n-unitary perfect polynomial over $\mathbb{F}_2$ is necessarily even. Moreover, we obtain a complete classification of all even 2^n-unitary perfect polynomials having at most three distinct irreducible factors. In particular, we characterize all such polynomials of the form

$$A = x^a(x+1)^b P^h,$$

where P is a Mersenne prime over $\mathbb{F}_2$ and $a, b, h \in \mathbb{N}$. As a consequence, several explicit infinite families of k-unitary perfect polynomials over $\mathbb{F}_2$ are obtained.

Keywords: sum of divisors; unitary divisors; multiplicative; polynomials; finite fields; characteristic 2

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1. Introduction

The study of perfect numbers and their analogues-unitary perfect and bi-unitary perfect numbers has a long and rich history in classical number theory. A classical result of Euclid [12] shows that if $2^p - 1$ is prime (a Mersenne prime), then $2^{p-1}(2^p - 1)$ is an even perfect number. Despite centuries of research, the existence of odd perfect numbers remains an open problem. Similarly, only a few unitary perfect numbers are known (such as $6, 60, 90, 87360, \ldots$), and the existence of infinitely many remains open. Importantly, no odd unitary perfect integer has been discovered.

Motivated by these classical problems, several authors [1–4,8,9] have investigated polynomial analogues of perfectness over finite fields. In particular, perfect, unitary perfect, and bi-unitary perfect polynomials over $\mathbb{F}_2$ have been studied extensively, leading to strong structural restrictions and complete classifications in certain cases.

Let $A \in \mathbb{F}_2[x]$ be a nonzero polynomial. A divisor B of A is unitary (resp. bi-unitary) if $gcd(B, A/B) = 1$ (resp. $gcd_u(B, A/B) = 1$, where gcd_u denotes the greatest common unitary divisor. The sum of divisors (resp. unitary divisors and bi-unitary divisors) of A is denoted by $\sigma(A)$(resp. $\sigma^*(A)$ and $\sigma^{**}(A)$). In symbols,

$$\sigma(A) = \sum_{B|A} B, \; \sigma^*(A) = \sum_{\substack{B|A \\ gcd(B,A/B)=1}} B, \text{ and } \sigma^{**}(A) = \sum_{\substack{B|A \\ gcd_u(B,A/B)=1}} B.$$

A polynomial $A \in \mathbb{F}_2[x]$ is perfect (resp. unitary perfect and bi-unitary perfect) if $\sigma(A) = A$ (resp. $\sigma^*(A) = A$ and $\sigma^{**}(A) = A$). Note that in studying perfect or unitary perfect polynomials, the distinction disappears when the polynomial is squarefree.

Also, A is called a bi-unitary superperfect if $\sigma^{**2}(A) = \sigma^{**}(\sigma^{**}(A)) = A$. Chehade et al. [7] gave all bi-unitary superperfect polynomials divisible by one or two irreducible polynomials over $\mathbb{F}_2$. Note that the functions σ, σ^* and σ^{**} are degree preserving and multiplicative while σ^{**2} is a degree preserving function but not a multiplicative one.

2. Preliminary

Throughout this paper, all polynomials are assumed to be monic in $\mathbb{F}_2[x]$. For a polynomial A, we denote by $\deg(A)$ its degree and by $\omega(A)$ the number of its distinct irreducible factors. Unless otherwise stated, the following notations are used.

- $\mathbb{N}$ (resp. $\mathbb{N}^*$) represents the set of non-negative (resp. positive) integers.
- $\overline{A}$ is the polynomial obtained from A with x replaced by $x+1$, that is $\overline{A}(x) = A(x+1)$.
- A^* is the inverse of the polynomial A with $\deg(A) = m$, in this sense $A^*(x) = x^m A(\frac{1}{x})$.

A polynomial of the form $T = 1 + x^a(x+1)^b$ with $\gcd(a,b) = 1$ is called a Mersenne polynomial. If such a polynomial is irreducible over $\mathbb{F}_2$, it is called a *Mersenne prime*. The first Mersenne primes over $\mathbb{F}_2$ are

$$T_1 = 1 + x + x^2, T_2 = 1 + x + x^3, T_3 = 1 + x^2 + x^3, T_4 = 1 + x + x^2 + x^3 + x^4, T_5 = 1 + x^3 + x^4.$$

A polynomial in $\mathbb{F}_2[x]$ is called *even* if it is divisible by x or $x+1$; otherwise, it is called *odd*.

Recently, the divisor function σ was generalized by summing $k-$th powers of divisors, giving rise to the notion of $k-$perfect polynomials. Chehade et al. [6] introduced a polynomial analogue of $\sigma_k(n)$, $n \in \mathbb{N}$, by defining for any nonzero polynomial $A \in \mathbb{F}_2[x]$ as follows: $\sigma_k(A) = \sum_{B|A} B^k$. If $A = \prod_{i=1}^{m} P_i^{\alpha_i}$, then

$$\sigma_k(A) = \prod_{i=1}^{m}\left(\frac{P_i^{k(\alpha_i+1)} - 1}{P_i^k - 1}\right).$$

Chehade et al. [6] also defined $k-$perfect polynomials over $\mathbb{F}_2$ as those polynomials A satisfying $\sigma_k(A) = A^k$. They showed that no odd 2^n-perfect polynomials exist over $\mathbb{F}_2$ and characterized all even 2^n-perfect polynomials over $\mathbb{F}_2$ that have the form $x^a(x+1)^b \prod_{i=1}^{m} P_i^{h_i}$, where each P_i is Mersenne and a, b, and h_i are positive integers.

2.1. $k-$Unitary Perfect Polynomials

Let A be a nonzero polynomial in $\mathbb{F}_2[x]$. Define $\sigma_k^*(A)$ to be the sum of the $k-$th powers of its distinct unitary divisors. That is, $\sigma_k^*(A) = \sum_{\substack{B|A \\ gcd(B,A/B)=1}} B^k$ and if $A = \prod_{i=1}^{m} P_i^{h_i}$, then

$$\sigma_k^*(A) = \prod_{i=1}^{m}\left(P_i^{k\alpha_i} + 1\right).$$

The function σ_k^* is multiplicative and degree preserving.
We now introduce the central concept investigated in this work, namely $k-$unitary perfect polynomials over $\mathbb{F}_2$.

Definition 1. *Let k be a positive integer. A polynomial A is called $k-$unitary perfect over $\mathbb{F}_2$ if*

$$\sigma_k^*(A) = A^k.$$

Example 1. *Let $A(x) = x^2(x+1)^3(x^2+x+1) \in \mathbb{F}_2[x]$, then*

$$\begin{aligned}\sigma_4^*(A(x)) &= (1 + x^{2(4)})(1 + (x+1)^{3(4)})(1 + (x^2+x+1)^4) \\ &= x^8(x+1)^{12}(x^2+x+1)^4 \\ &= A^4.\end{aligned}$$

Hence, A is a $4-$unitary perfect polynomial over $\mathbb{F}_2$.

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The unitary perfect polynomial is studied extensively by Beard and Habrin [1–3] and by Gallardo and Rahavandrainy [9] and [10].

When $k = 1$, Definition 1 reduces to the classical notion of unitary perfect polynomials. In this work, we focus primarily on the case $k = 2^n$ $(n \in \mathbb{N}^*)$, where the arithmetic of $\mathbb{F}_2$ plays a crucial role. We classify all 2^n-unitary perfect polynomials over $\mathbb{F}_2$ with at most three distinct irreducible factors, proving that no odd such polynomials exist and fully characterizing the even ones of the form $x^a(x+1)^b P^h$, where P is a Mersenne prime and a, b and h are positive integers.

Our main result is given in the following theorem.

Theorem 1. *Let $a, b, h \in \mathbb{N}$ and let P be a Mersenne prime in $\mathbb{F}_2[x]$. Then, $A = x^a(x+1)^b P^h$ is a 2^n-unitary perfect polynomial over $\mathbb{F}_2$ if and only if*

- $A = x^{2^t}(x+1)^{2^t}$ *if* $\omega(A) = 2$,
- $A \in \Big\{x^{3.2^t}(x+1)^{2^{t+1}}T_1^{2^t}, x^{2^{t+1}}(x+1)^{3.2^t}T_1^{2^t}, x^{3.2^t}(x+1)^{3.2^t}T_1^{2^{t+1}}, x^{5.2^t}(x+1)^{2^{t+2}}T_4^{2^t},$
  $x^{2^{t+2}}(x+1)^{5.2^t}T_5^{2^t}\Big\}$ *if* $\omega(A) = 3$,

for some positive integer t.

The family $\{x^{2^t}(x+1)^{2^t} : t \in \mathbb{N}\}$ may be regarded as an analogue of the family $\{x^{2^t-1}(x+1)^{2^t-1} : t \in \mathbb{N}\}$, which corresponds to the trivial instance of perfect polynomials over $\mathbb{F}_2$.

3. Useful Results

We consider the following families of polynomials arising in the computation of $\sigma^*(x^{p^t})$, $p \in \{3, 5, 7, 11\}$:

$$\begin{aligned} S_t &= 1 + x^{3^t} + x^{2.3^t}, \\ R_t &= 1 + x^{7^t} + x^{3.7^t}, \\ B_t &= 1 + x^{2.7^t} + x^{3.7^t}, \\ C_t &= 1 + x^{5^t} + x^{2.5^t} + x^{3.5^t} + x^{4.5^t}, \\ D_t &= 1 + x^{11^t} + x^{2.11^t} + \ldots + x^{10.11^t}. \end{aligned}$$

Note that the polynomials S_t, B_t, C_t, R_t and D_t are irreducible over $\mathbb{F}_2$ with $S_0 = T_1, R_0 = T_2, B_0 = T_3, C_0 = T_4$.

We need the following useful results. Since some of them are straightforward, their proofs are omitted.

Lemma 1. [9] *If $A = A_1A_2$ is unitary perfect over $\mathbb{F}_2$ and if* $\gcd(A_1, A_2) = 1$. *Then A_1 is unitary perfect if and only if A_2 is unitary perfect.*

Lemma 2. [9] *If $A(x)$ is unitary perfect over $\mathbb{F}_2$, then the polynomial $A(x+1)$ is also unitary perfect over $\mathbb{F}_2$.*

Lemma 3. [9] *If $A(x)$ is unitary perfect over $\mathbb{F}_2$, then A^{2^n} is also unitary perfect over $\mathbb{F}_2$, for any $n \in \mathbb{N}$.*

In [9], Gallardo and Rahavandrainy gave a complete list for all even perfect polynomials with 3 irreducible factors as given in the following lemma.

Lemma 4. *The complete list of even perfect polynomials over $\mathbb{F}_2$ with $\omega(A) = 3$ is of the form A^{2^n} or $\overline{A}^{2^n}$ where $A \in \{x^2(x+1)^3T_1, x^3(x+1)^3T_1^2, x^4(x+1)^5T_4\}$.*

The statement in part *iii)* of the following lemma is a result of Dickson (see [5, Lemma 2])

Lemma 5. i) *Any complete polynomial inverts into itself.*

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ii) *If* $1+x+\cdots+x^h = PQ$*, then either* $(P=P^*, Q=Q^*)$ *or* $(P=Q^*, Q=P^*)$*.*
iii) *If* $P=P^*$ *is Mersenne, then* $P \in \{T_1, T_4\}$*.*

Corollary 1. *If* $P=\overline{P}$ *is Mersenne, then* $P=T_1$*.*

Lemma 6. *([5], Lemmata 4, 5, 6 and Theorem 8) Let* $P, Q \in \mathbb{F}_2[x]$ *and let* $n, m \in \mathbb{N}$*.*
i) *If* $1+P+\cdots+P^{2n} = Q^m$*, then* $m \in \{0,1\}$*.*
ii) *If* $1+P+\cdots+P^{2n} = Q^m A$*, with* $m>1$ *and* $A \in \mathbb{F}_2[x]$ *is non-constant, then* $\deg(P) > \deg(Q)$*.*
iii) *If* $1+x+\cdots+x^{2n} = PQ$ *and* $P = 1+(x+1)+\cdots+(x+1)^{2m}$*, then* $n=4$*,* $P=T_1$ *and* $Q = P(x^3) = 1+x^3+x^6$*.*
iv) *If any irreducible factor of* $1+x+\cdots+x^{2n}$ *is Mersenne, then* $n \in \{1,2,3\}$*.*
v) *If* $1+x+\cdots+x^h = 1+(x+1)+\cdots+(x+1)^h$*, then* $h = 2^n-2$*, for some* $n \in \mathbb{N}$*.*

Lemma 7. *i)* *If* T_1 *divides* $1+x+\cdots+x^h$*, then* $h \equiv 2 \pmod 3$*.*
ii) *If* T_2 *or* T_3 *divides* $1+x+\cdots+x^h$*, then* $h \equiv 6 \pmod 7$*.*
iii) *If* T_4 *or* T_5 *divides* $1+x+\cdots+x^h$*, then* $h \equiv 4 \pmod 5$*.*

As a special case of ([11], Theorem 2.47), we have

Lemma 8. *The polynomial* $1+x+\cdots+x^m$ *is irreducible over* $\mathbb{F}_2$ *if and only if* $m+1$ *is a prime number and 2 is a primitive root in* $\mathbb{F}_{m+1}$*.*

Consequently, we get

Lemma 9. *i)* *The polynomial* $Q(x) = 1+x^5+\cdots+(x^5)^l$ *is irreducible over* $\mathbb{F}_2$ *if and only if* $l=4$*.*
ii) *The polynomial* $Q(x) = 1+x+\cdots+x^{3.2^r}$ *is irreducible over* $\mathbb{F}_2$ *if and only if* $r=2$*.*
iii) *The polynomial* $Q(x) = 1+x+\cdots+x^{5.2^r}$ *is irreducible over* $\mathbb{F}_2$ *if and only if* $r=1$*.*

Lemma 10. *[9] Any non-constant unitary perfect polynomial over* $\mathbb{F}_2$ *is divisible by* x *and by* $x+1$*. In particular, there is no odd unitary perfect polynomial over* $\mathbb{F}_2$*.*

Lemma 11. *If* P *is a Mersenne prime, then for any* $n \in \mathbb{N}^*$*,* $\sigma^*(P^n)$ *has always a factor* $x(x+1)$*.*

Lemma 12. *Let* P *be irreducible and let* $A = P^\alpha \in \mathbb{F}_2[x]$ *with* $\alpha \geq 1$*. Then* $\sigma_k^*(A) = (\sigma^*(A))^k$ *if and only if* $k = 2^n$*.*

Proof. $\sigma_k^*(A) = (\sigma^*(A))^k$ gives

$$1+P^{k\alpha} = \sigma_k^*(A) = (1+P^\alpha)^k.$$

In $\mathbb{F}_2[x]$, the binomial expansion of $(1+P^\alpha)^k$ reduces to $1+P^{k\alpha}$ if and only if all intermediate binomial coefficients vanish modulo 2, which occurs precisely when k is a power of 2. □

Corollary 2. *Let* $A = \prod_{i=1}^{r} P_i^{\alpha_i} \in \mathbb{F}_2[x]$ *where* P_i *is irreducible, then* $\sigma_{2^n}^*(A) = (\sigma^*(A))^{2^n}$*.*

The following corollary is a direct consequence of Corollary 2.

Corollary 3. $A = \prod_{i=1}^{m} P_i^{\alpha_i}$ *is unitary perfect over* $\mathbb{F}_2$ *iff* A *is* 2^n-*unitary perfect over* $\mathbb{F}_2$*.*

Lemma 13. *Let* $m \leq n$ *be positive integers and let* $A \in \mathbb{F}_2[x]$*, then* $\sigma_{2^m}^*(A)$ *divides* $\sigma_{2^n}^*(A)$*.*

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Proof.

$$\begin{aligned}\sigma^*_{2^n}(A) &= (\sigma^*(A))^{2^n}\\ &= (\sigma^*(A))^{2^m}(\sigma^*(A))^{2^{n-m}}\\ &= \sigma^*_{2^m}(A)(\sigma^*(A))^{2^{n-m}}.\end{aligned}$$

□

Hence if A is a multi-perfect polynomial over $\mathbb{F}_2$, (A divides $\sigma(A)$), then A is a 2^n-multi-perfect polynomial over $\mathbb{F}_2$.

Lemma 14. *Assume $\alpha \geq 1$, P is an irreducible polynomial and $A = P^\alpha \in \mathbb{F}_2[x]$, then $A \nmid \sigma^*_k(A)$.*

Proof. Assume that $A|\sigma^*_k(A)$, then there exists $Q \in \mathbb{F}_2[x]$ such that $\sigma^*_k(A) = QA$ with $deg(Q) < deg(A^k)$. So, $1 + P^{k\alpha} = QP^\alpha$ and $P^\alpha(Q + P^{k\alpha-\alpha}) = 1$. This means that P^α is a unit in $\mathbb{F}_2[x]$. Hence, $P^\alpha = 1$. This contradicts the fact that P is irreducible in $\mathbb{F}_2[x]$. □

Corollary 4. *Let $\alpha \geq 1$ and let $A = P^\alpha \in \mathbb{F}_2[x]$, then A is not $k-$unitary perfect, for any irreducible P.*

The proof of Corollary 4 can be directly obtained from Lemma 11 or from Corollary 14.
As a result of corollary 14 and lemma 4, we have $A \nmid \sigma^*_k(A)$ and hence $\sigma^*_k(A) \nmid A^k$.

Lemma 15. *Let $A_1, \ldots, A_m \in \mathbb{F}_2[x]$. Then*

$$\prod_{i=1}^m A_i(x+1) = \left(\prod_{i=1}^m A_i\right)(x+1).$$

Proof. Define the map $\varphi : \mathbb{F}_2[x] \to \mathbb{F}_2[x]$ by

$$\varphi(f(x)) = f(x+1).$$

Since substitution preserves addition and multiplication, φ is a ring automorphism of $\mathbb{F}_2[x]$. Hence,

$$\varphi\left(\prod_{i=1}^m A_i(x)\right) = \prod_{i=1}^m \varphi(A_i(x)) = \prod_{i=1}^m A_i(x+1).$$

By definition of φ, the left-hand side equals

$$\left(\prod_{i=1}^m A_i\right)(x+1),$$

which proves the result. □

Lemma 16. *If $A(x)$ is 2^n-unitary perfect over $\mathbb{F}_2$, then the polynomial $\overline{A}(x)$ is also 2^n-unitary perfect over $\mathbb{F}_2$.*

Proof. Let $A(x) = \prod_{i=1}^m P_i(x)^{a_i}$,where the $P_i(x) \in \mathbb{F}_2[x]$ are distinct irreducible polynomials. Since $A(x)$ is 2^n-unitary perfect, we have

$$\sigma^*_{2^n}(A) = \prod_{i=1}^m \left(1 + P_i^{2^n a_i}\right) = A^{2^n}.$$

Let $B(x) = A(x+1) = \prod_{i=1}^{m} P_i(x+1)^{a_i}$. Since the substitution $x \mapsto x+1$ is a ring automorphism of $\mathbb{F}_2[x]$, it preserves irreducibility and coprimality. Hence, the polynomials $P_i(x+1)$ are distinct irreducible polynomials in $\mathbb{F}_2[x]$, and the above factorization of $B(x)$ is its factorization into powers of distinct irreducibles. So,

$$\sigma^*_{2^n}(B) = \prod_{i=1}^{m}\left(1 + P_i(x+1)^{2^n a_i}\right).$$

By Lemma 15,

$$\prod_{i=1}^{m}\left(1 + P_i(x+1)^{2^n a_i}\right) = \left(\prod_{i=1}^{m}\left(1 + P_i^{2^n a_i}\right)\right)(x+1).$$

Since $\prod_{i=1}^{m}(1 + P_i^{2^n a_i}) = A^{2^n}$, it follows that

$$\sigma^*_{2^n}(B) = (A^{2^n})(x+1) = (A(x+1))^{2^n} = B(x)^{2^n}.$$

Therefore, $\overline{A}(x)$ is 2^n-unitary perfect over $\mathbb{F}_2$. □

Lemma 17. *If P is a Mersenne prime in $\mathbb{F}_2[x]$, then $x(x+1)|\sigma^*_k(P^n)$ for $n \in \mathbb{N}^*$, for all $k \in \mathbb{N}^*$.*

Proof. If n is even, put $n = 2^t u$ where u is odd and $t \in \mathbb{N}^*$.

$$\begin{aligned}\sigma^*_k(P^n) &= 1 + P^{kn} \\ &= 1 + P^{k2^t u} \\ &= \left(1 + P^{2^t}\right)\left(1 + P^{2^t} + \ldots + (P^{2^t})^{uk-1}\right) \\ &= (1+P)^{2^t}\left(1 + P + \ldots + P^{uk-1}\right)^{2^t}\end{aligned}$$

$1 + P$ is divisible by $x(x+1)$ then $x(x+1)|\sigma^*_k(P^n)$.
If n is odd, take $t = 0$. □

4. Factorization Patterns for $\sigma^*(x^{p^t})$

In this section, we study the unitary divisor sum function $\sigma^*(x^{p^t})$, p is a prime integer. Explicit factorizations and divisibility properties of $\sigma^*(x^{p^t})$ and related expressions for $P \in \{3,5,7,11\}$ obtained here reveal systematic appearances of Mersenne polynomials such as T_1 and T_4, which suggest a deeper underlying structure governing $\sigma^*(x^{p^t})$. These results provide essential tools for the classification results developed later. Some results are not directly invoked in the proofs of the main classification theorem; however, they are included to emphasize recurring factorization patterns of $\sigma^*(x^{p^t})$ and are useful in extending the classification to polynomials with larger values of $\omega(A)$.
The following lemma has a straightforward proof.

Lemma 18. *Let p be a prime integer, then*

i) $\sigma^*(x^p) = (1+x)\sigma(x^{p-1})$.
ii) $\sigma^*((1+x)^p) = x\sigma\left((1+x)^{p-1}\right)$.

Using Corollary 2 and Lemma 18, we get

Corollary 5. *Let t be a positive integer and let p be an odd prime, then*

i) $\sigma^*\left(x^{2^t p}\right) = (1+x)^{2^t}\left(\sigma\left(x^{p-1}\right)\right)^{2^t}$.

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ii) $\sigma^*\left((1+x)^{2^t p}\right) = x^{2^t}\left(\sigma\left((1+x)^{p-1}\right)\right)^{2^t}$.

Some values of $\sigma^*(P)$, P is Mersenne, are given in the below table:

P	$\sigma^*(P)$
T_1	$x(1+x)$,
T_2	$x(1+x)^2$,
T_3	$x^2(1+x)$,
T_4	$x(1+x)^3$,
T_5	$x^3(1+x)$.

Using $\sigma^*(T_1)$ and Corollary 2, we obtain

Lemma 19. *Let t be a positive integer, then $\sigma_2^*\left(T_1^{2^t-1}\right) = (x^2+x)^{2^t}$.*

Lemma 20. *Let t be a positive integer, then $\sigma_2^*\left(T_1^{2^t-1.3}\right) = (x^2+x)^{2^t}(1+x+x^4)^{2^t}$.*

Proof. The proof is done by induction. For $t=1$, we have $\sigma_2^*(T_1^3) = 1+(T_1^3)^2 = (x^2+x)^2(1+x+x^4)^2$. Hence, the statement is true for $t=1$. Now assume $\sigma_2^*\left(T_1^{2^t-1.3}\right) = (x^2+x)^{2^t}(1+x+x^4)^{2^t}$. Observe that

$$\begin{aligned}\sigma_2^*\left(T_1^{2^t.3}\right) &= 1+\left(T_1^{2^t.3}\right)^2\\ &= \left(1+T_1^{2^t.3}\right)^2\\ &= \left(\sigma_2^*\left(T_1^{2^t-1.3}\right)\right)^2\\ &= (x^2+x)^{2^{t+1}}(1+x+x^4)^{2^{t+1}}.\end{aligned}$$

The proof is now complete. □

Lemma 21. *$\sigma^*(x^{3^t})$ and $\sigma^*((1+x)^{3^t})$ are always divisible by T_1, $t\in\mathbb{N}^*$.*

Proof. Consider the polynomial x^3+1 in $\mathbb{F}_2[x]$. We have

$$x^3+1=(x+1)T_1.$$

Thus,

$$x^3+1\equiv 0\ (mod\ T_1),$$

and so

$$x^3\equiv -1\equiv 1\ (mod\ T_1).$$

For $t\in N^*$, we have

$$x^{3^t}\equiv (x^3)^{3^{t-1}}\equiv (1)^{3^{t-1}}\equiv 1\ (mod\ T_1).$$

This implies that

$$x^{3^t}+1\equiv 0\ (mod\ T_1).$$

Therefore, $1+x^{3^t}$ is a multiple of T_1. The case of $\sigma^*((1+x)^{3^t})$ is handled in a similar way. □

Lemma 22. $\sigma^*(x^{3^t}) = (1+x)S_0S_1...S_{t-1}$, $t\in\mathbb{N}^*$.

Proof. Since $\sigma^*(x^{3^t}) = 1 + x^{3^t}$, we prove by induction on t that $1 + x^{3^t} = (1+x)\prod_{k=0}^{t-1} S_k$. Let $P(t)$ be the statement:

$$P(t): \quad 1 + x^{3^t} = (1+x)\prod_{k=0}^{t-1} S_k.$$

For the case $t = 1$, $P(1) : 1 + x^{3^1} = 1 + x^3 = (1+x)S_0$. So $P(1)$ holds. Now, assume that the statement $P(t)$ is true for some integer $t \geq 1$:

$$1 + x^{3^t} = (1+x)\prod_{k=0}^{t-1} S_k.$$

So,

$$P(t+1) : 1 + x^{3^{t+1}} = (1+x)\prod_{k=0}^{t} S_k.$$

The right hand side of $P(t+1)$ is

$$\begin{aligned}(1+x)\left(\prod_{k=0}^{t-1} S_k\right) S_t &= (1 + x^{3^t})S_t \\ &= (1 + x^{3^t})(1 + x^{3^t} + x^{2\cdot 3^t}) \\ &= 1 + x^{3^{t+1}}.\end{aligned}$$

This is the left hand side of $P(t+1)$. Thus, the inductive step is complete. □

Corollary 6. *Let $A = x^{3^t} T_1 \in \mathbb{F}_2[x]$, then $\sigma^*(A) = x(1+x)^2 S_0 S_1 ... S_{t-1}$, $t \in \mathbb{N}^*$.*

Lemma 23. *T_4 is a factor of $\sigma^*(x^{5^t})$, for all $t \in \mathbb{N}^*$.*

Proof. We have $x^5 + 1 = (x+1)T_4$. Then, follow the same steps as in the proof of Lemma 21. □

A mimic of the preceding two lemmas gives

Lemma 24. *i) $\sigma^*(x^{7^t})$ is always divisible by $T_2 T_3$, $t \in \mathbb{N}^*$.*
ii) $\sigma^*(x^{11^t})$ is always divisible by D_0, $t \in \mathbb{N}^*$.

The proof of the following lemma follows by induction on t.

Lemma 25. *i) $\sigma^*(x^{5^t}) = (1+x)C_0 C_1 ... C_{t-1}$.*
ii) $\sigma^*(x^{7^t}) = (1+x)R_0 B_0 R_1 B_1 ... R_{t-1} B_{t-1}$.
iii) $\sigma^*(x^{11^t}) = (1+x)D_0 D_1 ... D_{t-1}$.

Corollary 7. *i) $\sigma^*(x^{5^t} T_1) = x(1+x)^2 C_0 C_1 ... C_{t-1}$.*
ii) $\sigma^*(x^{7^t} T_1) = x(1+x)^2 R_0 B_0 R_1 B_1 ... R_{t-1} B_{t-1}$.
ii) $\sigma^*(x^{11^t} T_1) = x(1+x)^2 D_0 D_1 ... D_{t-1}$.

5. Proof of Theorem 1

The proof proceeds by working on the possible values of $\omega(A)$. We first treat the case $\omega(A) = 2$, then handle the more delicate case $\omega(A) = 3$. For $\omega(A) = 3$, we show the exponent of the odd prime must be a power of 2 (Lemma 26) and that the irreducible polynomial must be from a specific set (Lemma 27). The final classification arises from the exponent comparison in the remaining cases (Lemmas 28–31).

5.1. Case $\omega(A) = 2$:

Assume $A = P^a Q^b$, where P and Q are irreducible in $\mathbb{F}_2[x]$ and a and b are positive integers. We begin by showing that odd unitary perfect polynomials over $\mathbb{F}_2$ do not exist.

We begin with a key observation. The following corollary establishes the nonexistence of odd unitary perfect polynomials over $\mathbb{F}_2$.

Corollary 8. *Every non constant 2^n−unitary perfect polynomial over $\mathbb{F}_2$ is divisible by $x(x+1)$.*

Proof. Follows directly from Lemma 10 and Corollary 3. □

Consequently, both irreducible factors of A must be linear, and therefore $A = x^a(x+1)^b$.

Proposition 1. *Let $A = P^a Q^b \in \mathbb{F}_2[x]$, then A is 2^n−unitary perfect over $\mathbb{F}_2$ if and only if A is of the form $(x^2+x)^{2^t}$, for some $t \in \mathbb{N}$.*

Proof. Sufficiency is immediate from the definition. For necessity, assume $A = x^a(x+1)^b$ hence we must have: $1 + x^a = (x+1)^a$ and $1 + (x+1)^b = x^b$. Hence, $a = b = 2^n$, for some $n \in \mathbb{N}$. □

The first part of Theorem 1 is done. Consequently the unitary perfect polynomials A with $\omega(A) = 2$ are exactly the perfect polynomials.

5.2. Case $\omega(A) = 3$:

Assume now that A has exactly three distinct irreducible factors. By Corollary 8, A over $\mathbb{F}_2$ must be even and has the form $x^a(x+1)^b P^c$, P is an odd irreducible polynomial. We begin by restricting the possible exponent of P.

Lemma 26. *If $A = x^a(x+1)^b P^c$ is a 2^n−unitary perfect polynomial over $\mathbb{F}_2$, then $c = 2^t$ for some $t \in \mathbb{N}$.*

Proof. Since A is a 2^n−unitary perfect polynomial,

$$\begin{aligned}\sigma^*_{2^n}(A) &= (1 + x^{a\cdot 2^n})(1 + (x+1)^{b\cdot 2^n})(1 + P^{c\cdot 2^n}) \\ &= A^{2^n}.\end{aligned}$$

The expression $(1 + x^{a\cdot 2^n})(1 + (x+1)^{b\cdot 2^n})(1 + P^{c\cdot 2^n})$ must have the factors x, $x+1$, and P. Considering the factor $\sigma^*_{2^n}(P^c) = (1 + P^{c\cdot 2^n}) = (1 + P^c)^{2^n}$. Assume $c = 2^t u$, where u is odd and $t \in \mathbb{N}$. So

$$\sigma^*(P^c) = \sigma^*(P^{2^t u}) = (\sigma^*(P^u))^{2^t} = (1 + P^u)^{2^t}$$

Since P is odd (neither divisible by x nor by $x+1$), then $1 + P^u$ is also not divisible by P unless u is a multiple of the characteristic of the field, which is 2. But u is odd. So P and $1 + P^u$ are coprime. Thus, the only possible prime factors of $1 + P^u$ are x and $x+1$ so $1 + P^u$ must be of the form $x^h(x+1)^k$ for some integers h, k.

But

$$\begin{aligned}1 + P^u &= (1 + P)(1 + P + \cdots + P^{u-1}) \\ &= x^h(x+1)^k.\end{aligned}$$

The two factors $(1 + P)$ and $(1 + P + \cdots + P^{u-1})$ are coprime and since their product must be of the form $x^h(x+1)^k$, the prime factors of each of these two factors must also be a subset of $\{x, x+1\}$. The only way for these two coprime polynomials to be made of factors of x and $x+1$ is for one of the factors to be equal to 1.

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If $u > 1$, the degree of the second factor,

$$\deg(1 + P + \cdots + P^{u-1}) = (u-1)\deg(P),$$

is greater than 0, so it cannot be 1. This forces the case where the second factor is 1, which only happens when the sum has a single term. Thus $u - 1 = 0$ or $u = 1$.
This implies that $c = 2^t$, which completes the proof. □

Lemma 27. *Let $a = 2^h r$, $b = 2^k s$ with r and s are odd. If $A = x^a(x+1)^b P^{2^t}$ is a 2^n-unitary perfect polynomial over $\mathbb{F}_2$, then $P \in \{T_1, T_4\}$.*

Proof. Using the identities

$$1 + x^a = (x+1)^{2^h}(1 + x + \cdots + x^{r-1})^{2^h}, \tag{1}$$

$$1 + (x+1)^b = x^{2^k}(1 + (x+1) + \cdots + (x+1)^{s-1})^{2^k}, \tag{2}$$

and

$$1 + P^{2^t} = (1+P)^{2^t}. \tag{3}$$

Lemma 6-i) implies that

$$1 + x + \cdots + x^{r-1},\ 1 + (x+1) + \cdots + (x+1)^{s-1} \in \{1, P\}.$$

By symmetry, this leads to two possible configurations:

$$\begin{aligned} &Case\ 1: 1 + x + \cdots + x^{r-1} = P,\ s = 1, \\ &Case\ 2: 1 + x + \cdots + x^{r-1} = P = 1 + (x+1) + \cdots + (x+1)^{s-1}. \end{aligned} \tag{4}$$

By Lemma 5-iii), we have: $P \in \{T_1, T_4\}$. □

Once the irreducible factor P is fixed, the equality $\sigma^*_{2^n}(A) = A^{2^n}$ rigidly determines all remaining exponents.

5.2.1. Subcase: $P = T_1$

Lemma 28. *Let r be an odd integer. If $A = x^{2^h r}(x+1)^{2^k} T_1^{2^t}$ is a 2^n-unitary perfect polynomial over $\mathbb{F}_2$, then $r = 3$, $k = h+1$ and $t = h$.*

Proof. From Case 1 in Lemma 27, we have $1 + x + \cdots + x^{r-1} = T_1$ which implies $r = 3$. $A^{2^n} = \sigma^*_{2^n}(A)$ yields

$$\begin{aligned} x^{2^h.3}(x+1)^{2^k} T_1^{2^t} &= (x+1)^{2^h} T_1^{2^h} x^{2^k}(x(x+1))^{2^t} \\ &= x^{2^k+2^t}(x+1)^{2^h+2^t} T_1^{2^h}. \end{aligned} \tag{5}$$

Comparing the exponents and degrees in 5 gives $t = h$ and $2^h + 2^h = 2^k$. So, $k = h+1$. □

5.2.2. Subcase: $P = T_4$

Lemma 29. *Let r be an odd integer. If $A = x^{2^h r}(x+1)^{2^k} T_4^{2^t}$ is a 2^n-unitary perfect polynomial over $\mathbb{F}_2$, then $r = 5$ and $k = h+2$ and $t = h$..*

Proof. Here $1 + x + \cdots + x^{r-1} = T_4$, so $r = 5$. Comparing exponents in $A = \sigma^*(A)$ yields $t = h$ and $2^h + 3.2^h = 2^k$, hence $k = h+2$. □

5.2.3. Case 2:

We now treat case 2. The symmetry between the factors x and $x+1$ forces identical exponent behavior.

Lemma 30. *Let r and s be odd integers. If $A = x^{2^h r}(x+1)^{2^k s}P^{2^t}$ is a 2^n-unitary perfect polynomial over $\mathbb{F}_2$, then $P = T_1$ and $r = s = 3$.*

Proof. Case 2 in Lemma 27 implies $1 + x + \cdots + x^{r-1} = P = 1 + (x+1) + \cdots + (x+1)^{s-1}$.

By Corollary 1, this forces $P = T_1$.Degree comparison then yields $r = s = 3$. □

Lemma 31. *If $A = x^{2^h.3}(x+1)^{2^k.3}T_1^{2^t}$ is a 2^n-unitary perfect polynomial over $\mathbb{F}_2$, then $k = h,\ t = h+1$.*

Proof. A is 2^n-unitary perfect. Under Case 2 of Lemma 27, symmetry between the factors x and $x+1$ forces $k = h$. Comparing the exponents of T_1 on both sides of the equality of $A^{2^n} = \sigma^*_{2^n}(A)$ then gives $t = h+1$. □

The proof of Theorem 1 is now complete.

Remark 1. *The classification of 2^n-unitary perfect polynomials A with a relatively large number $\omega(A)$ is likely to be highly intricate. The development of new methods will be necessary in order to achieve further progress in this direction.*

6. Conclusion

We completely classified all 2^n-unitary perfect polynomials over $\mathbb{F}_2$ with at most three distinct irreducible factors. We proved that no odd polynomial can satisfy the 2^n-unitary perfect condition. The classification shows that such polynomials must have a highly restricted structure, built from powers of x, $x+1$, and a single Mersenne prime P over $\mathbb{F}_2$.

These results extend previous work on perfect and unitary perfect polynomials and provide a foundation for further investigations involving more irreducible factors or analogous problems over other finite fields.

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