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Locally RSD-Generated Parametrized G1-Spline Surfaces Interpolating First Order Data over 3D Triangular Meshes

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Abstract

Given a triangular mesh in $\mathbb{R}^3$ with a family of points associated to its vertices resp. a vectors associated to its edges, we construct interpolating parametrized polynomial G1-spline surfaces by means of the method of reduced side derivatives (RSD) with a locally generated G1-correction over mesh edges. In the case of polynomial RSD shape functions, we establish polynomial edge corrections by means of an algorithm with independent interest for finding optimal GCD cofactors with lowest degree for arbitrary families of polynomials.

Keywords: triangular mesh in 3D; G1-spline; parametrized surface; reduced side derivatives; RSD interpolation, shape function, GCD cofactors

MSC: 65D07; 41A15; 65D15; 13A05

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1. Introduction

By a *triangular mesh in 3D* we mean a finite family $\mathcal{T} = \{\mathbf{T}_1, \ldots, \mathbf{T}_N\}$ of non-degenerate triangles $\mathbf{T}_i \subset \mathbb{R}^3$ such that the intersections $\mathbf{T}_i \cap \mathbf{T}_j$ $(i \neq j)$ are either empty or mesh points or common edges, and no three different mesh triangles meet in a common edge. A *$\mathcal{T}$-parametrized G_1-spline surface* is a continuous mapping $F : \bigcup_{i=1}^N \mathbf{T}_i \to \mathbb{R}^3$ whose subfunctions (the restrictions $F_i = F|_{\mathbf{T}_i}$) are $\mathcal{C}^\infty$-smooth with *G_1-coupling* along common edges (that is, given $i \neq j$ and $\mathbf{p} \in \mathbf{T}_i \cap \mathbf{T}_j$, the tangent vectors $F_i'(\mathbf{p}(\mathbf{q}-\mathbf{p}) = d/dt|_{t=0}F_i(\mathbf{p}+(t\mathbf{q}-\mathbf{p})$ $(\mathbf{q} \in \mathbf{T}_i)$ together with $F_j'(\mathbf{p}(\mathbf{q}-\mathbf{p})$ $(\mathbf{q} \in \mathbf{T}_j)$ do not span $\mathbb{R}^3$). It is well-known from classical differential geometry [7] that, in the above setting, if F is a homeomorphism with G_1-coupling and $\dim\big(F_i'(\mathbf{p})(\mathbf{T}_i - \mathbf{p})\big) = 2$ $(\mathbf{p} \in \mathbf{T}_i)$ in every triangle $\mathbf{T}_i$ then the figure range(F) is a $\mathcal{C}^1$-submanifold of $\mathbb{R}^3$.

Due to exigences of elaborating data of scanned surfaces, recently the construction of parametrized G_1-spline surfaces in 3D became a popular topics. It seems that one branch in the main stream consists of papers aiming to establish reasonable meshes with plane figures fitting to a set of 3D-points, while another branch concentrates in modifying algorithms with classical 1D- and 2D-splines in a 3D setting exploiting the use of large computing capacity, sometimes with compromises e.g by adding artificial new mesh points or modifying the underlying data (for typical examples see [2,4,5,6,12]).

In this paper we are going to apply our "minimalist" local $\mathcal{C}^1$-spline algorithm [9] extended in [10] to more shape functions called *RSD method* (method of reduced side derivatives, to be introduced in Section 2). By writing $\{\mathbf{p}_1, \ldots, \mathbf{p}_R\}$ for the family of mesh vertices, our purpose is to investigate the following problem with primary interest in polynomial solutions.

G1-Interpolation Problem. *Given two families $[\mathbf{f}_i]_{i=1}^R$ resp. $[\mathbf{g}_{i,j}]_{i,j=1}^R$ of vectors in $\mathbb{R}^3$, find a parametrized G_1-spline surface $F : \bigcup_{n=1}^N \mathbf{T}_n \to \mathbb{R}^3$ such that*

$$F(\mathbf{p}_i) = \mathbf{f}_i, \quad F'(\mathbf{p}_i)(\mathbf{p}_j - \mathbf{p}_i) = \mathbf{g}_{i,j} \quad \big(i,j \in [1,R],\ [\mathbf{p}_i, \mathbf{p}_j] \text{ is edge in some } \mathbf{T}_n\big). \tag{1}$$

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We shall proceed the following strategy: By introducing *extended barycentric weights* $\lambda_1,\dots,\lambda_R : \bigcup_{n=1}^N \mathbf{T}_n \to [0,1]$ and using any RSD family of shape functions we obtain a G0-spline map in the form $f = \sum_{i=1}^R \Psi_0(\lambda_i)\mathbf{f}_i + \sum_{i,j=1}^R \Psi_1(\lambda_i)\lambda_j \mathbf{g}_{i,j} + \sum_{(i,j,k)\in S_3} [\chi_0(\lambda_i,\lambda_j,\lambda_k)\mathbf{p}_i + \chi_1(\lambda_i,\lambda_j,\lambda_k)\mathbf{g}_{i,j}$ which automatically satisfies the initial conditions (1). To correct it to a G1-spline, we look for F in the form $F = f + \sum_{(i,j)\in\mathcal{I}} \frac{1}{2}\lambda_i^2\lambda_j^2\lambda_k [Z_{i,j}(\lambda_i) + Z_{i,j}(1-\lambda_j)]$ where the index set $\mathcal{I}$ consists of all triples (i,j,k) being such that $i<j$, $[\mathbf{p}_i,\mathbf{p}_j]$ is a double mesh edge (i.e., belonging to two different mesh triangles) and $\mathrm{Conv}\{\mathbf{p}_i,\mathbf{p}_j,\mathbf{p}_k\}$ is a mesh triangle. The splines F, f coincide along the mesh edges. The familiar determinant condition of G1-coupling along the common edge $[\mathbf{p}_i,\mathbf{p}_j$ of two adjacent subfunctions $F\big|\mathbf{T}_n$ and $F\big|\mathbf{T}_{\overline{n}}$ with $\mathbf{T}_n = \mathrm{Conv}\{\mathbf{p}_i,\mathbf{p}_j,\mathbf{p}_k\}$ resp. $\mathbf{T}_{\overline{n}} = \mathrm{Conv}\{\mathbf{p}_i,\mathbf{p}_j,\mathbf{p}_{\overline{k}}\}$ can be written in a form

$$\det\big[v(t) - t^2(1-t)^2 z(t), \overline{v}(t) - v(t), u(t)\big] \equiv 0 \tag{2}$$

where the terms $u(t), v(t), \overline{v}(t)$ are linear combinations of the vectors $\mathbf{f}_j, \mathbf{f}_i, \mathbf{g}_{\ell,m}$ $(\ell, m = i,j,k,\overline{k})$ with coefficients belonging to $\{\Psi_r(t), \Psi_r'(t), \Psi_r(1-t), \Psi_r'(1-t)\}$. We finish the paper with a study of the case with polynomial shape functions $\Psi_0(t), \Psi_1(t)$ such that $t^3\big|\Psi_r(t), \Psi_1(t)$ and $\Psi_0(t)+\Psi_0(1-t)$ like the functions Φ, Θ in [9], furthermore assuming that the families $G_\ell = \{\mathbf{g}_{\ell,m} : [\mathbf{p}_\ell, \mathbf{p}_m]$ is a mesh edge$\}$ $(\ell = 1,\dots,R)$ are coplanar like in the case when $\mathbf{g}_{\ell,m}$ is a tangent vector of a smooth surface at the point $\mathbf{p}_\ell$. Then we achieve a complete solution of the Problem, constructing a solution of (2) by means a of family $q^1(t), q^2(t), q^3(t)$ of cofactors for the GCD (greatest common divisor) $\rho(t)$ of the components $w^1(t), w^2(t), w^3(t)$ of $w(t) = [\overline{v}(t) - v(t)] \times u(t)$.

In general, given a family $p_1(t),\dots,p_k(t) \in \mathbb{F}[t]$ of polynomials over an arbitrary field $\mathbb{F}$, it is of independent interest to find cofactors (that is polynomials $q_1,\dots,q_K \in \mathbb{F}[t]$ with $\sum_{k=1}^K q_k(t)a_k(t) = \mathrm{GCD}(a_1,\dots,a_K)$) with lowest degrees possible. Actually one can choose $q_1,\dots,q_K$ above satisfying $\max_{k=1}^K \deg(q_k) \le \max_{k=1}^K \deg(p_k)$. Since we do not know a reference (cf. Remark 7), we give a proof for this fact and describe a related algorithmic construction.

2. Preliminaries

To establish standard notations, let $\mathbb{R}^n = \{\mathbf{x} : \mathbf{x} = [x_1,\dots,x_3],\ x_1,\dots,x_n \in \mathbb{R}\}$ denote the vector space of real n-tuples, equipped with the scalar product: $\langle \mathbf{x}|\mathbf{y}\rangle = \sum_k x_k y_k$ giving rise to the norm $\|\mathbf{x}\| = \langle \mathbf{x}|\mathbf{x}\rangle^{1/2}$ and the Euclidean distance $d_{\|.\|}(\mathbf{x},\mathbf{y}) = \|(\mathbf{x}-\mathbf{y}\|$. We shall use the notation $F'(\mathbf{x})\mathbf{u} = \frac{d}{dt}\big|_{t=0}[F(\mathbf{x}+t\mathbf{u}) - F(\mathbf{x})]$ for the Fréchet derivative of a function defined on some subset $\mathbf{D} \subset \mathbb{R}^n$ along the vector $\mathbf{u} \in \mathbb{R}^n$ whenever $\mathbf{x} + [-\varepsilon,\varepsilon]\mathbf{u} \subset \mathbf{D}$ for some $\varepsilon > 0$. It is well-known that the mapping is linear whenever F is continuously differentiable.

By a triangle with vertices $\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3 \in \mathbb{R}^n$ we mean their convex hull $\mathbf{T} = \mathrm{Conv}\{\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3\} = \{\sum_k t_k\mathbf{p}_k : [t_1,t_2,t_3] \in \Delta_3\}$ in terms of the unit 3-simplex $\Delta_3 = \{[t_1,t_2,t_3] : \sum_k t_k = 1,\ t_k \ge 0\}$. The tangent space $\{\sum_k \mathbf{p}_k : \sum_k t_k = 0\}$ and the supporting affine manifold (line or 2-plane) $\{\sum_k t_k\mathbf{p}_k : \sum_k t_k = 1\}$ of $\mathbf{T}$ will be denoted with $\mathrm{Tan}(\mathbf{T})$ and $\mathrm{Aff}(\mathbf{T})$, respectively. The triangle $\mathbf{T}$ is non-degenerate if $\dim(\mathrm{Aff}(\mathbf{T})) = 2$ that is when the vectors $\mathbf{p}_i - \mathbf{p}_j$ $(i,j = 1,2,3)$ are non-parallel. Given a non-degenerate triangle, the *normalized barycentric weights* [3] of its vertices are the functions $\lambda_{\mathbf{p}_i}^{\mathbf{T}} : \mathrm{Aff}(\mathbf{T}) \to \mathbb{R}$ unambiguously defined by the relations

$$\sum_k \lambda_{\mathbf{p}_x}^{\mathbf{T}}(\mathbf{x})\mathbf{p}_k = \mathbf{x}, \quad \sum_k \lambda_{\mathbf{p}_x}^{\mathbf{T}}(\mathbf{x}) = 1, \qquad (\mathbf{x} \in \mathrm{Aff}(\mathbf{T})).$$

The weights $\lambda_{\mathbf{p}_k}^{\mathbf{T}}$ are affine functions (i.e., satisfying the identity $\lambda_{\mathbf{p}_k}^{\mathbf{T}}(t\mathbf{x} + (1-t)\mathbf{y}) = t\lambda_{\mathbf{p}_k}^{\mathbf{T}}(\mathbf{x}) + (1-t)\lambda_{\mathbf{p}_k}^{\mathbf{T}}(\mathbf{y})$) with Fréchet derivatives being independent of the location which we denote with $G_{\mathbf{p}_k}^{\mathbf{T}}\mathbf{u}$. Namely $G_{\mathbf{p}_k}^{\mathbf{T}}\mathbf{u} = [\lambda_{\mathbf{p}_k}^{\mathbf{T}}]'(\mathbf{x})\mathbf{u} = \lambda_{\mathbf{p}_k}^{\mathbf{T}}(\mathbf{x}+\mathbf{u})$ $(\mathbf{u} \in \mathrm{Tan}(\mathbf{T}))$.

In the sequel we mainly restrict our considerations to settings in $\mathbb{R}^3$. We shall write $\mathbf{x}\times\mathbf{y} = [x_2y_3 - y_2x_3, x_3y_1 - y_3x_1, x_1y_2 - y_1x_2]$ for vectorial product in $\mathbb{R}^3$. In terms of the vectorial and scalar product in $\mathbb{R}^3$, the determinant formed by the components of three vectors $\mathbf{x},\mathbf{y},\mathbf{z} \in \mathbb{R}^3$ can be expressed

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as $\det[\mathbf{x},\mathbf{y},\mathbf{z}] = \langle \mathbf{x}|\mathbf{y}\times\mathbf{z}\rangle$. By a *triangular mesh* we mean a family of non-degenerate triagles with pairwise disjoint interior whose pairs are disjoint or meet in a common vertex or edge. An edge belonging to two different mesh triangles is said to be a *double edge*, the remaing edges are the single edges. A triangular mesh is *regular* if no three different members admit a common vertex., resp. *connected* if each of its members admits a double edge.

3. Mesh Structure, Data of First Order

Henceforth let $\mathcal{T} = \{\mathbf{T}_1,\dots,\mathbf{T}_N\}$ be an arbitrarily fixed connected regular triangular mesh in $\mathbb{R}^3$ with vertices resp edges ordered in the arrays $\mathcal{P} = [\mathbf{p}_1,\dots,\mathbf{p}_R] \subset \mathbb{R}^3$ resp. $\mathcal{E} = [\mathbf{E}_1,\dots,\mathbf{E}_M]$ where

$$\mathbf{T}_n = \mathrm{Conv}\{\mathbf{p}_{i_*(n,1)},\mathbf{p}_{i_*(n,2)},\mathbf{p}_{i_*(n,3)}\}, i(n,1)<i(n,2)<i(n,3) \quad (n=1,\dots,N), \tag{3}$$

$$\mathbf{E}_m = \mathrm{Conv}\{\mathbf{p}_{j_*(m,1)},\mathbf{p}_{j_*(m,2)}\}, j(m,1)<j(m,2) \quad (m=1,\dots,M) \tag{4}$$

with suitable index function $i_* : [1,N]\times[1,3] \to [1,R]$ resp. $j_* : [1,N]\times[1,M]$. We also assume that the indices of double edges precede those of the single ones: $\{\text{double edges}\} = \{\mathbf{E}_m : m = 1,\dots,M^*\}$. Three further index functions $n_*, k_* : [1,M]\times[1,2] \to [1,R]$ resp. $m_* : [1,N]\times[1,3] \to [1,M]$ will be used to describe edge adjacency:

$$n_*(m,1) = \min\{n : \mathbf{E}_m \subset \mathbf{T}_n\}, \quad n_*(m,2) = \max\{n : \mathbf{E}_m \subset \mathbf{T}_n\}, \tag{5}$$

$$k_*(m,\ell) = \left[k : \mathrm{Span}(\{\mathbf{p}_k\}\cup\mathbf{E}_m = \mathbf{T}_{n_*(m,\ell)}\right] \quad (\ell=1,2), \tag{6}$$

$$m_*(n,\ell) = \left[m : \mathbf{E}_m \text{ is the opposite edge of vertex } \mathbf{p}_{i_*(n,\ell)}\right] \text{ in } \mathbf{T}_n \quad (\ell=1,2,3). \tag{7}$$

In the sequel we write

$$\mathbf{T}=\bigcup_{n=1}^{N}\mathbf{T}_n,\ \ \mathbf{E}=\bigcup_{m=1}^{M}\mathbf{E}_m,\ \ \mathbf{P}=\{\mathbf{p}_1,\dots,\mathbf{p}_R\},\ \ \mathbf{F}=\{\mathbf{f}_1,\dots,\mathbf{f}_R\},\ \ \mathbf{p}_{i,j}=\mathbf{p}_j-\mathbf{p}_i, \tag{8}$$

$$\mathbf{u}_m = \sum_{\ell=1}^{3}\mathbf{p}_{n_*(m,1)} - \frac{3}{2}\sum_{\ell=1}^{2}\mathbf{p}_{n_*(m,\ell)}, \quad \overline{\mathbf{u}}_m = \sum_{\ell=1}^{3}\mathbf{p}_{n_*(m,)} - \frac{3}{2}\sum_{\ell=1}^{2}\mathbf{p}_{n_*(m,\ell)} \tag{9}$$

for the polyhedron formed by the mesh triangles, the skeleton of edges and the set of vertices, the the matrix of *edge vectors* and the *weight line vectors*, respectively.

Our later spline surface constructions will consist of families of curved images of the mesh triangles connecting point triples $\{\mathbf{f}_1,\mathbf{f}_2,\mathbf{f}_3\} \subset \mathbf{F}$ whenever $\mathrm{Conv}\{\mathbf{f}_1,\mathbf{f}_2,\mathbf{f}_3\} \in \mathcal{T}$. To prescribe tangent vectors for them at the vertices, henceforth we fix an arbitrary matrix

$$\mathbf{G} = \left[\mathbf{g}_{i,j} : i,j = 1,\dots,R\right], \qquad \mathbf{g}_{i,j} \in \mathbb{R}^3$$

with vector entries satisfying the geometric constrains. With the standard notations for the line segment $[\mathbf{p}_i,\mathbf{p}_j] = \mathrm{Conv}\{\mathbf{p}_i,\mathbf{p}_j\}$ resp the triangle $G_i^{j,k} = \mathrm{Conv}\{0,\mathbf{g}_{i,j},\mathbf{g}_{j,k}\}$,

$$\mathbf{g}_{i,j} = 0 \ \text{ if } \ i=j \ \text{ or } \ [\mathbf{p}_i,\mathbf{p}_j] \notin \mathcal{E}, \tag{10}$$

$$G_i^{j,k} \text{ is non-degenerate if } \mathrm{Conv}\{\mathbf{p}_i,\mathbf{p}_j,\mathbf{p}_k\} \in \mathcal{T}, \tag{11}$$

$$G_i^{j,k} \cap G_i^{j,\overline{k}} = [0,\mathbf{g}_{i,j}] \ \text{ whenever } \mathbf{E}_m = [\mathbf{p}_i,\mathbf{p}_j] \text{ is a double edge.} \tag{12}$$

Remark 1. These restrictions are natural in the sense that, for each mesh vertex $\mathbf{p}_i$, (10), (11) imply the existence of a plane $\mathbf{S}_i$ passing through the point $\mathbf{f}_i$ such that $\mathbf{g}_{i,j} \in \mathrm{Tan}(\mathbf{S}_i)$ $(i,j=1,\dots,R)$. The plane $\mathbf{S}_i$ will play the role of a guessed tangent plane of the surface interpolating the points in $\mathbf{F}$ by our construction. Condition (12) excludes "too twisted" surfaces.

Remark 2. The popular task of constructing surfaces passing through the mesh vertices, corresponds to the case $\mathbf{f}_i = \mathbf{p}_i$ $(i \in [1,R]$. Often only scanned data for the mesh points $\mathcal{P}$ with a triangularization (the

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family $\mathcal{T}$) are available and the tangent vectors $\mathbf{g}_{i,j}$ should be guessed. If we are given the tangent plane $\mathbf{S}_i$ (e.g., the scanner provides also a normal vector $\mathbf{n}_i$ to the scanned surface) there is a natural choice, namely the orthogonal projection of the edge vector $\mathbf{p}_{i,j} = \mathbf{p}_j - \mathbf{p}_i$ onto $\mathbf{S}_i$. Without further information on tangent planes, if the mesh triangles form a closed surface, a convenient guess for normal vectors is $\mathbf{n}_i = \mathbf{p}_{i,j_{\nu(i)}} \times \mathbf{p}_{i,j_1} + \sum_{k=1}^{\nu(i)} \mathbf{p}_{i,j_k} \times \mathbf{p}_{i,j_{k+1}}$ where $\mathbf{p}_{j_1}, \dots, \mathbf{p}_{j_{\nu(i)}}$ form a cycle of the neighboring vertices of $\mathbf{p}_i$ such that all the segments between consecutive elements are mesh edges.

Definition 1. Given any mesh vertex $\mathbf{p}_i \in \mathcal{P}$, define its *extended weights* $\lambda_k : \mathbf{T} \to [0,1]$ as the union of the functions $\lambda_{\mathbf{p}_k}^{\mathbf{T}_n}$ on the mesh triangles containing the point $\mathbf{p}_k$ as a vertex letting to vanish on the remaining mesh triangles. That is, in terms of restrictions,

$$\lambda_k \big| \mathbf{T}_n = \lambda_{\mathbf{p}_k}^{\mathbf{T}_n} \text{ if } \mathbf{p}_k \in \mathbf{T}_n; \quad \lambda_k \big| \mathbf{T}_n = 0 \text{ else.} \tag{13}$$

Notice that the functions λ_i are well-defined and continuous. This is clear outside the double edges since they consist of affine functions restricted to pairwise disjoint sets. Given any double edge $\mathbf{E}_m = \mathbf{T}_n \cap \mathbf{T}_{\overline{n}} = [\mathbf{p}_i, \mathbf{p}_j]$, we have the coincidence $\lambda_{\mathbf{p}_k}^{\mathbf{T}_n} | \mathbf{E}_m = \lambda_{\mathbf{p}_k}^{\mathbf{T}_n} | \mathbf{E}_m$. Indeed, in terms of the Kronecker-δ at the end points $\mathbf{p}_i, \mathbf{p}_j$ we have $\lambda_{\mathbf{p}_k}^{\mathbf{T}_n}(\mathbf{p}_\ell) = \delta_{k,\ell} = \lambda_{\mathbf{p}_k}^{\mathbf{T}_{\overline{n}}}(\mathbf{p}_\ell)$ $(\ell = i, j)$. Since the graph of an affine function defined on a triangle in $\mathbb{R}^3$ is a triangle in $\mathbb{R}^4$, the graphs of the subfunctions $\lambda_{\mathbf{p}_k}^{\mathbf{T}_n}, \lambda_{\mathbf{p}_k}^{\mathbf{T}_{\overline{n}}}$ of λ_k form two adjacent triangles in $\mathbb{R}^4$ meeting in the segment with end points $[\mathbf{p}_i, \delta_{k,i}]$ resp. $[\mathbf{p}_j, \delta_{k,j}]$ whence the continuity of λ_k is immediate.

Remark 3. (i) By definition $\mathcal{C}^1(\mathbf{T}_n, \mathbb{R}^3)$ is the family of all continuous functions $F : \mathbf{T}_n \to \mathbb{R}^3$ being continuously differentiable on the the interior $\mathbf{T}_n^o = \bigcup_{i=1}^R \{\mathbf{x} \in \mathbf{T}_n : \lambda_i(\mathbf{x}) > 0\}$ of $\mathbf{T}_n$ whose Fréchet derivatives $\big($as functions $\mathbf{T}_n^o \to \mathcal{L}(\mathrm{Tan}(\mathbf{T}_n), \mathbb{R}^3)\big)$ extend continuously to $\mathbf{T}_n$. It is an easy consequence of Whitney's embedding theorem [11] that any function $F \in \mathcal{C}^1(\mathbf{T}_n, \mathbb{R}^3)$ admits a continuously differentiable extension to $\mathrm{Aff}(\mathbf{T}_n)$

(ii) Recall that a *parametrized G_1-spline surface* in 3D over the mesh $\mathcal{T}$ is a continuous function $F : \mathbf{T} \to \mathbb{R}^3$ with subfunctions $F_n = F \big| \mathbf{T}_n \in \mathcal{C}^1(\mathbf{T}_n)$ such that any two submaps $F_n, F_{\overline{n}}$ $(n = j_*(m,1), \overline{n} = j_*(m,2))$ along a double edge $\mathbf{E}_m$ meet with tangent spaces not spanning the whole $\mathbb{R}^3$:

$$\dim \mathrm{Span}\Big(\{F_n'(\mathbf{x})\mathbf{u} : \mathbf{u} \in \mathrm{Tan}(\mathbf{T}_n)\} \cup \{F_{\overline{n}}'(\mathbf{u} : \mathbf{u} \in \mathrm{Tan}(\mathbf{T}_{\overline{n}})\} \Big) \le 2 \quad (\mathbf{x} \in \mathbf{E}_m). \tag{14}$$

Lemma 1. *In terms of the edge- resp. weight line vectors, the G_1-coupling relation (14) can be expressed in the analytic form*

$$\det\Big[F_n'(\mathbf{x})\mathbf{u}_m, F_{\overline{n}}'(\mathbf{x})\overline{\mathbf{u}}_m, F_n'(\mathbf{x})\mathbf{p}_{i,j}\Big] = 0 \qquad (\mathbf{x} \in \mathbf{E}_m = [\mathbf{p}_i, \mathbf{p}_j] = \mathbf{T}_n \cap \mathbf{T},\ n \ne \overline{n}) \tag{15}$$

Proof. This is an immediate consequence of the fact that $\mathrm{Tan}(\mathbf{T}_n) = \mathrm{Span}\{\mathbf{u}_m, \mathbf{p}_{i,j}\}$ and $\mathrm{Tan}(\mathbf{T}_{\overline{n}}) = \mathrm{Span}\{\overline{\mathbf{u}}_m, \mathbf{p}_{i,j}\}$ if $\mathbf{E}_m$ is a double edge with $n = n_*(m,1)$, $\overline{n} = n_*(m,2)$, $\mathbf{x} \in \mathbf{E}_m$ resp. $i = j_*(m,1)$, $j = j_*(m,2)$ and $\mathbf{x} \in \mathbf{E}_m = [\mathbf{p}_i, \mathbf{p}_j]$. □

4. Construction Lemma

The next observation describes the pattern of our later constructions.

Lemma 2. *Let $f : \mathbf{T} \to \mathbb{R}^3$ be a continuous map with subfunctions $f_n = f | \mathbf{T}_n \in \mathcal{C}^1(\mathbb{R}^3)$. Assume $z_1, \dots, z_M \in \mathcal{C}^1([0,1]^2, \mathbb{R}^3)$ are functions such that, for $m = 1, \dots, M$ we have*

$$t^2(1-t)^2 \det \begin{bmatrix} z_m(t, 1-t) \\ \overline{v}_m(t) - v_m(t) \\ u_m(t) \end{bmatrix} = \det \begin{bmatrix} v_m(t) \\ \overline{v}_m(t) \\ u_m(t) \end{bmatrix} \qquad (0 < t < 1); \tag{16}$$

$$v_m(t) = f_n'(\mathbf{x}_t^m)\mathbf{u}_m, \quad \overline{v}_m(t) = f_{\overline{n}}'(\mathbf{x}_t^m)\overline{\mathbf{u}}_m, \quad u_m(t) = f_n'(\mathbf{x}_t^m)\mathbf{p}_{j(m,1),j(m,2)} \tag{17}$$

with the indices

$$n = n_*(m,1), \ \ \overline{n} = n_*(m,2), \quad \mathbf{x}_t^m = t\mathbf{p}_{j_*(m,1)} + (1-t)\mathbf{p}_{j_*(m,2)} \ \ (0 \le t \le 1); \tag{18}$$

$$k = \sum_{\ell=1}^{3} i(n,\ell) - \sum_{\ell=1}^{2} j(m,\ell), \quad \overline{k} = \sum_{\ell=1}^{3} i(\overline{n},\ell) - \sum_{\ell=1}^{2} j(m,\ell) \tag{19}$$

and $\mathbf{u}_m, \overline{\mathbf{u}}_m$ *are the weight line vectors given in* (9)*. Then the function*

$$F = f - Z, \quad Z = \sum_{m=1}^{M} z_m\big(\lambda_{j(m,1)}, \lambda_{j(m,2)}\big)\lambda_{j(m,1)}^2 \lambda_{j(m,2)}^2 \big[\lambda_{k(m,1)} + \lambda_{k(m,2)}\big] \tag{20}$$

is a parametrized G_1*-spline surface over the mesh* $\mathcal{T}$*.*

Proof. Consider any mesh triangle $\mathbf{T}_n$ with edges $\mathbf{E}_{m_1} = [\mathbf{p}_{r_2}, \mathbf{p}_{r_3}]$, $\mathbf{E}_{m_2} = [\mathbf{p}_{r_3}, \mathbf{p}_{r_1}]$ resp. $\mathbf{E}_{m_3} = [\mathbf{p}_{r_1}, \mathbf{p}_{r_2}]$. Observe that the restriction $F_n = F|\mathbf{T}_n$ of F to $\mathbf{T}_n$ has the form

$$\begin{aligned} F_n =& f_n - \Big[z_{m_1}(\lambda_{r_2}, \lambda_{r_3})\lambda_{r_1}\lambda_{r_2}^2\lambda_{r_3}^2|_{\mathbf{T}_n} + \\ &+ z_{m_2}(\lambda_{r_3}, \lambda_{r_1})\lambda_{r_2}\lambda_{r_3}^2\lambda_{r_1}^2|_{\mathbf{T}_n} + z_{m_3}(\lambda_{r_1}, \lambda_{r_2})\lambda_{r_3}\lambda_{r_1}^2\lambda_{r_2}^2|_{\mathbf{T}_n}\Big]. \end{aligned}$$

Since each weight λ_{r_k} vanishes on the edge $\mathbf{E}_{m_k}$ $(k = 1,2,3)$, all products functions of the form $z_m(\lambda_r, \lambda_s)\lambda_q\lambda_r^2\lambda_s^2$ with $m \in \{m_1, m_2, m_3$ and $\{q,r,s\} = \{r_1, r_2, r_3\}$ belong to $\mathcal{C}^1(\mathbf{T}_n)$ and vanish along the edges of $\mathbf{T}_n$. Since the subfunctions $f_n, \lambda_r|\mathbf{T}_n$ $(r = 1, \ldots, R)$ belong to $\mathcal{C}^1(\mathbf{T}_n)$ by assumption, also $F_n \in \mathcal{C}^1(\mathbf{T}_n, \mathbb{R}^3)$. Thus $F : \mathbf{T} \to \mathbb{R}^3$ is a continuous function coinciding with f on the mesh edges.

To complete the proof we have to show the G_1-coupling of the subfunctions of F along the mesh edges. Suppose (without loss of generality) that $\mathbf{E}_m = \mathbf{E}_{m_3} = [\mathbf{p}_{r_1}, \mathbf{p}_{r_2}]$ is a duble edge between the triangles $\mathbf{T}_n = \mathrm{Conv}\{\mathbf{p}_{r_1}, \mathbf{p}_{r_2}, \mathbf{p}_{r_3}\}$ and $\mathbf{T}_{\overline{n}} = \mathrm{Conv}\{\mathbf{p}_{r_1}, \mathbf{p}_{r_2}, \mathbf{p}_{r_3}\}$. According to Lemma 1, the subfunctions F_n and $F_{\overline{n}}$ are G_1-coupled if and only if the determinant criterion (15) holds.

Let $\mathbf{x} = \mathbf{x}_t = t\mathbf{p}_{r_1} + (1-t)\mathbf{p}_{r_2}$ be a generic point on $\mathbf{E}_{m_3}$. Since the function λ_{r_3} vanishes on $[\mathbf{p}_{r_1}, \mathbf{p}_{r_2}]$, we have

$$\begin{aligned} F_n'(\mathbf{x}_t)\mathbf{u}_m &= f_n'(\mathbf{x}_t)\mathbf{u}_m - z_m\big(\lambda_{r_1}(\mathbf{x}_t), \lambda_{r_2}(\mathbf{x}_t)\big)[G_{r_3}\mathbf{u}]\lambda_{r_1}(\mathbf{x}_t)^2\lambda_{r_2}(\mathbf{x}_t)^2 = \\ &= f_n'(\mathbf{x}_t)\mathbf{u}_m - z_m(t, 1-t)[G_{r_3}\mathbf{u}_m]t^2(1-t)^2 = \\ &= u_m(t) - z_m(t,1-t)[G_{r_3}\mathbf{u}_m]t^2(1-t)^2. \end{aligned}$$

Similarly $F_{\overline{n}}'(\mathbf{x}_t)\mathbf{u}_m = \overline{u}_m(t) - z_m(t,1-t)[G_{r_3}\overline{\mathbf{u}}_m]t^2(1-t)^2$. Thus (9) holds if and only if

$$\begin{aligned} 0 = \det\begin{bmatrix} v_m(t) - t^2(1-t)^2 z_m(t,1-t) \\ \overline{v}_m(t) - t^2(1-t)^2 z_m(t,1-t) \\ u(t) \end{bmatrix} &= \\ = \det\begin{bmatrix} v_m(t) - t^2(1-t)^2 z_m(t,1-t) \\ \overline{v}_m(t) - v_m(t) \\ u_m(t) \end{bmatrix} &- \det\begin{bmatrix} t^2(1-t)^2 z_m(t,1-t) \\ \overline{v}_m(t) - v_m(t) \\ u_m(t) \end{bmatrix} \end{aligned}$$

which completes the proof. □

Remark 4. Notice that the statement imposes constrains on the corrector functions $z_m : [0,1]^2 \to \mathbb{R}^3$ only by the determinant condition 16) referring to the segment $\big\{(t, 1-t) : 0 \le t \le 1\big\}$. We can choose the values $z(t_1, t_2)$ for $(0 \le t_1, t_2, t_1 + t_2 < 0.$ rather freely which may influence heavily the behaviour of the spline-surface F outside the mesh edges.

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5. RSD Interpolation

Henceforth let $\Pi = [\Psi_1, \chi_0, \chi_1]$ be an arbitrarily fixed tuple of functions $\Psi_0, \Psi_1 \in \mathcal{C}^1\big([0,1]\big)$ resp. $\chi_0, \chi_1 \in \mathcal{C}^1\big([0,1]^3\big)$ such that

$$0 = \Psi_0(0) = \Psi_0'(0) = \Psi_1(0) = \Psi_1'(0) = \Psi_0'(1), \quad 1 = \Psi_0(1) = \Psi_1(1) \tag{21}$$

$$0 = \chi_k'(1,0,0) = \chi_k'(0,1,0) = \chi_k'(0,0,1) \quad (k = 1,2). \tag{22}$$

For arbitrary dimensional triagular meshes $\mathbf{T} \subset \mathbb{R}^d$ with arbitrary dimensional data $\mathbf{F} = \big[\mathbf{f}_n : n = 1,\dots,N\big]$, $\mathbf{G} = \big[\mathbf{g}_{i,j} : i,j = 1,\dots,R\big]$ in another space $\mathbb{R}^{\overline{d}}$ such that $\mathbf{g}_{i,i} = 0$, we define the associated *basic* Π*-interpolation splines* $\mathbf{T} \to \mathbb{R}^{\overline{d}}$ as the functions

$$f_\Pi^{\mathbf{T},\mathbf{F},\mathbf{G}} = \sum_{i=1}^{N} \Big[\Psi_0(\lambda_i)\mathbf{f}_i + \Psi_1(\lambda_i \sum_{j=1}^{N} \lambda_j \mathbf{g}_{i,j}\Big] + \\ + \sum_{(i,j,k)\in S_3} \Big[\chi_0(\lambda_i,\lambda_j,\lambda_k)\mathbf{f}_i + \chi_1(\lambda_i,\lambda_j,\lambda_k)\mathbf{g}_{i,j}\Big] \tag{23}$$

with $S_3 = \{\text{permutations of } 1,2,3\}$. Notice that under the hypothesis (21), (22), $f_\Pi^{\mathbf{F},\mathbf{T},\mathbf{G}}$ interpolates the data in $\mathbf{F},\mathbf{G}$ in the sense that

$$f_\Pi^{\mathbf{F},\mathbf{T},\mathbf{G}}(\mathbf{p}_1) = \mathbf{f}_i, \quad \big[f_\Pi^{\mathbf{F},\mathbf{T},\mathbf{G}}\big]_n'(\mathbf{p}_i)\mathbf{p}_{i,j} = \frac{d}{dt}\Big|_{t=0+} f_\Pi^{\mathbf{F},\mathbf{T},\mathbf{G}}\big((1-t)\mathbf{p}_1 + t\mathbf{p}_j\big) = \mathbf{g}_{i,j}$$

whenever $[\mathbf{p}_i, \mathbf{p}_j]$ is an edge of a mesh triangle $\mathbf{T}_n$.

Definition 2. We say that $\Pi = [\Psi_0, \Psi_1, \chi_0, \chi_1]$ is an *RSD tuple* if given any non-degenerate triangle $\mathbf{T} = \mathrm{Conv}\{\mathbf{p}_i, \mathbf{p}_j, \mathbf{p}_k\}$ in $\mathbb{R}^2$ (regarded as a mesh consisting of a single element), with 1-dimensional data $\mathbf{F} = \{\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3\} \subset \mathbb{R}$ resp. $\mathbf{G} = [\mathbf{g}_{i,j} : i,j = 1,2,3\} \subset \mathbb{R}$ with $\mathbf{g}_{i,i} = 0$, along any edge $[\mathbf{p}_i, \mathbf{p}_j]$ of $\mathbf{T}$, independently of the data $\mathbf{f}_k, \mathbf{g}_{k,i}, \mathbf{g}_{k,j}$ associated with the third vertex, we have

$$f_\Pi^{\mathbf{T},\mathbf{F},\mathbf{G}}\big(t\mathbf{p}_i + (1-t)\mathbf{p}_j\big) = \\ = \Psi_0(t)\mathbf{p}_i + \Psi_0(1-t)\mathbf{p}_j + \Psi_1(t)(1-t)\mathbf{g}_{i,j} + \Psi_1(1-t)t\mathbf{g}_{j,i}\,, \tag{24}$$

$$[f^{\mathbf{T},\mathbf{F},\mathbf{G}}]_\Pi'\big(t\mathbf{p}_i + (1-t)\mathbf{p}_j\big)\mathbf{u}_k = \\ = \Psi_1(t)\Big[\mathbf{g}_{i,k} - \frac{1}{2}\mathbf{g}_{i,j}\Big] + \Psi_1(1-t)\Big[\mathbf{g}_{j,k} - \frac{1}{2}\mathbf{g}_{j,i}\Big] \text{ with } \mathbf{u}_k = \mathbf{p}_k - \frac{1}{2}[\mathbf{p}_i + \mathbf{p}_j]\,. \tag{25}$$

Remark 5. The term RSD is an abbreviation for *reduced side derivative* named after the property described in (25). Motivated by the main result of [9], in [10] we introduced the concept of RSD tuples and proved that given any pair of functions $\Psi_0, \Psi_1 \in \mathcal{C}^1\big([0,1]\big)$ satisfying (21), one can find $\chi_0, \chi_1 \in \mathcal{C}^1\big([0,1]^3\big)$ with (22) such that $[\Psi_1, \Psi_0, \chi_0, \chi_1]$ be an RSD tuple.

In (25) we apply [10, Definition 3.2] with the weight line vectors $\mathbf{u}_m$. In accordance with [9, Theorem 1] and [10, Example 3.15ab], for later use we propose the following two convenient choices:

$$\Pi_0 = [\Phi, \Theta, 30t_1^2t_2^2t_3, 12t_1^2t_2^2t_3] \quad \text{resp.} \quad \Pi_1 = [\Phi, \Phi, 30t_1^2t_2^2t_3, 30t_1^2t_2^3t_3] \tag{26}$$

in terms of the shape functions

$$\Phi(t) = t^3(10 - 15t + 6t^2) \quad \text{and} \quad \Theta(t) = t^3(4 - 3t). \tag{27}$$

(a) Π_0 is the unique polynomial RSD tuple Π of minimal degrees with the *range shift property* $f_\Pi^{\mathbf{T}+\mathbf{v},\mathbf{F},\mathbf{G}} = f_\Pi^{\mathbf{T},\mathbf{F},\mathbf{G}} + \mathbf{v}$ $(\mathbf{v} \in \mathbb{R}^2)$. This follows from the classification in [8] of all locally generated constant preserving $\mathcal{C}^1$-spline procedures with polynomial shape functions.

(b) Π_1 is a polynomial RSD tuple obtained with affinity invariant procedure in the sense of [10]. Hence, it has range shift property along with the *coordinate stability* $f_{\Pi_1}^{\mathbf{T},\mathbf{F},\Delta\mathbf{T}}(\mathbf{x}) = \text{Identity}_{\mathbf{T}}$ where $\Delta(\mathbf{T}) = \big[\mathbf{p}_j - \mathbf{p}_i : i,j = 1,2,3\big]$.

Notice that due to linearity, given any tuple Π with range shift property (in particular the tuples Π_0, Π_1), the figure range$\big(F_{\Pi_1}^{\mathbf{T},[\mathbf{p}_j - \mathbf{p}_i : i,j=1,2,3]}\big)$ coincides with $\mathbf{T}$. Nevertheless Π_0 is not coordinate stable. Heuristically: we can expect to achieve better approximation by using procedures with Π_1 than with Π_0 if the side derivatives $\mathbf{g}_{i,j}$ are close to $\mathbf{g}_j - \mathbf{g}_i$.

Proposition 1. *Even in the general setting of $\mathcal{T}$ being a triangular mesh in $\mathbb{R}^d$ and $\mathbf{F}, \mathbf{G} \subset \mathbb{R}^{\overline{d}}$, if Π is and RSD tuple then the the subfunctions $f_\Pi^{\mathbf{T},\mathbf{F},\mathbf{G}}|\mathbf{T}_n$ of the related interpolation function* (23) *have also properties* (24),(25) *with the substitutions $i = i_*(n,1), j = i_*(n,2), k = i_*(n,3)$.*

Proof. The statement is an immediate consequence of the observations that, given any mesh triangle $\mathbf{T}_n$, its supporting plane $\text{Aff}(\mathbf{T}_n)$ is affinely equivalent to $\mathbb{R}^2$ and that one can verify 24),25) by checking the component functions $\mathbf{x} \mapsto \langle f_\Pi^{\mathbf{T},\mathbf{F},\mathbf{G}} | \mathbf{e}_\ell^d \rangle$ $(\ell = 1, \ldots, \overline{d})$ with the unit vectors $\mathbf{e}_\ell$ of $\mathbb{R}^{\overline{d}}$. □

6. RSD Corrections over Mesh Edges

We turn back to the setting in $\mathbb{R}^3$ and we are going to apply the construction in Lemma 2 with the RSD interpolation function $f : \mathbf{T} \to \mathbb{R}$ of the data. Concerning the derivative data $\mathbf{G} = \big[\mathbf{g}_{i,j} : i,j = 1,\ldots,R\big]$ we assume that there is an indexed family $\big[\mathbf{n}_i : i = 1,\ldots,R\big]$ of unit vectors (candidates for normal vectors at the mesh point for the parametrized surface to be constructed (cf. Remark 1) such that

$$\mathbf{g}_{i,j} \perp \mathbf{n}_i \quad (i,j = 1,\ldots,R). \tag{28}$$

Henceforth, for short, we write $f = f_\Pi^{\mathbf{T},\mathbf{F},\mathbf{G}}$ in terms of the weight values $(t_i, t_2, t_3) \in \Delta_3$ of a generic point in a mesh triangle:

$$\begin{aligned} f(\mathbf{x}^n_{t_1,t_2,t_3}) = \sum_{i=1}^3 \Big[\Psi_0(t_i)\mathbf{f}_i^n + \Psi_1(t_i)\sum_{j=1}^3 t_j \mathbf{g}_{i,j}^n\Big] + \quad & \Big(\mathbf{x}^n_{t_1,t_2,t_3} = \sum_{\ell=1}^3 t_\ell \mathbf{p}_\ell^n, \\ + \sum_{(i,j,k)\in S_3} \Big[\chi_0(t_i,t_j,t_k)\mathbf{p}_i^n + \chi_1(t_i,t_j,t_k)\mathbf{g}_{i,j}^n\Big], \quad & \mathbf{p}_\ell^n = \mathbf{p}_{i(n,\ell)},\ \mathbf{g}_{k,\ell}^n = \mathbf{q}_{i(n,k),i(n,\ell)}\Big). \end{aligned} \tag{29}$$

Given a double mesh edge $\mathbf{E}_m$ coupling the adjacent mesh triangles $\mathbf{T}^n, \mathbf{T}^{\overline{n}}$ with $n = n_*(m,1) < \overline{n} = n_*(m,2)$, we can express the the directional derivatives $v_m, \overline{v}_m, u_m, \overline{u}_m : [0,1] \to \mathbb{R}^3$ in Lemma 1 in terms of the shape functions Ψ_0, Ψ_1 and the directions as follows. With suitable indices $i,j,k,\overline{k} \in \{1,\ldots,R\}$ we can write

$$\begin{aligned} &\mathbf{E}_m = [\mathbf{p}_i, \mathbf{p}_j], \quad \mathbf{T}_n = \text{Conv}\{\mathbf{p}_i, \mathbf{p}_j, \mathbf{p}_k\}, \quad \mathbf{T}_{\overline{n}} = \text{Conv}\{\mathbf{p}_i, \mathbf{p}_j, \mathbf{p}_{\overline{k}}\}, \\ &\mathbf{u}_m = \mathbf{p}_k - \frac{1}{2}[\mathbf{p}_i + \mathbf{p}_j] = \mathbf{p}_{i,k} - \frac{1}{2}\mathbf{p}_{i,j} = \mathbf{p}_{j,k} - \frac{1}{2}\mathbf{p}_{j,i}, \\ &\overline{\mathbf{u}}_m = \mathbf{p}_{\overline{k}} - \frac{1}{2}[\mathbf{p}_i + \mathbf{p}_j] = \mathbf{p}_{i,\overline{k}} - \frac{1}{2}\mathbf{p}_{i,j} = \mathbf{p}_{j,\overline{k}} - \frac{1}{2}\mathbf{p}_{j,i} \,. \end{aligned}$$

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Due to the side derivative reduction property (25),

$$\begin{aligned} v_m(t) &= f'\big(t\mathbf{p}_i + (1-t)\mathbf{p}_j\big)\mathbf{u}_m = \\ &= \Psi_1(t)\Big[\mathbf{g}_{i,k} - \frac{1}{2}\mathbf{g}_{i,j}\Big] + \Psi_1(1-t)\Big[\mathbf{g}_{j,k} - \frac{1}{2}\mathbf{g}_{j,i}\Big]. \quad (30)\\ \overline{v}_m(t) &= \Psi_1(t)\Big[\mathbf{g}_{i,\overline{k}} - \frac{1}{2}\mathbf{g}_{i,j}\Big] + \Psi_1(1-t)\Big[\mathbf{g}_{j,\overline{k}} - \frac{1}{2}\mathbf{g}_{j,i}\Big], \quad (31)\\ u_m(t) &= f'\big(t\mathbf{p}_i + (1-t)\mathbf{p}_j\big)\mathbf{p}_{j,i} = -\frac{d}{dt}f\big(\mathbf{p}_i - t\mathbf{p}_{j,i}\big) = \\ &= \Psi_0'(t)(t)\mathbf{p}_i - \Psi_0'(1-t)\mathbf{p}_j + \big[\Psi_1'(t)(1-t) - \Psi_1(t)\big]\mathbf{g}_{i,j} - \\ &\qquad - \big[\Psi_1'(1-t)t - \Psi_1(1-t)\big]\mathbf{g}_{j,i}\,. \quad (32)\end{aligned}$$

Therefore, by setting

$$w_m(t) = u_m(t) \times \big[\overline{v}_m(t) - v_m(t)\big] \quad (33)$$

the determinant conditon (15) of G_1-coupling has the form

$$0 = \big\langle v_m(t) - t^2(1-t^2)z_m(t)\big|w_m(t)\big\rangle \qquad (0 \le t \le 1). \quad (34)$$

Geometrically, the parameter t above is the weight value $t = \lambda_i(\mathbf{x}_t) = 1 - \lambda_j(\mathbf{x}_j)$ of a generic point $\mathbf{x}_t = t_{\mathbf{p}_i} + (1-t)\mathbf{p}_j = 1 - \lambda_j$. Taking the algebraically more symmetric form $z(t) = \frac{1}{2}z\big(\lambda_i(\mathbf{x}_t)\big) + \frac{1}{2}z\big(1 - \lambda_j(\mathbf{x}_t)\big)$, we conclude the following characterization.

Theorem 1. *Given any RSD tuple* $\Pi = [\Psi_0, \Psi_1, \chi_0, \chi_1]$ *with a family of functions* $z_1, \ldots, z_M \in \mathcal{C}^1\big([0,1]^2, \mathbb{R}^3\big)$, *the map*

$$\begin{aligned} &F = f_\Pi^{\mathbf{T},\mathbf{P},\mathbf{G}} - Z_{[z_1,\ldots,z_m]}^{\mathbf{T},\mathbf{P},\mathbf{G}} \quad \text{where} \\ &Z_{[z_1,\ldots,z_m]}^{\mathbf{T},\mathbf{P},\mathbf{G}} = \sum_{m=1}^{M} z_m\big(\lambda_{j(m,1)}, \lambda_{j(m,2)}\big)\lambda_{j(m,1)}^2\lambda_{j(m,2)}^2\big[\lambda_{k(m,1)} + \lambda_{k(m,2)}\big]\end{aligned}$$

defined in terms the mesh $\mathcal{T}$ *with the structure described in* Section (3) *is a parametrized* G_1*-spline surface* $\mathbf{T} \to \mathbb{R}^3$ *satisfying the constraints* (1) *in the* G1-Interpolation Problem *whenever, in terms of the vector functions* (30), (31), (33) *we have*

$$t^2(1-t)^2\Big\langle z_m(t, 1-t)\Big|w_m(t)\Big\rangle = \Delta_m(t) \quad (0 \le t \le 1) \quad \text{with} \quad \Delta_m(t) = \det\begin{bmatrix} v_m(t) \\ \overline{v}_m(t) \\ u_m(t)\end{bmatrix}. \quad (35)$$

Remark 6. In terms of the index function (7), by setting $z_{M^*+1}, \ldots, z_M = 0$, the subfunction $Z_\Pi^{\mathbf{T},\mathbf{P},\mathbf{G}}\big|\mathbf{T}_n$ has the form

$$\begin{aligned} &Z_\Pi^{\mathbf{T},\mathbf{P},\mathbf{G}}\big(\mathbf{x}_{t_1,t_2,t_3}^n\big) = \sum_{\ell=1}^{3} z_m(t_1, t_2)t_1^{2-\delta_{1,\ell}}t_2^{2-\delta_{2,\ell}}t_3^{2-\delta_{3,\ell}} \qquad \big((t_1,t_2,t_3) \in \Delta_3\big); \\ &\text{whenever} \quad z_m = \Big[\text{solution of (35) if } m \le M^*, \quad 0 \text{ else}\Big] \end{aligned} \quad (36)$$

7. Criteria for RSD Solutions

Throughout the whole section, let $\Pi = [\Psi_0, \Psi_1, \chi_0, \chi_1]$ denote an arbitrarily fixed RSD tuple. For simplifying terminology, we use the term *divisibility* for functions in $\mathcal{C}^1\big([0,1]\big)$, meaning that f is divisible by g whenever $f(t) = q(t)g(t)$ $0 \le t \le 1)$ for some (unique) continuous function $g : [0,1[\to \mathbb{R}$ (being necessarily smooth on $(0,1)$).

We start with the following observation, which will be crucial when looking for polynomial solutions of the equations (35) to the G1-Interpolation Problem.

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Proposition 2. *If* $\Psi_0(t) + \Psi_0(1-t) = 1$ $(0 \le t \le 1)$ *and the shape functions* Ψ_0, Ψ *are divisible by* t^3 *then the determinant functions* $\Delta_m(t)$ *in* (35) *are divisible by* $t^2(1-t)^2$.

Proof. Fix an arbitrary edge index m and, for short, omit it for the terms $\Delta_m, v_m, \overline{v}_m, w$. Also we shorten the determinant expressions $\det[\cdots]$ in the form $|\cdots|$.

Start the argument recalling that, by assumption, the guessed tangent vectors issued from a mesh vertex are coplanar. In particular

$$\left|\mathbf{g}_{i,j}, \mathbf{g}_{i,k}, \mathbf{g}_{i,\overline{k}} = \left|\mathbf{g}_{j,i}, \mathbf{g}_{j,k}, \mathbf{g}_{j,\overline{k}}\right| = 0 \right. \tag{37}$$

for the terms appearing in (30), (31), (32). On the other hand, since by (30), (31) we simply have

$$\overline{v}(t) - v(t) = \Psi_1(t)[\mathbf{g}_{i,\overline{k}} - \mathbf{g}_{i,k}] + \Psi_1(1-t)[\mathbf{g}_{j,\overline{k}} - \mathbf{g}_{j,k}].$$

Furthermore the relation $\Psi_0(t) + \Psi_0(1-t) \equiv 1$ implies $\Psi_0'(t) - \Psi_0'(1-t) \equiv 0$ entailing that $\Psi_0'(t) = t^2(1-t)^2\eta(t)$ with the function $\eta(t) = t^{-2}(1-t)^{-2}[t^3\psi_0(t)]' = (1-t)^{-2}[3\psi_0(t) + t\psi_0'(t)]$ which is continuous on $[0,1)$. By the symmetry $\Psi_0'(t) = \Psi_0'(1-t)$, η is continuous also on $(0,-1]$ and hence on the whole closed interval $[0,1]$. Therefore we have

$$\Delta(t) = \begin{vmatrix} \Psi_1(t)[\mathbf{g}_{i,k} - 2^{-1}\mathbf{g}_{i,j}] + \Psi_1(1-t)[\mathbf{g}_{j,k} - 2^{-1}\mathbf{g}_{j,i}] \\ \Psi_1(t)[\mathbf{g}_{i,\overline{k}} - 2^{-1}\mathbf{g}_{i,j}] + \Psi_1(1-t)[\mathbf{g}_{j,\overline{k}} - 2^{-1}\mathbf{g}_{j,i}] \\ \Psi_0'(t)\mathbf{p}_{j,i} + \big[\Psi_1'(t)(1-t) - \Psi_1(t)\big]\mathbf{g}_{i,j} - \big[\Psi_1'(1-t)t - \Psi_1(1-t)\big]\mathbf{g}_{j,i} \end{vmatrix}.$$

$$= \Psi_0'(t)\widetilde{\Delta}_0(t) + \sum_{\ell_1=1}^{2}\sum_{\ell_2=0}^{1}\sum_{\ell_3=1}^{2} \Psi_1\big(\tau_{\ell_1}(t)\big)\Psi_1\big(\tau_{\ell_2}(t)\big)\widetilde{\Delta}_{\ell_1,\ell_2,\ell_3}(t) \tag{38}$$

with the functions $\tau_0(t) = t$, $\tau_1(t) = 1-t$ i.e., $\tau_\ell(t) = t^{\delta_{\ell,0}}(1-t)^{\delta_{\ell,1}}$ where

$$\widetilde{\Delta}_0 = \begin{vmatrix} \text{row 1 of } \Delta(t) \\ \text{row 2 of } \Delta(t) \\ \Psi_0'(t)\mathbf{p}_{j,i} \end{vmatrix}, \qquad \widetilde{\Delta}_{\ell_1,\ell_2,\ell_3}(t) = \begin{vmatrix} \text{term with } \Psi_1\big(\tau_{\ell_1}(t)\big) \\ \text{term with } \Psi_1\big(\tau_{\ell_2}(t)\big) \\ \text{terms with } \Psi_1\big(\tau_{\ell_3}(t), \Psi_1'\big(\tau_{\ell_3}(t)\big) \end{vmatrix}$$

We complete the proof with the observations that

(a) $\widetilde{\Delta}_0$ is divisible with $\Psi_0'(t)$ being divisible with $t^2(1-t)^2$;

(b) For $(\ell_1, \ell_2, \ell_3) \neq (0,0,0)$ or $(1,1,1)$, the determinant function $\widetilde{\Delta}_{\ell_1,\ell_2,\ell_3}(t)$ is divisible with $\Psi_1\big(\tau_{\ell_1}(t)\big)\Psi_1\big(\tau_{\ell_2}(t)\big)\big[\Psi_1(\tau_{\ell_3}(t)) + \Psi_1'\big(\tau_{\ell_3}(t)\big)\big[1 - \tau_{\ell_3}(t)\big]$. Here the term $\Psi_1\big(\tau_{\ell_1}(t)\big)\Psi_1\big(\tau_{\ell_2}(t)\big)\Psi_1\big(\tau_{\ell_3}(t)\big)$ is divisible with the product $\prod_{r=1}^{3}\tau_{\ell_r}^3 = t^{3[\delta_{0,\ell_1}+\delta_{0,\ell_2}+\delta_{0,\ell_3}]}(1-t)^{3[\delta_{1,\ell_1}+\delta_{1,\ell_2}+\delta_{1,\ell_3}]}$. Similarly $\Psi_1\big(\tau_{\ell_1}(t)\big)\Psi_1\big(\tau_{\ell_2}(t)\big)\Psi_1'\big(\tau_{\ell_3}(t)\big)\big[1 - \tau_{\ell_3}(t)\big]$ is divisible with $\tau_{\ell_1}(t)^3\tau_{\ell_2}(t)\tau_{\ell_3}(t)^2[1 - \tau_{\ell_3}(t)] = t^{3\delta_{0,\ell_1}+3\delta_{0,\ell_2}+2\delta_{0,\ell_3}+\delta_{1,\ell_3}}(1-t)^{3\delta_{1,\ell_1}+3\delta_{1,\ell_2}+2\delta_{1,\ell_3}+\delta_{1,\ell_3}}$. Here the sum of the exponents of t and $(1-t)$ equals $3+3+2+1 = 9$. i.e., both terms are divisible by a product $t^r(1-t)^{9-r}$ for some $0 \le r = r(\ell_1, \ell_2, \ell_3) \le 9$. Observe that, except for the cases $(\ell_1, \ell_2, \ell_3) = (0,0,0)$ or $(1,1,1)$, we have $2 \le r(\ell_1, \ell_2, \ell_3) \le 7$.

(c) $\widetilde{\Delta}_{0,0,0}(t) = \Psi_1(t)^2\big[\Psi_1(t) + \Psi_1'(t)(1-t)\big]\begin{vmatrix} \mathbf{g}_{i,k} - 2^{-1}\mathbf{g}_{i,j} \\ \mathbf{g}_{i,\overline{k}} - 2^{-1}\mathbf{g}_{i,j} \\ \mathbf{g}_{i,j} \end{vmatrix} = 0$ and $\widetilde{\Delta}_{1,1,1}(t) = \Psi_1(1-t)^2\big[\Psi_1(1-t) + \Psi_1'(1-t)t\big]\begin{vmatrix} \mathbf{g}_{j,k} - 2^{-1}\mathbf{g}_{j,i} \\ \mathbf{g}_{j,\overline{k}} - 2^{-1}\mathbf{g}_{j,i} \\ \mathbf{g}_{j,i} \end{vmatrix} = 0$ because the vector triples $\{\mathbf{g}_{i,k} - 2^{-1}\mathbf{g}_{i,j}, \mathbf{g}_{i,\overline{k}} - 2^{-1}\mathbf{g}_{i,j}, \mathbf{g}_{i,j}\}$ resp. $\{\mathbf{g}_{j,k} - \frac{1}{2}\mathbf{g}_{j,i}, \mathbf{g}_{j,\overline{k}} - \frac{1}{2}\mathbf{g}_{j,i}, \mathbf{g}_{j,i}\}$ are coplanar. □

8. Complete Polynomial RSD Solutions

Henceforth, until the end of the section, we assume that the terms in Π are polynomials. In particular we shall be interested in the extreme RSD tuples Π_0, Π_1 in (26) with the shape functions (27). Notice that

$$\Phi'(t) = 30t^2(1-t)^2, \quad \Phi(t) + \Phi(1-t) = 1 \quad \text{resp.} \quad \Theta'(t)(1-t) = 12t^2(1-t)^2. \tag{39}$$

We shall apply the following elementary facts from the theory of Euclidean resp. prime ideal rings [1] restricted to the setting of real polynomials:

F1. *If $p, q, r : \mathbb{R} \to \mathbb{R}$ are polynomial functions such that $p(t)q(t) = t^2 r(t)$ and $p(0), p(1) \neq 0$ then $t^2(1-t)^2 | q(t)$.*

F2. *If $p_1, \ldots, p_K, r : \mathbb{R} \to \mathbb{R}$ are polynomial functions then there exist polynomials (the so-called cofactors of r wrt. $q_1, \ldots, q_K$) such that $r = p_1 q_1 + \cdots + p_K q_K$ if and only if $\mathrm{GCD}(p_1, \ldots, p_K) | r$ i.e., the greatest common divisor of $\{p_1, \ldots, p_K\}$ is a divisor of r.*

Remark 7. The computer algebra packages MAPLE resp. WolframMathematica contain commands providing a cofactor representation $\mathrm{GCD}(p_1, p_2) = p_1 \phi_1(p_1, p_2) + p_2 \Phi_2(p_1, p_2)$ with the degree limitation $\max\{\deg(\phi_\ell(p_1, p_2)) : \ell = 1, 2\} \leq \max\{\deg(\phi_\ell(p_1, p_2) : \ell = 1, 2\}$. According to the reference in the packages, the construction of ϕ_1, ϕ_2 goes back to an early work [1] of Bézout, relying on a careful inspection of the steps of Euclidean division, restricted to the case of two polynomials. It seems, there is no analogous command for more polynomials. Our later discussion requires to calculate the GCD of three terms. Clearly we can produce a cofactor representation of the form by calculating consecutively the cofactors of $r = \mathrm{GCD}(p_1, p_2)$ and then the cofactors of $\mathrm{GCD}(r, p_3)$ with the standard routines ϕ_1, ϕ_2 we get a representation $\mathrm{GCD}(p_1, p_2, p_3) = p_1[q_{1,1}q_{2,1}] + p_2[q_{1,2}q_{2,1}] + p_3 q_{2,2}$ with $q_{1,\ell} = \phi_\ell(p_1, p_2)$, $q_{2,\ell} = \phi_\ell(r, p_3)$. Unfortunately, the degree limitation $\max\{\deg(q_{1,1}q_{2,1}), \deg(q_{1,2}q_{2,1}), \deg(q_{1,2})\}$ is no longer valid generally. (One can find several counter-examples of the form $p_1 = s_1 s_2$, $p_2 = s_2 s_3$, $p_3 = s_3 s_1$ with random coefficients). Nevertheless we can prove the following sharpened version of F1 suited for reducing remarkably the numerical costs involving algorithms with GCD of several polynomials.

F2*. *Given any family $p_1, \ldots, p_K$ of real polynomials (or even polynomials with coefficients in a generic field), we can choose $q_1, \ldots, q_K$ with $\max_{k=1}^K \deg(q_k) \leq \max_{k=1}^K \deg(q_k)$ such that $\sum_{k=1}^K p_k q_k = \mathrm{GCD}(p_1, \ldots, p_K)$.*

Since we do not know any reference, we include an Appendix with constructive proof which gives rise to a related algorithm in a straightforward manner.

Lemma 3. *Let $\mathbf{E}_m = [\mathbf{p}_i, \mathbf{p}_j]$ be a double edge being the intersection of the mesh triangles $\mathbf{T}_n = \mathrm{Conv}\{\mathbf{p}_i, \mathbf{p}_j, \mathbf{p}_k\}$ and $\mathbf{T}_{\overline{n}} = \mathrm{Conv}\{\mathbf{p}_i, \mathbf{p}_j, \mathbf{p}_{\overline{k}}\}$. Assume $\Psi_0, \Psi_1, \chi_0, \chi_1$ are polynomial maps and the lateral derivatives $v, \overline{v}, u$ in (30), (31), (32), (33) are polynomial functions. Then the determinant equation (35) admits a polynomial solution $z_m : \mathbb{R}^2 \to \mathbb{R}^3$ whenever $t^2(1-t)^2 | \Delta_m(t)$.*

Proof. Omitting the indices m without danger of confusion, let us write w^1, w^2, w^3 for the components of the polynomial vector function $w : \mathbb{R} \to \mathbb{R}$ and let $\rho = \mathrm{GCD}(w^1, w^2, w^3)$. Assume that $\Delta(t) = t^2(1-t)^2 \delta(t)$ for some polynomial $\delta : \mathbb{R} \to \mathbb{R}$.

Observe that due to hypothesis (12) on the vectors $\mathbf{g}_{r,s}$ $(r, s \in \{i, j, k, \overline{k}\})$, we have

$$\rho(0), \rho(1) \neq 0. \tag{40}$$

Proof by contradiction: The relation $0 = \rho(0) = \mathrm{GCD}(w^1, w^2, w^3)$ would imply $t | w^1, w^2, w^3$ whence $0 = w(0) = [\overline{v}(0) - v(0)] \times u(0) = [\mathbf{g}_{i,\overline{k}} - \mathbf{g}_{i,k}] \times \mathbf{g}_{i,j}$. This is impossible since, by supposing (40), we would have $\mathbf{g}_{i,\overline{k}} = \mathbf{g}_{i,k} + \gamma \mathbf{g}_{i,j}$ for some scalar $\gamma \in \mathbb{R}$ which would mean that the intersection of the

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triangles $\mathrm{Conv}\{\mathbf{p}_i, \mathbf{p}_j, \mathbf{p}_r\}$ $(r = k, \overline{k})$ would be a nondegenerate triangle. We conclude $\rho(1) \neq 0$ by arguing with the index change $i \leftrightarrow j$.

Consider the case $t^2(1-t)^2 \big| \Delta(t)$ i.e., $\Delta(t) = t^2(1-t)^2\delta$ for some polynomial $\delta : \mathbb{R} \to \mathbb{R}$. On the other hand, since $\rho = \mathrm{GCD}(w^1, w^2, w^3)$, we can write $w(t) = \rho(t)\overline{w}(t)$ with the polynomial function with components $\overline{w}^\ell = w^\ell(t)/\rho(t)$. By (33), we have $w(t) = v(t) \times w(t)$ and hence we get the identity

$$t^2(1-t)^2\delta(t) = \rho(t)\Big\langle v(t)\Big| ww(t)\Big\rangle.$$

According to **F1**, we see that necessarily $t^2(1-t)^2 \Big| \langle v(t) | \overline{w}(t)\rangle$ that is $\rho(t) \Big| \frac{\Delta(t)}{t^2(1-t)^2}$. According to **F2**, there are polynomials $q^1, q^2, q^3 : \mathbb{R} \to \mathbb{R}$ such that

$$\frac{\Delta(t)}{t^2(1-t)^2} = q^1(t)w^1(t) + q^2(t)w^2(t) + q^(t)w^1(t) = \langle q(t) | w(t)\rangle$$

which completes the proof. □

As an immediate corollary, we find the following polynomial solution of the G1-Interpolation Problem.

Theorem 2. *Given any polynomial RSD tuple Π, in particular $\Pi = \Pi_0$ or $\Pi = \Pi_1$, the map $F : \mathbf{T} \to \mathbb{R}^3$ in Theorem 1 applied with polynomial edge corrections $z_m(t_1, t_t2)$ such that*

$$\begin{aligned} &z_m(t, 1-t) = \left[\frac{\Delta_m(t)}{t^2(1-t)^2\rho(t)}\right] q_m(t) \qquad (m = 1, \ldots, M^*) \quad \text{where} \\ &\Delta_m(t) = \langle v_m(t) | w_m(i)\rangle, \quad \text{in terms of (30), (31), (32), (33) applied to } f = f_\Pi^{\mathbf{T},\mathbf{P},\mathbf{G}}, \\ &\rho_m(t) = \mathrm{GCD}(w_m^1(t), w_m^2(t), w_m^3(t)) \text{ with cofactors } q_m^1(t), q_m^2(t), q_m^3(t) \end{aligned} \tag{41}$$

is a parametrized G1-spline surface passing through the mesh points $\mathbf{p}_i$ with the lateral derivatives $F'(\mathbf{p}_i)\mathbf{p}_{i,j} = \mathbf{g}_{i,j}$ $([\mathbf{p}_i, \mathbf{p}_j] \in \mathcal{E})$ along mesh edges, which consists of polynomial submaps $F\big|\mathbf{T}_n$.

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Algorithm 1. Representation of $\mathrm{range}(F_\Pi^{\mathbf{T},\mathbf{F},\mathbf{G}})$ **with a polynomial RSD tuple** Π

Require: R, N, M^* for the number of mesh vertices, triangles resp. double edges;
the index functions i_*, j_*, n_*, k_*, m_* of the mesh structure in (3), (4), (5), (6), (7);
$[\mathbf{p}_i]_{i=1}^R$, $[\mathbf{f}_i]_{i=1}^R$, $[\mathbf{g}_{i,j}]_{i,j=1}^R$ for mesh vertices, data values resp. data vectors in (1);
polynomial RSD shape functions $\Psi_0, \Psi_1 \in \mathcal{C}^1([0,1])$, $\chi_0, \chi_1 \in \mathcal{C}^1([0,1]^3)$.

Ensure: List of functions $F_1, \overline{F1}, \dots, F_N, \overline{F}_N : \Delta_3 \to \mathbb{R}^3$ representing subfunctions $F_\Pi^{\mathbf{T},\mathbf{P},\mathbf{G}}|\mathbf{T}_n$
in the form $F_n(t_1,t_2,t_3) = F_\Pi^{\mathbf{T},\mathbf{P},\mathbf{G}}(\mathbf{x}^n_{t_1,t_2,t_3}) \quad ((t_1,t_2,t_2) \in \Delta_3)$
in terms of the local barycentric parametrization $\mathbf{x}^n_{t_1,t_2,t_3}$ in (29) of triangle $\mathbf{T}_n$.

Calculation: With auxiliary storages
$v_m, \overline{v}_m, u_m, w_m, q_m \quad (m \in [1,M])$ for polynomial maps $\mathbb{R} \to \mathbb{R}^3$;
$\zeta_m, \rho_m \quad m = 1, \dots, M)$ for polynomial functions.

STEP 1: Compute and store the basic approximations $f_\Pi^{T,T,G}|\mathbf{T}_n$

$$F_n \leftarrow \Big[(t_1,t_2,t_3) \mapsto f(\mathbf{x}^n_{t_1,t_2,t_3}) \text{ given in (29)}\Big] \quad (n = 1, \dots, N),$$

Substitutions $t_1 \to \lambda_{i_*(n,1)}$, $t_2 \to \lambda_{i_*(n,2)}$, $t_3 \to \lambda_{i_*(n,3)}$ in each F_n;

STEP 2: For $m = 1, \dots, m$, compute and save the edge correction functions

$$\zeta_m \leftarrow \left[t \mapsto \frac{\langle v_m(t)|w_m(t)\rangle}{t^2(1-t)^2\rho_m(t)}, \quad \begin{matrix} v_m, \overline{v}_m, w_m \text{ defined in (30), (31), (32), (33),} \\ \rho_m = [\text{GCD of the components of } w_m] \end{matrix} \right].$$

STEP 3: Using Algorithm 2, compute and save the GCD cofactors of the
components $w_m^1(t), w_m^2(t), w_m^3(t)$ of $w_m(t)$
$q_m \leftarrow [\mathrm{Cofactor}_\ell(w_m(t)) : \ell = 1,2,3]$.

OUTPUT$_1$: The subfunctions $F_\Pi^{\mathbf{T},\mathbf{F},\mathbf{G}}\Big|\mathbf{T}_n$ in storages F_n in terms of extended weights
computed consecutively along the double edges $\mathbf{E}_m$ $(m = 1, \dots, M^*)$
with corrections corresponding to
$z(t, 1-t) = \frac{1}{2}\zeta(t) + \frac{1}{2}\zeta(1-t)$ in Lemma 2:

$i \leftarrow j_*(m,1)$, $j \leftarrow j_*(m,2)$, $k \leftarrow k_*(m,1)$, $\overline{k} \leftarrow k_*(m,2)$, $n \leftarrow n_*(m,1)$, $\overline{n} \leftarrow j_*(m,2)$;

$$F_n \leftarrow F_n + \frac{1}{2} t_\ell \lambda_i^2 \lambda_j^2 \lambda_k \big[\zeta_m(\lambda_i) q_m(\lambda_i) + \zeta_m(1-\lambda_j) q_m(1-\lambda_j)\big],$$

$$F_{\overline{n}} \leftarrow F_{\overline{n}} + \frac{1}{2} t_\ell \lambda_i^2 \lambda_j^2 \lambda_{\overline{k}} \big[\zeta_m(\lambda_i) q_m(\lambda_i) + \zeta_m(1-\lambda_j) q_m(1-\lambda_j)\big];$$

OUTPUT$_2$: The subfunctions $F_\Pi^{\mathbf{T},\mathbf{F},\mathbf{G}}\Big|\mathbf{T}_n$ in storages F_n in terms of local weights

$$\overline{F}_n \leftarrow \Big[F_n \text{ with substitution } \lambda_{i_*(n,\ell} \to t_\ell \ (\ell = 1,2,3)\Big].$$

Appendix A GCD Cofactors with Low Degree

Let $\mathbb{F}$ denote an arbitrarily fixed field and let t be a fixed variable symbol. For short, write $\mathrm{Pol}_N(\mathbb{F})$ for the family of all polynomials $p = p(t) = a_0 + a_1 t + \cdots + a_N t^N$ (as formal sums) with coefficients from $\mathbb{F}$ having degree $\deg(p) = \max\{k, -\infty : a_k \neq 0\}$. For the polynomial division (Euclidean division) and its remainder term of $p, q \in \mathrm{Pol}(\mathbb{F}) = \bigcup_{N=1}^\infty \mathrm{Pol}_N(\mathbb{F})$ resp. the greatest common divisor of a family $\{a_1, \dots, a_K\} \subset \mathrm{Pol}(\mathbb{F})$ we write $p \div q$ with $\rho(p,q$ resp. $\mathrm{GCD}(a_1, \dots, a_K)$. Thus, by definition $p = [p \div q]q + \rho(p,q)$ where the main coefficient of $\mathrm{GCD}(a_1, \dots, a_K)$ has the value $1 \in \mathbb{F}$.

Remark 8. For later use, we recall the following elementary facts:

(a) If $p, q \in \mathrm{Pol}_N(\mathbb{F})$ with $\deg(p) \geq \deg(q) \geq 1$ then
$\deg(p \div q) = \deg(p) - \deg(q)$ and $\deg(\rho(p,q)) < \deg(q)$.

(b) If $p, q \in \mathrm{Pol}_N(\mathbb{F})$ with $\deg(p) \geq \deg(q) \geq 1$ then
$\{\text{Common divisors of } p \text{ and } q\} = \{\text{Common divisors of } q \text{ and } \rho(p,q)\}$.

(c) Given any family $a_1, \dots, a_K \in \mathrm{Pol}(\mathbb{F})$, we have
$\mathrm{GCD}(a_1, \dots, a_K) = \sum_{k=1}^K q_k a_k$ with suitable polynomials $q_1, \dots, q_k \in \mathrm{Pol}(\mathbb{F})$.

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In most popular computer algebra packages, there is a command performing an algorithm due to Bézout [1] providing cofactors $Q_1(a_1,a_2), Q_2(a_1,a_2) \in \mathrm{Pol}(\mathbb{F})$ such that $\mathrm{GCD}(a_1,a_2) = \sum_k Q_k(a_1,a_2)a_k$ with $\max_k \deg\big(Q_k(a_1,a_2)\big) \le \max_k \deg(a_k)$. As mentioned in Remark 7, it seems that no analogous algorithm (or related theoretical result) is available providing the GCD of three polynomials with sufficiently law dimensional cofactors. Below we are going to fill in this gap.

Lemma 4. *If $a_1,\dots,a_K \in \mathrm{Pol}(\mathbb{F})$ with $N = \max_{k=1}^K \deg(a_k) \ge 1$ and $R = \deg(\mathrm{GCD}(a_1,\dots,a_K))$ then we have $\mathrm{GCD}(a_1,\dots,a_K) = q_1a_1 + \cdots + q_Ka_K$. for some $q_1,\dots,q_K \in \mathrm{Pol}_{N-R-1}(\mathbb{F})$.*

Proof. Let $Q := \mathrm{GCD}(a_1,\dots,a_K)$, $K > 1$. Observe that for any family $q_1,\dots,q_K$ of polynomials we have $q_1a_1 + \cdots + q_Ka_K = Q$ if and only if $q_1[a_1/Q] + \cdots + q_K[a_K/Q] = 1$. Thus since $\deg(a_j/Q) = \deg(a_j) - \deg(Q) = \deg(a_j) - M$, and $\mathrm{GCD}(a_1/Q,\dots,a_K/Q) = \mathrm{GCD}(a_1 \div Q,\dots,a_K \div Q) = 1$, it suffices to restrict ourselves to the cases with $Q = 1$. That is it suffices to see the following statement:

$(*)$ *If the polynomials $a_1,\dots,a_K$ are relatively prime (i.e., $\mathrm{GCD}(a_1,\dots,a_K) = 1$) and $N = \deg(a_1) \ge \cdots \ge \deg(a_K) \ge 1$ then there exist $q_1,\dots,q_K$ of degree $\le N-1$ such that $\sum_{k=1}^K q_ka_k = 1$.*

The case $N=1$ is trivial: if $\mathrm{GCD}\{a_1,\dots,a_K\} = 1$ and $1 = \max_{k=1}^K \deg(a_K)$ then there are indices $m_1 \ne m_2$ such that $a_{m_\ell}(t) \equiv \alpha_\ell t + \beta_\ell$ $\ell = 1,2)$ with either $\alpha_1,\alpha_2 \ne 0$ or $\alpha_1,\beta_2 \ne 0 = \alpha_2$. In any case $1 \equiv \gamma_1 a_{m_1}(t) + \gamma_2 a_{m_2}(t)$ with suitable constants $\gamma_1,\gamma_2 \in \mathbb{F}$.

We proceed by induction: Let $N \ge 1$. Assume that given any polynomials $b_1,\dots,b_K$ with $1 \le \max_{k=1}^K \deg(b_k) \le N$ there exist $r_1,\dots,r_K \in \mathrm{Pol}_N(\mathbb{F})$ such that $1 \equiv \sum_{k=1}^K r_kb_k$.

Consider any sequence $a_1,\dots,a_K \in \mathrm{Pol}_{N+1}(\mathbb{F})$ with $\mathrm{GCD}(a_1,\dots,a_K) = 1$. Let $M := \min\{\deg(a_j) : a_j \not\equiv 0\}$ and let m denote an index such that $\deg(a_m) = M$. Notice that in the case of $M = 0$ we simply have $0 \ne a_m(t) \equiv \alpha \in \mathbb{F}$ and hence trivially $1 = \sum_{k=1}^K q_ka_k$ with $q_m \equiv \alpha^{-1}$ and $q_j \equiv 0$ for $j \ne m$.

In the remainder cases $M \ge 1$ we have the alternatives

(i) $N+1 > M \ge 1$ i.e., $N \ge \min\{\deg(a_j) : a_j \not\equiv 0\} = a_m$ for some index m;
(ii) $N+1 = M$ i.e., $\deg(a_j) = N+1$ for all indices j with $a_j \ne 0$.

In the case (i), define $b_m = a_m$, $b_j := \rho(a_j,a_m)$ for $j \ne m$ (in particular $b_j = 0$ if $a_j = 0$). According to Remark 8(b), $\mathrm{GCD}(a_1,\dots,a_K) = \mathrm{GCD}(b_1,\dots,b_K) = 1$. By Remark 8(a), also $\deg(b_j) < \deg(a_m) = M$ $(j \ne m)$ Thus $\max_j \deg(b_j) = \deg(a_m)$ and by the induction hypothesis, there are polynomials $r_1,\dots,r_K$ with degree$\le \deg(a_m)-1 = M-1$ such that

$$1 = r_1b_1 + \cdots + r_Kb_K = r_ma_m + \sum_{j\ne m} r_j[a_j - (a_j \div a_m)a_m] = \sum_{j=1}^K q_ja_j$$

$$\text{with } q_j = r_j \ (j \ne m), \quad q_m = r_m - \sum_{j\ne m} (a_j \div a_m) r_j.$$

Here we have

$$\deg(q_j) = \deg(r_j) \le \deg(a_m) - 1 = M - 1 \le N \ \ (j \ne m),$$
$$\deg(q_m) = \max\big\{\deg(r_m), \deg((a_j \div a_m)r_j) : m \ne j = 1,\dots,K-1\big\}.$$

Since, for $j \ne m$, $\deg(a_j \div a_m) = \deg(a_j) - \deg(a_m) \le N+1-\deg(a_m) = N+1-M$, we have $\deg(q_m) \le N+1-\deg(a_m)+\deg(r_j) \le N+1-\deg(a_m)+\deg(a_m)-1 = N$. It follows $\deg(q_j) \le N$ for all indices which completes the proof in case (i).

Case (ii): Let $\deg(a_j) = a_m = N+1$ for all indices with $a_j \ne 0$. Disregarding the trivial case $a_j = 0$ $(j \ne m)$ with $0 \ne a_m$, we can apply the arguments used in Case (i) to the sequence $\overline{a}_j := a_j$ $(j \ne m)$, $\overline{a}_m := \rho(a_n,a_m)$ with some index n such that $a_n \ne 0$, with the the conclusion that $\sum_{j=1}^K \overline{q}_j\overline{a}_j = 1$ for suitable $\overline{q}_1,\dots,\overline{q}_K \in \mathrm{Pol}_N(\mathbb{F})$. Since $\rho(a_n,a_m) = a_n - (a_n \div a_m)a_m$ where $\deg(a_n \div a_n) = \deg(a_n) - \deg(a_m) = 0$ that is $\overline{a}_m = a_n - \gamma a_m$ with some constant $\gamma \in \mathbb{F}$, we have $1 = \overline{q}_m(a_n - \gamma a_m) + \sum_{j\ne m} \overline{q}_ja_j = \sum_{j=1}^K q_ja_j$ with the polynomials $q_j := \overline{q}_j$ $(j \ne n,m)$, $q_n := \overline{q}_n + \overline{q}_m$, $q_m := -\gamma\overline{q}_m$ of degree $\le N$. $\square$

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Remark 9. Following the arguments in the proof of Lemma 4, we find GCD cofactors with degree$\leq N$ for a sequence $\overline{a}^0 = [a_1^0(t), \ldots, a_K^0(t)]$ of polynomials with degree N by a procedure which consists of decreasing the degree of some of the polynomials stepwise with multiplication with a suitable $(K \times K)$-matrix with polynomial entries.

Starting with $\overline{a}^0$ resp. $X_0 = \mathrm{Id}_K$ repeat the operations $\mathcal{N}, \mathcal{R}, \mathcal{D}$ realized by multiplications from the right with the $(K \times K)$-matrices $N_{\overline{p}}, R_{\overline{p}}, D_{\overline{p}}$ given below until we achieve a sequence of the form $\overline{a}^s = [0, \ldots, 0, a_K^s(t)]$.

$$
\begin{aligned}
&\mathcal{N}, \mathcal{R}, \mathcal{D} : \mathbb{F}[t]^K \to \mathbb{F}[t]^K \text{ operations on } K\text{-tuples of polynomials,}\\
&\qquad \mathcal{N} : [a_k(t)]_{k=1}^K \mapsto [a_k(t)/\mathrm{maincoeff}(a_k)]_{k=1}^K \qquad \textit{normalization},\\
&\qquad \mathcal{N}(\overline{p}) = \overline{p} N_{\overline{p}}, \quad N_{\overline{p}} = [\delta_{i,j}/\mathrm{maincoeff}_*(p_j)]_{i,j=1}^K\\
&\qquad\qquad \text{where } \mathrm{maincoeff}\big(\textstyle\sum_{k=0}^N \alpha_k t^k\big) = \alpha_N \text{ if } \alpha_N \neq 0 \text{ resp. } \mathrm{maincoeff}(0) = 0,\\
&\qquad \mathcal{R} : [a_k(t)]_{k=1}^K \mapsto [a_{\sigma(k)}(t)]_{k=1}^K \qquad \textit{reordering},\\
&\qquad\qquad \text{where } \sigma = \sigma_{\overline{p}} \text{ is an index permutation with the effect}\\
&\qquad\qquad |\deg(a_\pi(1))| \geq |\deg(a_\pi(2))| \geq \cdots \geq |\deg(a_\pi(K))|, \ (\deg(0) = -\infty);\\
&\qquad \mathcal{R}\overline{p} = \overline{p} R_{\overline{p}}, \quad R_{\overline{p}} = [\delta_{i,\sigma_{\overline{p}}(j)}]_{i,j=1}^K.\\
&\qquad \mathcal{D} : [0, \ldots, 0, a_L(t), \ldots, a_K(t)] \mapsto [0, \ldots, 0, a_L^*(t), a_{L+1}, \ldots, a_K(t)] \ \ (a_L \neq 0)\\
&\qquad\qquad \textit{degree decreasing} \text{ with } \ a_L^*(t) = a_L(t) - t^{\deg(a_L) - \deg(a_{L+1})} a_{L+1}(t),\\
&\qquad\qquad \text{and leaving the sequences of the form } [0, \ldots, 0, a_K(t)] \text{ invariant;}\\
&\qquad \text{resp. } D_{\overline{p}} = \mathrm{Id} \text{ if } L = K.
\end{aligned}
$$

The procedure terminates after at most $S = \sum_{k=1}^K \deg(a_k^0)$ steps because the sum of the degrees of the non-zero polynomials in any sequence containing more than on non-zero members is decreased by 1 after each application of $\mathcal{D}$. The GCD of the non-zero polynomials in any sequence remains invariant after each substep. Hence for the values $\overline{p}_s, X_s$ of stores $\overline{p}$ resp. X_s at the end of STEP(s) we have $\overline{p}_{s+1} = \overline{a}^0 X_s$. Thus in STEP($S^*$) of the termination, we have $\overline{p} = \overline{p}_{S^*} = [0, 0, \ldots, \mathrm{GCD}] = \overline{a}^0 X = \sum_{k=1}^K a_k^0 X_{K,k}$.

Algorithm 2. Construction of GCD cofactors with low degree

Require: $K \in \{2, 3, \ldots\}$ for the number of polynomials for GCD calculation;
$\overline{a}^0 = [a_1^0(t), a_2(t), \ldots, a_K^0(t)]$, list of polynomials in the variable t

Ensure: $\mathrm{GCD}(\overline{a}_1^0, \ldots, a_K^0)$ and a list $\overline{q} = [q_1(t), \ldots, q_n(t)]$ of polynomials
such that $\max_{k=1}^K \deg(q_k) \leq \max_{k=1}^K \deg(a_k)$ and $\sum_{k=1}^K a_k q_k = \mathrm{GCD}(a_1, \ldots, a_K)$.

Calculation: With auxiliary stores $\overline{p}$ for K-vectors resp. X, N, R, D for $(K \times K)$-matrices.

STEP(0): $\overline{p} \leftarrow \overline{a}^0$, $X_0 \leftarrow \mathrm{Id}_K$;
STEP(s+1): $\overline{q} \leftarrow N_{\overline{p}} \overline{p}$, $X \leftarrow X N_{\overline{p}}$,
$\overline{p} \leftarrow R_{\overline{p}} \overline{p}$, $X \leftarrow X R_{\overline{p}}$,
$\overline{p} \leftarrow D_{\overline{p}} \overline{r}$, $X \leftarrow X D_{\overline{p}}$;
STOP if $p_1(t) = \cdots = p_{K-1}(t) \equiv 0$.
OUTPUT: $p_K(t)$ as the GCD of $a_1^0(t), \ldots, a_K^0(t)$,
$X_{K,1}(t), X_{K,2}(t), \ldots, X_{K,K}(t)$ as its cofactors wrt. $a_1^0(t), \ldots, a_K^0(t)$.

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