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Posted Date: 2 August 2023

doi: 10.20944/preprints202305.0311.v2

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Article

Formal Calculation

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Abstract: Formal Calculation introduces a way to calculate various nested sums.It uses an auxiliary Form for calculation and provides results in three forms.Besides computation, it is also a powerful tool for analysis and studies various numbers in a unified way. This article contains many results of two types of Stirling numbers, associated Stirling numbers and Eulerian numbers. Formal Calculation provides a method for obtaining combinatorial identities. Applying it to the Gaussian coefficient is a new way for the analysis of q - Binomial. In the process of analysis, this paper makes a great generalization of Euler numbers and polynomials, Wilson's theorem and Wolstenholme's theorem, revealing that they are just special cases. Finally, this article introduces a theorem on symmetry.

Keywords: Formal Calculation; nested sums; Gaussian coefficient; Stirling number; associated Stirling numbers; Eulerian number and polynomial; Wolstenholme theorem

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1. Introduction

Formal Calculation has gone through 3 years of development, and this article contains its summary and latest achievements. The concept is introduced in [1–4] .

Definition 1. *Recursive define the operator ∇^p,p is an integer.*

$$\nabla^0 f(n) = f(n), \sum_{n=0}^{N-1} \nabla^1 f(n+1) = f(N), \sum_{n=0}^{N-1} f(n+1) = \nabla^{-1} f(N)$$

Definition 2. *Recursive define SUM(N,PS,PT),abbreviated as SUM(N).$K_i, D_i \in$ Ring with identity elements.*

$$SUM(N, [K_1 : D_1], [T_1 = 1]) = \sum_{n=0}^{N-1} (K_1 + nD_1)$$

$$SUM(N, [K_1 : D_1, K_2 : D_2], [T_1, T_2 = T_1 + 2 - p]) = \sum_{n=0}^{N-1} (K_2 + nD_2) \nabla^p SUM(n+1, [K_1 : D_1], [T_1])$$

$$If\ f(N) = \sum A_i \binom{N_i}{M_i}\ and\ M_i\ is\ not\ changed\ with\ N, then\ \nabla^p f(N) = \sum A_i \binom{N_i - p}{M_i - p}$$

$[K_1 : D, K_2 : D ... K_M : D]$ is abbreviated as $[K_1, K_2 ... K_M] : D$,$[K_1, K_2 ... K_M] : 1$ is abbreviated as $[K_1, K_2 ... K_M]$.
By default,this paper use:
PS=$[K_1 : D_1, K_2 : D_2 ... K_M : D_M]$,PT=$[T_1, T_2 ... T_M]$,PS1=[PS,$K_{M+1} : D_{M+1}$],PT1=[PT,$T_{M+1}$]
This is actually nested summation. For example:

$$SUM(N, PS, [1, 2, 3 ... M]) = \sum_{n=0}^{N-1} \prod_{i=1}^{M} (K_i + nD_i)$$

$$SUM(N, PS, [1, 3, 5 ... 2M - 1]) = \sum_{n_M=0}^{N-1} (K_M + n_M D_M) ... \sum_{n_2=0}^{n_3} (K_2 + n_2 D_2) \sum_{n_1=0}^{n_2} (K_1 + n_1 D_1)$$

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$$SUM(N,PS,[1,2,4]) = \sum_{n_3=0}^{N-1}(K_3+n_3D_3)\sum_{n=0}^{n_3}(K_1+nD_1)(K_2+nD_2)$$

$$SUM(N,PS,[1,3,4]) = \sum_{n_3=0}^{N-1}(K_3+n_3D_3)(K_2+n_3D_2)\sum_{n=0}^{n_3}(K_1+nD_1)$$

The following use K to represent the set$[K_1, K_2...K_M]$, T to represent the set$[T_1, T_2...T_M]$. Use the Form: $(K_1+T_1)(K_2+T_2)...(K_M+T_M) = \sum\prod_{i=1}^{M} X_i, X_i = T_i\, or\, K_i$

Definition 3. *X(T)=Number of* $\{X_1, X_2...X_M\} \in T$

Definition 4. X_{T-1}*=Number of* $\{X_1, X_2...X_{i-1}\} \in T$*,*$X_{K-1}$*=Number of* $\{X_1, X_2...X_{i-1}\} \in K$

Obviously:$X_{T-1} + X_{K-1} = i-1$

Theorem 1.1. [2] SUM(N,PS,PT)=

$$Form_1 \to \sum_{g=0}^{M} H_1(g)\binom{N+T_M-M}{N-1-g} = \sum_{g=0}^{M} H_1(g)\binom{N+T_M-M}{T_M-M+1+g}, B_i = \begin{cases}(T_i-X_{K-1})D_i, X_i=T_i\\ K_i+X_{T-1}D_i, X_i=K_i\end{cases}$$

$$Form_2 \to \sum_{g=0}^{M} H_2(g)\binom{N+T_M-M+g}{N-1} = \sum_{g=0}^{M} H_2(g)\binom{N+T_M-M+g}{T_M-M+1+g}, B_i = \begin{cases}(T_i-X_{K-1})D_i, X_i=T_i\\ K_i+(X_{K-1}-T_i)D_i, X_i=K_i\end{cases}$$

$$Form_3 \to \sum_{g=0}^{M} H_3(g)\binom{N+T_M-g}{N-1-g} = \sum_{g=0}^{M} H_3(g)\binom{N+T_M-g}{T_M+1}, B_i = \begin{cases}-K_i+(T_i-X_{T-1})D_i, X_i=T_i\\ K_i+X_{T-1}D_i, X_i=K_i\end{cases}$$

The factors of $\prod X_i$ cannot be exchanged.$H_i(g)$, short for $H_i(g,PS,PT)$,is also defined above= $\sum_{X(T)=g}\prod_{i=1}^{M} B_i$

The theorem is proved by induction.There have three forms because:

$$\begin{aligned}&\sum_{n=0}^{N-1} n\binom{n+K}{M} = (M+1)\binom{N+K}{M+2} + (M-K)\binom{N+K}{M+1}\\ &= (M+1)\binom{N+K+1}{M+2} - (1-K)\binom{N+K}{M+1} = (M-K)\binom{N+K+1}{M+2} + (1+K)\binom{N+K}{M+2}\end{aligned}$$

Definition 5. $F_M^{K=\{K_1,K_2...K_M\}}$*=∑(M-dictinct products,factors∈ K),*F_M^N *is short for* $F_M^{\{1,2...N\}}$*,*$F_0^K = 0$

Definition 6. $E_M^{K=\{K_1,K_2...K_M\}}$*=∑(M-products,factors∈ K),*E_M^N *is short for* $E_M^{\{1,2...N\}}$*,*$E_0^K = 0$

Theorem 1.2. $\nabla SUM(N,PS,[1,2...M]) = \prod_{i=1}^{M}(K_i+(N-1)D_i) = \prod_{i=1}^{M}(K_i+nD_i)$

Theorem 1.3. In SUM(N,[...PS...],[...T+1,T+2...T+M...]),K_i can exchange orders.

Theorem 1.4. $SUM(N,[L_1,L_2...L_Q,PS],[L_1,L_2...L_Q,PT]) = \prod_{i=1}^{Q} L_i SUM(N,PS,PT) \to T_1$ can great than 1,$T_i \in \mathbb{N}$

Theorem 1.5. $SUM(N,[1,1...1],[1,2...M]) = SUM(N,[1,1...1],[2,3...M]) = 1^M + 2^M + ... + N^M$

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Theorem 1.6. $SUM(N,[1,1...1],[1,3...2M-1]) = SUM(N,[1,1...1],[3,5...2M-1])$

$$= \sum_{\lambda_1+..+\lambda_N=M,\lambda_i\geq 0} 1^{\lambda_1}2^{\lambda_2}...N^{\lambda_N} = \sum_{1\leq i_1\leq i_2\leq ...\leq i_M\leq N} i_1i_2...i_M = E_M^N = S_2(N+M,N)$$

S_2 is Stirling numbers of the second kind.

Theorem 1.7. $SUM(N,[1,2...M],[1,3...2M-1]) = SUM(N,[2,3...M],[3,5...2M-1])$

$$= \sum_{1\leq i_1<i_2<...<i_M\leq N+M-1} i_1i_2...i_M = F_M^{N+M-1} = S_1(N+M,N)$$

S_1 is unsigned Stirling numbers of the first kind.

Example 1.1:

$$SUM(N,[1,2,3],[1,3,5]) \rightarrow Form = (1+T_1)(2+T_2)(3+T_3)$$

$$\sum_{X(T)=1}\prod X_i = 1\times 2\times T_3 + 1\times T_2\times 3 + T_1\times 2\times 3$$

$H_1(1) = 1\times 2\times (T_3 - X_{K-1}) + 1\times (T_2 - X_{K-1})\times (3+X_{T-1}) + T_1\times (2+X_{T-1})\times (3+X_{T-1})$
$= 1\times 2\times (5-2) + 1\times (3-1)\times (3+1) + 1\times (2+1)\times (3+1) = 26$

$$SUM(N,[1,2,3],[1,3,5]) = 1\times 3\times 5\binom{N+2}{6} + 35\binom{N+2}{5} + 26\binom{N+2}{4} + 1\times 2\times 3\binom{N+2}{3}$$

$$SUM(N,[2,3],[3,5]) = 3\times 5\binom{N+3}{6} + (2\times 4+3\times 4)\binom{N+3}{5} + 2\times 3\binom{N+3}{4}$$

It also can be calculated in the Ring with identity elements.

$SUM(N,[\begin{pmatrix} 7 & 3 \\ 1 & 2 \end{pmatrix} : \begin{pmatrix} 1 & 3 \\ 4 & 2 \end{pmatrix}, \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} : \begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix}],[1,3])$

$= \begin{pmatrix} 15 & 27 \\ 30 & 48 \end{pmatrix}\binom{N+1}{4} + \begin{pmatrix} 44 & 70 \\ 28 & 38 \end{pmatrix}\binom{N+1}{3} + \begin{pmatrix} 17 & 13 \\ 4 & 5 \end{pmatrix}\binom{N+1}{2}$

$H_1(1) = 1\times \begin{pmatrix} 1 & 3 \\ 4 & 2 \end{pmatrix}[\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} + \begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix}] + \begin{pmatrix} 7 & 3 \\ 1 & 2 \end{pmatrix}(3-1)\begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 44 & 70 \\ 28 & 38 \end{pmatrix}$

2. Property

2.1. Equivalence of three forms

By definition:

1. $H_1(g,PS1,PT1) = H_1(g-1)(T_{M+1} - [M-(g-1)])D_{M+1} + H_1(g)(K_{M+1} + gD_{M+1})$
2. $H_2(g,PS1,PT1) = H_2(g-1)(T_{M+1} - [M-(g-1)])D_{M+1} + H_2(g)(K_{M+1} + [M-g-T_{M+1}]D_{M+1})$
3. $H_3(g,PS1,PT1) = H_3(g-1)(-K_{M+1} + (T_{M+1} - [g-1])D_{M+1}) + H_3(g)(K_{M+1} + gD_{M+1})$

Using these relationships and induction can prove:

Theorem 2.1. $H_1(g) = \sum_{k=g}^{M} H_2(k)\binom{k}{g} = \sum_{k=0}^{g} H_3(k)\binom{M-k}{M-g}$ [3]

Inversion$\rightarrow$

Theorem 2.2. $H_2(g) = \sum_{k=g}^{M}(-1)^{k+g}H_1(k)\binom{k}{g}, H_3(g) = \sum_{k=0}^{g}(-1)^{k+g}H_1(k)\binom{M-k}{M-g}$

Calculation with 2.1 $\rightarrow$

Theorem 2.3. $\sum_{g=0}^{M} H_1(g) = \sum_{g=0}^{M} H_2(g)2^g = \sum_{g=0}^{M} H_3(g)2^{M-g}$

Theorem 2.4. $\sum_{g=0}^{M} H_1(g)\binom{A}{B-g} = \sum_{g=0}^{M} H_2(g)\binom{A+g}{B} = \sum_{g=0}^{M} H_3(g)\binom{A+M-g}{B-g}, A, B \in \mathbb{N}$

This $\rightarrow$ $Form_1 = Form_2 = Form_3.A = B \rightarrow$

Theorem 2.5. $\sum_{g=0}^{M} H_1(g)\binom{A}{g} = \sum_{g=0}^{M} H_2(g)\binom{A+g}{g} = \sum_{g=0}^{M} H_3(g)\binom{A+M-g}{M}, A, B \in \mathbb{N}$

Induction [3]$\rightarrow$

Theorem 2.6. $\sum_{g=0}^{M} H_1(g)g\binom{A+1}{B-g} = \sum_{g=0}^{M} H_2(g)g\binom{A+g}{B-1} = \sum_{g=0}^{M} \{H_3(g)g\binom{A+M-g}{B-g} + M \times H_3(g)\binom{A+M-g}{B-1-g}\}$

2.2. Property of H(g)

$\prod X_i = (\prod X_i \in T)(\prod X_i \in K)$.In some cases, H(g) is easy to calculate.

Definition 7. $H(g, T)$, *short for* $H(g, T, PS, PT) = \prod_{X_i \in T} B_i, H(g, \sum T) = \sum \prod_{X_i \in T} B_i$
Also define $H(g, K), H(g, \sum K)$

Theorem 2.7. If $D_i = 1$ and $T_i + 1 = T_{i+1}$,$H_1(g, T) = \prod_{i=1}^{g} T_i$. $H_1(g, T, [K_1, K_2...K_M], [1, 2...M]) = g!$

Theorem 2.8. $H_1(g, \sum K, [1, 1...1], PT) = E_{M-g}^{g+1}$

Theorem 2.9. If $D_i = 1$,$H_1(g, \sum K) = F_{M-g}^K E_0^g + F_{M-g-1}^K E_1^g + ... + F_0^K E_{M-g}^g$

Theorem 2.10. If $D_i = 1$,$H_1(g, \sum T) = F_g^T E_0^{M-g} - F_{g-1}^T E_1^{M-g} + ... + (-1)^g F_0^T E_g^{M-g}$

Theorem 2.11. If $D_i = 1$ and $K_{i+1} - K_i = T_{i+1} - T_i = 1$,$H_1(g)$=$\binom{M}{g} T_1...T_g \times K_{g+1}...K_M$

Theorem 2.12. If PS=PT,$H_1(g) = \prod_{i=1}^{M} T_i \binom{M}{N}, H_2(M) = H_3(0) = \prod_{i=1}^{M} T_i, H_2(g < M) = H_3(g > 0) = 0$

Theorem 2.13. $H_1(g, [AD : D, PS], [A, PT]) = AD(H_1(g-1) + H_1(g)) \rightarrow H_1(g, [1, PS], [1, PT]) = H_1(g-1) + H_1(g)$

Definition 8. $E_p^q \odot ([T_1, T_2....T_M], C)$
$= \sum_{\lambda_1+\lambda_2+...+\lambda_q=p, \lambda_i \geq 0} 1^{\lambda_1}2^{\lambda_2}...q^{\lambda_q}(T_1 + \lambda_1 C)(T_2 + \lambda_1 C + \lambda_2 C)...(T_{q-1} + \lambda_1 C + \lambda_2 C + ... + \lambda_{q-1} C)$

[5] has proved:$\left\langle {M \atop g} \right\rangle = \left\langle {M \atop M-1-g} \right\rangle = E_{M-g-1}^{g+1} \odot ([1, 1...1], 1])$
$\left\langle {M \atop g} \right\rangle$ is Eulerian numbers.Worpitzky identity:$N^M = \sum_{g=0}^{M-1} \left\langle {M \atop g} \right\rangle \binom{N+g}{M}$

$$\left\langle {5 \atop 2} \right\rangle = E_2^3 \odot ([1, 1...], 1) = \sum_{\lambda_1+\lambda_2+\lambda_3=2} 1^{\lambda_1}2^{\lambda_2}3^{\lambda_3}(1+\lambda_1)(1+\lambda_1+\lambda_2)$$

$$= 1^2 2^0 3^0(1+2)(1+2) + 1^0 2^2 3^0(1+0)(1+0+2) + 1^0 2^0 3^2(1+0)(1+0+0)$$

$$+1^1 2^1 3^0(1+1)(1+1+1) + 1^1 2^0 3^1(1+1)(1+1+0) + 1^0 2^1 3^1(1+0)(1+0+1) = 66$$

Theorem 2.14. $H_1(g, [1, 1...1], PT = [T_i = T_1 + (C+1)(i-1)]) = E_{M-g}^{g+1} \odot (PT, C)$

Theorem 2.15. $H_3(g, [1, 1...1], PT = [T_i = T_1 + (C+1)(i-1)]) = E_{M-g}^{g+1} \odot ([T_i = T_1 - 1 + C(i-1)], C+1)$

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2.3. Shape of numbers

In this section,if not specifically mentioned,$T_1 = 1, T_{i+1} - T_i = 1$ *or* 2.
To calculate $\sum\limits_{1 \le K_1 < K_2 < ... < K_M \le N} K_1 K_2 ... K_M$ (*),products needs to be divided into 2^{M-1} categories.
There are **M-1** intervals between factors.If the interval=1,define as **continuity**. If the interval>1,define as **discontinuity**. Continuities, Discontinuities and their Positions, is defined as **Shape**. So there have 2^{M-1} Shapes.From the definition of nested sum:

$$\sum_{1 \le K_1 < K_2 \le N} K_1 K_2 = SUM(N, [1,2], [1,2]) + SUM(N-1, [1,3], [1,3])$$

$$\sum_{1 \le K_1 < K_2 < K_3 \le N} K_1 K_2 K_3 =$$
$$SUM(N, [1,2,3], [1,2,3]) + SUM(N-1, [1,2,4], [1,2,4]) + SUM(N-1, [1,3,4], [1,3,4]) + SUM(N-2, [1,3,5], [1,3,5])$$

Definition 9. *PB(PT)=Number of* $T_i + 1 < T_{i+1}$*=Number of discontinuities*

$(*) = \sum\limits_{All\ of\ the\ Shapes} SUM(N - PB(PT), PT, PT)$.From 2.12 we can obtain a simple formula:

Theorem 2.16. $SUM(N, PT, PT) = \prod\limits_{i=1}^{M} T_i \binom{N+T_M}{M+1}, T_i \in \mathbb{N}$

This is an important conclusion,generalizes the famous formula $\sum\limits_{n=0}^{N-1} \binom{n}{M} = \binom{N}{M+1}$. It was discovered during the calculation of (*) and leads to the birth of Formal Calculation. There are two ways to understand SUM(N,PT,PT),one is nested sum,another is the intervals between factors.

Theorem 2.17. Number of Products in SUM(N,PT,PT) =$\binom{N+PB(PT)}{PB(PT)+1}$

Definition 10. $MIN_g(M) = \sum\limits_{PB(PT)=g} \prod\limits_{i=1}^{M} T_i = 1 \times \sum\limits_{PB(PT)=g} \prod\limits_{i=2}^{M} T_i$

This is the sum of the products of PTs with the same number of discontinuities.

$$(*) = \sum_{g=0}^{M-1} MIN_g(M) \binom{N}{g+1}$$

By definition:

Theorem 2.18. $MIN_g(M) = \sum \frac{(M+g)!}{i_1 i_2 ... i_g}, 2 \le i_1 < i_2 < ... < i_g \le M + g - 1, i_{j+1} - i_j \ge 2$

Based on the concept of Shape rather than 2.18, it is easier to understand.eg:

$$MIN_1(3) = (124) + (134), MIN_2(4) = (12357) + (12457) + (13457) + (12467) + (13467) + (13567)$$

Here (...) is products.
From the definition of nested sum,there exists general classification principles:

Theorem 2.19. $SUM(N, [K_1 : D_1, K_2 : D_2 ... K_M : D_M], [T_1, T_2 ... T_M])$

$$= SUM(N, PS, [T_1 ... T_i, T_{i+1} - 1 ... T_M - 1]) + SUM(N-1, [K_1 : D_1 ... K_i : D_i, K_{i+1} + D_{i+1} : D_{i+1} ... K_M + D_M : D_M], PT)$$

$$eg : SUM(N, [1,2,3], [1,3,5]) = SUM(N, [1,2,3], [1,2,4]) + SUM(N-1, [1,3,4], [1,3,5])$$
$$= SUM(N, [1,2,3], [1,2,3]) + SUM(N-1, [1,2,4], [1,2,4]) + SUM(N-1, [1,3,4], [1,3,4]) + SUM(N-2, [1,3,5], [1,3,5])$$
$$(*) = SUM(N, [1,2...M], [1,3...2M-1]). It's\ exactly\ 1.7.$$

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We can also define Shape for general PS:

$$\prod (K_i + \lambda_i D_i), \begin{cases} \lambda_i = \lambda_{i+1}, continuity \\ \lambda_i < \lambda_{i+1}, discontinuity \end{cases}$$

Definition 11. $MIN_g(PS) = K_1 \times \sum_{PB(\prod (K_i + \lambda_i D_i)) = g} \prod_{i=2}^{M} (K_i + \lambda_i D_i), \lambda_1 = 0, \lambda_{i+1} - \lambda_i = 0 \; or \; 1$

By definition of H(g):

Theorem 2.20. $K_1 \times H_1(g, [K_2 : D_2 ... K_M : D_M], [\frac{K_2}{D_2} + 1, \frac{K_3}{D_3} + 2 ... \frac{K_M}{D_M} + M - 1]) = MIN_g(PS)$

Theorem 2.21. $K_1 \times H_2(g, [D_2 : D_2 ... D_M : D_M], [\frac{K_2}{D_2} + 1, \frac{K_3}{D_3} + 2 ... \frac{K_M}{D_M} + M - 1]) = (-1)^{M-1-g} MIN_g(PS)$

2.4. H(g) and Associated Stirling Numbers

Associated Stirling Numbers of the first kind $S_{1,r}(n,k)$ is defined as the number of permutations of a set of n elements having exactly k cycles, all length >= r.

1. $S_{1,r}(n,k) = \frac{n!}{k!} \sum_{i_1 + i_2 + ... + i_k = n, i_j \geq r} \frac{1}{i_1 i_2 ... i_k}$
2. $S_{1,r}(n+1,k) = n S_{1,r}(n,k) + (n)_{r-1} S_{1,r}(n-r+1, k-1), n \geq kr$
3. $\sum \frac{(n-1)!}{i_1 i_2 ... i_{k-1}}, r \leq i_1 < i_2 < ... < i_{k-1} \leq n-r, i_{j+1} - i_j \geq r$ [6]

Derived from 2 and definition of H(g) or 3 and 2.18:

Theorem 2.22. $MIN_g(M) = S_{1,2}(M+g+1, g+1) = \frac{(M+g+1)!}{(g+1)!} \sum_{i_1 + i_2 + ... + i_{g+1} = M+g+1, i_j \geq 2} \frac{1}{i_1 i_2 ... i_{g+1}}$

Table 1. Table of $MIN_g(M) = S_{1,2}(M+g+1, g+1)$.

	g=0	g=1	g=2	g=3	g=4	g=5	g=6
M=1	1						
M=2	2	3					
M=3	6	20	15				
M=4	24	130	210	105			
M=5	120	924	2380	2520	945		
M=6	720	7308	26432	44100	34650	10395	
M=7	5040	64224	303660	705320	866250	540540	135135

Associated Stirling Numbers of the second kind $S_{2,r}(n,k)$ is defined as the number of permutations of a set of n elements having exactly k blocks, all length >= r.

1. $S_{2,r}(n,k) = \frac{n!}{k!} \sum_{i_1 + i_2 + ... + i_k = n, i_j \geq r} \frac{1}{i_1! i_2! ... i_k!}$
2. $S_{2,r}(n+1,k) = k S_{2,r}(n,k) + \binom{n}{r-1} S_{2,r}(n-r+1, k-1), n \geq kr$

Derived from 2:

Theorem 2.23. $H_2(g, [1, 1 ... 1], [3, 5 ... 2M-1]) = S_{2,2}(M+g+1, g+1) = \frac{(M+g+1)!}{(g+1)!} \sum_{i_1 + ... + i_{g+1} = M+g+1, i_j \geq 2} \frac{1}{i_1! ... i_{g+1}!}$

Table 2. Table of $H_2(g,[1,1...1],[3,5...2M-1]) = S_{2,2}(M+g+1,g+1)$.

	g=0	g=1	g=2	g=3	g=4	g=5	g=6
M=1	1						
M=2	1	3					
M=3	1	10	15				
M=4	1	25	105	105			
M=5	1	56	490	1260	945		
M=6	1	119	1918	9450	17325	10395	
M=7	1	246	6825	56980	190575	270270	135135

$$SUM(N,[2,3...M],[3,5...2M-1]) = S_1(N+M,N) = S_{1,1}(N+M,N)$$
$$= \sum_{g=0}^{M-1} S_{1,2}(M+g+1,g+1)\binom{N+M}{M+1+g} = \sum_{g=1}^{M} S_{1,2}(M+g,g)\binom{N+M}{M+g}$$
$$S_{1,1}(N+M,N) \to = \frac{(N+M)!}{N!}\sum_{i_1+i_2+...+i_N=N+M, i_j\geq 1}\frac{1}{i_1 i_2 ... i_N}$$
$$= \frac{(N+M)!}{N!}\sum_{g=1}^{M}\Big[\sum_{i_1+...+i_N=N+M, i_j\geq 1, Number\ of\ i_j>1=g}\frac{1}{i_1 i_2 ... i_N}\Big] = \sum_{g=1}^{M} f(*)$$
$$f(*) = \frac{(N+M)!}{N!}\binom{N}{N-g}\sum_{i_1+...+i_g=g+M, i_j\geq 1}\frac{1}{i_1 i_2 ... i_g} = \binom{N+M}{M+g} S_{1,2}(M+g,g)$$

Use the same way:

Theorem 2.24. $S_{1,r}(rN+M,N) = \sum_{g=1}^{M}\frac{1}{r^{N-g}}\frac{(rN-rg)!}{(N-g)!}\binom{rN+M}{rg+M} S_{1,r+1}(rg+M,g)$

Theorem 2.25. $S_{2,r}(rN+M,N) = \sum_{g=1}^{M}\frac{1}{(r!)^{N-g}}\frac{(rN-rg)!}{(N-g)!}\binom{rN+M}{rg+M} S_{2,r+1}(rg+M,g)$

2.5. Table of H(g)

Table 3. Table of H(g).

PS	PT	$H_1(g)$	$H_2(g)$	$H_3(g)$
$[1,1...1]$	$[1,2...M]$	$g!E_{M-g}^{g+1} = g!S_2(M+1,g+1)$	$(-1)^{M-g}g!S_2(M,g)$	$\left\langle {M \atop g} \right\rangle$
$[1,1...1]$	$[2,3...M]$	$(g+1)!S_2(M,g+1)$	$(-1)^{M-1-g}(g+1)!S_2(M,g+1)$	$\left\langle {M \atop g} \right\rangle$
$[1,1...1]$	$[1,3...2M-1]$	$E_{M-g}^{g+1}\odot(PT,1)$	$(-1)^{M-g}MIN_{g-1}(M)$	$E_{M-g}^{g+1}\odot([0,1...],2)$
$[1,1...1]$	$[3,5...2M-1]$	$S_{2,2}(M+1+g,g+1)$	$(-1)^{M-1-g}MIN_g(M)$	$E_{M-1-g}^{g+1}\odot([2,3...],2)$
$[1,2...M]$	$[1,3...2M-1]$	$MIN_{g-1}(M)+MIN_g(M)$	$1\times(-1)^{M-g}E_{M-g}^{g}\odot([3,5...],1)$	$1\times E_g^{M-g}\odot([2,3...],2)$
$[2,3...M]$	$[3,5...2M-1]$	$MIN_g(M)$	$(-1)^{M-1-g}S_{2,2}(M+1+g,g+1)$	$E_g^{M-g}\odot([2,3...],2)$

3. Application

3.1. Number analysis

Theorem 3.1. $\prod(K_1+nD_i)$ can be decomposed into three forms by 1.1 and ∇.

Theorem 3.2. SUM(N,PS,PT)=SUM(N,[1,1...1,PS],[1,1...1,PT]) expand $\sum_{g=0}^{M}(...) = \sum_{g=0}^{M+(number\ of\ 1\ added)}(...)$

PS=[1,1...1],PT=[1,2...M], 2.1→ $H_1(M) = \sum_{g=0}^{M} H_3(g), H_1(1) = \sum_{g=1}^{M} H_2(g)g$

Theorem 3.3. $\sum_{g=0}^{M} \left\langle {M \atop g} \right\rangle = M!, \sum_{g=1}^{M} (-1)^{M-g} g \times g! S_2(M,g) = 2^M - 1$

PS=[1,1...1],PT=[2,3...M], 2.1→ $\sum_{k=0}^{M} (-1)^{M-k} k! S_2(M,k) = 1$

Theorem 3.4. $g!S_2(M,g) = \sum_{k=g}^{M} (-1)^{M-k} k! S_2(M,k) \binom{K-1}{g-1} = \sum_{k=0}^{g-1} \left\langle {M \atop k} \right\rangle \binom{M-1-k}{M-g}, 1 \leq g \leq M$

SUM(N,[2,3...M],[3,5...2M-1]),SUM(N,[1,1...1],[3,5...2M-1]), 2.1 →

Theorem 3.5. $S_{1,2}(M+g,g) = \sum_{k=g}^{M} (-1)^{M-k} S_{2,2}(M+k,k) \binom{K-1}{g-1}, S_{2,2}(M+g,g) = \sum_{k=g}^{M} (-1)^{M-k} S_{1,2}(M+k,k) \binom{K-1}{g-1}$

SUM(N,[1,1...1],[2,3...M]), 2.3 →

Theorem 3.6. $\sum_{g=1}^{M} g!S_2(M,g) = \sum_{g=1}^{M} (-1)^{M-g} g! S_2(M,g) 2^{g-1} = \sum_{g=1}^{M} \left\langle {M \atop M-g} \right\rangle 2^{M-g}$

SUM(N,[1,1...1],[3,5...2M-1]),SUM(N,[2,3...M],[3,5...2M-1]), 2.3 →

Theorem 3.7. $\sum_{g=1}^{M} S_{2,2}(M+g,g) = \sum_{g=1}^{M} (-1)^{M-g} S_{1,2}(M+g,g) 2^{g-1}, \sum_{g=1}^{M} S_{1,2}(M+g,g) = \sum_{g=1}^{M} (-1)^{M-g} S_{2,2}(M+g,g) 2^{g-1}$

SUM(N,[1,1...1],[1,2...M]), 2.6 →

Theorem 3.8. $\sum_{g=1}^{M} g!S_2(M,g)(g-1)\binom{A+1}{g} = \sum_{g=1}^{M} (-1)^{M-g} g! S_2(M,g)(g-1)\binom{A+g-1}{g}$

PS=PT=[1,2...M],$H_1(g) = H_1(M-g) = M!\binom{M}{g}$, 2.9 → $g!(F^K_{M-g}E^g_0 + F^K_{M-g-1}E^g_1 + ... + F^K_0 E^g_{M-g})$

Theorem 3.9. $M!\binom{M}{g} = g! \sum_{i=0}^{M-g} S_1(M+1, g+1+i) S_2(g+i, g) = (M-g)! \sum_{i=0}^{g} S_1(M+1, M+1-i) S_2(M-i, M-g)$

$H_1(g, [1,1...1], [1,2...M]) = g!S_2(M+1, g+1), F_i^{\{1,1...1\}} = \binom{M}{i}$, 2.9 →

Theorem 3.10. $S_2(M+1, g+1) = \sum_{i=0}^{M-g} S_2(g+i, g) \binom{M}{g+i} = \sum_{i=0}^{M-g} S_2(M-i, g) \binom{M}{i}$

$H_1(g, [1,2...M], [1,2...M]) = H_1(g, [M, M-1...1], [1,2...M]) = M!\binom{M}{g}$ 2.10 →
$H_1(g) = \frac{M!}{g!}(F^M_g E^{M-g}_0 - F^M_{g-1} E^{M-g}_1 + ... + (-1)^g F^M_0 E^{M-g}_g)$

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Theorem 3.11. $g!\binom{M}{g} = \sum_{i=0}^{g} S_1(M+1, M+1-g+i)S_2(M-g+i, M-g)(-1)^i$

$H_1(g, [K+i], [T+i])$, 2.11 $\rightarrow \binom{M}{g} T_1 ... T_g \times K_{g+1} ... K_M$, 2.9$\rightarrow T_1 ... T_g(...)$, 2.10$\rightarrow K_{g+1} ... K_M(...)$

Theorem 3.12. $\prod_{i=g+1}^{M}(K+i)\binom{M}{g} = F_{M-g}^{\{K+i\}}E_0^g + ... + F_0^{\{K+i\}}E_{M-g}^g, \prod_{i=1}^{g}(T+i)\binom{M}{g} = F_g^{\{T+i\}}E_0^{M-g} + ... + (-1)^g F_0^{\{T+i\}}E_g^{M-g}$

SUM(1,[1,1...1],[3,5...2M-1])=1,SUM(1,[2,3...M],[3,5...2M-1])=M!,= $SUM(1) = \sum_{g=0}^{M-1} H_2(g)\binom{M+g+1}{M+g+1} \rightarrow$

Theorem 3.13. $\sum_{g=0}^{M-1}(-1)^{M-1-g}MIN_g(M) = \sum_{g=1}^{M}(-1)^{M-g}S_{1,2}(M+g, g) = 1, \sum_{g=1}^{M}(-1)^{M-g}S_{2,2}(M+g, g) = M!$

2.20 and 2.21 $\rightarrow$

Theorem 3.14. $\sum_{g=0}^{M-1}(-1)^{M-1-g}MIN_g(PS) = K_1\prod_{i=2}^{M} D_i$

$SUM(N, [T, T...T], [T, T...T]) = T^M\binom{N+T}{T+1} = \sum_{g=0}^{M} T^M\binom{M}{g}\binom{N+T-M}{T-M+1+g} \rightarrow$

Theorem 3.15. $\sum_{g=0}^{M}\binom{M}{g}\binom{N+T-M}{T-M+1+g} = \binom{N+T}{T+1}, T \in \mathbb{N}$

3.2. Congruences

P is prime.K_i is any integer.

Theorem 3.16. $(P, D) = 1, SUM(P, [K_1, K_2...K_M] : D, [1, 2...M]) = \sum_{g=0}^{M} H_1(g)\binom{P}{1+g} \equiv \begin{cases} 0 \pmod{P}, M<P-1 \\ -1 \pmod{P}, M=P-1 \end{cases}$

Proof. If M=P-1,$SUM(P) \equiv H_1(P-1)\binom{P}{P} \equiv H_1(P-1) \equiv (P-1)!D^{P-1} \equiv -1 \pmod{P}$ $\square$

If a product has a factor $\equiv 0 \pmod{P}$,ignore it;change factors to its minimum positive residue,we can obtain many congruences.Wilson's Theorem is just a special case . $A, B, C \in \mathbb{N}$

$$1^A2^B + 2^A3^B + ... + (P-2)^A(P-1)^B \equiv 1^A3^B + 2^A4^B + ... + (P-3)^A(P-1)^B + (P-1)^A1^B \equiv \begin{cases} 0 \pmod{P}, A+B<P-1 \\ -1 \pmod{P}, A+B=P-1 \end{cases}$$

$$1^A2^B3^C + 2^A3^B4^C + ... + (P-3)^A(P-2)^B(P-1)^C \equiv \begin{cases} 0 \pmod{P}, A+B+C<P-1 \\ -1 \pmod{P}, A+B+C=P-1 \end{cases}$$

Theorem 3.17. $\sum_{0<K_i,K_j<P,K_i\neq K_j} {K_1}^{\lambda_1}{K_2}^{\lambda_2}...{K_q}^{\lambda_q} \equiv \begin{cases} -1 \pmod{P}, \lambda_1+\lambda_2+...+\lambda_q=P-1 \\ 0 \pmod{P}, \lambda_1+\lambda_2+...+\lambda_q<P-1 \end{cases}, \lambda_i \in \mathbb{N}$

Theorem 3.18. $MIN_g(M) \equiv 0 \pmod{P}, 0 < g < M-1, g+M = P-1$

Wolstenholme's Theorem is also a special case.P>3.

1. Wolstenholme's Theorem:$(P-1)!\sum_{n=1}^{P-1}\frac{1}{n} = \sum_{0<K_i,K_j<P,K_i\neq K_j} K_1K_2...K_{P-2} \equiv 0 \pmod{P^2}$
2. $\sum_{n=1}^{P-1} n^{P-2} \equiv 0 \pmod{P^2}$

They are two extremes.In fact,there have:

Theorem 3.19. $\sum\limits_{0<K_i,K_j<P,K_i\neq K_j} {K_1}^{C_1}{K_2}^{C_2}...{K_q}^{Cq} \equiv 0 \pmod{P^2}, C_1+C_2+...+C_q = P-2, C_i > 0$

Proof.
If $X \pmod{P^2}$ and $X+Y \pmod{P^2}$ then $Y \pmod{P^2}$.The Sum has symmetry.
For $\sum AB^{P-3}$,pair each AB^{P-3} with $(P-A)B^{P-3}$ to PB^{P-3},$(P-A)B^{P-3} \in \sum AB^{P-3}$ or $\sum B^{P-2}$.
$\sum(Paired\ Prodcuts) = x\sum AB^{P-3} + y\sum B^{P-2} = zP\sum B^{P-3}, 0 < x,y < P$.
$2 \to y\sum B^{P-2} \equiv 0 \pmod{P^2}; P\sum B^{P-3} \equiv 0 \pmod{P^2}$.So $\sum AB^{P-3} \equiv 0 \pmod{P^2}$.
Similarly:

$$\sum(Paired\ Prodcuts\ of\ ABC^{P-4} + B^2C^{P-4}) = x\sum ABC^{P-4} + y\sum B^2C^{P-4} \equiv 0 \pmod{P^2}, 0 < x,y < P$$

$$x\sum ABC^{P-4} + y\sum B^2C^{P-4}, symmetry \to$$
$$\equiv z\sum(A+B)BC^{P-4} \equiv z\sum ABC^{P-4} \pmod{P}, 0 < z < P \to$$
$$\sum ABC^{P-4} \equiv \sum B^2C^{P-4} \equiv 0 \pmod{P^2}$$

Prove the conclusion in a similar way... □

Theorem 3.20. $E_{P-2}^{P-1} = S_2(2P-3, P-1) \equiv 0 \pmod{P^2}; E_{P-2}^{P} = S_2(2P-2, P) \equiv 0 \pmod{P^2}$

$$eg: S_2(7,4) = 350 \equiv 0 \pmod{25}, S_2(8,5) = 1050 \equiv 0 \pmod{25}$$
$$eg: S_2(11,6) = 179487 \equiv 0 \pmod{49}, S_2(12,7) = 627396 \equiv 0 \pmod{49}$$

4. Combinatorial Identities

Definition 12. *R-FOLD SUM:*$\sum_{(r)}^{N} f(k) = \sum\limits_{k_r=1}^{N} ... \sum\limits_{k_2=1}^{k_3} \sum\limits_{k_1=1}^{k_2} f(k_1) = \sum\limits_{k_r=0}^{N-1} ... \sum\limits_{k_2=0}^{k_3} \sum\limits_{k_1=0}^{k_2} f(k_1+1)$

By nested sum:

Theorem 4.1. $\sum_{(r)}^{N} \nabla^p SUM(k, PS, PT) = \nabla^{p-r} SUM(N, PS, PT)$

Theorem 4.2. $\sum\limits_{k_r=1}^{N} ... \sum\limits_{k_2=1}^{k_3} \sum\limits_{k_1=1}^{k_2} \nabla SUM(k_r, PS, [1,2...M]) = SUM(N, PS, [T_i = i + r - 1])$

Proof.

$$PT = [T_i = i + r - 1]$$
$$PS1 = [1:0, 1:0...1:0, PS], PT1 = [1,3...2(r-1)-1, 1+2(r-1), 2+2(r-1)...M+2(r-1)]$$
$$B_{i<r} = \begin{cases} (T_i - X_{T-1})D_i{=}0, X_i \in T \\ K_i + X_{K-1}D_i{=}1, X_i \in K \end{cases} \to H_1(g > M, PS1, PT1) = 0, H_1(g \leq M, PS1, PT1) = H_1(g, PS, PT)$$
$$SUM(N, PS1, PT1) = \sum_{g=0}^{M} H_1(g, PS, PT)\binom{N+r-1}{r+g} = \sum_{k_r=1}^{N} \nabla SUM(k_r, PS, [1,2...M])... \sum_{k_2=1}^{k_3}\sum_{k_1=1}^{k_2} 1$$

□

Theorem 4.3. $\sum\limits_{k_x=1}^{N} ... \sum\limits_{k_2=1}^{k_3} \sum\limits_{k_1=1}^{k_2} \nabla SUM(k_r, PS, [1,2...M]) = \nabla^{r-x} SUM(N, PS, [T_i = i + (r-1)])$

$$eg: (*) \sum_{k_r=1}^{N} ... \sum_{k_2=1}^{k_3} \sum_{k_1=1}^{k_2} \binom{k_i}{j} = \frac{1}{j!}\nabla^{i-r} SUM(N, [0, -1, -2... -(j-1)], [T_n = n + (i-1)])$$

$$H_1(g<j)=0, T_j-j=i-1\rightarrow$$

$(*) = \frac{1}{j!}\nabla^{i-r}H_1(j)\binom{(N+1)+(i-1)}{j+1+(i-1)} = \frac{1}{j!}\nabla^{i-r}\frac{(j+i-1)!}{(i-1)!}\binom{N+i}{j+i} = \binom{j+i-1}{i-1}\binom{N+i-(i-r)}{j+i-(i-r)} = \binom{j+i-1}{i-1}\binom{N+r}{j+r}$ [7]

Using induction to prove:

Theorem 4.4. $\sum\limits_{0\le n_1\le ....\le n_M\le N-1}(K+n_1D_1 + ... + n_MD_M) = \nabla^{\binom{M}{2}}SUM(N,[K : \frac{D_1+2D_2+....+MD_M}{\binom{M+1}{2}},[\binom{M+1}{2}])$

$$=(D_1+2D_2+....+MD_M)\binom{N+M-1}{M+1}+K\binom{N+M-1}{M}$$

$$\sum_{0\le n_1\le n_2\le ....\le n_M\le N-1}(K+n_1+n_2+....+n_M)=\binom{M+1}{2}\binom{N+M-1}{M+1}+K\binom{N+M-1}{M}$$

$\sum\limits_{0\le n_1\le ....\le n_p\le n}(n_1+....+n_p)=\sum\limits_{n_p=0}^{n}\sum\limits_{n_{p-1}=0}^{n_p}...\sum\limits_{n_1=0}^{n_2}(n_1+....+n_p)=\binom{p+1}{2}\binom{n+p}{p+1}=\frac{pn}{2}\binom{n+p}{p}$. [7]

1.2 , 1.3 , 1.4 can be used to derive combinatorial identities.

Theorem 4.5. $\binom{n+A}{A}\binom{n+M+B}{M} =$

$$\sum_{g=0}^{M}\binom{A+g}{g}\binom{M+B}{M-g}\binom{n+A}{A+g}=\sum_{g=0}^{M}(-1)^{M-g}\binom{A+g}{g}\binom{A-B}{M-g}\binom{n+A+g}{A+g}=\sum_{g=0}^{M}\binom{A-B}{g}\binom{M+B}{M-g}\binom{n+A+M-g}{A+M}$$

Proof.

$$original=\frac{1}{A!M!}\nabla SUM(N,[1,2...A,B+1,B+2....B+M],[1,2...A,A+1,A+2....A+M])$$
$$=\frac{1}{M!}\nabla SUM(N,[B+1,B+2....B+M],[A+1,A+2....A+M])$$
$$H_1(g)=\binom{M}{g}(A+1)....(A+g)(B+g+1)....(B+M)=\binom{M}{g}g!\binom{A+g}{g}(M-g)!\binom{B+M}{M-g}$$

Using a similar method to obtain $H_2(g), H_3(g)$

□

Theorem 4.6. $\binom{n+X}{A}\binom{n+Y}{M}=\sum\limits_{g=0}^{A}\binom{M+g}{g}\binom{M+X-Y}{A-g}\binom{n+Y}{M+g}, 0\le Y\le M$

Proof.

$$original=\frac{1}{A!M!}\nabla SUM(N,[X,X-1...X-A+1,Y,Y-1...Y-M+1],[1,2...A+M])$$
$$=\frac{1}{A!M!}\nabla SUM(N,[1,2...Y,0,-1,-2...-(M-Y)+1,X,X-1...X-A+1],[1,2...A+M])$$
$$=\frac{Y!}{A!M!}\nabla SUM(N,[0,-1,-2...-(M-Y)+1,X,X-1...X-A+1],[Y+1,Y+2...A+M])$$
$$If\ H_1(g)\neq 0\ then\ X_1,X_2...X_{M-Y}\in T\rightarrow H_1(g<M-Y)=0, Let\ C=A+M-Y$$
$$H_1(g\ge M-Y)\neq 0\rightarrow Number\ of\ K=\binom{C-(M-Y)}{C-g}\rightarrow H_1(g\ge M-Y,\sum K)=\binom{C-(M-Y)}{C-g}[X+M-Y]_{C-g}$$
$$H_1(g\ge M-Y)=\binom{C-(M-Y)}{C-g}[X+M-Y]_{C-g}[Y+1]^g$$
$$original=\frac{Y!}{M!A!}\sum_{g=M-Y}^{C}\binom{C-(M-Y)}{C-g}[X+M-Y]_{C-g}[Y+1]^g\binom{n+Y}{Y+g}, -q=M-Y-g\rightarrow$$
$$=\frac{Y!}{M!A!}\sum_{q=0}^{A}\binom{A}{A-q}\binom{X+M-Y}{A-q}(A-q)!\binom{M+q}{M-Y+q}(M-Y+q)!\binom{n+Y}{M+q}$$

□

PS=[0,-1,-2...-(B-1),A-(M-B-1):-1,A-(M-B-1)+1:-1...A:-1],PT=[1,2...M]→

Theorem 4.7. $\binom{n}{B}\binom{A-n}{M-B} = \sum_{g=0}^{M-B}(-1)^{M-B-g}\binom{A-M+g}{g}\binom{M-g}{B}\binom{n}{M-g}, n \geq 0$

PS=[0,-1,-2...-A+1,0,-1,-2...-B+1],PT=[1,2...A+B]→

Theorem 4.8. $\binom{n}{A}\binom{n}{B} = \sum_{g=0}^{B}\binom{B}{g}\binom{A+B-g}{B-g}\binom{n}{A+B-g}$. record at [8]: (6.44)

Theorem 4.9. $\prod_{i=1}^{M}(A+2i+n) = \binom{n+A}{A}^{-1}\sum_{g=0}^{M}(2(M-g)-1)!!\binom{2M-g}{g}[A+1]^g\binom{n+A+g}{A+g}$ A ∈ ℕ or A = 0

Proof.

PS=[A+2,A+4...A+2M],PT=[A+1,A+2...A+M]

$$H_2(g,\sum K) = SUM(g+1,[1,3...2(M-g)-1],[1,3...2(M-g)-1])) = (2(M-g)-1)!!\binom{2M-g}{g}, H_2(g,T) = [A+1]^g$$

$$\binom{n+A}{A}\prod_{i=1}^{M}(A+2i+n) = \frac{1}{A!}\nabla SUM(N,[1,2...A,PS],[1,2...A,PT]) = \nabla SUM(N,PS,PT)$$

□

Theorem 4.10. SUM(N,[A+1,A+3...A+2M-1],[1,3...2M-1])=$\sum_{g=0}^{M}[A]^{M-g}(2g-1)!!\binom{M+g}{2g}\binom{N+M-1+g}{M+g}$

Proof.

$$H_2(g,\sum T) = SUM(M-g+1,[1,3...2g-1],[1,3...2g-1])) = (2g-1)!!\binom{M+g}{2g}, H_2(g,K) = [A]^{M-g}$$

□

Theorem 4.11. SUM(N,[A,A+1...A+M-1]:2,[1,3...2M-1])=$\binom{M+N-1}{M}[A+M+N-2]_M$

Proof.

$$SUM(g+1,[A,A+1...A+M-1-g]:2,[1,3...2(M-g)-1])$$
$$= H_1(g,\sum K,[A,A+1...A+M-1],[1,2...M]) = \binom{M}{g}[A+M-1]_{M-g}$$

□

$$SUM(N,[1,2...M]:2,[1,3...2M-1]) = M!\binom{N+M-1}{M}^2, M = 1 \to 1+3+...+(2N-1) = N^2$$

4.11 and 4.5 →

Theorem 4.12. $\frac{1}{M!}H(g,[A+1,A+2...A+M]:2,[1,3...2M-1]) = H(g,[A+1,A+2...A+M],[M+1,M+2...2M])$

$$This \to H_1(g,[A+1,A+2...A+M]:2,[1,3...2M-1]) = M!\binom{M+g}{g}\binom{M+A}{M-g}$$

$$Definition\ of\ H_1(g) \to$$

$$H_1(1) = 2\{(A+1)...(A+M-1)\times M + (A+1)...(A+M-2)\times(M-1)\times(A+M+2)\times(M-1)+...\} \to$$

 doi:10.20944/preprints202305.0311.v2

Theorem 4.13. $\sum_{n=0}^{M-1}(M-n)\prod_{k=1}^{M-1}(A+k+3[\frac{k+n}{M}])=\frac{(M+1)!}{2}\binom{M+A}{M-1}$.[n] is truncates integer.

Theorem 4.14. $SUM(N,[A+2,A+4...A+2M]:3,[1,3...2M-1]) = \sum_{g=0}^{M}\binom{A+N-1+g}{g}\binom{N+M-1-g}{M-g}\frac{[M+N-g]^M}{2^{M-g}}$

Proof.

$$\text{PS1=[A+2,A+4...A+2M],PT1=[A+1,A+2...A+M]}$$
$$H_1(g,PS1,PT1)=[A+1]^g SUM(g+1,[A+2,A+4...A+2(M-g)]:3,[1,3...2(M-g)-1]))$$
$$=\sum_{k=g}^{M}H_2(g,PS1,PT1)\binom{k}{g},4.9\rightarrow=\sum_{k=g}^{M}(2(M-k)-1)!!\binom{2M-k}{k}[A+1]^k\binom{k}{g}$$

□

5. Matrix of SUM(N)

Consider H(g) as variables,list SUM(N),SUM(N+1)...SUM(N+M),we can obtain a (M+1) ×(M+1) matrix.
Let $P=N+T_M-M, Q=N-1$,define A(P,Q,M), respectively corresponding to the three forms.

$$A_1(P,Q,M)=\begin{pmatrix}\binom{P}{Q} & \cdots & \binom{P}{Q-M}\\ \vdots & \ddots & \vdots\\ \binom{P+M}{Q+M} & \cdots & \binom{P+M}{Q}\end{pmatrix}$$

$$A_2(P,Q,M)=\begin{pmatrix}\binom{P}{Q} & \cdots & \binom{P+M}{Q}\\ \vdots & \ddots & \vdots\\ \binom{P+M}{Q+M} & \cdots & \binom{P+2M}{Q+M}\end{pmatrix}$$

$$A_3(P,Q,M)=\begin{pmatrix}\binom{P+M}{Q} & \cdots & \binom{P}{Q-M}\\ \vdots & \ddots & \vdots\\ \binom{P+2M}{Q+M} & \cdots & \binom{P+M}{Q}\end{pmatrix}$$

Theorem 5.1. $\| A_1(P,Q,M) \|=\| A_2(P,Q,M) \|=\| A_3(P,Q,M) \|, \| A(P,Q,M) \|=\| A(P,P-Q,M) \|$ [3]

Theorem 5.2. $\| A(P,0,M) \|=1, \| A(P,1,M) \|=\binom{P+M}{1+M}, \| A(P,Q>1,M) \|=\prod_{g=0}^{Q-1}\frac{\binom{P+M-g}{1+M}}{\binom{1+M-g}{1+M}}$ [3]

If SUM(N) or ∇SUM(N) is easy to obtained,then H(g) can be calculated with Cramer's law.Below,$T_M\geq M$

Theorem 5.3. $H_1(g) = \sum_{k=1}^{g+1}(-1)^{g+1+k}\binom{T_M-M+1+g}{T_M-M+k}SUM(k) = \sum_{k=1}^{g+1}(-1)^{g+1+k}\binom{T_M-M+g}{T_M-M+k-1}\nabla SUM(k)$

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$$SUM(N,[1,2...M],[1,2...M]) = M!\binom{N+M}{M+1} \to \binom{M}{g} = \sum_{k=1}^{g+1}(-1)^{g+1+k}\binom{1+g}{k}\binom{k+M}{M+1}$$

$$SUM(N,[2,3...M],[3,5...2M-1]) = S_1(N+M,N) \to MIN_g(M) = \sum_{k=1}^{g+1}(-1)^{g+1+k}\binom{M+1+g}{M+k}S_1(M+k,k)$$

$$(1)SUM(N,[1,1...1],[2,3...M]) = N^M \to S_2(M,g) = \frac{1}{g!}\sum_{k=0}^{g}(-1)^{g+k}\binom{g}{k}k^M = \frac{1}{g!}\sum_{k=0}^{g}(-1)^k\binom{g}{k}(g-k)^M$$

Theorem 5.4. $z(k) = \sum_{i=1}^{k}(-1)^{i+k}\binom{k}{i}\nabla SUM(k), H_2(g) = \sum_{k=g+1}^{M+1}(-1)^{g+k-1}\binom{k-1}{g}z(k)$

$$SUM(N,[1,1...1],[2,3...M]) = N^M \to z(k) = \sum_{i=1}^{k}(-1)^{i+k}\binom{k}{i}i^M = k!S_2(M,k)$$

$$\to g!S_2(M,g) = \sum_{k=g}^{M}(-1)^{M+k}\binom{k-1}{g-1}k!S_2(M,k) \text{ 3.4}$$

Theorem 5.5. $H_3(g) = \sum_{k=1}^{g+1}(-1)^{g+1+k}\binom{2+T_M}{g+1-k}SUM(k) = \sum_{k=1}^{g+1}(-1)^{g+1+k}\binom{1+T_M}{g+1-k}\nabla SUM(k)$

$$(2)SUM(N,[1,1...1],[2,3...M]) = N^M \to \left\langle {M \atop g} \right\rangle = \sum_{k=1}^{g+1}(-1)^{1+g+k}\binom{M+1}{g+1-k}k^M = \sum_{k=0}^{g}(-1)^k\binom{M+1}{k}(g+1-k)^M$$

(1)(2) are already known formula.

6. Eulerian polynomials and Beyond

In this section,$q \neq 0, q \neq 1$.
Inductive proof:

Theorem 6.1. $\sum_{n=0}^{N-1}q^n\binom{n+K}{M} = q^N\sum_{g=0}^{M}(-1)^g\frac{\binom{N+K-1-g}{M-g}}{(q-1)^{g+1}} + \frac{q^{M-K}}{(1-q)^{M+1}}$

Definition 13. $A_q^M = \sum_{k=0}^{M}(1-q)^{M-k}q^kS_2(M,k)k!, A_q^0 = 1, A_q^1 = q$

Table 4. Table of A_q^M.

	M=0	M=1	M=2	M=3	M=4	M=5	M=6	OEIS
A_2^M	1	2	6	26	150	1082	9366	A000629
A_3^M	1	3	12	66	480	4368	47712	A123227
A_4^M	1	4	20	132	1140	12324	160020	A201355

Use 3.4 $\to$

Theorem 6.2. $A_q^M = q\sum_{k=0}^{M}(q-1)^{M-k}S_2(M,k)k!$

$$n^M = \nabla SUM(N,[1,1...1],[2,3...M]) = \sum_{g=0}^{M}S_2(M,g)g!\binom{n}{g} = \sum_{g=0}^{M}(-1)^{M-g}S_2(M,g)g!\binom{n+g}{g} = \sum_{g=0}^{M}\left\langle {M \atop g} \right\rangle\binom{n+g}{M}$$

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$$(1)\sum_{n=0}^{N-1} q^n n^M = \sum_{g=0}^{M} S_2(M,g)g! \sum_{n=0}^{N-1} q^n \binom{n}{g}$$

$$= \sum_{g=0}^{M} S_2(M,g)g!\{q^N \sum_{k=0}^{g} (-1)^k \frac{\binom{N-1-k}{g-k}}{(q-1)^{k+1}} + \frac{q^g}{(1-q)^{g+1}}\}$$

$$= \frac{q^N}{(q-1)^{M+1}} \sum_{g=0}^{M} S_2(M,g)g! \sum_{k=0}^{M} (-1)^k \binom{N-1-k}{g-k} (q-1)^{M-k} + \frac{\sum_{g=0}^{M} S_2(M,g)g!(1-q)^{M-g} q^g}{(1-q)^{M+1}}$$

$$= \frac{q^N}{(q-1)^{M+1}} \sum_{k=0}^{M} (-1)^k (q-1)^{M-k} \sum_{g=0}^{M} S_2(M,g)g! \binom{N-1-k}{g-k} + \frac{A_q^M}{(1-q)^{M+1}}$$

$$= \frac{q^N}{(q-1)^{M+1}} \sum_{k=0}^{M} (-1)^k (q-1)^{M-k} \nabla^k (N-1)^M + \frac{A_q^M}{(1-q)^{M+1}} (*)$$

Use the $From_2$ of n^M in (1),the first part of (*) keep same, we can obtain 6.2. Use the $From_3$:

Theorem 6.3. $A_q^M = \sum_{g=0}^{M} \left\langle {M \atop g} \right\rangle q^{M-g} = \sum_{g=0}^{M} \left\langle {M \atop g} \right\rangle q^{1+g}$

$$n^M = \nabla SUM(N, [1, 1...1], [1, 2...M]) = \sum_{g=0}^{M} S_2(M+1, g+1)g! \binom{n-1}{g} \to$$

Theorem 6.4. $A_q^M = \sum_{k=0}^{M} (1-q)^{M-k} q^{k+1} S_2(M+1, k+1)k!, M > 0$

$$S_2(M+1, k+1) = \frac{1}{(k+1)!} \sum_{j=0}^{k+1} (-1)^{j+k+1} \binom{k+1}{j} j^{M+1} = \frac{1}{k!} \sum_{j=0}^{k} (-1)^{j+k} \binom{k}{j} (j+1)^M$$

$$\nabla^k (N-1)^M = \sum_{j=0}^{k} (-1)^j \binom{k}{j} (N-1-j)^M = \sum_{j=0}^{k} (-1)^j \binom{k}{j} \sum_{g=0}^{M} \binom{M}{g} N^g (j+1)^{M-g} (-1)^{M-g}$$

$$= \sum_{g=0}^{M} (-1)^{M-g-k} \binom{M}{g} N^g \sum_{j=0}^{k} (-1)^{j+k} \binom{k}{j} (j+1)^{M-g} = \sum_{g=0}^{M} (-1)^{M-g-k} \binom{M}{g} N^g S_2(M-g+1, k+1)k! \to$$

$$(*) = \frac{q^N}{(q-1)^{M+1}} \sum_{k=0}^{M} (-1)^k (q-1)^{M-k} \{\sum_{g=0}^{M} (-1)^{M-g-k} \binom{M}{g} N^g S_2(M-g+1, k+1)k!\} + \frac{A_q^M}{(1-q)^{M+1}}$$

$$= \frac{q^N}{(q-1)^{M+1}} \sum_{g=0}^{M} (-1)^{M-g} (q-1)^g \binom{M}{g} N^g \sum_{k=0}^{M} (q-1)^{M-k-g} S_2(M-g+1, k+1)k! + \frac{A_q^M}{(1-q)^{M+1}} (**)$$

$$N = 0, (*) = 0 \to \frac{q^N}{(q-1)^{M+1}} (...) + \frac{A_q^M}{(1-q)^{M+1}} = 0 \to$$

Theorem 6.5. $A_q^M = \sum_{k=0}^{M} (q-1)^{M-k} S_2(M+1, k+1)k!$

$$A_q^{M-g} = \sum_{k=0}^{M} (q-1)^{M-k-g} S_2(M-g+1, k+1)k! \ \ From\ (**) \to$$

 doi:10.20944/preprints202305.0311.v2

Theorem 6.6. $\sum\limits_{n=0}^{N-1} q^n n^M = \frac{q^N}{(q-1)^{M+1}} \sum\limits_{g=0}^{M} (-1)^{M-g}(q-1)^g \binom{M}{g} A_q^{M-g} N^g + \frac{A_q^M}{(1-q)^{M+1}}$

Table 5. Table of $\sum\limits_{n=0}^{N-1} q^n n^M$.

$\sum\limits_{n=0}^{N-1} q^n n = \frac{q^N((q-1)N-q)+q}{(q-1)^2}$
$\sum\limits_{n=0}^{N-1} 2^n = 2^N - 1$
$\sum\limits_{n=0}^{N-1} 2^n n = 2^N(N-2)+2$
$\sum\limits_{n=0}^{N-1} 2^n n^2 = 2^N(N^2-4N+6)-6$
$\sum\limits_{n=0}^{N-1} 2^n n^3 = 2^N(N^3-6N^2+18N-26)+26$
$\sum\limits_{n=0}^{N-1} 2^n n^4 = 2^N(N^4-8N^3+36N^2-104N+150)-150$
$\sum\limits_{n=0}^{N-1} 2^n n^5 = 2^N(N^5-10N^4+60N^3-260N^2+750N-1082)+1082$
$\sum\limits_{n=0}^{N-1} 3^n = \frac{3^N-1}{2}$
$\sum\limits_{n=0}^{N-1} 3^n n = \frac{3^N(2N-3)+3}{4}$
$\sum\limits_{n=0}^{N-1} 3^n n^2 = \frac{3^N(4N^2-12N+12)-12}{8}$
$\sum\limits_{n=0}^{N-1} 3^n n^3 = \frac{3^N(8N^3-36N^2+72N-66)+66}{16}$
$\sum\limits_{n=0}^{N-1} 3^n n^4 = \frac{3^N(16N^4-96N^3+288N^2-528N+480)-480}{32}$
$\sum\limits_{n=0}^{N-1} 4^n n^0 = \frac{4^N-1}{3}$
$\sum\limits_{n=0}^{N-1} 4^n n^1 = \frac{4^N(3N-4)+4}{9}$
$\sum\limits_{n=0}^{N-1} 4^n n^2 = \frac{4^N(9N^2-24N+20)-20}{27}$
$\sum\limits_{n=0}^{N-1} 4^n n^3 = \frac{4^N(27N^3-108N^2+180N-132)+132}{81}$

We know that Eulerian polynomials: $A_M(t) : \sum\limits_{i=0}^{\infty} t^i i^M = \frac{tA_M(t)}{(1-t)^{M+1}}, A_M(t) = \sum\limits_{g=0}^{M-1} \left\langle {M \atop g} \right\rangle t^g$
$|q| < 1, \lim\limits_{N\to\infty} \sum\limits_{n=0}^{N-1} q^n n^M = \frac{A_q^M}{(1-t)^{M+1}} \to A_t^M = tA_M(t)$. There have 5 expressions for $A_M(t)$. Eulerian Numbers and Polynomials is just a special case,we can handler:

$$\sum_{n=0}^{N-1} q^n \nabla^p SUM(n+Y, PS, PT), X = T_M - M - p$$

$$= \begin{cases} \sum\limits_{g=0}^{M} H_1(g) \sum\limits_{n=0}^{N-1} q^n \binom{n+Y+X}{X+1+g} = \sum\limits_{g=0}^{M} H_1(g)\{q^N \sum\limits_{k=0}^{X+1+g} (-1)^k \frac{\binom{N+Y+X-1-k}{X+1+g-k}}{(q-1)^{k+1}} + \frac{q^{1+g-Y}}{(1-q)^{X+2+g}}\} \\ \sum\limits_{g=0}^{M} H_2(g) \sum\limits_{n=0}^{N-1} q^n \binom{n+Y+X+g}{X+1+g} = \sum\limits_{g=0}^{M} H_2(g)\{q^N \sum\limits_{k=0}^{X+1+g} (-1)^k \frac{\binom{N+Y+X+g-1-k}{X+1+g-k}}{(q-1)^{k+1}} + \frac{q^{1-Y}}{(1-q)^{X+2+g}}\} \\ \sum\limits_{g=0}^{M} H_3(g) \sum\limits_{n=0}^{N-1} q^n \binom{n+Y+X+M-g}{X+1+M} = \sum\limits_{g=0}^{M} H_3(g)\{q^N \sum\limits_{k=0}^{X+1+M} (-1)^k \frac{\binom{N+Y+X+M-g-1-k}{X+1+M-k}}{(q-1)^{k+1}} + \frac{q^{1+g-Y}}{(1-q)^{X+2+M}}\} \end{cases}$$

$$Y = 0 \to$$

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Theorem 6.7. $\sum_{n=0}^{M} H_1(g)(1-q)^{M-g}q^{g+1} = q\sum_{n=0}^{M} H_2(g)(1-q)^{M-g} = \sum_{n=0}^{M} H_3(g)q^{g+1}$

Here q can take any value,which is magical.$q = 0.5 \to 2.3$

Definition 14. $A_q(PS, PT) = 6.7$

Theorem 6.8. $\sum_{n=0}^{N-1} q^n \nabla^p SUM(n+Y, PS, PT) = \frac{q^N}{(q-1)^{M+1}} \sum_{k=0}^{M} (q-1)^{M-k}(-1)^k \nabla^{p+K-1} SUM(N+Y-2) + \frac{A_q(PS,PT)q^{-Y}}{(1-q)^{T_M-p+2}}$

Theorem 6.9. $\sum_{n=0}^{\infty} q^n \nabla^p SUM(n+Y, PS, PT) = \frac{A_q(PS,PT)q^{-Y}}{(1-q)^{T_M+2-p}}$

$$eg : A_q^M([d, d...d] : d, [1, 2...M]) = d^M A_q^M$$

$$\sum_{n=0}^{N-1} q^n (nd)^M = \frac{q^N}{(q-1)^{M+1}} \sum_{g=0}^{M} (-1)^{M-g}(q-1)^g \binom{M}{g} A_q^{M-g}([d, d...d] : d, [1, 2...M])(nd)^g + \frac{d^M A_q^M}{(1-q)^{M+1}}$$

Many results of [9,10] can be obtained by this.

7. Formal Calculation of Gaussian Coefficients

7.1. Basic concepts

Gaussian Coefficients: $\left[{N \atop M}\right]_q = \frac{(q^N-1)(q^{N-1}-1)...(q^{N-M+1}-1)}{(q^M-1)(q^{M-1}-1)...(q^1-1)}$, $q \neq 0, 1$,abbreviated as G_M^N

1. $G_0^N = 1, G_{M<0,M>N}^N = 0, G_M^N = G_{N-M}^N$
2. $G_M^N = q^M G_M^{N-1} + G_{M-1}^{N-1} = G_M^{N-1} + q^{N-M} G_{M-1}^{N-1}$
3. $\sum_{n=0}^{N-1} q^n G_M^{n+K} = q^{M-K} G_{M+1}^{N+K}$
4. $G_K^M = \sum_{w \in \Omega(0^{M-K}, 1^K)} q^{inv(w)}$,that is,all words $w = w_1 w_2 \ldots w_M$ with M-K zeroes and K ones, and inv($\cdot$) denotes the inversion statistic defined

The Formal Calculation was obtained by [3]. It use $q^n(K_i + G_1^n D_i)$ instead of $K_i + q^n D_i$.

Definition 15. *Recursive define the operator ∇_q^p,p is an integer.*

$$\nabla_q^0 f(n) = f(n), \sum_{n=0}^{N-1} q^n \nabla_q^1 f(n+1) = f(N), \sum_{n=0}^{N-1} q^n f(n+1) = \nabla_q^{-1} f(N)$$

Definition 16. *Recursive define $SUM_q(N, PS, PT)$,abbreviated as $SUM_q(N)$.*

$$SUM_q(N, [K_1 : D_1], [T_1 = 1]) = \sum_{n=0}^{N-1} q^n (K_1 + G_1^n D_1)$$

$$SUM_q(N, [K_1 : D_1, K_2 : D_2], [T_1, T_2 = T_1 + 2 - p]) = \sum_{n=0}^{N-1} q^n (K_2 + G_1^n D_2) \nabla^p SUM_q(n+1, [K_1 : D_1], [T_1])$$

Theorem 7.1. $\sum_{n=0}^{N-1} q^n G_1^n G_M^{n+K}, M > 0, M \geq K$
$= q^{2(M-K)+1} G_1^{M+1} G_{M+2}^{N+K} + q^{M-K} G_1^{M-K} G_{M+1}^{N+K}$
$= q^{M-2K-1} G_1^{M+1} G_{M+2}^{N+K+1} + q^{M-K}(G_1^{M-K} - q^{-K-1} G_1^{M+1}) G_{M+1}^{N+K}$
$= (q^{2(M-K)+1} G_1^{M+1} - q^{2M-K+2} G_1^{M-K}) G_{M+2}^{N+K} + q^{M-K} G_1^{M-K} G_{M+2}^{N+K+1}$

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Use this and induction to prove:

Theorem 7.2. $SUM_q(N, PS, PT)$=

$$Form_1 \rightarrow \sum_{g=0}^{M} H_1^q(g) G_{N-1-g}^{N+T_M-M} = \sum_{g=0}^{M} H_1^q(g) G_{T_M-M+1+g}^{N+T_M-M}, B_i = \begin{cases} q^{1+(T_i-T_{i-1})X_{T-1}} G_1^{T_i-X_{K-1}} D_i, X_i=T_i \\ q^{(T_i-T_{i-1}-1)X_{T-1}}(K_i+G_1^{X_{T-1}} D_i), X_i=K_i \end{cases}$$

$$Form_2 \rightarrow \sum_{g=0}^{M} H_2^q(g) G_{N-1}^{N+T_M-M+g} = \sum_{g=0}^{M} H_2^q(g) G_{T_M-M+1+g}^{N+T_M-M+g}, B_i = \begin{cases} q^{-(T_i-X_{K-1})} G_1^{T_i-X_{K-1}} D_i, X_i=T_i \\ K_i - q^{-(T_i-X_{K-1})} G_1^{T_i-X_{K-1}} D_i, X_i=K_i \end{cases}$$

$$Form_3 \rightarrow \sum_{g=0}^{M} H_3^q(g) G_{N-1-g}^{N+T_M-g} = \sum_{g=0}^{M} H_3^q(g) G_{T_M+1}^{N+T_M-g}, B_i = \begin{cases} q^{1+(T_i-T_{i-1}-1)X_{T-1}} \{(q^{X_{T-1}} G_1^{T_i} - q^{T_i} G_1^{X_{T-1}}) D_i - K_i q^{T_i}\}, X_i=T_i \\ q^{(T_i-T_{i-1}-1)X_{T-1}}(K_i+G_1^{X_{T-1}} D_i), X_i=K_i \end{cases}$$

$$H^q(g) = \sum_{X(T)=g} \prod_{i=1}^{M} B_i, \lim_{q \to 1} SUM_q(N) = SUM(q), \lim_{q \to 1} H^q(g) = H(g)$$

7.2. Property

Theorem 7.3. $\nabla_q^1 SUM_q(N, PS, [1, 2...M]) = \prod_{i=1}^{M} (K_i + D_i G_1^n)$

Theorem 7.4. In $SUM_q(N, [...PS...], [...T+1, T+2...T+M...])$,$K_i$ can exchange orders.

$Form_2$ is simplest,$X = T_M - M - p$, 3 $\rightarrow$

Theorem 7.5. $\nabla_q^p SUM_q(N) = \sum_{g=0}^{M} H_1^q(g) G_{X+1+g}^{N+X} q^{-gp} = \sum_{g=0}^{M} H_2^q(g) G_{X+1+g}^{N+X+g} = \sum_{g=0}^{M} H_3^q(g) G_{X+M+1}^{N+X+M-g} q^{-gp}$

Definition 17. $n_{q-} = q^{-n} G_1^n, n_{q+} = q^n G_1^n, n_{q-}! = n_{q-}...2_{q-}1_{q-}, 0_{q-}! = 0, n_{q+}!$ *is similar*

$Form_2 \rightarrow$

Theorem 7.6. $SUM_q(N, [L_{1_{q-}}, L_{2_{q-}}...L_{Q_{q-}}, PS], [L_1, L_2...L_Q, PT]) = \prod_{i=1}^{Q} L_{i_{q-}} SUM_q(N, PS, PT)$,$T_1$ can great than 1,$T_i \in \mathbb{N}$

Theorem 7.7. $SUM_q(N, [T_{1_{q-}}, T_{2_{q-}}...T_{M_{q-}}], [T_1, T_2...T_M]) = \prod_{i=1}^{M} T_{i_{q-}} G_{T_M+1}^{N+T_M}$

By definition and 4

Theorem 7.8. *In* $H^q(g)$, $\sum_{X_i \in K} \prod q^{X_T} = G_{M-g}^M = G_g^M, X_T =$ *number of* $\{X_1, X_2...X_i\} \in T$

Induction $\rightarrow$:

Theorem 7.9. $PT = [1, 2....M], q^{-\binom{g+1}{2}} H_1^q(g) = q^{\binom{g+1}{2}} \sum_{k=g}^{M} H_2^q(k) G_g^k$

Theorem 7.10. $H_1^q(0) = \sum_{k=0}^{M} H_2^q(k)$

Except for 7.9 and 7.10, other situations are relatively complex and no conclusions similar to 2.1 have been drawn.

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7.3. Application

Theorem 7.11. $\sum\limits_{0 \le \lambda_1 \le \lambda_2 \le \ldots \le \lambda_M \le N} q^{\lambda_1+\lambda_2+\ldots+\lambda_M} = G_M^{N+M} = G_N^{N+M}$ [7]

Proof. $SUM_q(N,[1,1\ldots1]:0,[1,3\ldots2M-1]) \to H_2^q(g>0)=0, H_2^q(0)=1 \to \sum\limits_{\lambda_M=0}^{N-1} q^{\lambda_M} \ldots \sum\limits_{\lambda_1=0}^{\lambda_2} q^{\lambda_1} = G_M^{N+M-1}$ □

Theorem 7.12. $\sum\limits_{1 \le \lambda_1 < \lambda_2 < \ldots < \lambda_M \le N} q^{\lambda_1+\lambda_2+\ldots+\lambda_M} = q^{\binom{M+1}{2}} \sum\limits_{g=0}^{M} (-1)^g q^{\frac{g^2}{2}} \begin{bmatrix} M \\ g \end{bmatrix}_q \begin{bmatrix} N+M-g \\ 2M \end{bmatrix}_{q^{\frac{1}{2}}}$

Proof.

$$SUM_q(N,[q^2:(q-1)q^2, q^4:(q-1)q^4 \ldots q^{2M}:(q-1)q^{2M}],[1,3\ldots2M-1]) = \sum_{g=0}^{M} H_3^q(g) G_{2M}^{N+2M-1-g}$$

$$= \sum_{\lambda_M=0}^{N-1} q^{2(\lambda_M+M)} \ldots \sum_{\lambda_2=0}^{\lambda_3} q^{2(\lambda_2+2)} \sum_{\lambda_1=0}^{\lambda_2} q^{2(\lambda_1+1)} = \sum_{1 \le \lambda_1 < \lambda_2 < \ldots < \lambda_M \le N+M-1} q^{2(\lambda_1+\lambda_2+\ldots+\lambda_M)} = *$$

$$B_i = \begin{cases} q^{1+X_{T-1}}\{(q^{X_{T-1}}G_1^{T_i} - q^{T_i}G_1^{X_{T-1}})D_i - K_i q^{T_i}\} = -q^{2i+1+2X_{T-1}} = -q^{-1+2i+2X_T}, X_i = T_i \\ q^{X_{T-1}}(K_i + G_1^{X_{T-1}}D_i) = q^{X_{T-1}}(q^{2i} + G_1^{X_{T-1}} q^{2i}(q-1)) = q^{2X_{T-1}+2i} = q^{2i+2X_T}, X_i = K_i \end{cases}$$

$$H_3^q(g) = (-1)^g q^{2(1+2+\ldots+M)+2(1+2+\ldots+g)-g} \sum \prod q^{2X_T} = (-1)^g q^{M(M+1)+g^2} \begin{bmatrix} M \\ g \end{bmatrix}_{q^2}$$

$$* = q^{M(M+1)} \sum_{g=0}^{M} (-1)^g q^{g^2} \begin{bmatrix} M \\ g \end{bmatrix}_{q^2} G_{2M}^{N+2M-1-g}$$

□

Theorem 7.13. $G_{M+1}^{N+M} = q^{-\binom{M+1}{2}} \sum\limits_{g=0}^{M} [q^{\binom{g+1}{2}+g(M+1)} \sum\limits_{1 \le \lambda_1 < \lambda_2 < \ldots < \lambda_{M-g} \le M} q^{\lambda_1+\lambda_2+\ldots+\lambda_{M-g}}] G_{1+g}^N$

Proof.

$$SUM_q(N,[1_{q-},2_{q-}\ldots M_{q-}],[1,2\ldots M]) = M_{q-}!G_{M+1}^{N+M}$$

$$= SUM_q(N,[M_{q-}\ldots 2_{q-},1_{q-}],[1,2\ldots M]) = \sum_{g=0}^{M} H_1^q(g) G_{1+g}^N$$

$$B_i = \begin{cases} q^{1+X_{T-1}} G_1^{T_i - X_{k-1}} = q^{1+X_{T-1}} G_1^{i-X_{k-1}} = q^{1+X_{T-1}} G_1^{1+X_{T-1}}, X_i = T_i \\ K_i + G_1^{X_{T-1}} = q^{-(M+1-i)} G_1^{M+1-i} + G_1^{X_{T-1}} = \frac{q^{X_{T-1}} - q^{-(M+1-i)}}{q-1} = q^{-(M+1-i)} G_1^{M+1-i+X_{T-1}} = q^{-(M+1-i)} G_1^{M-X_{K-1}}, X_i = K_i \end{cases}$$

$$H_1^q(g,T) = g_{q+}!, H_1^q(g,\sum K) = G_1^M G_1^{M-1} \ldots G_1^{g+1} q^{-(M+1)(M-g)} \sum_{1 \le \lambda_1 < \lambda_2 < \ldots < \lambda_{M-g} \le M} q^{\lambda_1+\lambda_2+\ldots+\lambda_{M-g}}$$

$$\frac{H_1^q(g,T)}{M_{q-}!} = \frac{q^{\binom{g+1}{2}} q^{-(M+1)(M-g)} \sum\limits_{1 \le \lambda_1 < \ldots < \lambda_{M-g} \le M} q^{\lambda_1+\ldots+\lambda_{M-g}}}{q^{-(1+2+\ldots+M)}}$$

□

$$eg: G_4^{n+3} = q^{12} G_4^n + q^5(q^1+q^2+q^3) G_3^n + q^{-1}(q^1q^2+q^1q^3+q^2q^3) G_2^n + q^{-6}(q^1q^2q^3) G_1^n$$

Definition 18. $E_{qM}^{N} = \sum\limits_{1 \le \lambda_1 < \ldots < \lambda_M \le N} G_1^{\lambda_1} \ldots G_1^{\lambda_M} = S_2^q(N+M,N)$

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Theorem 7.14. $(G_1^N)^M = (1+q+...+q^{n-1})^M = \sum_{g=1}^{M} g_{q+}!S_2^q(M,g)G_g^N q^{-g}$

Proof.

$$\nabla_q^1 SUM(N,[1_{q-},1_{q-}...1_{q-}],[1,2...M]) = \prod(\frac{1}{q}+\frac{q^n-1}{q-1}) = q^{-M}(G_1^{n+1})^M = q^{-M}(G_1^N)^M$$

$$= q^{-1}\nabla_q^1 SUM(N,[1_{q-},1_{q-}...1_{q-}],[2,3...M]) = q^{-1}\sum_{g=0}^{M-1} H_1^q(g)G_{1+g}^N q^{-g}$$

$$B_i = \begin{cases} q^{1+X_{T-1}}G_1^{(i+1)-X_{K-1}} = q^{X_T}G_1^{1+X_T}, X_i\in T \\ q^{-1}+G_1^{X_{T-1}} = q^{-1}G_1^{1+X_{T-1}}, X_i\in K \end{cases}$$

$$H_1^q(g,T) = q^{-(g+1)}(g+1)_{q+}!, H_1^q(g,\sum K) = q^{-(M-1-g)}E_{qM-1-g}^{g+1}$$

$$(G_1^N)^M = q^{M-1}\sum_{g=0}^{M-1} q^{-(g+1)}(g+1)_{q+}q^{-(M-1-g)}S_2^q(M,g+1)!G_{1+g}^N q^{-g}$$

□

$SUM_q(N,[K_i+1:D_i(q-1)],[1,2...M]) \rightarrow$

Theorem 7.15. $\sum_{n=0}^{N-1} q^n \prod_{i=1}^{M}(K_i+D_1q^n) = \sum_{g=0}^{M} H_1^q(g)G_{1+g}^N, B_i = \begin{cases} q^{X_T}(q^{X_T}-1)D_i, X_i\in T \\ K_i+q^{X_T}D_i, X_i\in K \end{cases}$

induction $\rightarrow$

Theorem 7.16.

$$\sum_{n=0}^{N-1}\prod_{i=1}^{M}(K_i+D_1q^n) = \sum_{g=1}^{M} f(g)G_g^N + N\prod K_i, f(g) = \prod B_i = \begin{cases} 1, X_i\in T, X_{T-1}=0 \\ q^{X_{T-1}}(q^{X_{T-1}}-1)D_i, X_i\in T, X_{T-1}>0 \\ K_i, X_i\in K, X_{T-1}=0 \\ K_i+q^{X_{T-1}-1}D_i, X_i\in K, X_{T-1}>0 \end{cases}$$

Theorem 7.17. $q^{Mn} = [1]\sum_{g=0}^{M} G_g^M \prod_{i=0}^{g-1}(q^n-q^i) = [2]q^{-M}\sum_{g=0}^{M} q^g \begin{bmatrix} M \\ g \end{bmatrix}_{q^{-1}} \prod_{i=1}^{g}(q^n-q^{-i}) = [3]\sum_{g=0}^{M} G_g^M(-1)^g q^{\binom{g}{2}} G_M^{n+M-g}$

Proof.

$$q^{Mn} = \nabla_q^1 SUM(N,[1,1...1]:q-1,[1,2...M])$$

$$= \sum_{g=0}^{M} H_1^q(g)G_g^n q^{-g} = \sum_{g=0}^{M} H_2^q(g)G_g^{n+g} = \sum_{g=0}^{M} H_3^q(g)G_M^{n+M-g}q^{-g}$$

$$B_i = \begin{cases} q^{1+X_{T-1}}G_1^{i-X_{K-1}}(q-1) = q^{X_T}(q^{X_T}-1), X_i\in T \\ 1+G_1^{X_{T-1}}(q-1) = q^{X_{T-1}} = q^{X_T}, X_i\in K \end{cases}, H_1^q(g,T) = (q-1)^g g_{q+}!, H_1^q(g,\sum K) = G_g^M \rightarrow [1]$$

$$B_i = \begin{cases} q^{-(1+X_{T-1})}(q^{1+X_{T-1}}-1), X\in T \\ q^{-(1+X_{T-1})} = q^{-1}q^{-X_T}, X\in K \end{cases}, H_2^q(g,T) = (q-1)^g g_{q-}!, H_2^q(g,\sum K) = q^{-(M-g)}\begin{bmatrix} M \\ g \end{bmatrix}_{q^{-1}} \rightarrow [2]$$

$$B_i = \begin{cases} -q^{1+X_{T-1}} = -q^{X_T}, X\in T \\ q^{X_{T-1}} = q^{X_T}, X\in K \end{cases}, H_3^q(g,T) = (-1)^g\binom{g+1}{2}, H_3^q(g,\sum K) = G_g^M \rightarrow [3]$$

□

Theorem 7.18. $\sum_{k_r=1}^{N}...\sum_{k_2=1}^{k_3}\sum_{k_1=1}^{k_2} q^{k_1+k_2+...+k_r}\nabla_q^p SUM_q(k_1,PS,PT) = \nabla_q^{p-r}SUM_q(N,PS,PT)$

 doi:10.20944/preprints202305.0311.v2

Theorem 7.19. $\sum_{k_x=1}^{N} \dots \sum_{k_2=1}^{k_3} \sum_{k_1=1}^{k_2} q^{k_1+k_2+\dots+k_x} \nabla_q^1 SUM_q(k_r, PS, [1,2...M]) = \nabla_q^{x-r} SUM_q(N, PS, [T_i = i + (r-1)])$

Theorem 7.20. $(K+qD)^M = (q-1)D \sum_{g=0}^{M-1} (K+D)^g (K+qD)^{M-1-g} + (K+D)^M$

Proof.

$$PS = [K+D, K+D...K+D] : (q-1)D, PT = [1,2...M]$$

$$B_i = \begin{cases} q^{X_T}(q^{X_T}-1)D, X\in T \\ K+Dq^{X_T}, X\in K \end{cases}, H_1^q(1) = q^1(q^1-1)D \sum_{a+b=M-1,a,b\geq 0} (K+D)^a (K+qD)^b, H_1^q(0) = (K+D)^M$$

$$SUM_q(2) - SUM_q(1) = H_1^q(1) + H_1^q(0)G_1^2 - H_1^q(0) = H_1^q(1) + qH_1^q(0)$$

$$= q\prod_{i=1}^{M} (K+D+G_1^1(q-1)D) = q(K+qD)^M$$

□

7.4. Matrix of $SUM_q(N)$

Similar to Section "Matrix of SUM(N)",Let $P = N + T_M - M, Q = N - 1$,define $A^q(P,Q,M)$, respectively corresponding to the three forms.

$$A_1^q(P,Q,M) = \begin{pmatrix} G_Q^P & \cdots & G_{Q-M}^P \\ \vdots & \ddots & \vdots \\ G_{Q+M}^{P+M} & \cdots & G_Q^{P+M} \end{pmatrix}$$

$$A_2^q(P,Q,M) = \begin{pmatrix} G_Q^P & \cdots & G_Q^{P+M} \\ \vdots & \ddots & \vdots \\ G_{Q+M}^{P+M} & \cdots & G_{Q+M}^{P+2M} \end{pmatrix}$$

$$A_3^q(P,Q,M) = \begin{pmatrix} G_Q^{P+M} & \cdots & G_{Q-M}^P \\ \vdots & \ddots & \vdots \\ G_{Q+M}^{P+2M} & \cdots & G_Q^{P+M} \end{pmatrix}$$

Theorem 7.21. $\| A_2^q(P,Q,M) \| = \| A_2^q(P,P-Q,M) \| = \frac{G_Q^P}{G_0^P} \frac{G_{Q+1}^{P+2}}{G_1^{P+2}} \dots \frac{G_{Q+M}^{P+2M}}{G_M^{P+2M}} q^{(P+1)+2(P+2)+\dots+M(P+M)}$

Theorem 7.22. $\| A_2^q(P,0,M) \| = q^{(P+1)+2(P+2)+\dots+M(P+M)}, \| A_2^q(P,1,M) \| = G_{M+1}^{P+M} q^{(P+1)+2(P+2)+\dots+M(P+M)}$

Theorem 7.23.
$\| A_{1,3}^q(P,Q,M) \| = \frac{G_Q^P}{G_0^P} \frac{G_{Q+1}^{P+2}}{G_1^{P+2}} \dots \frac{G_{Q+M}^{P+2M}}{G_M^{P+2M}} q^{Q\binom{M+1}{2}}, \| A_{1,3}^q(P,0,M) \| = 1, \| A_{1,3}^q(P,1,M) \| = G_{M+1}^{P+M} q^{\binom{M+1}{2}}$

8. Multi-parameter Formal Calculation

2-parameters Formal Calculation calculate nested sum of $K_i + \binom{n}{1} D_{1,i} + \binom{n}{2} D_{2,i}$.

$$SUM(N, [K_1], [T_{1,1} = 1 : D_{1,1}], [T_{2,1} = 1 : D_{2,1}]) = \sum_{n=0}^{N-1} (K_1 + D_{1,1}n + D_{2,1}\binom{n}{2})$$

 doi:10.20944/preprints202305.0311.v2

$$SUM(N,[K_1,K_2],[T_{1,1}:D_{1,1},T_{1,2}=T_{1,1}+2-p:D_{1,2}],[T_{2,1}:D_{2,1},T_{2,2}=T_{1,2}:D_{2,2}])$$
$$=\sum_{n=0}^{N-1}(K_2+D_{1,2}n+D_{2,2}\tbinom{n}{2})\nabla^p SUM(N,[K_1],[T_{1,1}:D_{1,1}],[T_{2,1}:D_{2,1}])$$

Recursive define $SUM(N,PS,PT_1,PT_2)$. There is always $T_i=T_{1,i}=T_{2,i}$
Use the Form: $(K_1+T_{1,1}+T_{2,1})(K_2+T_{1,2}+T_{2,2})...(K_M+T_{1,M}+T_{2,M})$
Use T_1 to represent the set $\{T_{1,1},T_{1,2}...T_{1,M}\}$,$T_2$ to represent the set $\{T_{2,1},T_{2,2}...T_{2,M}\}$.

Definition 19. *$X(PT_1)$=Number of $\{X_1...X_M\}\in T_1$, $X(PT_2)$=Number of $\{X_1...X_M\}\in T_2$,$X(PT)$=$X(PT_1)$+2$X(PT_2)$*

Definition 20. X_{PT1}=*Number of* $\{X_1...X_i\}\in T_1$,X_{PT2}=*Number of* $\{X_1...X_i\}\in T_2$,$X_{PT}=X_{PT1}+2X_{PT2}$

Use induction to prove:

Theorem 8.1. $SUM(N,PS,PT_1,PT_2)$

$$=\sum_{g=0}^{2M}H_1(g)\binom{N+T_M-M}{T_M-M+1+g},B_i=\begin{cases}K_i+X_{PT}D_{1,i}+\binom{X_{PT}}{2}D_{2,i},X_i=K_i\\(T_i-i+X_{PT})D_{1,i}+(T_i-i+X_{PT})(X_{PT}-1)D_{2,i},X_i=T_{1,i}\\\binom{T_i-i+X_{PT}}{2}D_{2,i},X_i=T_{2,i}\end{cases}$$

$$=\sum_{g=0}^{2M}H_2(g)\binom{N+T_M-M+g}{T_M-M+1+g},B_i=\begin{cases}K_i-(T_i-i+X_{PT}+1)D_{1,i}+\binom{T_i-i+X_{PT}+2}{2}D_{2,i},X_i=K_i\\(T_i-i+X_{PT})D_{1,i}-(T_i-i+X_{PT})(T_i-i+X_{PT}+1)D_{2,i},X_i=T_{1,i}\\\binom{T_i-i+X_{PT}}{2}D_{2,i},X_i=T_{2,i}\end{cases}$$

$$=\sum_{g=0}^{2M}H_3(g)\binom{N+T_M+M-g}{T_M+M+1},B_i=\begin{cases}K_i+X_{PT-1}D_{1,i}+\binom{X_{PT-1}}{2}D_{2,i},X_i=K_i\\-2K_i+(T_i+i-1-2X_{PT-1})D_{1,i}+(T_i+i-X_{PT-1})D_{2,i},X_i=T_{1,i}\\K_i-(T_i+i-1-X_{PT-1})D_{1,i}+\binom{T_i+i-X_{PT-1}}{2}D_{2,i},X_i=T_{2,i}\end{cases}$$

According to this way, it can be extended to multi-parameter $SUM(N)$ and $SUM_q(N)$.This formula is very complex and has not been studied in terms of analysis yet.

9. A theorem of symmetry

In this section,$T_i\geq i$.
From 2.12→ $H_1(g,PT,PT)=\prod_{i=1}^{M}T_i\binom{M}{g}=\prod_{i=1}^{M}T_i\binom{M}{g,M-g}$.Promoted it:
The Set $\{T_1,T_2...T_M\}$ come from p Source:$S_1,S_2...S_p$,$T_i\in S_j\ means\ T_i\ come\ from\ soure\ j$

Definition 21. $Diff(S_x,S_x)=0,Diff(S_x,S_y)=-Diff(S_y,S_x)=1,x\neq y$

Definition 22. $Diff(T_i,T_j)=Diff(S_x,S_y),T_i\in S_x,T_j\in S_y$

Definition 23. $W(g_1,g_2...g_p,[T_1,T_2....T_M])=\sum_{g_1+g_2+...+g_p=M,g_i\geq 0}\prod_{i=1}^{M}(T_i+\sum_{j<i}Diff(T_j,T_i))$
In set T, g_1 come from S_1,g_2 comes from S_2...,g_M comes from S_M

[1] has proved:

Theorem 9.1. $W(g_1,g_2...g_p,[T_1,T_2....T_M])=\prod_{i=1}^{M}T_i\binom{M}{g_1,g_2,...,g_M}$

Promote to Gaussian coefficient:

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Definition 24.

$$W_q(g_1,g_2...g_p,[T_1,T_2....T_M]) = \sum_{g_1+g_2+...+g_p=M,g_i\geq 0}\prod_{i=1}^{M} G_1^{T_i+\sum_{j<i} Diff(T_j,T_i)} q^{\sum_{j<i,Diff(T_j,T_i)=-1} 1}$$

Definition 25. $G^M_{g_1,g_2\cdots g_p} = \frac{(q^M-1)(q^{M-1}-1)...(q^1-1)}{\prod_{i=0}^{p}(q^{g_i}-1)(q^{g_i-1}-1)...(q^i-1)}, g_1+g_2+...+g_p = M$

Theorem 9.2. $W_q(g_1,g_2...g_p,[T_1,T_2....T_M]) = (\prod_{i=1}^{M} G_1^{T_i})G^M_{g_1,g_2\cdots g_p}, T_i \geq i$

Proof.
When g=2,it clearly holds when M=1 or g_1=0 or g_1=M.
when M=2:$G_1^{T_1}G_2^{T_2+1} + G_1^{T_1}G_2^{T_2-1}q = G_1^{T_1}G_1^{T_2}G_1^2$
When M+1,prove by induction,PT1=[PT,T_{M+1}]

$$\begin{aligned}W_q(g_1,g_2+1,PT1) &= W_q(g_1,g_2,PT)G_1^{T_{M+1}+g_1} + W_q(g_1-1,g_2+1,PT)G_1^{T_{M+1}-(g_2+1)}q^{g_2+1}\\ &= (\prod_{i=1}^{M} G_1^{T_i})G^M_{g_1,g_2}G_1^{T_{M+1}+g_1} + (\prod_{i=1}^{M} G_1^{T_i})G^M_{g_1-1,g_2+1}G_1^{T_{M+1}-(g_2+1)}q^{g_2+1}\end{aligned}$$

Just need to prove:

$$G^M_{g_1}G_1^{T_{M+1}+g_1} + G^M_{g_1-1}G_1^{T_{M+1}-(M-g_1+1)}q^{M-g_1+1} = G_1^{T_{M+1}}G^{M+1}_{g_1} = G_1^{T_{M+1}}(q^{g_1}G^M_{g_1} + G^M_{g_1-1})(*)$$

$$\begin{aligned}G_1^{T_{M+1}}(q^{g_1}G^M_{g_1} + G^M_{g_1-1}) \times \frac{(q^{M-g_1+1}-1)}{G^M_{g_1}} &= G_1^{T_{M+1}}(q^{g_1}(q^{M-g_1+1}-1) + (q^{g_1}-1))\\ &= (q^{T_{M+1}-1}+...+q+1)(q^{M+1}-1) \quad (1)\end{aligned}$$

$$Left \times \frac{(q^{M-g_1+1}-1)}{G^M_{g_1}} = (q^{M-g_1+1}-1)G_1^{T_{M+1}+g_1} + (q^{g_1}-1)G_1^{T_{M+1}-(M-g_1+1)}q^{M-g_1+1}$$

$$= (q^{M-g_1+1}-1)(q^{T_{M+1}+g_1-1}+...+q+1) + (q^{g_1}-1)(q^{T_{M+1}-1}+...+q^{M-g_1+2}+q^{M-g_1+1}) = (2)$$

$(1)-(2) = 0 \rightarrow$It's holds when g=2.

$$W_q(g_1,g_2+g_3,[T_1,T_2....T_M]) = (\prod_{i=1}^{M} G_1^{T_i})G^{g_1+g_2+g_3}_{g_1,g_2+g_3}$$

Every product has g_2+g_3 factors come from $Source_2$,divide them to $g_2 \times Source_2 + g_3 \times Source_3$
g_1-factors are invariant,g_2+g_3)-factors are variant.

$$\sum\prod(variant\ factors) = W_q(g_2,g_3,[X_1,X_2....X_{g_2+g_3}]) = \prod_{i=1}^{g_2+g_3} G_1^{X_i}G^{g_2+g_3}_{g_2,g_3}$$

$$W_q(g_1,g_2,g_3,[T_1,T_2....T_M]) = (\prod_{i=1}^{M} G_1^{T_i})G^{g_1+g_2+g_3}_{g_1,g_2+g_3}G^{g_2+g_3}_{g_2,g_3} = (\prod_{i=1}^{M} G_1^{T_i})G^{g_1+g_2+g_3}_{g_1,g_2,g_3}$$

□

 doi:10.20944/preprints202305.0311.v2

References

1. Ji, P. Subdivide the Shape of Numbers and a Theorem of Ring. *Open Access Library Journal* **2020**, *7*, 1-14. doi: 10.4236/oalib.1106719.
2. Ji, P. Redefining the Shape of Numbers and Three Forms of Calculation. *Open Access Library Journal* **2021**, *8*, 1-22. doi: 10.4236/oalib.1107277.
3. Ji, P. Further Study of the Shape of the Numbers and More Calculation Formulas. *Open Access Library Journal* **2021**, *8*,1-27. doi: 10.4236/oalib.1107969.
4. Ji, P. Application and Popularization of Formal Calculation. *Open Access Library Journal* **2022**, *9*, 1-21. doi: 10.4236/oalib.1109483.
5. Deng-Ji, Q. A New Explicit Expression for the Eulerian Numbers. *Journal of Qingdao University of Science and Technology: Natural Science Edition* **2012**, *33.4*.
6. Deng-Ji, Q. Associated Stirling number of the first kind. *Journal of Qingdao University of Science and Technology: Natural Science Edition* **2015**, *36.5*.
7. Butler, S.; Karasik, P. A note on nested sums. *Journal of Integer Sequences* **2010**, *13.4*.
8. Gould, H. Combinatorial Identites. *From the seven unpublished manuscript of H.W. Gould. Edited by Jocelyn Quaintance* **2010**.
9. Tingyao Xiong, Hung-Ping Tsao, J.I.H. General Eulerian Numbers and Eulerian Polynomials. *Hindawi Publishing Corporation Journal of Mathematics* **2015**, *Vol 18*.
10. Wünsche, A. Generalized Eulerian Numbers. *Advances in Pure Mathematics* **2018**, *2018, 8*, 335–361.