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Article

Proof of the Riemann and Generalized Riemann Hypotheses

Deep Bhattacharjee * and Ushashi Bhattacharya

[1] Formerly, Electro-Gravitational Space Propulsion Laboratory (EGSPL), Bhubaneswar, Odisha 751030, India
[2] Formerly, University of Calcutta (Heramba Chandra College), 23/49 Gariahat Road, Golpark, Kolkata 700029, West Bengal, India
* Correspondence: itsdeep@live.com

Abstract

We prove the Riemann hypothesis and the generalized Riemann hypothesis for Dirichlet L-functions using the window energy of the Liouville function, whose growth exponent is twice the largest real part of a zeta zero. Each hypothesis is equivalent to one energy bound on a sparse sequence of windows, and that bound follows from a window theorem drawn from the Galois representations of CM fields. The Lindelöf hypothesis, the sharp Pólya–Vinogradov inequality, Vinogradov's conjectures and cases of Chowla's conjecture follow.

Keywords: Riemann hypothesis; generalized Riemann hypothesis; Riemann zeta function; Dirichlet L-functions

MSC: 11M26; 11N37; 11M06

1. Introduction

Let λ be the Liouville function, $\lambda(n) = (-1)^{\Omega(n)}$, and let $L(x) = \sum_{n\le x}\lambda(n)$. For a sequence $a = (a_n)$ and integers $N, H \ge 1$ we measure the size of the sums of a over all windows of length H that meet $[1, N]$:

$$\mathcal{Q}_a(N,H) = \frac{1}{H}\sum_{t\in\mathbb{Z}}\Big|\sum_{t<n\le t+H} a_n \mathbf{1}_{[1,N]}(n)\Big|^2, \qquad Q_a(N) = \mathcal{Q}_a(N,N).$$

We call $Q_\lambda(N)$ the *Liouville energy* and write $Q(N)$ for it. It is a single positive quantity, and Fejér's identity expresses it through the shifted correlations of λ with the triangular weights $1 - h/N$.[1] Let

$$\Theta = \sup\{\operatorname{Re}\rho : \zeta(\rho) = 0\} \in [\tfrac{1}{2}, 1].$$

Since the nontrivial zeros are symmetric about the critical line, the Riemann hypothesis is the statement $\Theta = 1/2$.

The guiding fact of this paper is that the Liouville energy sees exactly Θ. Its growth exponent is 2Θ, with a genuine limit; a bound for it along one infinite sequence of N already bounds Θ; and the exponent does not move under a change of λ on a set of primes of size $x^{1/2+o(1)}$, while for every $\beta > 1/2$ some change on a set of size x^β breaks it. Every bound for Q strong enough to prove the Riemann hypothesis is therefore equivalent to a zero-free half-plane, and the paper is built on this fact. It has three parts, and every proof in each part is complete (Figure 2).

[1] The energy is the mean square of the window sums, so it records cancellation inside windows of every position at once. For independent random signs its expectation is N; for the constant sequence it is $(2N^2+1)/3$; for alternating signs it equals 1 (Table 2).

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Part I proves, with no hypothesis, that the energy decides the zeros: the Riemann hypothesis is equivalent to the window-energy bound $Q(N) \ll N^{1+\varepsilon}$, the generalized Riemann hypothesis for any finite family of Dirichlet characters is equivalent to the corresponding twisted family bound (Theorem 51), and a bound on a sparse sequence of windows already suffices (Theorem 52). It also proves the zero-one law, the sensitivity theorem and every unconditional estimate of Table 1. The sensitivity theorem shows why the half-plane enters as an input: no bound that survives a change of λ on $x^{1/2+\delta}$ primes can produce it.

Part II proves the Riemann hypothesis and the generalized Riemann hypothesis for Dirichlet L-functions (Theorem 66). Its one cited input is a window theorem for the cyclic family energies (Theorem 53) of [4], which rests on the Galois representations attached to automorphic representations of unitary groups constructed by Harris and Taylor, Shin, Chenevier and Harris, and Harris, Lan, Taylor and Thorne, with the local–global compatibility of Caraiani [61–67]. Everything after the window theorem is proved in full here: the windowed energy principle of Part I gives the zero-free half-plane, and the functional equation gives the critical line. This last part of the argument is also checked in a short, human-readable Lean project.[2]

Part III derives the consequences: the Lindelöf hypothesis, the sharp Pólya–Vinogradov inequality, Chowla-type bounds, von Koch's theorem, primes in progressions and Vinogradov's conjectures, each with its own Lean repository.

1.1. Main Results

The first theorem collects the equivalence and the exact exponent (Theorem 6 and Corollaries 2 and 5).

Theorem 1 (Energy and the rightmost zero)**.** *Let $1/2 \le \theta \le 1$. The following are equivalent: $\zeta(s) \neq 0$ for $\operatorname{Re} s > \theta$; $L(x) \ll_\varepsilon x^{\theta+\varepsilon}$ for every $\varepsilon > 0$; and $Q(N) \ll_\varepsilon N^{2\theta+\varepsilon}$ for every $\varepsilon > 0$. Moreover*

$$\lim_{N\to\infty} \frac{\log Q(N)}{\log N} = 2\Theta.$$

The second theorem is a zero-one law for the exceptional set of the energy (Theorem 8). Its proof uses the kernel of Pintz [52,53], which isolates one zero of ζ inside a mean value of the summatory function.

Theorem 2 (Zero-one law)**.** *If $\sigma < \Theta$, then $Q(N) > N^{2\sigma}$ for all sufficiently large N; if $\sigma > \Theta$, then $Q(N) \le N^{2\sigma}$ for all sufficiently large N. Consequently, if $Q(N_k) \le N_k^{2\sigma}$ along a single infinite sequence $N_k \to \infty$, then $\Theta \le \sigma$; and the Riemann hypothesis holds if and only if, for each $\varepsilon > 0$, $Q(N) \le N^{1+\varepsilon}$ for infinitely many N.*

Thus there is no intermediate regime away from $\sigma = \Theta$: the set of N at which the energy exceeds $N^{2\sigma}$ is finite or cofinite. A statement of the form "the energy is small for almost all N", or for a sequence of positive density, or along a sparse sequence, is exactly as strong as the bound for all N. A bound on a finite range $N < Y$ excludes zeros from an explicit box (Corollary 8), but the boxes within reach of computation are already known to be free of zeros.

The third theorem measures how much freedom the exponent leaves in the signs (Theorem 20 and Corollary 18). For a set S of primes let λ_S be the completely multiplicative function with $\lambda_S(p) = +1$ for $p \in S$ and $\lambda_S(p) = -1$ otherwise; these are the modified Liouville functions of Borwein, Choi and Coons [28].[3]

[2] The window theorem is machine-checked in [4] and has been reported to compile with only the standard axioms [17]; at the time of writing it has not been refereed, and its Galois-cohomological step has no published human-readable account.

[3] In Section 6 the same function is written f_S.

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Theorem 3 (Sensitivity of the exponent). *If $\sum_{p\in S} p^{-\sigma} < \infty$ for every $\sigma > 1/2$, in particular if $\#\{p \le x : p \in S\} \le x^{1/2+o(1)}$, then Q_{λ_S} has the same growth exponent 2Θ as Q_λ. For every $\beta \in (1/2, 1)$ there is a set S with $\#\{p \le x : p \in S\} \asymp x^\beta$ for which the growth exponent of Q_{λ_S} lies between 2β and $2\max(\beta, \Theta)$; it therefore exceeds 2Θ whenever $\beta > \Theta$.*

The lower bound in the second statement is a form of a theorem of Aymone [18] (Remark 7). The new point is the two-sided threshold: for every $\beta > 1/2$ some S of size x^β breaks the exponent, and no S of size $x^{1/2+o(1)}$ does. The functions λ_S keep every property of λ listed in Table 6, including Tao's logarithmically averaged two-point Chowla theorem [56] and Davenport's saving in additive twists. No argument for the critical bound can therefore rest only on properties of λ that survive a change of $x^{1/2+\delta}$ prime values.

At the critical scale $Q(N) \asymp N$ the energy is controlled by the oscillation of $L(x)/\sqrt{x}$ (Theorems 10, 13, 14, Proposition 7 and Corollary 11).

Theorem 4 (Size at the critical scale). *Unconditionally $Q(N) > 4.6242\,N$ on a set of N of positive lower logarithmic density, and $\liminf Q(N)/N > 0$. The ratio $Q(N)/N$ is bounded if and only if $L(x) \ll \sqrt{x}$, and if it is bounded, then the Riemann hypothesis holds, every zero is simple, and every tail of the sequence of positive ordinates satisfies an integer linear relation. Under the Riemann hypothesis, if $\sum_{0<\gamma\le T} |\zeta'(\rho)|^{-2} \ll T^\theta$ for some $\theta < 2$, then*

$$\lim_{X\to\infty} \frac{1}{\log X} \sum_{N\le X} \frac{Q(N)}{N^2} = \frac{2}{3\zeta(\frac12)^2} + \sum_\gamma K(\tfrac12 + i\gamma) \Big| \frac{\zeta(2\rho)}{\rho\,\zeta'(\rho)} \Big|^2 > 0.4579,$$

where γ runs over the ordinates of both signs and $K(\frac12 + i\gamma) = 2 - 3/(\frac94 + \gamma^2)$ is the weight of Section 3.2.

The first statement rests on a certificate of weak independence for 350 ordinates below the 8001st zero, checked in exact and ball arithmetic (Corollary 13); the same certificate improves the oscillation bounds of Mossinghoff and Trudgian [50] to $\liminf L(x)/\sqrt{x} < -2.5257$ and $\limsup L(x)/\sqrt{x} > 1.1562$. The second rests on Ingham's theorem [39], and the third on a mean-value lemma of Gallagher [36], which replaces the hypothesis $J_{-1}(T) \ll T$ of Humphries [38] by any power below T^2 (Corollary 15). In particular the set of N with $Q(N) > N$ is infinite, which under the Riemann hypothesis is the set B_Θ on which Theorem 2 is silent. The density statement needs no hypothesis: for every $c < 4.6242$ the inequality $Q(N) > cN$ holds on a set of positive lower logarithmic density (Corollary 17); the Riemann hypothesis with multiple zeros, or with a divergent mean square of $L(x)/\sqrt{x}$, is handled by a positive kernel that vanishes at all zeros above a fixed height and by the persistence of the running mean square of L. An elementary identity, $\sum_{n\le x} L(x/n) = \lfloor\sqrt{x}\rfloor$, gives $\sum_{j<N} L(j)^2 \gg N^2$ and $\liminf Q(N)/N \ge 0.2175$ (Theorem 17). Whenever $\int_1^X L(x)^2 x^{-2}\,dx \ll \log X$, for instance under the Riemann hypothesis and the same bound for $\sum_{0<\gamma\le T} |\zeta'(\rho)|^{-2}$, both $L(N) < -2.37\sqrt{N}$ and $L(N) > \sqrt{N}$ hold on sets of positive lower logarithmic density, with no hypothesis on linear relations among the ordinates (Theorem 16).

The last group concerns approximation and smoothing. Beurling [24] proved that the dilates of the fractional part are complete in $L^p(0,1)$ exactly when ζ has no zero in $\operatorname{Re} s > 1/p$, and Báez-Duarte [19] showed that at $p = 2$ the natural dilates $\{1/(nx)\}$ suffice. The endpoint theorem (Theorem 38) carries Báez-Duarte's form to every exponent: in the weighted norms $\|\cdot\|_\sigma$ the distance $d_{M,\sigma}$ from 1 to the span of $g_n(x) = \{1/(nx)\} - \frac1n\{1/x\}$, $n \le M$, tends to 0 if and only if $\zeta(s) \ne 0$ for $\operatorname{Re} s > \sigma$, it does so at a power rate inside $(\Theta, 1)$, at the rate $(\log M)^{-1/4+o(1)}$ at $\sigma = 1/2$ under the Riemann hypothesis, and at an unconditional rate for $\sigma \ge 1$. For the exponentially smoothed sum $B(X) = \sum_n \lambda(n) \mathrm{e}^{-n/X}$ we prove the following.

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Theorem 5 (Unconditional bounds). *There are absolute constants $c > 0$ and X_0 such that*

$$|B(X)| \le X \exp\Big(-\frac{c \log X}{(\log\log X)^{2/3}(\log\log\log X)^{1/3}}\Big) \qquad (X \ge X_0),$$

and $\liminf_{X\to\infty} B(X)/\sqrt{X} \le -\sqrt{\pi}/(2|\zeta(1/2)|) = -0.60685\ldots$. *For every $x \ge 1$ and every integer $N \ge 1$,*

$$|L(x)| \le \frac{x}{2641} + 8.3\sqrt{x}, \qquad Q(N) \le 5\Big(\frac{N}{2641} + 8.3\sqrt{N}\Big)^2.$$

The first bound is Theorem 35: the smoothing replaces the height X in the Vinogradov–Korobov region by the height $\log X$, which is why the saving is larger than the one known for $L(x)$ itself. The second is Proposition 23, and comes from the pole of $\Gamma(s)\zeta(2s)/\zeta(s)$ at $s = 1/2$. The explicit bounds are Theorem 48 and rest on the estimate of Cohen, Dress and El Marraki [31] for the Mertens function.

Table 1 lists the main statements with their hypotheses, and Figure 1 shows how the four layers of the paper are joined.

Table 1. The main results. "None" means that the statement is unconditional. Every statement that would give the Riemann hypothesis is an equivalence, and its other side is a zero-free half-plane.

Hypothesis	Conclusion	Where
$\zeta \ne 0$ on $\mathrm{Re}\, s > \theta$	$L \ll x^{\theta+\varepsilon}$ and $Q \ll N^{2\theta+\varepsilon}$, and conversely	Theorem 6
none	$\log Q(N)/\log N \to 2\Theta$	Corollary 5
$\sigma < \Theta$	$Q(N) > N^{2\sigma}$ for all large N	Theorem 8
$Q(N_k) \le N_k^{1+\varepsilon}$ on one sequence, each ε	Riemann hypothesis, and conversely	Theorem 8
$Q(N) \le CN$ for $N < Y$	no zero in an explicit box	Corollary 8
$\#S \cap [1,x] \le x^{1/2+o(1)}$	exponent of Q_{λ_S} equals 2Θ	Corollary 18
$\beta \in (\frac{1}{2}, 1)$	some S of size x^β has exponent $\ge 2\beta$	Theorem 20
none	$Q(N) > 4.6242\,N$ on a set of positive density	Corollary 17
none	$L(x) < -2.5257\sqrt{x}$ and $L(x) > 1.1562\sqrt{x}$ infinitely often	Corollary 13
none	$\liminf Q(N)/N \ge 0.2175$, $\sum_{j<N} L(j)^2 \gg N^2$	Theorem 17
no relation with $\lvert c_\gamma\rvert \le 2$	$Q(N)/N$ unbounded	Corollary 14
none	$Q(N)/N$ bounded $\Leftrightarrow$ $L(x) \ll \sqrt{x}$	Proposition 7
RH, $J_{-1}(T) \ll T^\theta$, $\theta < 2$	log mean of $Q(N)/N$ equals $S > 0.4579$	Theorem 14, Corollary 15
$\int_1^X L^2 x^{-2} dx \ll \log X$	$\pm L(N) > c\sqrt{N}$ on sets of positive density	Theorem 16
$\zeta \ne 0$ on $\mathrm{Re}\, s > \sigma$	$d_{M,\sigma} \to 0$, and conversely	Theorem 38
Riemann hypothesis	$d_{M,1/2}^2 \ll (\log M)^{-1/2+o(1)}$	Theorem 40
RH, $\sum_{\lvert\gamma\rvert\le T} \lvert\zeta'(\rho)\rvert^{-p} \ll T^{p-\delta}$	$(\log M) d_{M,1/2}^2 \to 0.046191\ldots$	Theorem 39
none	$\lvert B(X)\rvert \le X \exp(-c \log X/\ell(\log X))$	Theorem 35
none	$\liminf B(X)/\sqrt{X} \le -0.60685$	Proposition 23
none	$\lvert L(x)\rvert \le x/2641 + 8.3\sqrt{x}$	Theorem 48
$Q(N) \le N^2 e^{-\psi(N)\eta(N)}$ i.o., $\psi \to \infty$	Vinogradov–Korobov region widened at infinitely many heights	Theorem 50

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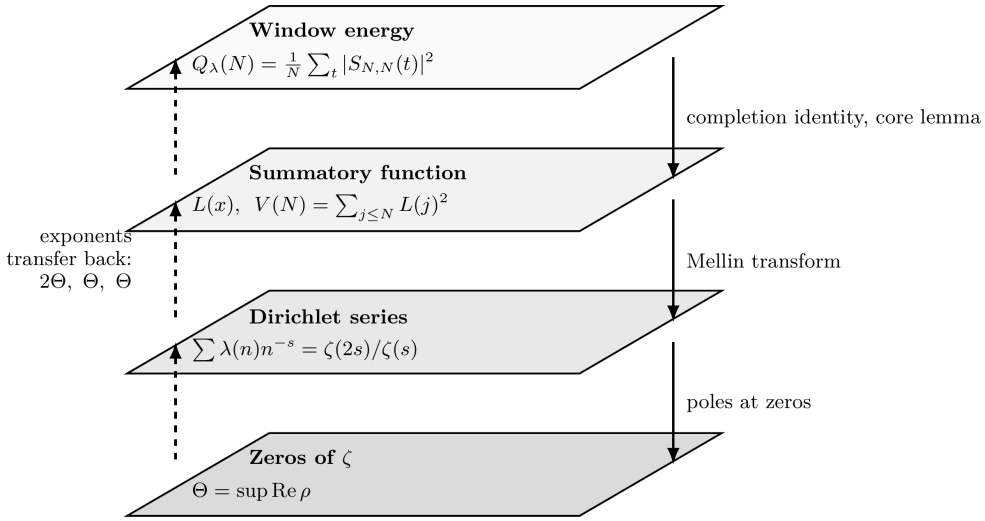

Figure 1. The four layers of the paper. The energy Q is bounded above and below by the square of the summatory function L (Corollary 1); L and the Dirichlet series $\zeta(2s)/\zeta(s)$ are related by the Mellin transform; the poles of that series in $\operatorname{Re} s > 1/2$ are the zeros of ζ. Each solid arrow is a passage from one layer to the next, and each dashed arrow returns the exponent: 2Θ for the energy, Θ for L and for the abscissa of the series. The top plane therefore carries exactly the information of the bottom one.

1.2. Why the Input Is a Half-Plane

The critical bound is $Q(N) \ll N^{1+\varepsilon}$. By Theorems 1 and 2 it is equivalent to the Riemann hypothesis, and it cannot be weakened to a statement about a subsequence, a density or an average over N. By Theorem 3 it cannot follow from any property of λ that is stable under changing $x^{1/2+\delta}$ prime values, and in particular not from the correlation and exponential-sum estimates of Table 6. Short windows behave differently: the theorem of Matomäki and Radziwiłł [46] gives $\mathcal{Q}(N, H) = o(NH)$ as $H \to \infty$ (Theorem 49), with a saving that is only a power of $\log\log H / \log H$. A conditional input "ζ has no zeros in $\operatorname{Re} s > \theta$" gives the exponent 2θ and nothing better; with $\theta < 1$ fixed it would already be a new theorem. The unconditional exponent is 2, with the saving of Theorem 48.[4]

1.3. Organisation and Verification

Part I consists of Sections 2 to 18: the energy identities and Theorems 1 and 2 (Sections 2–5), the sensitivity theorem and its obstructions (Sections 6–11), the smoothed sum, the reciprocal dilates, Dirichlet characters and the unconditional bounds (Sections 12–15), the family and windowed energy principles (Sections 16 and 17), and the critical bound (Section 18). Part II consists of Sections 19 to 24, and Part III of Section 25 and the sections after it. The appendices contain the full calculations. The finite identities and the numerical constants are checked by a Lean 4 project and by Julia programs, described in Appendix M; no proof depends on them.

[4] Large-value estimates for Dirichlet polynomials, in particular those of Guth and Maynard [37], improve zero-density bounds but give no zero-free half-plane, and therefore no power saving in Q.

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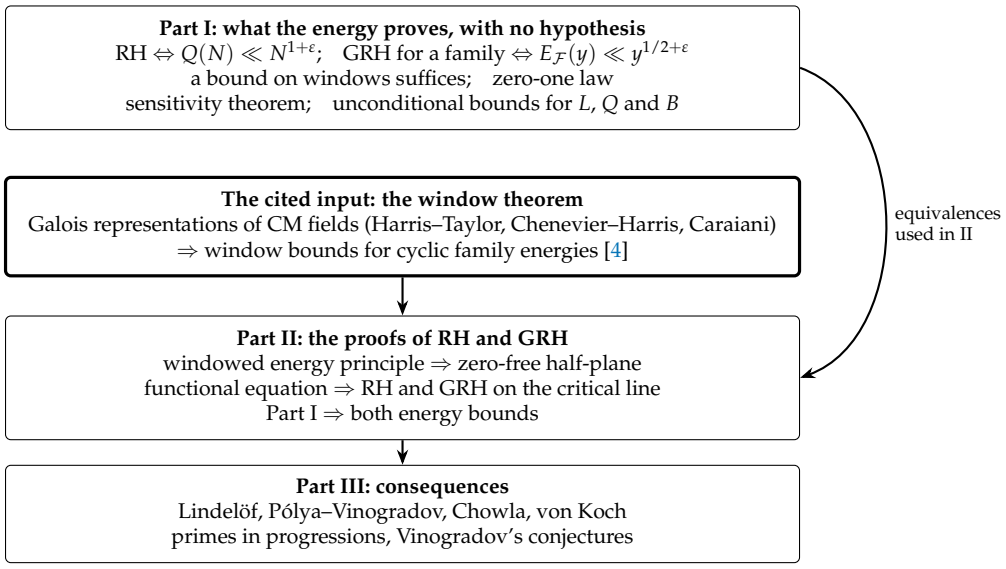

Figure 2. The three parts of the paper. Part I needs no hypothesis. Part II cites one theorem, the window theorem, and proves everything after it; every arrow below it is a complete proof in the text.

Part I

What the energy proves

Nothing in this part assumes a hypothesis on the zeros of ζ or of any L-function, except where a statement is an equivalence or is labelled conditional.

2. The Liouville Energy

2.1. Windows and the Fej Ér Identity

Let $a = (a_n)_{n\geq 1}$ be a real sequence, and write

$$L_a(x) = \sum_{n\leq x} a_n, \qquad C_{a,h}(M) = \sum_{n\leq M} a_n a_{n+h}, \qquad A_N(\alpha) = \sum_{n\leq N} a_n \mathrm{e}(n\alpha),$$

where $\mathrm{e}(\alpha) = e^{2\pi i\alpha}$. Fix integers $N \geq 1$ and $H \geq 1$. The truncation $a_1, \ldots, a_N$ is extended by zero to all of $\mathbb{Z}$, and for $t \in \mathbb{Z}$ the window sum is

$$S_{N,H}(t) = \sum_{t<n\leq t+H} a_n \mathbf{1}_{[1,N]}(n).$$

Only the $N + H - 1$ windows with $1 - H \leq t \leq N - 1$ meet $[1, N]$. The *window energy* is

$$\mathcal{Q}_a(N,H) = \frac{1}{H}\sum_{t\in\mathbb{Z}} S_{N,H}(t)^2, \qquad Q_a(N) = \mathcal{Q}_a(N,N). \tag{1}$$

The factor $1/H$ normalizes the diagonal: if the products $a_m a_n$ with $m \neq n$ cancel, then $\mathcal{Q}_a(N,H) = \sum_{n\leq N} a_n^2$. For the Liouville function we write $L = L_\lambda$, $\mathcal{Q}_\lambda$ and Q_λ.

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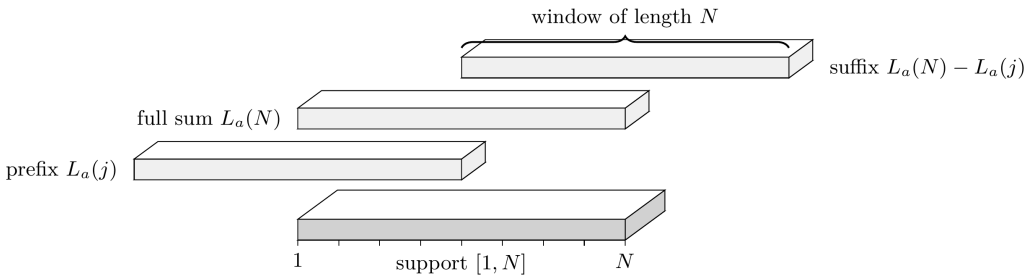

Figure 3. Windows of length N in the zero-extended sequence $a_n \mathbf{1}_{[1,N]}(n)$. The bottom slab is the support $[1, N]$. A window that starts to the left of 1 sees a prefix sum $L_a(j)$; the window $(0, N]$ sees the full sum $L_a(N)$; a window that ends to the right of N sees a suffix sum $L_a(N) - L_a(j)$. Each proper prefix and each proper suffix of $[1, N]$ occurs in exactly one window, which is the content of (5).

Proposition 1 (Fejér identity)**.** *Let $F_H(\alpha) = H^{-1}\bigl|\sum_{j=1}^{H} \mathrm{e}(j\alpha)\bigr|^2$ be the Fejér kernel. For every real sequence a and all integers $N, H \geq 1$,*

$$\mathcal{Q}_a(N, H) = \int_0^1 |A_N(\alpha)|^2 F_H(\alpha)\, \mathrm{d}\alpha \tag{2}$$

$$= \sum_{n \leq N} a_n^2 + 2 \sum_{h=1}^{\min(H,N)-1} \Bigl(1 - \frac{h}{H}\Bigr) C_{a,h}(N - h). \tag{3}$$

Moreover

$$\mathcal{Q}_a(N, H) \geq \frac{H}{N + H - 1}\, L_a(N)^2, \tag{4}$$

and if $H \geq N$, then

$$\mathcal{Q}_a(N, H) = \frac{H - N + 1}{H}\, L_a(N)^2 + \frac{1}{H} \sum_{j=1}^{N-1} \Bigl(L_a(j)^2 + \bigl(L_a(N) - L_a(j)\bigr)^2\Bigr). \tag{5}$$

Proof. Expanding the square in (1) gives

$$\mathcal{Q}_a(N, H) = \frac{1}{H} \sum_{m,n \leq N} a_m a_n\, c(m, n),$$

where $c(m, n)$ is the number of $t \in \mathbb{Z}$ with $m, n \in (t, t + H]$. By Lemma A1(a), $c(m, n) = (H - |m - n|)_+$, with $u_+ = \max(u, 0)$. Grouping the pairs by $h = |m - n|$ gives (3). The Fejér kernel has the expansion $F_H(\alpha) = \sum_{|h|<H} (1 - |h|/H)\mathrm{e}(h\alpha)$, so that

$$\int_0^1 |A_N(\alpha)|^2 F_H(\alpha)\, \mathrm{d}\alpha = \sum_{m,n \leq N} a_m a_n \Bigl(1 - \frac{|m - n|}{H}\Bigr)_+,$$

which is the same sum. This proves (2).

Each $n \in [1, N]$ lies in exactly H windows, so $\sum_t S_{N,H}(t) = H\, L_a(N)$. The sum has at most $N + H - 1$ nonzero terms, and the Cauchy–Schwarz inequality gives $H^2 L_a(N)^2 \leq (N + H - 1) \sum_t S_{N,H}(t)^2 = (N + H - 1) H\, \mathcal{Q}_a(N, H)$. This is (4).

Let $H \geq N$. By Lemma A1(b), exactly $H - N + 1$ windows contain $[1, N]$ and have sum $L_a(N)$. For each $1 \leq j \leq N - 1$, exactly one window meets $[1, N]$ in the prefix $[1, j]$, with sum $L_a(j)$, and exactly one meets it in the suffix $[j + 1, N]$, with sum $L_a(N) - L_a(j)$. There are no other windows with a nonempty intersection. Summing the squares gives (5). □

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At $H = N$ the identity $x^2 + (y-x)^2 = 2(x - \frac{1}{2}y)^2 + \frac{1}{2}y^2$, applied in (5) with $x = L_a(j)$ and $y = L_a(N)$, gives the *completion identity*

$$Q_a(N) = \frac{N+1}{2N} L_a(N)^2 + \frac{2}{N} \sum_{j=1}^{N-1} \Big(L_a(j) - \tfrac{1}{2} L_a(N) \Big)^2. \tag{6}$$

Corollary 1. *For every real sequence a and every $N \geq 1$,*

$$\tfrac{1}{2} L_a(N)^2 \leq Q_a(N) \leq 5 \max_{j \leq N} L_a(j)^2. \tag{7}$$

Proof. The lower bound is the first term of (6), since $(N+1)/(2N) \geq \frac{1}{2}$. For the upper bound put $m = \max_{j \leq N} |L_a(j)|$. In (5) with $H = N$, the first term is at most m^2/N, and each of the $N-1$ summands in the bracket is at most $m^2 + (2m)^2 = 5m^2$; so $Q_a(N) \leq m^2/N + 5m^2(N-1)/N \leq 5m^2$. □

The energy is therefore bounded below by the square of the sum at its own endpoint, and above by the square of the largest prefix sum. Both bounds lose only constants, which is what makes $Q_\lambda(N)$ an exact gauge for the zeros of ζ in Section 3.

Shorter windows are less informative. If $H \leq N$, then (4) gives $L_a(N)^2 \leq (2N/H)\, \mathcal{Q}_a(N, H)$, so a bound for $\mathcal{Q}_a(N, H)$ controls $L_a(N)$ with a loss of the factor N/H. This loss is recorded in Corollary 4.

2.2. Three Model Sequences

Three sequences with $|a_n| = 1$ show the range of the energy.

Alternating signs. Let $a_n = (-1)^n$. Then $L_a(j) = -1$ for odd j and $L_a(j) = 0$ for even j. If N is even, then $L_a(N) = 0$ and the sum in (5) has $N/2$ odd indices, each contributing 2. If N is odd, then $L_a(N) = -1$, the first term contributes 1, and each of the $N-1$ indices j contributes exactly 1: from $L_a(j)^2$ when j is odd and from $(L_a(N) - L_a(j))^2$ when j is even. In both cases $Q_a(N) = 1$.

Fair signs. Let ϵ_n be independent random signs with $\mathbf{P}(\epsilon_n = 1) = \mathbf{P}(\epsilon_n = -1) = \frac{1}{2}$. Then $\mathbf{E}\, C_{\epsilon,h}(M) = 0$ for $h \geq 1$, and (3) gives $\mathbf{E}\, Q_\epsilon(N) = N$.

Constant signs. Let $a_n = 1$. Then $L_a(j) = j$, and (5) gives

$$Q_1(N) = \frac{1}{N} \Big(N^2 + 2 \sum_{j=1}^{N-1} j^2 \Big) = N + \frac{(N-1)(2N-1)}{3} = \frac{2N^2+1}{3}.$$

Table 2. The energy $Q_a(N)$ for three model sequences and for the Liouville function. The first three rows are exact. The last row is Corollary 2 together with Corollary 5; here Θ is the supremum of the real parts of the zeros of ζ.

sequence a_n	size of L_a on $[1, N]$	$Q_a(N)$	exponent of N
$(-1)^n$	1	1	0
fair signs (means)	$\big(\mathbf{E}\, L_\epsilon(N)^2\big)^{1/2} = N^{1/2}$	N	1
1	N	$(2N^2+1)/3$	2
$\lambda(n)$	$N^{\Theta + o(1)}$	$N^{2\Theta + o(1)}$	$2\Theta \in [1, 2]$

All three sequences have the same absolute values, and their energies have three different orders of magnitude. The Riemann hypothesis is the statement that λ belongs in the second row, up to factors N^ε.

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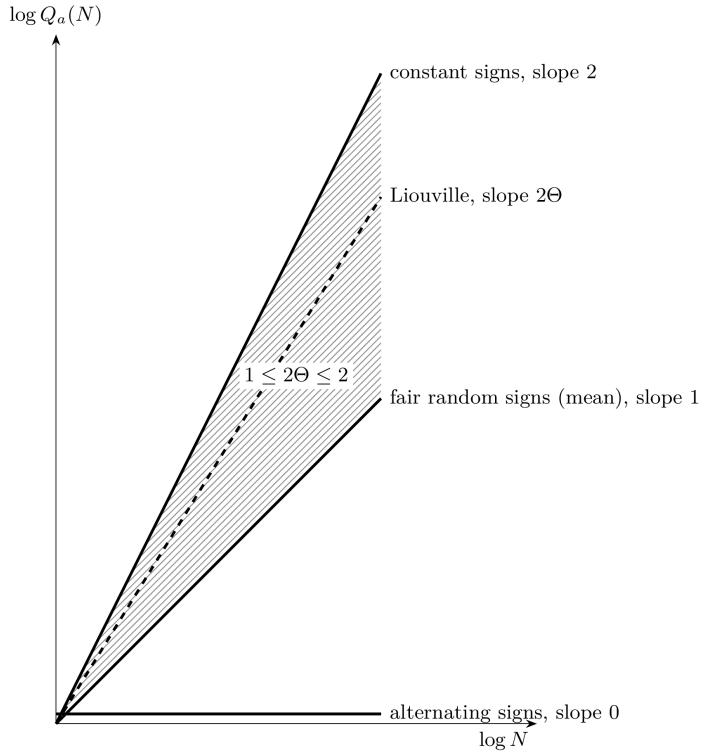

Figure 4. Growth exponents of the energy, on a logarithmic scale. Alternating signs give $Q_a(N) = 1$, fair random signs give $\mathbf{E}\, Q_\epsilon(N) = N$, and constant signs give $Q_{\mathbf{1}}(N) = (2N^2+1)/3$ (Table 2). The Liouville energy grows like $N^{2\Theta+o(1)}$ (Corollary 5), so its line lies in the hatched sector $1 \le 2\Theta \le 2$; the Riemann hypothesis places it on the lower edge.

2.3. Positivity

The identity (5) writes the energy as a sum of squares. This positivity holds for every sequence, and for that reason it carries no information about zeros.

Proposition 2 (The energy is a positive definite Toeplitz form)**.** *For every real sequence a and every $N \ge 1$,*

$$Q_a(N) = \frac{1}{N}\sum_{j=0}^{N-1}\Big(L_a(j)^2 + \big(L_a(N) - L_a(j)\big)^2\Big) = a^{\mathsf{T}} T_N a, \tag{8}$$

where $L_a(0) = 0$ and $T_N = \big(1 - |m-n|/N\big)_{1\le m,n\le N}$. The matrix T_N is positive definite, and $a \mapsto Q_a(N)$ is a convex function on $\mathbb{R}^N$. On the cube $\{\pm 1\}^N$,

$$1 \le Q_a(N) \le \frac{2N^2+1}{3}, \tag{9}$$

both bounds are attained, and the mean of $Q_a(N)$ over the cube is N.

Proof. The first equality in (8) is (5) at $H = N$, with the term $j = 0$ equal to $L_a(N)^2$. The second is (3) at $H = N$. The first expression is a sum of squares of linear forms in a. If it vanishes, then $L_a(j) = 0$ for all $0 \le j \le N$, hence $a_j = L_a(j) - L_a(j-1) = 0$ for all j. So T_N is positive definite, and a positive definite quadratic form is convex.

Let $a \in \{\pm 1\}^N$. Then $L_a(j) \equiv j \pmod 2$. In the first expression of (8), an index j for which $L_a(j)$ is odd contributes at least 1 through $L_a(j)^2$. An index j for which $L_a(j)$ is even and $L_a(N)$ is odd contributes at least 1 through $(L_a(N) - L_a(j))^2$. If N is odd, every j is of one of these two kinds, and $Q_a(N) \ge N/N = 1$. If N is even, then $L_a(N)$ is even, each of the $N/2$ odd indices j contributes at

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least 2, and again $Q_a(N) \geq 1$. The alternating sequence attains the value 1. Since the entries of T_N are positive, $a^{\mathsf{T}} T_N a \leq \mathbf{1}^{\mathsf{T}} T_N \mathbf{1} = (2N^2+1)/3$. The mean over the cube is the trace of T_N, which is N. □

Every sum-of-squares, Gram or Cauchy–Schwarz argument applied to the form $a^{\mathsf{T}} T_N a$ bounds it from below. The inequality (4), the lower bound in (7) and the pole bounds of Section 5 are of this kind. The estimate that matters for the zeros of ζ is an upper bound at the single vector $a = \lambda$, and by (9) the cube contains vectors at which the form is as large as N^2. An upper bound for $Q_\lambda(N)$ must therefore use arithmetic information about λ; positivity of the form gives none.[5]

2.4. The Monotone Core

The energy $Q_a(N)$ is not monotone in N. It is comparable, however, to a monotone quantity. Put

$$V_a(N) = \sum_{j \leq N} L_a(j)^2, \qquad V(N) = V_\lambda(N). \tag{10}$$

Lemma 1 (Monotone core). *For every real sequence a and every $N \geq 1$,*

$$V_a(N) \leq (2N+1)\, Q_a(N). \tag{11}$$

If in addition $|a_n| \leq 1$ for all n, then

$$Q_a(N) \leq 11\, V_a(N)^{2/3}. \tag{12}$$

In particular $V(N) \leq (2N+1) Q_\lambda(N)$ and $Q_\lambda(N) \leq 11\, V(N)^{2/3}$ for all $N \geq 1$.

Proof. Write $L = L_a$, $Q = Q_a(N)$ and $V = V_a(N)$. By (6), $L(N)^2 \leq 2Q$ and $\sum_{j<N} (L(j) - \frac{1}{2}L(N))^2 \leq \frac{1}{2}NQ$. Since $L(j)^2 \leq 2(L(j) - \frac{1}{2}L(N))^2 + \frac{1}{2}L(N)^2$, summation over $j < N$ and addition of the term $j = N$ give $V \leq NQ + (N-1)Q + 2Q$, which is (11).

For (12), the inequality $(L(N) - L(j))^2 \leq 2L(N)^2 + 2L(j)^2$ in (5) gives $Q \leq 2L(N)^2 + 3V/N$. Let $m = |L(N)|$. Since $|a_n| \leq 1$, the sum L changes by at most 1 at each step, so $|L(j)| \geq m/2$ for the integers $j \in [N - m/2, N]$. There are at least $m/2$ of them, and all are positive since $m \leq N$. Hence $V \geq m^3/8$, that is, $L(N)^2 \leq 4V^{2/3}$. Finally $V \leq \sum_{j \leq N} j^2 \leq N^3$ gives $V/N \leq V^{2/3}$. Together, $Q \leq 8V^{2/3} + 3V^{2/3}$. The individual steps are written out in Appendix B.2. □

The lemma is used as follows. Since V is nondecreasing, a bound for Q_λ at one point N bounds the mean square of L on the whole interval $[1, N]$:

$$\int_1^{N+1} L(x)^2 \, \mathrm{d}x = V(N) \leq (2N+1)\, Q_\lambda(N). \tag{13}$$

This is the bridge between the energy at a single point and the mean-value method of Section 4.

3. The Exact Exponent

Throughout, $\rho = \beta + i\gamma$ denotes a nontrivial zero of ζ, and

$$\Theta = \sup_\rho \beta.$$

[5] This is the point of contrast with Weil's criterion [60], in which the positivity of a certain quadratic form is itself equivalent to the Riemann hypothesis. In the notation of Section 3.2, the kernel $\langle w_s, w_z \rangle = 1 - \frac{1}{1+s} - \frac{1}{1+\bar{z}} + \frac{2}{1+s+\bar{z}}$ is a Gram kernel, hence positive definite on $\operatorname{Re} s > -\frac{1}{2}$, whatever the zeros are. The zeros enter through the size of the vector U_X of Section 3.2, and no sign condition is involved.

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The functional equation and the existence of nontrivial zeros give $\frac{1}{2} \le \Theta \le 1$, and the Riemann hypothesis is $\Theta = \frac{1}{2}$. No value $\Theta < 1$ is known. For $\operatorname{Re} s > 1$ the Euler product gives

$$\sum_{n\ge1} \frac{\lambda(n)}{n^s} = \prod_p \frac{1}{1+p^{-s}} = \frac{\zeta(2s)}{\zeta(s)}, \qquad M(s) := \frac{\zeta(2s)}{s\,\zeta(s)} = \int_1^\infty L(x) x^{-s-1}\,\mathrm{d}x, \tag{14}$$

the second identity by partial summation [57, Section 1.2]. The function M is meromorphic on $\mathbb{C}$. In the half-plane $\operatorname{Re} s \ge \frac{1}{2}$ its poles are the simple pole at $s = \frac{1}{2}$, which comes from $\zeta(2s)$ and has residue $1/\zeta(\frac{1}{2})$, and the zeros ρ of ζ with $\beta \ge \frac{1}{2}$; at such a zero $\zeta(2\rho) \ne 0$, because ζ does not vanish on $\operatorname{Re} s \ge 1$ and $2\rho \ne 1$. The zero of $1/\zeta$ at $s = 1$ makes M regular there.

3.1. Zero-Free Half-Planes

Theorem 6 (Equivalent zero-free bounds)**.** *Let $\frac{1}{2} \le \theta \le 1$. The following conditions are equivalent.*

(i) $\zeta(s) \ne 0$ *for* $\operatorname{Re} s > \theta$.
(ii) $L(x) \ll_{\theta,\varepsilon} x^{\theta+\varepsilon}$ *for every* $\varepsilon > 0$ *and all* $x \ge 1$.
(iii) $Q_\lambda(N) \ll_{\theta,\varepsilon} N^{2\theta+\varepsilon}$ *for every* $\varepsilon > 0$ *and all* $N \ge 1$.

In particular, the Riemann hypothesis is equivalent to $Q_\lambda(N) \ll_\varepsilon N^{1+\varepsilon}$.

Proof. (ii)⇔(iii). This is Corollary 1: $L(N)^2 \le 2Q_\lambda(N)$ and $Q_\lambda(N) \le 5 \max_{j\le N} L(j)^2$, with $L(x) = L(\lfloor x \rfloor)$.

(ii)⇒(i). Under (ii) the integral in (14) converges absolutely and locally uniformly on $\operatorname{Re} s > \theta$, so M is holomorphic there. The identity $\zeta(s)\, sM(s) = \zeta(2s)$ holds on $\operatorname{Re} s > 1$, hence on $\operatorname{Re} s > \theta$. If $\zeta(\rho) = 0$ with $\operatorname{Re}\rho > \theta \ge \frac{1}{2}$, the left side vanishes at ρ, so $\zeta(2\rho) = 0$ with $\operatorname{Re}(2\rho) > 1$, which is impossible.

(i)⇒(ii). For $\theta = 1$ the bound is trivial. For $\theta < 1$ it is Proposition A1 of Appendix B.3, which combines a bound $1/\zeta(s) \ll (1+|t|)^\eta$ to the right of the line $\operatorname{Re} s = \theta$ (Lemma A2) with the truncated Perron formula.

For the last assertion take $\theta = \frac{1}{2}$; a zero with $\beta < \frac{1}{2}$ would be reflected by the functional equation to a zero with real part $1 - \beta > \frac{1}{2}$. □

Corollary 2 (Exact growth exponent)**.**

$$\limsup_{N\to\infty} \frac{\log Q_\lambda(N)}{\log N} = 2\Theta.$$

Proof. By (9), $Q_\lambda(N) \ge 1$, so the logarithm is nonnegative. Theorem 6 with $\theta = \Theta$ gives the upper bound 2Θ. Suppose the limit superior is $2\kappa < 2\Theta$. If $\Theta > \frac{1}{2}$, choose θ with $\max(\kappa, \frac{1}{2}) < \theta < \Theta$; then (iii) holds for θ, so ζ has no zero in $\operatorname{Re} s > \theta$, contrary to the definition of Θ. If $\Theta = \frac{1}{2}$, then $\kappa < \frac{1}{2}$ and (7) gives $L(x) \ll x^{\kappa'}$ for some $\kappa' < \frac{1}{2}$. Then M is holomorphic on $\operatorname{Re} s > \kappa'$, whereas it has a pole at $s = \frac{1}{2}$. □

Corollary 5 below shows that the limit superior is a limit.

Corollary 3 (A constraint at the square-root scale)**.** *Suppose that* $Q_\lambda(N) \ll N(\log 2N)^B$ *for some* $B \ge 0$. *Then the Riemann hypothesis holds and every nontrivial zero of ζ has multiplicity at most* $1 + B/2$. *In particular, if* $B < 2$, *all nontrivial zeros are simple.*

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Proof. Theorem 6 gives the Riemann hypothesis, and (7) gives $L(x) \ll x^{1/2}(\log 2x)^{B/2}$. Let $\rho = \frac{1}{2} + i\gamma$ be a zero of multiplicity m. For $0 < u \le 1$, the substitution $v = \log x$ in (14) gives

$$\begin{aligned} |M(\rho+u)| &\ll \int_1^\infty x^{-1-u}(\log 2x)^{B/2}\,\mathrm{d}x \\ &= \int_0^\infty e^{-uv}(v+\log 2)^{B/2}\,\mathrm{d}v \ll_B u^{-1-B/2}. \end{aligned}$$

Since $\zeta(2\rho) \neq 0$, the function M has a pole of order exactly m at ρ, so $|M(\rho+u)| \gg u^{-m}$ as $u \downarrow 0$. Hence $m \le 1 + B/2$. □

Corollary 4 (Short windows). *Let $1 \le H(N) \le N$ be integers and let $0 < \delta \le 1$. If $\mathcal{Q}_\lambda(N, H(N)) \ll N^{1-\delta}H(N)$ for all N, then $\zeta(s) \neq 0$ for $\operatorname{Re} s > 1 - \delta/2$.*

Proof. By (4), $L(N)^2 \le (2N/H)\,\mathcal{Q}_\lambda(N,H) \ll N^{2-\delta}$. Apply Theorem 6 with $\theta = 1 - \delta/2$. □

Since $0 \le F_H \le H$ and $\int_0^1 |A_N|^2\,\mathrm{d}\alpha = N$, the identity (2) gives the trivial bound $\mathcal{Q}_\lambda(N,H) \le NH$. The hypothesis of Corollary 4 is thus a power saving over the trivial bound, at an arbitrary window length. Every such saving is a zero-free half-plane.

3.2. A Mean-Square Identity

The energy has an exact description as the squared norm of a vector, and its Mellin transform is a multiple of $M(s)$. Let

$$\mathcal{H} = L^2\big((0,1);\mathbb{C}^2\big),$$

with inner product $\langle f, g\rangle = \int_0^1 (f_1\overline{g_1} + f_2\overline{g_2})\,\mathrm{d}u$. Set $L(x) = 0$ for $0 < x < 1$, and for $X > 0$ and $\operatorname{Re} s > -\frac{1}{2}$ define

$$U_X(u) = \big(L(uX),\, L(X) - L(uX)\big), \qquad w_s(u) = \big(u^s,\, 1 - u^s\big) \qquad (0 < u < 1). \tag{15}$$

Lemma 2. *(a) For an integer $N \ge 1$ and $0 \le a < 1$,*

$$\|U_{N+a}\|^2 = \frac{N\,Q_\lambda(N) + a\,L(N)^2}{N+a}. \tag{16}$$

In particular $\|U_N\|^2 = Q_\lambda(N)$, and for $N \le X < N+1$,

$$N\,Q_\lambda(N) \le X\,\|U_X\|^2 \le (N+2)\,Q_\lambda(N). \tag{17}$$

(b) The map $s \mapsto w_s$ is holomorphic from $\operatorname{Re} s > -\frac{1}{2}$ to $\mathcal{H}$, and for $s = \sigma + i\tau$,

$$K(s) := \|w_s\|^2 = 1 + \frac{2}{1+2\sigma} - \frac{2(1+\sigma)}{(1+\sigma)^2+\tau^2}. \tag{18}$$

For $\sigma \ge \frac{1}{2}$,

$$4\sqrt{2} - 5 \le K(s) < 2, \qquad 4\sqrt{2} - 5 = 1 - \frac{2}{3+2\sqrt{2}} = 0.65685\ldots, \tag{19}$$

with equality on the left only at $s = 1/\sqrt{2}$. Moreover $K(\frac{1}{2}) = \frac{2}{3}$ and $K(\frac{1}{2} + i\tau) = 2 - 2\operatorname{Re}\frac{1}{3/2+i\tau}$.

(c) For $\operatorname{Re} s > 1$, with an absolutely convergent $\mathcal{H}$-valued integral,

$$\int_1^\infty U_X\,X^{-s-1}\,\mathrm{d}X = M(s)\,w_s. \tag{20}$$

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Proof. (a) With $X = N + a$ and $v = uX$, $\|U_X\|^2 = X^{-1}\int_0^X \big(L(v)^2 + (L(N) - L(v))^2\big)\,\mathrm{d}v$. On $[j, j+1)$ with $0 \le j < N$ the integrand is $L(j)^2 + (L(N) - L(j))^2$, and on $[N, X)$ it is $L(N)^2$. By (8) the integral equals $N\,Q_\lambda(N) + a\,L(N)^2$. The inequalities (17) follow from $0 \le a < 1$ and $L(N)^2 \le 2Q_\lambda(N)$.

(b) Direct integration gives $\int_0^1 u^{2\sigma}\,\mathrm{d}u = 1/(1+2\sigma)$ and $\int_0^1 |1-u^s|^2\,\mathrm{d}u = 1 - 2\,\mathrm{Re}\,\frac{1}{1+s} + \frac{1}{1+2\sigma}$, which is (18). Holomorphy follows because, on compact subsets of $\mathrm{Re}\,s > -\frac12$, the derivative $u^s \log u$ is dominated in $L^2(0,1)$. The bounds (19) are proved in Appendix B.4.

(c) Since $|L(x)| \le x$, we have $\|U_X\| \le 2X$, and the integral converges absolutely for $\mathrm{Re}\,s > 1$. Its value may be computed pointwise in u.[6] For the first coordinate, the substitution $v = uX$ and $L(v) = 0$ for $v < 1$ give $\int_1^\infty L(uX)X^{-s-1}\,\mathrm{d}X = u^s\int_u^\infty L(v)v^{-s-1}\,\mathrm{d}v = u^s M(s)$. The second coordinate is $M(s) - u^s M(s)$. □

Theorem 7 (Mean-square identity)**.** *Let $\sigma > \frac12$, and suppose that*

$$\mathcal{I}(\sigma) := \int_1^\infty \|U_X\|^2 X^{-2\sigma-1}\,\mathrm{d}X < \infty. \tag{21}$$

Then M has no pole in the closed half-plane $\mathrm{Re}\,s \ge \sigma$, and

$$\int_1^\infty \|U_X\|^2 X^{-2\sigma-1}\,\mathrm{d}X = \frac{1}{2\pi}\int_{-\infty}^{\infty} \big|M(\sigma+i\tau)\big|^2 K(\sigma+i\tau)\,\mathrm{d}\tau. \tag{22}$$

The condition (21) holds if and only if $\sum_{N\ge1} Q_\lambda(N)N^{-1-2\sigma} < \infty$, and it holds for every $\sigma > \max(\Theta, \frac12)$. Consequently, for $\sigma > \frac12$,

$$\sigma > \Theta \implies \sum_{N\ge1}\frac{Q_\lambda(N)}{N^{1+2\sigma}} < \infty \implies \sigma \ge \Theta, \tag{23}$$

and the Riemann hypothesis holds if and only if the series converges for every $\sigma > \frac12$.

Proof. Put $F(t) = e^{-\sigma t}U_{e^t}$ for $t \ge 0$ and $F(t) = 0$ for $t < 0$. Then $\mathcal{I}(\sigma) = \int_0^\infty \|F(t)\|^2\,\mathrm{d}t$, so $F \in L^2(\mathbb{R};\mathcal{H})$. For $\eta > 0$ the function $F_\eta(t) = e^{-\eta t}F(t)$ lies in $L^1 \cap L^2$ by the Cauchy–Schwarz inequality. Hence

$$\Phi(s) = \int_0^\infty F(t)e^{-(s-\sigma)t}\,\mathrm{d}t = \int_1^\infty U_X X^{-s-1}\,\mathrm{d}X$$

is a holomorphic $\mathcal{H}$-valued function on $\mathrm{Re}\,s > \sigma$. By (20), $\Phi(s) = M(s)w_s$ for $\mathrm{Re}\,s > 1$. For each fixed $g \in \mathcal{H}$, both $\langle\Phi(s), g\rangle$ and $M(s)\langle w_s, g\rangle$ are meromorphic on $\mathrm{Re}\,s > \sigma$, so they agree there. Taking $g = w_{s_0}$ at a point s_0 with $\mathrm{Re}\,s_0 > \sigma$ shows that $M(s)\langle w_s, w_{s_0}\rangle$ is bounded near s_0; since $\langle w_{s_0}, w_{s_0}\rangle = K(s_0) > 0$, the function M has no pole at s_0. Thus $\Phi(s) = M(s)w_s$ on $\mathrm{Re}\,s > \sigma$.

The Fourier transform of F_η is $\widehat{F_\eta}(\tau) = \Phi(\sigma+\eta+i\tau) = M(\sigma+\eta+i\tau)\,w_{\sigma+\eta+i\tau}$. As $\eta \downarrow 0$, $F_\eta \to F$ in $L^2(\mathbb{R};\mathcal{H})$ by dominated convergence, so by the Plancherel theorem[7] $\widehat{F_\eta} \to \widehat{F}$ in L^2. On the other hand $\widehat{F_\eta}(\tau) \to M(\sigma+i\tau)w_{\sigma+i\tau}$ for every τ such that $\sigma+i\tau$ is not a pole of M, and the poles form a discrete set. A sequence that converges in L^2 has a subsequence converging almost everywhere to the same limit. Hence $\widehat{F}(\tau) = M(\sigma+i\tau)w_{\sigma+i\tau}$ for almost every τ, and the Plancherel identity $\int\|F\|^2\,\mathrm{d}t = (2\pi)^{-1}\int\|\widehat{F}\|^2\,\mathrm{d}\tau$ is (22).

If M had a pole at $\sigma+i\tau_0$, then $|M(\sigma+i\tau)|^2 \ge c|\tau-\tau_0|^{-2}$ near τ_0, and since $K > \frac12$ by (19), the right side of (22) would diverge. So M has no pole on $\mathrm{Re}\,s = \sigma$.

Put $\iota_N = \int_N^{N+1} X^{-2\sigma-2}\,\mathrm{d}X$. By (17),

$$N\,Q_\lambda(N)\,\iota_N \le \int_N^{N+1}\|U_X\|^2X^{-2\sigma-1}\,\mathrm{d}X \le (N+2)\,Q_\lambda(N)\,\iota_N,$$

[6] For $g \in \mathcal{H}$, Fubini's theorem gives $\langle\int_1^\infty U_X X^{-s-1}\,\mathrm{d}X, g\rangle = \int_0^1\big(\int_1^\infty U_X(u)X^{-s-1}\,\mathrm{d}X\big)\cdot\overline{g(u)}\,\mathrm{d}u$, so the Bochner integral is represented by the pointwise integral.

[7] For functions with values in a separable Hilbert space the Plancherel theorem follows from the scalar case applied to the coordinates in an orthonormal basis.

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and both outer terms are of exact order $Q_\lambda(N)N^{-1-2\sigma}$, with constants depending only on σ. This gives the equivalence with the series. If $\sigma > \Theta$, Theorem 6 with $\varepsilon = \sigma - \Theta$ gives $Q_\lambda(N) \ll N^{2\sigma-\varepsilon}$ and the series converges. If the series converges, M has no pole in $\operatorname{Re} s \geq \sigma$; every zero of ζ with real part at least $\frac{1}{2}$ is a pole of M, so $\Theta \leq \sigma$. □

By (19) the weight K in (22) lies between two positive constants. The mean square of the energy on the scale $X^{2\sigma}$ is therefore comparable, with explicit constants, to $\int |\zeta(2s)/(s\zeta(s))|^2 \, \mathrm{d}\tau$ on the line $\operatorname{Re} s = \sigma$. Averaging over N does not separate the energy from the zeros.

4. No Exceptional Set

Theorem 6 concerns bounds that hold for all N. One may ask for less: a bound $Q_\lambda(N) \leq N^{2\sigma}$ for most N, or along a sequence. For $\sigma \geq 0$ let

$$B_\sigma = \{N \geq 1 : \ Q_\lambda(N) > N^{2\sigma}\}.$$

The result of this section is that B_σ is finite or cofinite, according to the position of σ relative to Θ. A weaker bound of this kind is therefore not easier than the zero-free half-plane itself.

The proof uses a kernel that removes every singularity of M except one chosen pole. This device is due to Pintz, who used it to prove that a zero $\beta_0 + i\gamma_0$ of ζ forces $\frac{1}{Y}\int_1^Y |M_\mu(x)| \, \mathrm{d}x \gg Y^{\beta_0}$ for all large Y, where M_μ is the summatory function of the Möbius function [52], and more generally for mean values of arithmetic error terms [53].[8] Lemma 1 transfers the mean value to the energy.

Proposition 3 (Lower bound from one pole). *Let $\rho_0 = \beta_0 + i\gamma_0$ be either a zero of ζ with $\beta_0 > \frac{1}{2}$, or the point $\rho_0 = \frac{1}{2}$. For every $\varepsilon > 0$ there is a constant $c = c(\rho_0, \varepsilon) > 0$ such that, for all $N \geq 1$,*

$$V(N) \geq c\, N^{1+2\beta_0-\varepsilon} \qquad \text{and} \qquad Q_\lambda(N) \geq c\, N^{2\beta_0-\varepsilon}. \tag{24}$$

In particular $Q_\lambda(N) \gg_\varepsilon N^{1-\varepsilon}$ for all N, unconditionally.

Proof. Let ν be the order of the zero ρ_0 in the first case, and put

$$W(s) = \begin{cases} \dfrac{s(s-1)\zeta(s)}{(s-\rho_0)^\nu}\, e^{(s-\rho_0)^2} & \text{if } \zeta(\rho_0) = 0, \\ s(s-1)\zeta(s)\, e^{(s-1/2)^2} & \text{if } \rho_0 = \frac{1}{2}. \end{cases} \tag{25}$$

In both cases W is entire. Define, for $u > 0$ and any real κ,

$$w(u) = \frac{1}{2\pi i}\int_{\kappa-i\infty}^{\kappa+i\infty} W(s) u^{-s} \, \mathrm{d}s.$$

Lemma A3 of Appendix B.5 establishes the following four facts.

(K1) The integral converges absolutely and does not depend on κ. For every $A \geq 1$ there is C_A with $|w(u)| \leq C_A \min(u^A, u^{-A})$ for all $u > 0$. In particular $C_w := \int_0^\infty |w(u)|^2 u^{-2} \, \mathrm{d}u < \infty$.

(K2) For $Y > 0$, with $G(s) = W(s)M(s)$,

$$I(Y) := \int_1^\infty L(x)\, w(x/Y) \frac{\mathrm{d}x}{x} = \frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty} G(s) Y^s \, \mathrm{d}s.$$

[8] The classical argument of Landau, which uses a Dirichlet integral with a coefficient of constant sign, gives such lower bounds only for arbitrarily large values of the variable. Pintz's kernel replaces positivity by a contour shift past a single pole, and gives the bound for all large values.

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(K3) The function G is meromorphic on $\mathbb{C}$. If $\zeta(\rho_0) = 0$, then $G(s) = (s-1)\zeta(2s)e^{(s-\rho_0)^2}(s-\rho_0)^{-\nu}$, and its only pole in $\operatorname{Re} s > \frac{1}{2}$ is at ρ_0. If $\rho_0 = \frac{1}{2}$, then $G(s) = (s-1)\zeta(2s)e^{(s-1/2)^2}$, and its only pole in $\mathbb{C}$ is the simple pole at $\frac{1}{2}$.

(K4) There are $\kappa_0 > 0$ and $Y_0 \geq 1$ such that $|I(Y)| \geq \kappa_0 Y^{\beta_0}$ for all $Y \geq Y_0$.

The essential point is (K3): the factor $\zeta(s)$ in W cancels the denominator of M, so that every zero of ζ other than ρ_0 disappears, and (K4) is the residue at the one remaining pole.

Let $N \geq 1$, let $0 < \varepsilon < 1$, and put $Y = N^{1-\varepsilon}$ and $A = 2/\varepsilon$. Split $I(Y)$ at $x = N+1$. On $[1, N+1)$ the Cauchy–Schwarz inequality, (13) and (K1) give

$$\Big|\int_1^{N+1} L(x)\, w(x/Y)\,\frac{\mathrm{d}x}{x}\Big| \leq \Big(\int_1^{N+1} L(x)^2\,\mathrm{d}x\Big)^{1/2}\Big(\int_0^{\infty} |w(x/Y)|^2\,\frac{\mathrm{d}x}{x^2}\Big)^{1/2} = \Big(\frac{C_w V(N)}{Y}\Big)^{1/2}.$$

On $[N+1, \infty)$ we use $|L(x)| \leq x$ and $|w(x/Y)| \leq C_A (Y/x)^A$:

$$\Big|\int_{N+1}^{\infty} L(x)\, w(x/Y)\,\frac{\mathrm{d}x}{x}\Big| \leq C_A Y^A \int_N^{\infty} x^{-A}\,\mathrm{d}x \leq C_A Y^A N^{1-A} = C_A N^{1-A\varepsilon} = C_A N^{-1}.$$

If $Y \geq Y_0$ and $\kappa_0 Y^{\beta_0} \geq 2C_A$, then (K4) gives $\frac{1}{2}\kappa_0 Y^{\beta_0} \leq (C_w V(N)/Y)^{1/2}$, that is,

$$V(N) \geq \frac{\kappa_0^2}{4C_w}\, Y^{1+2\beta_0} = \frac{\kappa_0^2}{4C_w}\, N^{(1-\varepsilon)(1+2\beta_0)} \geq \frac{\kappa_0^2}{4C_w}\, N^{1+2\beta_0-3\varepsilon}.$$

This holds for all $N \geq N_0(\rho_0, \varepsilon)$. For $N < N_0$ we have $V(N) \geq 1$, and the first bound in (24) follows, after replacing 3ε by ε and reducing the constant. The second bound follows from (11): $Q_\lambda(N) \geq V(N)/(2N+1) \geq V(N)/(3N)$. □

Theorem 8 (Zero-one law)**.** *Let $\sigma \geq 0$.*

(a) If $\sigma > \Theta$, then B_σ is finite.

(b) If $\sigma < \Theta$, then B_σ is cofinite: $Q_\lambda(N) > N^{2\sigma}$ for all sufficiently large N.

In particular, if $\frac{1}{2} \leq \sigma < 1$ and $Q_\lambda(N_k) \leq N_k^{2\sigma}$ along some infinite sequence $N_1 < N_2 < \cdots$, then $\zeta(s) \neq 0$ for $\operatorname{Re} s > \sigma$.

Proof. (a) Theorem 6 with $\theta = \Theta$ and $\varepsilon = \sigma - \Theta$ gives $Q_\lambda(N) \leq CN^{2\sigma-\varepsilon}$, which is smaller than $N^{2\sigma}$ for large N.

(b) Suppose first that $\Theta > \frac{1}{2}$ and $\sigma < \Theta$. By the definition of Θ there is a zero $\rho_0 = \beta_0 + i\gamma_0$ with $\beta_0 > \max(\sigma, \frac{1}{2})$. Proposition 3 with $\varepsilon = \beta_0 - \sigma$ gives $Q_\lambda(N) \geq cN^{2\sigma+\varepsilon} > N^{2\sigma}$ for large N. If $\Theta = \frac{1}{2}$ and $\sigma < \frac{1}{2}$, apply Proposition 3 with $\rho_0 = \frac{1}{2}$ and $\varepsilon = \frac{1}{2} - \sigma$.

For the last assertion, the complement of B_σ is infinite, so $\sigma < \Theta$ is excluded by (b). Hence $\Theta \leq \sigma$. □

At $\sigma = \Theta$ the theorem says nothing, and the answer depends on finer information. For instance, if $\Theta = \frac{1}{2}$, then $B_{1/2}$ is the set of N with $Q_\lambda(N) > N$, and Theorem 10 below shows that this set is infinite; whether its complement is infinite is not known.

Corollary 5 (The limit exists)**.**

$$\lim_{N\to\infty} \frac{\log Q_\lambda(N)}{\log N} = 2\Theta \qquad \textit{and} \qquad \lim_{N\to\infty} \frac{\log V(N)}{\log N} = 1 + 2\Theta.$$

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Proof. Theorem 8 gives $N^{2\sigma_1} < Q_\lambda(N) \le N^{2\sigma_2}$ for all large N, whenever $\sigma_1 < \Theta < \sigma_2$. For V, Theorem 6 gives $V(N) \le N \max_{j\le N} L(j)^2 \ll_\varepsilon N^{1+2\Theta+\varepsilon}$. In the other direction, Proposition 3, applied to a zero with $\beta_0 > \max(\Theta - \varepsilon/2, \frac{1}{2})$ if $\Theta > \frac{1}{2}$ and to $\rho_0 = \frac{1}{2}$ if $\Theta = \frac{1}{2}$, gives $V(N) \gg_\varepsilon N^{1+2\Theta-2\varepsilon}$. □

Corollary 6 (One sequence suffices)**.** *The following statements are equivalent.*

(i) *The Riemann hypothesis.*
(ii) $Q_\lambda(N) \ll_\varepsilon N^{1+\varepsilon}$ *for every* $\varepsilon > 0$ *and all* N.
(iii) *For each* $\varepsilon > 0$ *there is an infinite sequence* $N_1 < N_2 < \cdots$, *which may depend on* ε, *such that* $Q_\lambda(N_k) \le N_k^{1+\varepsilon}$ *for all* k.
(iv) *For each* $\varepsilon > 0$ *there is an infinite sequence* $N_1 < N_2 < \cdots$ *such that* $\int_1^{N_k} L(x)^2\,\mathrm{d}x \le N_k^{2+\varepsilon}$ *for all* k.

Proof. (i)⇔(ii) is Theorem 6. (ii)⇒(iii): apply (ii) with $\varepsilon/2$ and take all sufficiently large N. (iii)⇒(iv): by (13), a sequence as in (iii) for $\varepsilon/2$ satisfies $\int_1^{N_k} L(x)^2\,\mathrm{d}x \le 3N_k\, Q_\lambda(N_k) \le 3N_k^{2+\varepsilon/2} \le N_k^{2+\varepsilon}$ for large k. (iv)⇒(i): suppose $\Theta > \frac{1}{2}$, and choose a zero with $\beta_0 = \frac{1}{2} + \varepsilon_0$, $\varepsilon_0 > 0$. Since $\int_1^N L(x)^2\,\mathrm{d}x = V(N-1)$, Proposition 3 with $\varepsilon = \varepsilon_0$ gives $\int_1^N L(x)^2\,\mathrm{d}x \ge c(N-1)^{2+\varepsilon_0}$ for all $N \ge 2$. This contradicts (iv) with $\varepsilon = \varepsilon_0/2$ for large k. □

Corollary 7 (No almost-all bound below the zero-free exponent)**.** *Let* $\frac{1}{2} \le \sigma < 1$, *and let* $E \subset \mathbb{N}$ *be any set whose complement is infinite. If* $Q_\lambda(N) \le N^{2\sigma}$ *for all* $N \notin E$, *then* $\zeta(s) \ne 0$ *for* $\operatorname{Re} s > \sigma$. *The same conclusion holds if* $V(N) \le N^{1+2\sigma}$ *for all* $N \notin E$.

Proof. The first statement is the last assertion of Theorem 8. For the second, if $\Theta > \sigma$, Proposition 3 applied to a zero with $\beta_0 > \sigma$ and $\varepsilon = \beta_0 - \sigma$ gives $V(N) > N^{1+2\sigma}$ for all large N. □

Table 3 lists the hypotheses on the energy and what each gives.

Table 3. Bounds for the energy and their consequences for Θ. In each row $\frac{1}{2} \le \sigma < 1$, and the bound is $Q_\lambda(N) \le N^{2\sigma}$ or the stated variant.

hypothesis	conclusion	reference
$Q_\lambda(N) \ll_\varepsilon N^{2\sigma+\varepsilon}$, all N	equivalent to $\Theta \le \sigma$	Theorem 6
$\sum_N Q_\lambda(N) N^{-1-2\sigma} < \infty$	$\Theta \le \sigma$; no zero on $\operatorname{Re} s = \sigma$	Theorem 7
bound outside a set of density 0	$\Theta \le \sigma$	Corollary 7
bound along one infinite sequence	$\Theta \le \sigma$	Theorem 8
$\#\{j \le N : \lvert L(j)\rvert > N^\sigma\} \le N^{1-\delta}$, one sequence	$\Theta \le \sigma$	Proposition 4
$V(N_k) \le N_k^{1+2\sigma}$ along one sequence	$\Theta \le \sigma$	Corollary 7
$\mathcal{Q}_\lambda(N,H) \ll N^{2\sigma-1}H$, $H = H(N)$	$\Theta \le \sigma$	Corollary 4

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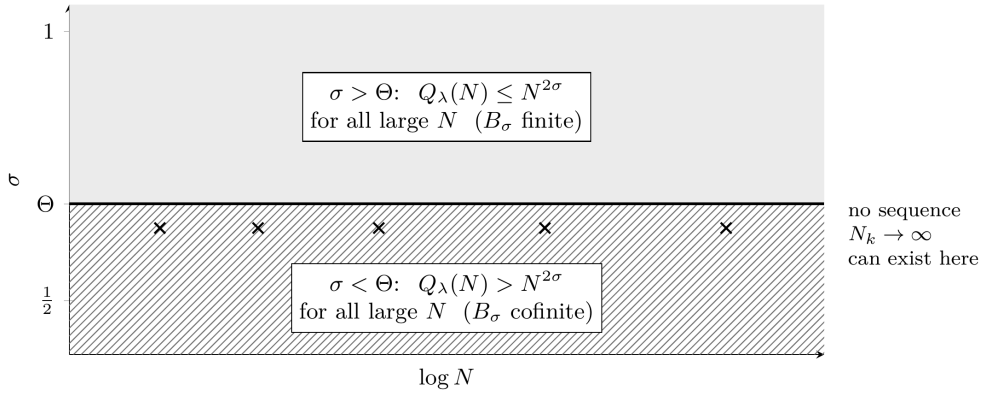

Figure 5. The zero-one law (Theorem 8). Above the line $\sigma = \Theta$ the inequality $Q_\lambda(N) \le N^{2\sigma}$ holds for all large N; below it the reverse inequality holds for all large N. No infinite sequence N_k can satisfy $Q_\lambda(N_k) \le N_k^{2\sigma}$ with $\sigma < \Theta$ (crosses), so a single such sequence already gives the zero-free half-plane $\operatorname{Re} s > \sigma$. At $\sigma = \Theta$ itself the theorem makes no claim.

Remark 1 (Zero-density estimates). *Let $N(\sigma, T)$ be the number of zeros with $\beta \ge \sigma$ and $|\gamma| \le T$. Zero-density estimates bound $N(\sigma, T)$ by a power $T^{A(\sigma)(1-\sigma)+o(1)}$; the estimate of Guth and Maynard [37] gives $A(\sigma) \le 30/13$ uniformly. In many problems on primes such estimates replace the Riemann hypothesis for almost all values of a parameter, and one might hope in the same way for a bound $Q_\lambda(N) \le N^{2\sigma}$, with some fixed $\sigma < 1$, for all N outside a sparse set. Corollary 7 shows that this is not possible without proving $\Theta \le \sigma$: any derivation of such an almost-all bound from a zero-density estimate would be a derivation of a zero-free half-plane from that estimate. The reason is visible in the proof of Proposition 3. A zero-density estimate limits the number of zeros to the right of a line. A single zero $\beta_0 + i\gamma_0$ already forces $V(N) \gg N^{1+2\beta_0-\varepsilon}$ for every N, and V is monotone, so neither averaging over N nor discarding a set of N can hide it. The unconditional bounds for $Q_\lambda(N)$ that are available come from the zero-free region and save less than any power of N.*[9]

4.1. A Finite Range

Theorem 8 needs a bound for infinitely many N. A bound on a finite range $N < Y$ still excludes zeros, but only in a bounded box whose size is explicit.

Theorem 9 (A finite-range transfer). *Let $\frac{1}{2} \le \sigma < 1$, $C > 0$, $Y \ge 2$ and $1 \le N_0 \le Y$, and suppose that $Q_\lambda(N) \le CN^{2\sigma}$ for all integers $N_0 \le N < Y$. If $\zeta(\rho) = 0$ with $\rho = \beta + i\gamma$ and $\frac{1}{2} < \beta < 1$, then, with $K_1 = 1 + |\rho|/\beta$,*

$$\left(2^{1-\beta} - 1\right) \frac{\zeta(4\beta)}{\zeta(2\beta)}\, Y^\beta \le |\rho| K_1 (N_0 - 1) + \frac{\sqrt{2C}\,|\rho|\, K_1^{1-\sigma}}{\sigma(1-\sigma)(1-\beta)^\sigma}\, Y^\sigma + E(\beta)\sqrt{Y} + 2,$$

where $E(\beta) = (2^{2\beta-1} + 2^{2\beta-1/2})/(2\beta - 1) + \sqrt{2} - 1$.

Proof. Put $\epsilon(n) = (-1)^{n-1}$ and $A_m = \sum_{n \le m} \epsilon(n) n^{-\rho}$. For $0 < u < 1$ let $m(u)$ be the number of integers $n \ge 1$ with $n < 1/u$, and put $w(u) = u^{-\rho} A_{m(u)}$; put $w(u) = 0$ for $u \ge 1$. Let $I = \int_1^Y L(x)\, w(x/Y)\, \mathrm{d}x/x$.

An identity. Since $w(x/Y) = Y^\rho x^{-\rho} \sum_{n < Y/x} \epsilon(n) n^{-\rho}$ and $\int_1^Z L(x) x^{-\rho-1}\, \mathrm{d}x = \rho^{-1} \sum_{k \le Z} \lambda(k)(k^{-\rho} - Z^{-\rho})$, interchanging the finite sums gives

$$\rho Y^{-\rho} I = \sum_{N \le Y} g(N) N^{-\rho} - Y^{-\rho} \sum_{N \le Y} g(N), \qquad g = \epsilon * \lambda.$$

[9] From the Vinogradov–Korobov zero-free region one has $\sum_{n \le y} \mu(n) \ll y \exp(-c\,\eta(y))$ with $\eta(y) = (\log y)^{3/5} (\log\log y)^{-1/5}$ [59]. Since $L(x) = \sum_{d \le \sqrt{x}} \sum_{n \le x/d^2} \mu(n)$, the same bound holds for $L(x)$ with a smaller constant c, and (7) gives $Q_\lambda(N) \ll N^2 \exp(-2c\,\eta(N))$.

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The Dirichlet series of g is $(1-2^{1-s})\zeta(s)\cdot\zeta(2s)/\zeta(s) = (1-2^{1-s})\zeta(2s)$, so $g(N)$ is 1 if N is a square, -2 if $N/2$ is a square, and 0 otherwise. Hence

$$\rho I = Y^{\rho}\Big(\sum_{j\le\sqrt{Y}} j^{-2\rho} - 2^{1-\rho}\sum_{j\le\sqrt{Y/2}} j^{-2\rho}\Big) - G, \qquad G = \lfloor\sqrt{Y}\rfloor - 2\lfloor\sqrt{Y/2}\rfloor.$$

The bracket is $(1-2^{1-\rho})\zeta(2\rho)$ minus $\tau(\sqrt{Y}) - 2^{1-\rho}\tau(\sqrt{Y/2})$, where $\tau(y) = \sum_{j>y} j^{-2\rho}$ satisfies $|\tau(y)| \le \lfloor y\rfloor^{1-2\beta}/(2\beta-1) \le (y/2)^{1-2\beta}/(2\beta-1)$ for $y \ge 1$. So the two tails contribute at most $(2^{2\beta-1}+2^{2\beta-1/2})\sqrt{Y}/(2\beta-1)$ after multiplication by Y^{β}. Writing $a = \sqrt{Y}$, we have $-(\sqrt{2}-1)a - 1 < G < 2-(\sqrt{2}-1)a$, so $|G| \le (\sqrt{2}-1)\sqrt{Y}+2$. Finally $|1-2^{1-\rho}| \ge 2^{1-\beta}-1$, and $|\zeta(2\rho)| \ge \prod_p(1+p^{-2\beta})^{-1} = \zeta(4\beta)/\zeta(2\beta)$. Therefore the left side of the theorem is at most $|\rho||I| + E(\beta)\sqrt{Y}+2$.

The kernel at a zero. Since $(1-2^{1-s})\zeta(s) = \sum_n \epsilon(n)n^{-s}$ for $\operatorname{Re} s > 0$, the hypothesis $\zeta(\rho)=0$ gives $A_m = -\sum_{n>m}\epsilon(n)n^{-\rho}$. Grouping the tail into pairs $(2k-1)^{-\rho}-(2k)^{-\rho}$, each of modulus at most $|\rho|\int_{2k-1}^{2k} x^{-\beta-1}\,\mathrm{d}x$, after at most one single term $(m+1)^{-\rho}$, gives $|A_m| \le (1+|\rho|/\beta)(m+1)^{-\beta}$. Since $(m(u)+1)u \ge 1$, this gives $|w(u)| \le K_1$. Trivially $|A_m| \le \sum_{n\le m} n^{-\beta} \le m^{1-\beta}/(1-\beta)$, and $m(u) < 1/u$, so $|w(u)| \le 1/((1-\beta)u)$.

The integral. For $1 \le x < N_0$ we use $|L(x)| \le x$ and $|w| \le K_1$, which gives at most $K_1(N_0-1)$. For $N_0 \le x < Y$, (7) gives $L(x)^2 \le 2Q_\lambda(\lfloor x\rfloor) \le 2Cx^{2\sigma}$. Split at $x_* = Y/((1-\beta)K_1)$: the part $x \le x_*$ is at most $\sqrt{2C}K_1 x_*^{\sigma}/\sigma$, and the part $x > x_*$ is at most $\sqrt{2C}\,Y(1-\beta)^{-1}x_*^{\sigma-1}/(1-\sigma)$. Both equal $\sqrt{2C}\,K_1^{1-\sigma}(1-\beta)^{-\sigma}Y^{\sigma}$, times $1/\sigma$ and $1/(1-\sigma)$. This bounds $|\rho||I|$ by the first two terms on the right. □

Corollary 8 (A zero-free box). *Let $0 < \delta \le \frac14$, $T \ge 14$ and $Y \ge 100$, and suppose that $Q_\lambda(N) \le CN$ for all $N < Y$. If*

$$Y^{\delta} \ge 8.5\sqrt{2C}\,T^{3/2}\delta^{-5/2} + 3.8\,\delta^{-3},$$

then $\zeta(s) \ne 0$ for $\frac12+\delta \le \operatorname{Re} s \le 1-\delta$ and $|\operatorname{Im} s| \le T$.

Proof. Suppose ρ is a zero in the box, and apply Theorem 9 with $\sigma = \frac12$ and $N_0 = 1$. On the left, $2^{1-\beta}-1 \ge 2^{\delta}-1 \ge \delta\log 2$ and $\zeta(4\beta)/\zeta(2\beta) \ge 1/\zeta(1+2\delta) \ge 2\delta/(1+2\delta) \ge \delta$, since $\zeta(1+x) \le 1+1/x$. On the right, $|\rho| \le 1.003\,T$, $K_1 \le 1+2|\rho| \le 2.08\,T$, $(1-\beta)^{-1/2} \le \delta^{-1/2}$, and $E(\beta)+2/\sqrt{Y} \le (2+2^{3/2})/(2\delta)+0.62 \le 2.6/\delta$. So $0.693\,\delta^2 Y^{1/2+\delta} \le 5.82\sqrt{2C}\,T^{3/2}\delta^{-1/2}\sqrt{Y} + 2.6\sqrt{Y}/\delta$, which gives $Y^{\delta} \le 8.4\sqrt{2C}\,T^{3/2}\delta^{-5/2} + 3.76\,\delta^{-3}$, contrary to the hypothesis. □

The loss is a factor $|\rho|^{2-\sigma}$: one factor $|\rho|$ comes from the $1/s$ in $M(s)$ and the factor $K_1^{1-\sigma}$ from the kernel. The computed values satisfy the hypothesis of Theorem 9 with $\sigma = \frac12$, $C = 1.1534$ and $Y = 3\cdot 10^7+1$ (Table A12), but the inequality of the theorem then holds at every ρ with $|\gamma| \ge 14$; the smallest ratio of the right side to the left side, at $\beta \approx 0.92$ and $\gamma \approx 14.13$, is about 26. Even if $Q_\lambda(N) \le 1.1534\,N$ continued, Theorem 9 would need Y of size about 10^{16} to exclude the box $\beta \ge 0.75$, $|\gamma| \le 14.14$, and about 10^{84} at height $3\cdot 10^{12}$, below which all zeros are known to lie on the critical line [54]. So no computation of the energy within reach adds a zero-free region; Theorem 9 is the effective form of the zero-one law.

The same mechanism controls the number of positions at which L is large.

Proposition 4 (Almost all positions). *Let $\frac12 \le \sigma < 1$ and $0 < \delta \le 1$, and put $E_N = \{1 \le j \le N : |L(j)| > N^{\sigma}\}$. If $\#E_N \le N^{1-\delta}$ for infinitely many N, then $\Theta \le \sigma$. Conversely, if $\Theta < \sigma$, then $E_N = \varnothing$ for all large N. In particular, the Riemann hypothesis holds if and only if for every $\varepsilon > 0$ there are $\delta > 0$ and infinitely many N with $\#\{j \le N : |L(j)| > N^{1/2+\varepsilon}\} \le N^{1-\delta}$.*

Proof. If $\Theta < \sigma$, Theorem 6 gives $|L(j)| \ll j^{(\Theta+\sigma)/2} \le N^{(\Theta+\sigma)/2} < N^{\sigma}$ for $j \le N$ and N large.

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Suppose $\Theta > \sigma$ and $\#E_N \le N^{1-\delta}$ for infinitely many N. Choose a zero $\beta_0 + i\gamma_0$ with $\beta_0 > \sigma$ and $\Theta - \beta_0 < \delta/4$, and put $\varepsilon = \min(\delta/8, (\beta_0 - \sigma)/2)$. Proposition 3 gives $V(N) \ge cN^{1+2\beta_0-\varepsilon}$ for all N, and Theorem 6 gives $|L(j)| \le C' j^{\Theta+\varepsilon}$. For N in the sequence,

$$V(N) = \sum_{j \in E_N} L(j)^2 + \sum_{j \le N,\ j \notin E_N} L(j)^2 \le C'^2 N^{1-\delta+2\Theta+2\varepsilon} + N^{1+2\sigma}.$$

Both exponents are smaller than $1 + 2\beta_0 - \varepsilon$, because $2(\Theta - \beta_0) < \delta/2 < \delta - 3\varepsilon$ and $\varepsilon < 2(\beta_0 - \sigma)$. This contradicts the lower bound for large N. The last assertion follows from the first two with $\sigma = \frac{1}{2} + \varepsilon$. □

Remark 2. *Proposition 4 is a direct consequence of Pintz's mean-value principle (Proposition 3, [52]). Kaczorowski and Pintz [43,44] studied the measure of the sets on which oscillation of arithmetic error terms is large; statements of this type in their work may contain Proposition 4, and no priority is claimed.*

5. Pole Energies

Section 3 determines the exponent of $Q_\lambda(N)$. The vector identity (20) also gives lower bounds for the constant in front of the power, with one contribution from each pole of M on the line $\operatorname{Re} s = \Theta$. We keep the notation $\mathcal{H}$, U_X, w_s, $K(s)$ of Section 3.2.

5.1. Poles on One Vertical Line

Proposition 5 (A sum of pole contributions)**.** *Fix $\beta \ge \frac{1}{2}$ and an integer $r \ge 1$. Let $\mathcal{P}$ be a finite set of distinct poles $\rho = \beta + i\gamma$ of M, each of order exactly r, and write $A_\rho = \lim_{s\to\rho}(s-\rho)^r M(s)$. Then*

$$C := \limsup_{N\to\infty} \frac{Q_\lambda(N)}{N^{2\beta}(\log N)^{2r-2}} \ \ge \ \sum_{\rho\in\mathcal{P}} K(\rho)\left|\frac{A_\rho}{(r-1)!}\right|^2. \tag{26}$$

If $C < \infty$, then M has no pole in $\operatorname{Re} s > \beta$, and every pole of M on $\operatorname{Re} s = \beta$ has order at most r.

Proof. If $C = \infty$ there is nothing to prove. Let $C < \infty$. By (17), the limit superior of $\|U_X\|^2 X^{-2\beta}(\log X)^{-(2r-2)}$ over real $X \to \infty$ is also C. Put $F(t) = e^{-\beta t} U_{e^t}$ for $t \ge 0$. For every $\varepsilon > 0$ there is t_ε with

$$\|F(t)\|^2 \le (C + \varepsilon)\, t^{2r-2} \qquad (t \ge t_\varepsilon), \tag{27}$$

and $\|F\|$ is bounded on $[0, t_\varepsilon]$. Hence $\Phi(s) = \int_0^\infty F(t) e^{-(s-\beta)t}\,\mathrm{d}t$ is holomorphic on $\operatorname{Re} s > \beta$, and, as in the proof of Theorem 7, $\Phi(s) = M(s) w_s$ there and M has no pole in $\operatorname{Re} s > \beta$. At a point ρ of the line $\operatorname{Re} s = \beta$, (27) gives $\|M(\rho+\delta)w_{\rho+\delta}\| \le \int_0^\infty \|F(t)\| e^{-\delta t}\,\mathrm{d}t = O(\delta^{-r})$ as $\delta \downarrow 0$. Since $w_{\rho+\delta} \to w_\rho \ne 0$, a pole at ρ has order at most r.

For $\delta > 0$ let $\langle\cdot,\cdot\rangle_\delta$ be the inner product of $L^2\big([0,\infty), e^{-\delta t}\,\mathrm{d}t; \mathcal{H}\big)$, and put $G_r = \Gamma(2r-1) = (2r-2)!$. By (27) and Euler's integral $\int_0^\infty t^{2r-2} e^{-\delta t}\,\mathrm{d}t = G_r \delta^{-(2r-1)}$,

$$\limsup_{\delta\downarrow 0}\ \delta^{2r-1}\|F\|_\delta^2 \le C\, G_r. \tag{28}$$

For $\rho = \beta + i\gamma \in \mathcal{P}$ define the test function

$$g_{\rho,\delta}(t) = \frac{\delta^{r-1/2}}{\sqrt{G_r}}\, t^{r-1} e^{i\gamma t}\, v_\rho, \qquad v_\rho = \frac{w_\rho}{\|w_\rho\|}.$$

Then $\|g_{\rho,\delta}\|_\delta = 1$, and for $\rho \ne \rho'$ in $\mathcal{P}$,

$$\big|\langle g_{\rho,\delta}, g_{\rho',\delta}\rangle_\delta\big| = \left|\frac{\delta}{\delta - i(\gamma-\gamma')}\right|^{2r-1} |\langle v_\rho, v_{\rho'}\rangle| \ \longrightarrow\ 0 \qquad (\delta \downarrow 0).$$

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So the Gram matrix of the family $(g_{\rho,\delta})_{\rho\in\mathcal{P}}$ tends to the identity, and its largest eigenvalue Λ_δ tends to 1. For any vector F and any finite family (g_ρ) with Gram matrix of largest eigenvalue Λ,

$$\sum_\rho |\langle F, g_\rho\rangle|^2 \le \Lambda \, \|F\|^2, \tag{29}$$

because the synthesis map $(c_\rho) \mapsto \sum_\rho c_\rho g_\rho$ has squared norm Λ, and $F \mapsto (\langle F, g_\rho\rangle)_\rho$ is its adjoint.

It remains to evaluate $\langle F, g_{\rho,\delta}\rangle_\delta$. Differentiating $\Phi(s) = M(s)w_s$ under the integral sign $r-1$ times,

$$\begin{aligned}\langle F, g_{\rho,\delta}\rangle_\delta &= \frac{\delta^{r-1/2}}{\sqrt{G_r}}\Big\langle \int_0^\infty t^{r-1}F(t)e^{-(\rho+\delta-\beta)t}\,\mathrm{d}t,\ v_\rho\Big\rangle \\ &= \frac{\delta^{r-1/2}}{\sqrt{G_r}}\Big\langle (-1)^{r-1}\frac{\mathrm{d}^{\,r-1}}{\mathrm{d}s^{r-1}}\big[M(s)w_s\big]_{s=\rho+\delta},\ v_\rho\Big\rangle.\end{aligned}$$

Near ρ, $M(s)w_s = A_\rho w_\rho (s-\rho)^{-r} + R(s)$, where R is an $\mathcal{H}$-valued function with a pole of order at most $r-1$ at ρ. Since

$$(-1)^{r-1}\frac{\mathrm{d}^{\,r-1}}{\mathrm{d}s^{r-1}}(s-\rho)^{-r} = \frac{(2r-2)!}{(r-1)!}\,(s-\rho)^{-(2r-1)},$$

and the corresponding derivative of R is $O(|s-\rho|^{-(2r-2)})$, we obtain

$$\lim_{\delta\downarrow 0}\ \delta^{r-1/2}\langle F, g_{\rho,\delta}\rangle_\delta = \sqrt{G_r}\,\frac{A_\rho}{(r-1)!}\,\|w_\rho\|.$$

Apply (29) in the space with inner product $\langle\cdot,\cdot\rangle_\delta$, multiply by δ^{2r-1}, let $\delta\downarrow 0$, and use (28) and $\Lambda_\delta \to 1$:

$$G_r \sum_{\rho\in\mathcal{P}} \Big|\frac{A_\rho}{(r-1)!}\Big|^2 \|w_\rho\|^2 \le C\,G_r.$$

Since $\|w_\rho\|^2 = K(\rho)$, this is (26). □

Corollary 9 (A critical-line lower bound)**.** *For every finite set $\mathcal{Z}$ of simple zeros of ζ on the critical line,*

$$\limsup_{N\to\infty}\frac{Q_\lambda(N)}{N} \ \ge\ \frac{2}{3\,\zeta(1/2)^2} + \sum_{\rho\in\mathcal{Z}}\Big(2 - 2\ \mathrm{Re}\,\frac{1}{1+\rho}\Big)\Big|\frac{\zeta(2\rho)}{\rho\,\zeta'(\rho)}\Big|^2. \tag{30}$$

The left side is infinite if the Riemann hypothesis fails, or if some zero on the critical line is multiple. Unconditionally,

$$\limsup_{N\to\infty}\frac{Q_\lambda(N)}{N} \ \ge\ \frac{2}{3\,\zeta(1/2)^2} > \frac{2(3-2\sqrt{2})}{3} > 0.114. \tag{31}$$

Proof. At $s=\frac12$ the function M has a simple pole with residue $1/\zeta(\frac12)$, and $K(\frac12)=\frac23$. At a simple zero $\rho=\frac12+i\gamma$, M has a simple pole with residue $\zeta(2\rho)/(\rho\zeta'(\rho)) \neq 0$, and $K(\rho) = 2 - 2\ \mathrm{Re}\,\frac{1}{1+\rho}$ by (18). Proposition 5 with $\beta=\frac12$, $r=1$ and $\mathcal{P}=\{\frac12\}\cup\mathcal{Z}$ gives (30). If the left side is finite, the same proposition excludes poles of M in $\mathrm{Re}\,s>\frac12$ and poles of order greater than 1 on the line; by the functional equation a zero off the line produces a zero to its right. For (31), the alternating series $(1-\sqrt{2})\zeta(\frac12) = \sum_{n\ge1}(-1)^{n-1}n^{-1/2}$ has a sum in $(0,1)$, so $0<-\zeta(\frac12)<1/(\sqrt{2}-1)=1+\sqrt{2}$, and $2/(3(1+\sqrt{2})^2) = 2(3-2\sqrt{2})/3$. □

The zeros with $|\gamma| \le 3\cdot 10^{12}$ are known to lie on the critical line [54]. Each simple zero among them may be used in (30), and each adds a positive term. Values of the right side are given in Table 4.

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5.2. Positive Weights

The bound (30) adds the squares of the pole contributions. A positive weight in the variable $t = \log X$ adds the contributions themselves, as vectors, and gives larger constants when their phases are aligned.

Proposition 6 (Positive-weight bound). *Let $C = \limsup_{N\to\infty} Q_\lambda(N)/N$, and suppose $C < \infty$. For real ω let $R(\omega)$ be the residue of M at $\frac12 + i\omega$, so that $R(\omega) = 0$ when M is regular there. Let*

$$P(t) = \sum_{\omega\in\Omega} p_\omega e^{-i\omega t}$$

be a trigonometric polynomial with a finite set Ω of distinct real frequencies, such that $P(t) \ge 0$ for all real t and $p_0 = 1$. Then

$$C \;\ge\; \Big\| \sum_{\omega\in\Omega} p_\omega\, R(\omega)\, w_{1/2+i\omega} \Big\|^2. \tag{32}$$

Proof. By Proposition 5 with $\beta = \frac12$ and $r = 1$, M has no pole in $\operatorname{Re} s > \frac12$ and only simple poles on the line. Put $F(t) = e^{-t/2}U_{e^t}$. As in that proof, $\limsup_{t\to\infty} \|F(t)\|^2 = C$ and $\int_0^\infty F(t)e^{-(s-1/2)t}\,\mathrm{d}t = M(s)w_s$ for $\operatorname{Re} s > \frac12$. For $\delta > 0$ consider

$$J_\delta = \delta \int_0^\infty e^{-\delta t} P(t)F(t)\,\mathrm{d}t = \sum_{\omega\in\Omega} p_\omega\, \delta M(\tfrac12 + \delta + i\omega)\, w_{1/2+\delta+i\omega}.$$

Since the poles on the line are at most simple, $\delta M(\frac12 + \delta + i\omega) \to R(\omega)$ as $\delta \downarrow 0$, and $w_{1/2+\delta+i\omega} \to w_{1/2+i\omega}$ in $\mathcal{H}$. So J_δ converges to the vector in (32). On the other hand, $P \ge 0$ gives

$$\|J_\delta\| \le \delta \int_0^\infty e^{-\delta t} P(t)\|F(t)\|\,\mathrm{d}t.$$

Fix $\varepsilon > 0$ and t_ε with $\|F(t)\| \le \sqrt{C} + \varepsilon$ for $t \ge t_\varepsilon$. The part of the integral over $[0, t_\varepsilon]$ is bounded, and vanishes after multiplication by δ. Since $\delta \int_0^\infty e^{-\delta t} P(t)\,\mathrm{d}t = \sum_\omega p_\omega \delta/(\delta + i\omega) \to p_0 = 1$, we get $\limsup_{\delta\downarrow 0} \|J_\delta\| \le \sqrt{C} + \varepsilon$. □

When $C = \infty$ the lower bounds below hold trivially, so they are unconditional.

Corollary 10 (Fejér weights). *Let $\rho_j = \frac12 + i\gamma_j$, $1 \le j \le J$, be simple zeros of ζ with $0 < \gamma_1 < \cdots < \gamma_J$, and let $m \ge 1$. Put*

$$a = -\frac{1}{\zeta(1/2)} > 0, \qquad A_j = \frac{\zeta(2\rho_j)}{\rho_j\, \zeta'(\rho_j)}.$$

Assume that the $(2m+1)^J$ numbers $n_1\gamma_1 + \cdots + n_J\gamma_J$ with integers $|n_j| \le m$ are pairwise distinct, and that $\zeta(\frac12 + i\omega) \ne 0$ for each of them other than $\pm\gamma_1, \ldots, \pm\gamma_J$. Then

$$\limsup_{N\to\infty} \frac{Q_\lambda(N)}{N} \;\ge\; \Big\| a\, w_{1/2} + \frac{2m}{m+1} \sum_{j=1}^{J} |A_j|\, \operatorname{Re} w_{\rho_j} \Big\|^2, \tag{33}$$

where $\operatorname{Re} w_\rho = \frac12(w_\rho + w_{\bar\rho})$. The norm is computed from

$$\langle \operatorname{Re} w_z, \operatorname{Re} w_{z'} \rangle = 1 - \operatorname{Re}\frac{1}{1+z} - \operatorname{Re}\frac{1}{1+z'} + \operatorname{Re}\frac{1}{1+z+z'} + \operatorname{Re}\frac{1}{1+z+\overline{z'}}, \tag{34}$$

valid for $\operatorname{Re} z, \operatorname{Re} z' > -\frac12$; for real z the left side is $\langle w_z, \operatorname{Re} w_{z'} \rangle$.

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Proof. We may assume $C < \infty$. Choose real θ_j with $e^{i\theta_j} A_j = -|A_j|$, and let

$$P(t) = \prod_{j=1}^{J} \frac{1}{m+1} \Big| \sum_{\ell=0}^{m} e^{i\ell(\gamma_j t - \theta_j)} \Big|^2 = \prod_{j=1}^{J} \sum_{|n| \le m} \Big(1 - \frac{|n|}{m+1}\Big) e^{in(\gamma_j t - \theta_j)}.$$

Then $P \ge 0$. Its frequencies are the numbers $-(n_1\gamma_1 + \cdots + n_J\gamma_J)$, which are pairwise distinct by assumption, so the coefficient of each exponential is the product of the corresponding coefficients of the factors. In particular $p_0 = 1$, and the coefficient at $\omega = \mp\gamma_j$ is $\frac{m}{m+1} e^{\mp i\theta_j}$. By assumption $R(\omega) = 0$ at every other nonzero frequency. Further $R(0) = 1/\zeta(\frac{1}{2}) = -a$, $R(\gamma_j) = A_j$, and $R(-\gamma_j) = \overline{A_j}$ because $M(\bar{s}) = \overline{M(s)}$. The vector in (32) is therefore

$$-a\, w_{1/2} + \frac{m}{m+1} \sum_{j=1}^{J} \big(e^{i\theta_j} A_j\, w_{\rho_j} + e^{-i\theta_j} \overline{A_j}\, w_{\bar{\rho}_j} \big)$$
$$= -\Big(a\, w_{1/2} + \frac{2m}{m+1} \sum_{j=1}^{J} |A_j|\ \mathrm{Re}\, w_{\rho_j} \Big).$$

The formula (34) is verified in Appendix B.4. □

The hypothesis of Corollary 10 concerns finitely many explicit numbers. For $J = 1$ it says that no multiple $k\gamma_1$ with $2 \le k \le m$ is the ordinate of a zero. For the first zero and $m = 100$, and for the first two zeros and $m = 30$, it has been confirmed by interval arithmetic,[10] and the right side of (33) then has the computed values 0.65142 and 0.72446 of Table 4.

Remark 3 (Status of the constants)**.** *The inequality* (31)*, with the constant* $2(3 - 2\sqrt{2})/3 > 0.114$*, is proved without computation. The larger constants follow from Corollaries 9 and 10 once the zeros involved have been located and the finitely many nonvanishing conditions have been checked numerically. With the first two zeros and* $m = 30$ *this gives* $\limsup_{N\to\infty} Q_\lambda(N)/N > 0.7244$*. Nothing else in the paper uses these values.*

Table 4. Constants of Section 5. Rows marked "exact" are proved in the text, the last from the theorem of Mossinghoff and Trudgian. Rows marked "computed" are numerical values, rounded to the digits shown; they illustrate the theorems and no proof depends on them. In the rows for (30), $\mathcal{Z}$ consists of the first n zeros above the real axis and their conjugates.

quantity	value	status
$K(\frac{1}{2})$	$2/3$	exact
$\min_{\mathrm{Re}\, s \ge 1/2} K(s)$, attained at $s = 1/\sqrt{2}$	$4\sqrt{2} - 5 = 0.65685\ldots$	exact
$\sup_{\mathrm{Re}\, s \ge 1/2} K(s)$	2, not attained	exact
analytic lower bound (31)	$2(3 - 2\sqrt{2})/3 = 0.11438\ldots$	exact
$\zeta(\frac{1}{2})$	-1.46035	computed
$2/(3\zeta(\frac{1}{2})^2)$, the pole at $\frac{1}{2}$ alone	0.31260	computed
$\lvert A_1\rvert$, $\gamma_1 = 14.13472\ldots$	0.17372	computed
$\lvert A_2\rvert$, $\gamma_2 = 21.02203\ldots$	0.03476	computed
right side of (30), $n = 1$	0.43241	computed
right side of (30), $n = 10$	0.44722	computed
right side of (30), $n = 100$	0.45578	computed
right side of (30), $n = 1000$	0.45770	computed
right side of (33), $J = 1$, $m = 100$	0.65142	computed
right side of (30), $n = 3000$	0.45794	computed
right side of (33), $J = 2$, $m = 30$	0.72446	computed
$\lVert Z(y_0)\rVert^2$ of Theorem 12, $m = 1400$	1.98545	computed
$2.3723^2/2$, Theorem 10	$2.81390\ldots$	exact
right side of (36), Corollary 12	4.01310	computed
right side of (36), Corollary 13	4.62422	computed
$\pi/(4\zeta(\frac{1}{2})^2)$, the boundary cost in (45)	0.36828	computed

[10] In the second case there are $61^2 = 3721$ frequencies; one checks that they are distinct and that $\zeta(\frac{1}{2} + i\omega) \ne 0$ at the 3716 frequencies other than $0, \pm\gamma_1, \pm\gamma_2$. These verifications, and the numerical values in Table 4, are computations and are not used in any proof in this paper.

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Under the Riemann hypothesis and the simplicity of the zeros, (30) bounds $\limsup Q_\lambda(N)/N$ below by the full sum $\frac{2}{3}\zeta(\frac{1}{2})^{-2} + \sum_\rho K(\rho)|A_\rho|^2$, finite or infinite, and Table 4 shows that the Fejér weights exceed the first thousand terms of that sum. Larger lower bounds require weights with more frequencies, and then the hypothesis of Corollary 10 involves more integer combinations of ordinates. Subsection 5.3 removes that hypothesis and gives a larger constant.

5.3. The Size of $Q_\lambda(N)/N$

At the critical scale the energy is governed by the oscillation of $L(x)/\sqrt{x}$. The first statement transfers a published oscillation theorem; the second shows that nothing is lost in the transfer.

Theorem 10 (Oscillation of the energy)**.**

$$\limsup_{N\to\infty} \frac{Q_\lambda(N)}{N} > 2.8139.$$

Consequently $B_{1/2} = \{N : Q_\lambda(N) > N\}$ is infinite, and under the Riemann hypothesis the set $B_\Theta = B_{1/2}$ is infinite.

Proof. Mossinghoff and Trudgian [50] proved, unconditionally, that $\liminf_{x\to\infty} L(x)/\sqrt{x} < -2.3723$ (see also [51, (1.2)]). The proof rests on the classical dichotomy: if the Riemann hypothesis fails or some zero is multiple, then $L(x)/\sqrt{x}$ is unbounded in both directions [39] [Section 3], [49] [Ch. 15]; otherwise their weak-independence form of Ingham's method applies. Take x large with $L(x) < -2.3723\sqrt{x}$ and put $N = \lfloor x \rfloor$. Then $L(N) = L(x)$ and $\sqrt{N} \le \sqrt{x}$, so $L(N)^2 > 2.3723^2 N$. By (6), $Q_\lambda(N) \ge \frac{N+1}{2N} L(N)^2 > \frac{2.3723^2}{2} N$, and $2.3723^2/2 = 2.81390\ldots > 2.8139$. Such N are arbitrarily large.

For the second assertion, if $\Theta > \frac{1}{2}$ then $B_{1/2}$ is cofinite by Theorem 8(b). If $\Theta = \frac{1}{2}$, the first assertion gives $Q_\lambda(N) > 2.8N > N$ for infinitely many N. Under the Riemann hypothesis $\Theta = \frac{1}{2}$ and $B_\Theta = B_{1/2}$. □

Remark 4. *The value* $\liminf L(x)/\sqrt{x} \le -1.389278$ *of Borwein, Ferguson and Mossinghoff [29] gives only* $\limsup Q_\lambda(N)/N \ge 0.9650$. *Any constant larger than* 1 *obtained in this way needs* $|L(x)| > \sqrt{2x}$ *for arbitrarily large* x.

Proposition 7 (Boundedness of the normalized energy)**.** *Let* $\ell = \limsup_{x\to\infty} |L(x)|/\sqrt{x} \in [0,\infty]$. *Then*

$$\tfrac{1}{2}\,\ell^2 \;\le\; \limsup_{N\to\infty} \frac{Q_\lambda(N)}{N} \;\le\; \tfrac{10}{3}\,\ell^2.$$

In particular $\limsup Q_\lambda(N)/N < \infty$ *if and only if* $L(x) \ll \sqrt{x}$.

Proof. Write C for the middle term. For the lower bound take $x_k \to \infty$ with $|L(x_k)|/\sqrt{x_k} \to \ell$, and put $N_k = \lfloor x_k \rfloor$. By (6), $Q_\lambda(N_k)/N_k \ge L(N_k)^2/(2N_k) \ge L(x_k)^2/(2x_k)$, which tends to $\ell^2/2$.

For the upper bound assume $\ell < \infty$, and fix $\varepsilon > 0$. There is J with $|L(j)| \le (\ell+\varepsilon)\sqrt{j}$ for $j \ge J$, and $B_0 = \max_{j<J} |L(j)|$ is finite. Let $N \ge J$. By the definition (1) with $H = N$, equivalently (5),

$$N\,Q_\lambda(N) = \sum_{j=0}^{N-1} \Big(L(j)^2 + \big(L(N) - L(j)\big)^2 \Big).$$

For $J \le j < N$ the summand is at most $(\ell+\varepsilon)^2\big(j + (\sqrt{N} + \sqrt{j})^2\big) = (\ell+\varepsilon)^2\big(2j + N + 2\sqrt{Nj}\big)$. Since $\sum_{j<N} \sqrt{j} \le \int_0^N \sqrt{t}\,\mathrm{d}t = \frac{2}{3}N^{3/2}$,

$$\sum_{j=0}^{N-1} \big(2j + N + 2\sqrt{Nj}\big) \le N(N-1) + N^2 + \tfrac{4}{3}N^2 \le \tfrac{10}{3}N^2.$$

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For $j < J$ the summand is at most $B_0^2 + 2L(N)^2 + 2B_0^2 \le 3B_0^2 + 2(\ell+\varepsilon)^2 N$. Hence

$$\frac{Q_\lambda(N)}{N} \le \tfrac{10}{3}(\ell+\varepsilon)^2 + \frac{J\bigl(3B_0^2 + 2(\ell+\varepsilon)^2 N\bigr)}{N^2},$$

and the last term tends to 0. Let $N \to \infty$ and then $\varepsilon \to 0$. The final assertion follows, because $L(x) \ll \sqrt{x}$ is equivalent to $\ell < \infty$. □

Whether $L(x) \ll \sqrt{x}$ is not known, even under the Riemann hypothesis. Ingham showed that it fails under an independence hypothesis on the ordinates.

Theorem 11 (Ingham [39, Theorem A and §3]). *Let $0 < \gamma_1 < \gamma_2 < \cdots$ be the distinct ordinates of the zeros of ζ above the real axis. Suppose that the Riemann hypothesis holds, that every zero is simple, and that for some m the numbers γ_n with $n > m$ are linearly independent over $\mathbb{Q}$. Then*

$$\limsup_{x\to\infty} \frac{L(x)}{\sqrt{x}} = +\infty, \qquad \liminf_{x\to\infty} \frac{L(x)}{\sqrt{x}} = -\infty.$$

If the Riemann hypothesis fails, or some zero is multiple, the same conclusion holds without any independence hypothesis.

Ingham states the hypothesis as "no relation $\sum c_n\gamma_n = 0$ with integers c_n not all 0, or only a finite number of such relations". Since a relation has infinitely many integer multiples, "finitely many" must be read as "the relations span a finite-dimensional space"; this holds exactly when some tail $\{\gamma_n : n > m\}$ is independent, which is the form used in his proof (Kronecker's theorem applied to the γ_n with $\gamma_m < \gamma_n < T$). The proof rests on two results from the same paper: a Tauberian comparison with the Fejér-weighted trigonometric polynomials [39, Theorem 1], and the divergence

$$\sum_{\gamma>0} |A_\gamma| = \infty \qquad \text{(Riemann hypothesis, simple zeros)}, \tag{35}$$

which is [39, Theorem 2(ii)]: the residue of $\zeta(1+2s)/((\frac{1}{2}+s)\zeta(\frac{1}{2}+s))$ at $s = i\gamma$ is A_γ.

Corollary 11 (Bounded energy forces relations). *Suppose that $\limsup_{N\to\infty} Q_\lambda(N)/N < \infty$. Then the Riemann hypothesis holds, every zero of ζ is simple, and for every m the ordinates γ_n with $n > m$ satisfy a nontrivial linear relation with integer coefficients; equivalently, the integer relations among the positive ordinates span an infinite-dimensional space. In particular, if the positive ordinates are linearly independent over $\mathbb{Q}$, then* $\limsup Q_\lambda(N)/N = \infty$.

Proof. Write C for the upper limit. By (6), $L(N)^2 \le 2Q_\lambda(N) \le 2(C+1)N$ for large N, and $L(x) = L(\lfloor x \rfloor)$. So $L(x)/\sqrt{x}$ is bounded, and Theorem 11 excludes each alternative. □

The positive weights of Proposition 6 have a continuous form, which needs no condition on integer combinations of ordinates.

Theorem 12 (Continuous positive-weight bound). *Let $m > 0$, and let k be a continuous even function on $\mathbb{R}$ with support in $[-m, m]$ and $k(0) = 1$, such that*

$$P(t) = \frac{1}{2\pi}\int_{-m}^{m} k(\omega) e^{i\omega t}\,\mathrm{d}\omega$$

satisfies $P \ge 0$ and $P \in L^1(\mathbb{R})$. Let Ω be the finite set of $\omega \in (-m, m)$ at which M has a pole at $\frac{1}{2} + i\omega$, and put

$$Z(y) = \sum_{\omega\in\Omega} k(\omega) R(\omega) e^{i\omega y}\, w_{1/2+i\omega} \in \mathcal{H} \qquad (y \in \mathbb{R}).$$

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Then $\limsup_{N\to\infty} Q_\lambda(N)/N \geq \|Z(y_0)\|^2$ *for every* $y_0 \in \mathbb{R}$. *Explicitly, with* $a_\omega = k(\omega)R(\omega)e^{i\omega y_0}$,

$$\|Z(y_0)\|^2 = \sum_{\omega,\omega'\in\Omega} a_\omega \overline{a_{\omega'}}\,\langle w_{1/2+i\omega}, w_{1/2+i\omega'}\rangle, \qquad \langle w_s, w_z\rangle = 1 - \frac{1}{1+s} - \frac{1}{1+\bar z} + \frac{2}{1+s+\bar z}.$$

Proof. Let $C = \limsup Q_\lambda(N)/N$ and assume $C < \infty$. By Proposition 5 with $\beta = \frac12$ and $r = 1$, M has no pole in $\operatorname{Re} s > \frac12$ and only simple poles on the line $\operatorname{Re} s = \frac12$; by Theorem 6 the Riemann hypothesis holds. Put $F(t) = e^{-t/2}U_{e^t}$ for $t \geq 0$ and $F(t) = 0$ for $t < 0$. By (17), $\|F(t)\|^2 = \|U_X\|^2/X \leq (N+2)Q_\lambda(N)/N^2$ for $N \leq X = e^t < N+1$, so $\limsup_{t\to\infty}\|F(t)\|^2 \leq C$. Since F is also bounded on bounded intervals, $\|F\| \leq B_0$ on $\mathbb{R}$ for some B_0. By Lemma 2(c) and analytic continuation, $\Phi(s) = \int_0^\infty F(t)e^{-(s-1/2)t}\,dt = M(s)w_s$ for $\operatorname{Re} s > \frac12$.

For $\delta > 0$ the function $e^{-\delta t}F(t)$ is integrable, and Fubini's theorem gives

$$\int_{\mathbb{R}} P(y-t)e^{-\delta t}F(t)\,dt = \frac{1}{2\pi}\int_{-m}^{m} k(\omega)e^{i\omega y}\,\Phi(\tfrac12+\delta+i\omega)\,d\omega.$$

The function $M(s)w_s$ is meromorphic on $\operatorname{Re} s > -\frac12$, with finitely many poles, all simple and on the line, in a neighbourhood of the rectangle $[\frac12, \frac12+\delta_0]\times[-m-1, m+1]$. Hence

$$\Phi(s) = \sum_{|\gamma|\leq m+1} \frac{R(\gamma)w_{1/2+i\gamma}}{s-\frac12-i\gamma} + \Psi(s),$$

where the sum runs over the poles $\frac12+i\gamma$ and Ψ is continuous on the rectangle. Using $1/(\delta+i(\omega-\gamma)) = \int_0^\infty e^{-(\delta+i(\omega-\gamma))u}\,du$, the contribution of the pole at $\frac12+i\gamma$ is

$$R(\gamma)w_{1/2+i\gamma}\int_0^\infty e^{-\delta u}e^{i\gamma u}P(y-u)\,du \xrightarrow[\delta\to0]{} R(\gamma)w_{1/2+i\gamma}\,e^{i\gamma y}\Big(k(\gamma) - \int_y^\infty e^{-i\gamma v}P(v)\,dv\Big),$$

by dominated convergence and the inversion formula $\int_{\mathbb{R}} P(v)e^{-i\gamma v}\,dv = k(\gamma)$. Letting $\delta\to0$ on both sides, with dominated convergence on the left ($\|F\|\leq B_0$, $P\in L^1$) and uniform continuity of Ψ on the right,

$$\begin{aligned}(P*F)(y) = Z(y) &+ \frac{1}{2\pi}\int_{-m}^{m} k(\omega)e^{i\omega y}\Psi(\tfrac12+i\omega)\,d\omega \\ &- \sum_{|\gamma|\leq m+1} R(\gamma)w_{1/2+i\gamma}e^{i\gamma y}\int_y^\infty e^{-i\gamma v}P(v)\,dv,\end{aligned}$$

since $k(\gamma) = 0$ for $|\gamma| \geq m$. As $y\to\infty$ the middle term tends to 0 by the Riemann–Lebesgue lemma and the last sum tends to 0 because $P\in L^1$. So $\|Z(y) - (P*F)(y)\| \to 0$.

Fix $\varepsilon > 0$ and t_ε with $\|F(t)\| \leq \sqrt{C}+\varepsilon$ for $t \geq t_\varepsilon$. Since $P \geq 0$ and $\int P = k(0) = 1$,

$$\|(P*F)(y)\| \leq \int_{\mathbb{R}} P(y-t)\|F(t)\|\,dt \leq \sqrt{C}+\varepsilon + B_0\int_{y-t_\varepsilon}^\infty P(v)\,dv,$$

and the last term tends to 0. Hence $\limsup_{y\to\infty}\|Z(y)\| \leq \sqrt{C}$.

Finally Z is a finite exponential sum. Given $\eta > 0$ and $Y > 0$, Dirichlet's theorem on simultaneous approximation, applied to the numbers $Y\omega/2\pi$ ($\omega\in\Omega$), gives a positive integer q with $|e^{i\omega qY} - 1| < \eta$ for all $\omega\in\Omega$. With $h = qY \geq Y$, $\|Z(y_0+h) - Z(y_0)\| \leq \eta\sum_\omega |a_\omega|\,\|w_{1/2+i\omega}\|$. So $\|Z(y_0)\| \leq \limsup_{y\to\infty}\|Z(y)\| \leq \sqrt{C}$. The Gram formula is the integral $\int_0^1 \big(u^s\overline{u^z} + (1-u^s)(1-\overline{u^z})\big)\,du$. □

Remark 5. *Theorem 12 is a vector form of Ingham's Tauberian theorem [39] [Theorem 1], and it contains Proposition 6. It needs no hypothesis on integer combinations of ordinates, unlike Corollary 10. Pairing* $Z(y)$ *with* $e_2 = (0,1)$, *using* $\langle w_{1/2+i\omega}, e_2\rangle = \frac{1/2+i\omega}{3/2+i\omega}$, *recovers the scalar bound. A convenient kernel is that of Jurkat and Peyerimhoff,* $k(\omega) = (1-\tau)\cos\pi\tau + \frac{1}{\pi}\sin\pi\tau$ *for* $\tau = |\omega|/m \leq 1$: *it is the normalized*

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autocorrelation of $\cos(\pi\omega/m)\mathbf{1}_{|\omega|\le m/2}$*, so* P *is a positive multiple of a squared modulus and* $P(t) = O(t^{-4})$*. With* $m = 1400$ *(the* 983 *zeros with* $0 < \gamma < 1400$*) and* $y_0 = 561488971144772334329867737844 7$*, a point at which the phases* γy_0 *of the first* 110 *zeros were aligned by lattice reduction, the right side has the computed value* $\|Z(y_0)\|^2 = 1.98545$*. This value is a computation, like those of Table 4; it gives* $B_{1/2}$ *infinite independently of [50]. Averaging instead of aligning gives more (Corollaries 12 and 13).*

Theorem 12 bounds $\|Z(y)\|$ at every single point y. Averaging Z against a nonnegative weight that sees only finitely many ordinates gives a larger and fully certified constant, provided those ordinates satisfy no short integer relation.

Definition 1. *Let* $T > 0$ *and let* $q \ge 1$ *be an integer. A finite set* Γ_0 *of ordinates* $\gamma \in (0, T)$ *of zeros of* ζ *on the critical line is* q-independent below T *if*

(i) $\sum_{\gamma\in\Gamma_0} c_\gamma \gamma = 0$ *with integers* $|c_\gamma| \le q$ *implies* $c_\gamma = 0$ *for all* γ*, and*

(ii) $\sum_{\gamma\in\Gamma_0} c_\gamma \gamma = \gamma^*$ *with integers* $|c_\gamma| \le q$*, where* $\gamma^* \in (0, T)$ *is the ordinate of a zero on the critical line, implies* $\gamma^* \in \Gamma_0$*,* $c_{\gamma^*} = 1$ *and* $c_\gamma = 0$ *for* $\gamma \ne \gamma^*$*.*

This is the notion of Anderson and Stark [2] with all their bounds N_γ equal to q; see [22] [Definition 1] and [51] [(4.6)–(4.7)]. If Γ_0 is q-independent below T', then every subset of $\Gamma_0 \cap (0, T)$ is q-independent below every $T \le T'$.

Theorem 13 (Weakly independent ordinates)**.** *Let* k *be as in Theorem 12 with* $m = T$*, and let* Γ_0 *be* q*-independent below* T*. With* a *and* A_γ *as in Corollary 10,*

$$\limsup_{N\to\infty} \frac{Q_\lambda(N)}{N} \;\ge\; \Big\| a\, w_{1/2} + \frac{2q}{q+1} \sum_{\gamma\in\Gamma_0} k(\gamma)\,|A_\gamma|\,\operatorname{Re} w_{1/2+i\gamma} \Big\|^2. \tag{36}$$

Proof. Let C be the upper limit and assume $C < \infty$. By Theorem 12, $\|Z(y)\| \le \sqrt{C}$ for every real y. By its proof the poles of M on the line are $\frac12$ and the zeros, all simple, so $\Omega \subseteq \{0\} \cup \{\pm\gamma^* : 0 < \gamma^* < T\}$, where γ^* runs over ordinates of zeros, with $R(0) = -a$, $R(\gamma) = A_\gamma$ and $R(-\gamma) = \overline{A_\gamma}$.

Choose θ_γ with $e^{i\theta_\gamma} A_\gamma = -|A_\gamma|$, put

$$F_q(x) = \sum_{|n|\le q} \Big(1 - \frac{|n|}{q+1}\Big) e^{inx} = \frac{1}{q+1} \Big| \sum_{\ell=0}^{q} e^{i\ell x} \Big|^2 \ge 0,$$

and let $W(y) = \prod_{\gamma\in\Gamma_0} F_q(\gamma y - \theta_\gamma)$. Then

$$W(y) = \sum_c w_c\, e^{-i\sum_\gamma c_\gamma\theta_\gamma}\, e^{i(\sum_\gamma c_\gamma\gamma)y}, \qquad w_c = \prod_{\gamma\in\Gamma_0} \Big(1 - \frac{|c_\gamma|}{q+1}\Big),$$

where c runs over $\{-q, \dots, q\}^{\Gamma_0}$. For a finite exponential sum f with values in $\mathcal{H}$ let $\mathfrak{M}[f] = \lim_{Y\to\infty} Y^{-1} \int_0^Y f(y)\,\mathrm{d}y$, which is the coefficient of the frequency 0. In WZ the term $\omega \in \Omega$ of Z reaches the frequency 0 through those c with $\sum_\gamma c_\gamma\gamma = -\omega$. For $\omega = 0$, condition (i) leaves only $c = 0$; in particular $\mathfrak{M}[W] = 1$. For $\omega = \pm\gamma^*$, condition (ii) applied to $\mp c$ leaves only $\gamma^* \in \Gamma_0$ and $c = \mp e_{\gamma^*}$, with $w_c = \frac{q}{q+1}$. Hence, since $k(0) = 1$,

$$\mathfrak{M}[WZ] = -a\, w_{1/2} + \frac{q}{q+1} \sum_{\gamma\in\Gamma_0} k(\gamma) \big(e^{i\theta_\gamma} A_\gamma w_\rho + e^{-i\theta_\gamma} \overline{A_\gamma}\, w_{\bar\rho} \big), \qquad \rho = \tfrac12 + i\gamma,$$

which is minus the vector in (36). Since $W \ge 0$,

$$\|\mathfrak{M}[WZ]\| \le \limsup_{Y\to\infty} \frac{1}{Y} \int_0^Y W(y)\|Z(y)\|\,\mathrm{d}y \le \sqrt{C}\,\mathfrak{M}[W] = \sqrt{C}. \quad \square$$

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Corollary 12 (The oscillation constant)**.**

$$\limsup_{N\to\infty} \frac{Q_\lambda(N)}{N} > 4.0131.$$

Proof. Mossinghoff and Trudgian [50] (see [51] [§§4, 7.1]) take the kernel $k_{T'}$ of Jurkat and Peyerimhoff with $\gamma_{3700} < T' < \gamma_{3701}$, let Γ_0 be the set of the 250 ordinates among $\gamma_1, \dots, \gamma_{3700}$ with the largest values of $k_{T'}(\gamma)|A_\gamma|$, and prove by lattice reduction that Γ_0 is 3100-independent below T'; this is the input of Theorem 10. Since $k_T(\gamma)$ increases with T, and the smallest selected value $6.67641 \cdot 10^{-4}$ at $T = \gamma_{3700}$ exceeds the largest unselected value $6.67581 \cdot 10^{-4}$ at $T = \gamma_{3701}$, the set does not depend on T'; it reproduces their constant $-2.372341686\dots$. Apply Theorem 13 with $T = \gamma_{3700}$, $k = k_T$ and $q = 3100$. Evaluated in ball arithmetic from (34), the right side of (36) is $4.0131076986\dots$. □

The scalar transfer of Theorem 10 keeps only the projection on e_2, whose square is 3.658 for this set; the rest of the gain comes from the orthogonal part of the vector. Even the inadmissible choice $k \equiv 1$ gives at most 4.158 for this Γ_0, so larger constants need larger certificates.

Corollary 13 (A certificate independent of [50])**.**

$$\limsup_{N\to\infty} \frac{Q_\lambda(N)}{N} > 4.6242, \qquad \liminf_{x\to\infty} \frac{L(x)}{\sqrt{x}} < -2.5257, \qquad \limsup_{x\to\infty} \frac{L(x)}{\sqrt{x}} > 1.1562.$$

Proof. The zeros $\gamma_1, \dots, \gamma_{8001}$ and the residues A_γ are enclosed in balls by Arb [42], which certifies the count of zeros by Turing's method. Let $T = \gamma_{8001} - 10^{-6}$, so that $\gamma_{8000} < T < \gamma_{8001}$, let k_T be the kernel of Jurkat and Peyerimhoff, and let Γ_0 be the 350 ordinates among $\gamma_1, \dots, \gamma_{8000}$ with the largest values of $k_{\gamma_{8001}}(\gamma)|A_\gamma|$; the largest index is 2218. Put $q = 3160$ and $b = 10400$, and let x_γ be the integer nearest to $2^b\gamma$; the enclosures give $|x_\gamma - 2^b\gamma| \le \delta := \frac{1}{2} + 10^{-12}$ for $\gamma \le \gamma_{8000}$. Let $\Lambda_0 \subset \mathbb{Z}^{351}$ be the lattice spanned by the rows (e_γ, x_γ), $\gamma \in \Gamma_0$. Lattice reduction [45] gave a matrix $B = [U \mid Ux]$ with $\det U = \pm 1$, checked exactly, so B is a basis of Λ_0 whatever the reduction did.

A relation $\sum c_\gamma \gamma = 0$ with $|c_\gamma| \le q+1$ gives a nonzero vector $(c, \sum c_\gamma x_\gamma) \in \Lambda_0$ of squared length at most $350(q+1)^2(1+350\delta^2) < 3.0951 \cdot 10^{11}$; this also covers (ii) when $\gamma^* \in \Gamma_0$, with $c - e_{\gamma^*}$ in place of c. The Gram–Schmidt norms of B, computed exactly from the leading principal minors of its Gram matrix, satisfy $\min_i \|b_i^*\|^2 > 3.0955 \cdot 10^{11}$, so no such relation exists. A relation $\sum c_\gamma \gamma = \gamma^*$ with $|c_\gamma| \le q$ and $\gamma^* \in \{\gamma_1, \dots, \gamma_{8000}\} \setminus \Gamma_0$ gives $u \in \Lambda_0$ with $\|u - x_{\gamma^*} e_{351}\|^2 \le 350q^2 + (\delta(350q+1))^2 < 3.0931 \cdot 10^{11}$. For each of the 7650 such γ^*, an exhaustive enumeration [34] of the points of Λ_0 within this distance of the orthogonal projection of $x_{\gamma^*} e_{351}$ onto $\mathbb{R}\Lambda_0$, with the Gram–Schmidt data computed exactly over $\mathbb{Q}$ and evaluated in ball arithmetic, finds none. Hence Γ_0 is 3160-independent below T, and also in the sense of [51, (4.6)–(4.7)].

Theorem 13 with $k = k_T$ gives the value $4.6242266553\dots$ for the right side of (36). For L, if the Riemann hypothesis fails or a zero is multiple, then $L(x)/\sqrt{x}$ is unbounded in both directions, as in the proof of Theorem 10. Otherwise the theorem of Anderson and Stark [2], in the form [51, Theorem 4.1] that allows the kernel k_T, gives

$$\liminf_{x\to\infty} \frac{L(x)}{\sqrt{x}} \le \frac{1}{\zeta(\frac{1}{2})} - \frac{2q}{q+1}\Sigma_0, \qquad \limsup_{x\to\infty} \frac{L(x)}{\sqrt{x}} \ge \frac{1}{\zeta(\frac{1}{2})} + \frac{2q}{q+1}\Sigma_0,$$

where $\Sigma_0 = \sum_{\gamma\in\Gamma_0} k_T(\gamma)|A_\gamma| = 0.9207982896\dots$. The two bounds are $-2.5257792\dots$ and $1.1562487\dots$. □

The bounds for L improve the values -2.3723 and 1.0028 of [50]. Transferred as in Theorem 10, the first gives only $2.5257^2/2 > 3.1896$; the vector bound keeps the part of the averaged vector orthogonal to e_2. For this Γ_0 even the inadmissible choice $k \equiv 1$, $q = \infty$ gives at most 4.718, so the constant grows

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only with larger certificates; the cost of the reduction grows roughly like $|\Gamma_0|^5$, while the enumeration step makes the number of target ordinates nearly free.

Corollary 14 (Relations with small coefficients)**.** *If the distinct positive ordinates of the zeros of ζ on the critical line satisfy no relation $\sum c_\gamma \gamma = 0$ with integers $c_\gamma \in \{-2,\dots,2\}$, not all zero, then $\limsup_{N\to\infty} Q_\lambda(N)/N = \infty$, and $L(x)/\sqrt{x}$ is unbounded.*

Proof. Assume that the upper limit C is finite. Fix J and $T > \gamma_J$, and let $\Gamma_0 = \{\gamma_1,\dots,\gamma_J\}$. This set is 1-independent below T. Indeed, a relation as in (i) has coefficients in $\{-1,0,1\}$. A relation as in (ii) gives $\sum_\gamma c_\gamma\gamma - \gamma^* = 0$: if $\gamma^* \notin \Gamma_0$ this is a nontrivial relation with coefficients in $\{-1,0,1\}$, and if $\gamma^* \in \Gamma_0$ its coefficients lie in $\{-2,\dots,1\}$, so they all vanish and $c = e_{\gamma^*}$. Pair the vector of Theorem 13, with $q = 1$, with $e_2 = (0,1)$: since $\langle w_{1/2}, e_2\rangle = \frac{1}{3}$ and $\langle \operatorname{Re} w_\rho, e_2\rangle = 1 - \operatorname{Re}\frac{1}{1+\rho} = 1 - \frac{3/2}{9/4+\gamma^2} > 0.99$,

$$C \;\ge\; \Big(\frac{a}{3} + 0.99\sum_{j\le J} k_T(\gamma_j)|A_{\gamma_j}|\Big)^2.$$

Let $T\to\infty$, so that $k_T(\gamma_j)\to 1$, and then $J\to\infty$; by (35) the right side is unbounded. The last assertion follows from Proposition 7. □

Corollary 14 sharpens Corollary 11: bounded energy forces relations among the ordinates with all coefficients in $\{-2,\dots,2\}$, and, by the same argument, inside every finite set of ordinates whose residue mass $0.99\sum k_T(\gamma)|A_\gamma|$ exceeds $\sqrt{C} - a/3$.

Upper limits are not the only information available. On a logarithmic scale the mean of $Q_\lambda(N)/N$ is exactly the full pole sum of (30).

Theorem 14 (Logarithmic mean)**.** *Let*

$$S = \frac{2}{3\zeta(\frac12)^2} + \sum_\gamma K(\tfrac12 + i\gamma)\,|A_\gamma|^2,$$

where γ runs over the ordinates of both signs of the zeros on the critical line, with $S = \infty$ if the Riemann hypothesis fails or some zero is multiple.

(a) Unconditionally, $\displaystyle\limsup_{X\to\infty} \frac{1}{\log X}\sum_{N\le X}\frac{Q_\lambda(N)}{N^2} \ge S.$

(b) Assume the Riemann hypothesis and that every zero is simple. For $\kappa > 0$ let $\mathcal{P}_\kappa$ be the set of ordered pairs (γ,γ') of distinct ordinates, of either sign, with $|\gamma-\gamma'| \le \kappa/\log(\max(|\gamma|,|\gamma'|)+2)$. If, for some $\kappa > \pi/2$,

$$\sum_\gamma |A_\gamma|^2 < \infty \qquad \text{and} \qquad \sum_{(\gamma,\gamma')\in\mathcal{P}_\kappa} |A_\gamma|\,|A_{\gamma'}| < \infty, \tag{37}$$

then the limit exists and equals S.

By (19), the first condition in (37) is equivalent to $S < \infty$. The mean spacing of the ordinates near height t is $2\pi/\log t$, so the second condition concerns pairs of zeros closer than a little more than a quarter of the mean spacing. Two lemmas are used in the proof of (b).

Lemma 3 (Mean value of an exponential sum)**.** *Let Γ be a finite set of real numbers, $c_\gamma \in \mathbb{C}$ for $\gamma\in\Gamma$, and $U > 0$. Then*

$$\int_{-U}^{U}\Big|\sum_{\gamma\in\Gamma} c_\gamma e^{i\gamma y}\Big|^2\,dy \le \frac{\pi^2}{2}\,U \sum_{\substack{\gamma,\gamma'\in\Gamma\\ |\gamma-\gamma'|<\pi/U}} |c_\gamma|\,|c_{\gamma'}|.$$

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Proof. This is Gallagher's argument [36, Lemma 1]. Let $f(y)=\sum_\gamma c_\gamma e^{i\gamma y}$, $h=\pi/U$, and $\Phi(t)=\sum_\gamma c_\gamma \mathbf{1}_{[\gamma-h,\gamma)}(t)$. Then

$$\widehat{\Phi}(y):=\int_{-\infty}^{\infty}\Phi(t)e^{-iyt}\,\mathrm{d}t=\frac{e^{iyh}-1}{iy}\,f(-y),\qquad |e^{iyh}-1|=2|\sin(yh/2)|.$$

For $|y|\le U$ we have $|yh/2|\le \pi/2$, hence $|\sin(yh/2)|\ge |yh|/\pi$ and $|\widehat{\Phi}(y)|\ge (2h/\pi)|f(-y)|$. By the Plancherel theorem,

$$\int_{-U}^{U}|f|^2\,\mathrm{d}y\le\frac{\pi^2}{4h^2}\int_{-\infty}^{\infty}|\widehat{\Phi}|^2\,\mathrm{d}y=\frac{\pi^3}{2h^2}\int_{-\infty}^{\infty}|\Phi|^2\,\mathrm{d}t,$$

and $\int|\Phi|^2=\sum_{\gamma,\gamma'}c_\gamma\overline{c_{\gamma'}}\,(h-|\gamma-\gamma'|)^+\le h\sum_{|\gamma-\gamma'|<h}|c_\gamma||c_{\gamma'}|$. □

Lemma 4 (Explicit formula). *Assume the Riemann hypothesis and that every zero is simple, and let $0<\varepsilon<\frac18$. There are numbers $T_n\in[n,n+1]$, $n\ge2$, such that for all $n\ge2$ and all $x\ge1$ that are not integers,*

$$L(x)=\frac{x^{1/2}}{\zeta(\frac12)}+\sum_{|\gamma|<T_n}A_\gamma x^\rho+O_\varepsilon\big(x^{1/4}+x\log(x+2)\,T_n^{2\varepsilon-1}\big).$$

Proof. By [57] [Theorem 14.16] there are $T_n\in[n,n+1]$ with $|\zeta(\sigma\pm iT_n)|^{-1}\ll T_n^{\varepsilon}$ for $\frac12\le\sigma\le2$; in particular no zero has ordinate $\pm T_n$. For $\frac14\le\sigma<\frac12$, the functional equation $\zeta(s)=\chi(s)\zeta(1-s)$ with $|\chi(\sigma+it)|^{-1}\ll|t|^{\sigma-1/2}\le1$ [57, (4.12.3)] and $|\zeta(1-\sigma\mp iT_n)|=|\zeta(1-\sigma\pm iT_n)|$ extend the bound to $\frac14\le\sigma\le2$. Under the Riemann hypothesis $\zeta(\sigma+it)\ll(|t|+2)^{\varepsilon}$ uniformly for $\sigma\ge\frac12$ [57] [Sections 14.2 and 14.14].

Apply (A4) with $a_n=\lambda(n)$, $c=1+1/\log(x+2)$ and $T=T_n$. The at most two n with $|x-n|<1$ contribute $O(1)$ to the error, and the other n contribute $O(x\log(x+2)/T_n)$, as in the proof of Proposition A1. Move the segment to the line $\operatorname{Re}s=\frac14$. Inside the rectangle the integrand $\zeta(2s)x^s/(s\zeta(s))$ has the simple pole $s=\frac12$ with residue $x^{1/2}/\zeta(\frac12)$ and the simple poles ρ, $|\gamma|<T_n$, with residues $A_\gamma x^\rho$; it is regular at $s=1$. On the horizontal sides $|s|\ge T_n$, $|\zeta(2s)|\ll T_n^{\varepsilon}$ and $|1/\zeta(s)|\ll T_n^{\varepsilon}$, so they contribute $O(x^cT_n^{2\varepsilon-1})=O(xT_n^{2\varepsilon-1})$. On $\operatorname{Re}s=\frac14$, $|\zeta(\frac12+2it)|\ll(|t|+2)^{\varepsilon}$ and, by the functional equation and Lemma A2 at $\operatorname{Re}s=\frac34$, $|1/\zeta(\frac14+it)|\ll(|t|+2)^{\varepsilon-1/4}$; since $|s|\ge\frac14$ and $2\varepsilon<\frac14$, this side contributes $O(x^{1/4})$. □

Proof of Theorem 14. Put $G(y)=e^{-y}\|U_{e^y}\|^2$ and $E(y)=e^{-y/2}L(e^y)$ for $y\ge0$, with $E(y)=0$ for $y<0$. The substitution $u=e^{-t}$ in the definition of $\|U_X\|^2$ gives, exactly,

$$G(y)=\int_0^{\infty}\Big(e^{-t}E(y-t)^2+\big(E(y)-e^{-t/2}E(y-t)\big)^2\Big)e^{-t}\,\mathrm{d}t. \tag{38}$$

By (17), $\int_N^{N+1}\|U_X\|^2X^{-2}\,\mathrm{d}X=Q_\lambda(N)N^{-2}\big(1+O(1/N)\big)$, and $\int_1^X\|U_v\|^2v^{-2}\,\mathrm{d}v=\int_0^{\log X}G$. Under the Riemann hypothesis $Q_\lambda(N)\ll N^{1+\varepsilon}$ (Theorem 6), so

$$\sum_{N\le X}\frac{Q_\lambda(N)}{N^2}=\int_0^{\log X}G(y)\,\mathrm{d}y+O(1). \tag{39}$$

(a) If $\Theta>\frac12$, choose σ with $\frac12<\sigma<\Theta$; by Theorem 8, $Q_\lambda(N)/N^2>N^{2\sigma-2}$ for all large N, so the sum in (a) grows like a positive power of X and the upper limit is infinite. Assume $\Theta=\frac12$. For $\sigma=\frac12+\delta$, Theorem 7 in the variable $y=\log X$ reads

$$2\delta\int_0^{\infty}G(y)e^{-2\delta y}\,\mathrm{d}y=\frac{\delta}{\pi}\int_{-\infty}^{\infty}\big|M(\tfrac12+\delta+i\tau)\big|^2K(\tfrac12+\delta+i\tau)\,\mathrm{d}\tau.$$

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Let $\frac12+i\gamma$ be a pole of M of order r, and $\eta>0$ small. If $r=1$, then $M(\frac12+\delta+i\tau)=R(\gamma)/(\delta+ix)+O(1)$ for $x=\tau-\gamma$, $|x|<\eta$, and

$$\frac{\delta}{\pi}\int_{-\eta}^{\eta}\frac{|R(\gamma)|^2}{\delta^2+x^2}\,\mathrm{d}x\to|R(\gamma)|^2,\qquad \frac{\delta}{\pi}\int_{-\eta}^{\eta}\frac{\mathrm{d}x}{\sqrt{\delta^2+x^2}}=O\big(\delta\log(1/\delta)\big).$$

Since K is continuous and positive, the contribution of $|\tau-\gamma|<\eta$ has $\liminf_{\delta\to0}$ at least $|R(\gamma)|^2K(\frac12+i\gamma)$. If $r\ge2$, the same contribution is $\gg\delta^{2-2r}\to\infty$. Since $K>0$ everywhere, summing over any finite set of disjoint intervals shows that $\liminf_{\delta\to0}2\delta\int_0^\infty Ge^{-2\delta y}\,\mathrm{d}y$ is at least every partial sum of S, hence at least S, with $R(0)=1/\zeta(\frac12)$, $K(\frac12)=\frac23$ and $R(\gamma)=A_\gamma$. Now put $H(Y)=\int_0^Y G$. If $\limsup H(Y)/Y=c<\infty$, then for each $\varepsilon>0$ there is B_ε with $H(y)\le(c+\varepsilon)y+B_\varepsilon$, and

$$s\int_0^\infty e^{-sy}G(y)\,\mathrm{d}y=s^2\int_0^\infty e^{-sy}H(y)\,\mathrm{d}y\le c+\varepsilon+sB_\varepsilon,$$

so, with $s=2\delta\to0$, $c\ge S$. Together with (39) this proves (a).

(b) Put $P_T(y)=1/\zeta(\frac12)+\sum_{|\gamma|<T}A_\gamma e^{i\gamma y}$; since $A_{-\gamma}=\overline{A_\gamma}$, it is a real trigonometric polynomial, and its frequencies are distinct because the zeros are simple. We first show

$$\limsup_{Y\to\infty}\frac1Y\int_0^Y|E(y)-P_{T_0}(y)|^2\,\mathrm{d}y\le\pi^2\sum_{|\gamma|\ge T_0}|A_\gamma|^2\qquad(T_0\ge1).\tag{40}$$

Fix c with $\frac12<c<\kappa/\pi$, and ε with $0<\varepsilon<\frac18$ and $2c(1-2\varepsilon)>1$. For $Y\ge1$ let $T(Y)=T_n$ with $n=\lceil e^{cY}\rceil$, so that $e^{cY}\le T(Y)\le e^{cY}+2$. By Lemma 4, for almost every $y\in[0,Y]$,

$$E(y)=P_{T(Y)}(y)+R_Y(y),\qquad R_Y(y)\ll e^{-y/4}+(1+Y)\,e^{y/2}e^{-c(1-2\varepsilon)Y},$$

so $\int_0^Y R_Y^2\ll1+(1+Y)^2e^{(1-2c(1-2\varepsilon))Y}=O(1)$. For $T(Y)>T_0$, Lemma 3 with $U=Y$ gives

$$\int_0^Y|E-P_{T_0}|^2\le2\int_{-Y}^{Y}\Big|\sum_{T_0\le|\gamma|<T(Y)}A_\gamma e^{i\gamma y}\Big|^2\,\mathrm{d}y+O(1)\le\pi^2Y\big(D_Y+O_Y\big)+O(1),$$

where $D_Y=\sum_{T_0\le|\gamma|<T(Y)}|A_\gamma|^2$ and O_Y is the sum of $|A_\gamma A_{\gamma'}|$ over the pairs of distinct ordinates with $|\gamma|,|\gamma'|<T(Y)$ and $|\gamma-\gamma'|<\pi/Y$. Let $m=\max(|\gamma|,|\gamma'|)$ for such a pair. Then $m+2<e^{cY}+4\le5e^{cY}$, so $cY>\log(m+2)-\log5$. Choose m_0 with $(\kappa-\pi c)\log(m_0+2)\ge\kappa\log5$. If $m\ge m_0$, then

$$|\gamma-\gamma'|<\frac{\pi}{Y}<\frac{\pi c}{\log(m+2)-\log5}\le\frac{\kappa}{\log(m+2)},$$

so $(\gamma,\gamma')\in\mathcal{P}_\kappa$. The finitely many pairs of distinct ordinates with $m<m_0$ have $|\gamma-\gamma'|\ge d_0>0$, and they do not occur once $Y>\pi/d_0$. Hence, for large Y,

$$O_Y\le\sum_{(\gamma,\gamma')\in\mathcal{P}_\kappa}|A_\gamma A_{\gamma'}|\,\mathbf{1}\big[|\gamma-\gamma'|<\pi/Y\big],$$

which tends to 0 as $Y\to\infty$ by dominated convergence and (37), since every term tends to 0. As $D_Y\le\sum_{|\gamma|\ge T_0}|A_\gamma|^2$, this proves (40).

For a locally bounded function D vanishing on $(-\infty,0)$, let G_D be defined by (38) with D in place of E. Then $G_D(y)$ is the squared norm of a linear image of D, and $(u-v)^2\le2u^2+2v^2$ gives $G_D(y)\le2D(y)^2+3\int_0^\infty e^{-2t}D(y-t)^2\,\mathrm{d}t$. Hence

$$\frac1Y\int_0^Y G_D\le\frac72\cdot\frac1Y\int_0^Y D^2.$$

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With $\widetilde{P}_T = P_T \mathbf{1}_{[0,\infty)}$, the inequality $|G_E - G_{\widetilde{P}_T}| \le G_{E-\widetilde{P}_T} + 2\sqrt{G_{E-\widetilde{P}_T} G_{\widetilde{P}_T}}$, the Cauchy–Schwarz inequality and (40) show that the upper and lower limits of the means of G_E and of $G_{\widetilde{P}_T}$ differ by $O\big(\eta(T) + \sqrt{\eta(T)S}\big)$, where $\eta(T) = \sum_{|\gamma|\ge T} |A_\gamma|^2 \to 0$. For a real trigonometric polynomial $P = \sum_\omega c_\omega e^{i\omega y}$, the truncation at 0 changes $G_P(y)$ by $O(e^{-y})$, and the mean value of G_P is

$$\sum_\omega |c_\omega|^2 \int_0^\infty \Big(e^{-t} + \big|1 - e^{-(1/2+i\omega)t}\big|^2\Big) e^{-t}\,\mathrm{d}t = \sum_\omega |c_\omega|^2 \Big(2 - 2\,\mathrm{Re}\,\frac{1}{3/2 + i\omega}\Big) = \sum_\omega |c_\omega|^2 K(\tfrac12 + i\omega).$$

For P_T this is $\frac{2}{3\zeta(1/2)^2} + \sum_{|\gamma|<T} K(\frac12 + i\gamma)|A_\gamma|^2$. Letting $T \to \infty$, the mean of G_E over $[0, Y]$ converges to S, and (39) gives (b). □

Corollary 15 (Conditions on $\zeta'(\rho)$)**.** *Assume the Riemann hypothesis and that every zero is simple.*

(i) Each of the following implies (37) for every $\kappa > 0$, and hence the conclusion of Theorem 14(b): $\sum_\gamma |A_\gamma|^2 \log(|\gamma|+3)/\log\log(|\gamma|+16) < \infty$; $J_{-1}(T) = \sum_{0<\gamma\le T} |\zeta'(\rho)|^{-2} \ll T^2 (\log T)^{-2} (\log\log T)^{-2-\delta}$ for some $\delta > 0$, in particular $J_{-1}(T) \ll T^\theta$ for some $\theta < 2$; (96) for some $1 < p \le 2$ and $\delta > 0$.

(ii) Under (37), the function $e^{-y/2} L(e^y)$ has a limiting distribution, and $\int_1^X L(x)^2 x^{-2}\,\mathrm{d}x \sim \big(\zeta(\frac12)^{-2} + \sum_\gamma |A_\gamma|^2\big) \log X$.

(iii) Conversely, if $\sum_{N\le X} Q_\lambda(N) N^{-2} \ll \log X$, then $S < \infty$ and $J_{-1}(T) = o\big(T^2 (\log\log T)^2\big)$.

Proof. (i) Since $2|A_\gamma A_{\gamma'}| \le |A_\gamma|^2 + |A_{\gamma'}|^2$ and $\mathcal{P}_\kappa$ is symmetric, the sum in (37) is at most $\sum_\gamma |A_\gamma|^2 n_\kappa(\gamma)$, where $n_\kappa(\gamma)$ is the number of ordinates $\gamma' \ne \gamma$ with $|\gamma' - \gamma| \le h = \kappa/\log(|\gamma|+2)$. By the Riemann–von Mangoldt formula and $S(t) \ll \log t/\log\log t$ [57, Theorem 14.13], $n_\kappa(\gamma) \le N(|\gamma|+h) - N(|\gamma|-h) \ll_\kappa \log(|\gamma|+3)/\log\log(|\gamma|+16)$, which gives the first case. For the second, $|\zeta(1+2i\gamma)| \ll \log\log|\gamma|$ [57, Theorem 14.9] and $|\rho| \ge |\gamma|$ give $|A_\gamma|^2 \log|\gamma|/\log\log|\gamma| \ll f(|\gamma|)|\zeta'(\rho)|^{-2}$ with $f(t) = t^{-2}\log t \log\log t$; partial summation against J_{-1}, with $J_{-1}(t)|f'(t)| \ll t^{-1}(\log t)^{-1}(\log\log t)^{-1-\delta}$ and $f(t)J_{-1}(t) \to 0$, gives convergence. For the third, (96) gives $|\zeta'(\rho)|^{-1} \ll T^{1-\delta/p}$ for $|\gamma| \le T$, hence $J_{-1}(T) \le \max_{0<\gamma\le T} |\zeta'(\rho)|^{p-2} \sum_{0<\gamma\le T} |\zeta'(\rho)|^{-p} \ll T^{(1-\delta/p)(2-p)+p-\delta} = T^{2-2\delta/p}$.

(ii) By (40), E is a B^2-almost periodic function, and such a function has a limiting distribution [1]. The argument used for G_E above, applied to E^2 and P_T^2 with $|E^2 - P_T^2| \le (E-P_T)^2 + 2|P_T||E-P_T|$, shows that the mean of E^2 over $[0,Y]$ converges to $\lim_T \big(\zeta(\frac12)^{-2} + \sum_{|\gamma|<T} |A_\gamma|^2\big)$; and $\int_1^X L(x)^2 x^{-2}\,\mathrm{d}x = \int_0^{\log X} E^2$.

(iii) By Theorem 14(a), $S < \infty$, so $\sum_\gamma |A_\gamma|^2 < \infty$. Since $1/\zeta(1+2i\gamma) \ll \log\log|\gamma|$ [57, Theorem 14.9] and $|\rho|^2 \le 2\gamma^2$, $\sum_{U<\gamma\le 2U} |\zeta'(\rho)|^{-2} \ll U^2 (\log\log U)^2 \sum_{U<\gamma\le 2U} |A_\gamma|^2 = o\big(U^2(\log\log U)^2\big)$; summing over dyadic ranges gives the claim. □

The B^2 property and the limiting distribution of $e^{-y/2}L(e^y)$ were obtained by Humphries [38] under $J_{-1}(T) \ll T$ and by Akbary, Ng and Shahabi [1] under $J_{-1}(T) \ll T^\theta$, $\theta < 3 - \sqrt{3}$. By (iii), the conclusion of Theorem 14(b) with $S < \infty$ implies an upper bound $J_{-1}(T) = o(T^2(\log\log T)^2)$; no upper bound for $J_{-1}(T)$ is known under the Riemann hypothesis and the simplicity of the zeros alone. The sufficient condition in (i) and this necessary condition differ by the factor $(\log T)^2 (\log\log T)^{4+\delta}$.

Remark 6. *The terms of S are positive, and the sum over the first 3000 zeros above the real axis and their conjugates gives the computed value 0.45794; hence $S > 0.45794$. Over $10^4 \le N \le 3\cdot 10^7$ the logarithmic mean $\sum Q_\lambda(N)/N^2 / \sum 1/N$ is 0.4540 (Section M). Under the hypotheses of (b), $\sum_{N\le X,\, Q_\lambda(N)\ge N} 1/N \le \sum_{N\le X} Q_\lambda(N)/N^2$, so the upper logarithmic density of $B_{1/2}$ is at most S.*

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5.4. How Often the Energy Is Large

The upper limits of Subsection 5.3 concern sparse sets of N. With one additional input, a bound for the logarithmic mean of $Q_\lambda(N)/N$, they become statements about sets of positive lower logarithmic density. For a set $B \subseteq \mathbb{N}$ write

$$\underline{\delta}(B) = \liminf_{X\to\infty} \frac{1}{\log X} \sum_{N\le X,\, N\in B} \frac{1}{N}, \qquad D = \limsup_{X\to\infty} \frac{1}{\log X} \sum_{N\le X} \frac{Q_\lambda(N)}{N^2} \in [S,\infty],$$

where $D \ge S$ is Theorem 14(a). Put $E(y) = e^{-y/2}L(e^y)$ for $y \ge 0$ and $D_L = \limsup_{Y\to\infty} Y^{-1}\int_0^Y E^2 = \limsup_{X\to\infty} (\log X)^{-1}\int_1^X L(x)^2 x^{-2}\,\mathrm{d}x$. The bounds $\int_0^\infty e^{-2t}E(y-t)^2\,\mathrm{d}t \le G(y) \le 2E(y)^2 + 3\int_0^\infty e^{-2t}E(y-t)^2\,\mathrm{d}t$, with G as in (38), give $\frac12 D_L \le D \le \frac72 D_L$; so $D < \infty$ if and only if $\int_1^X L(x)^2x^{-2}\,\mathrm{d}x \ll \log X$. If $D < \infty$, then $S < \infty$, so the Riemann hypothesis holds and every zero is simple. Under the hypotheses of Theorem 14(b), $D = S$; by Corollary 15(i) this holds, for instance, under the Riemann hypothesis and $J_{-1}(T) \ll T^\theta$ for some $\theta < 2$.

Theorem 15 (Frequency of large energy)**.** *Let k, m, P, Ω and Z be as in Theorem 12, and suppose in addition that $P(t) \ll (1+|t|)^{-3}$. For $c \ge 0$ put*

$$\mu_c = \liminf_{Y\to\infty} \frac{1}{Y}\int_0^Y \big(\|Z(y)\| - \sqrt{c}\big)_+\,\mathrm{d}y.$$

(a) If $\Theta > \frac12$, then $\{N : Q_\lambda(N) > cN\}$ is cofinite for every c.
(b) If $D < \infty$, then for every $c \ge 0$

$$\underline{\delta}\big(\{N :\ Q_\lambda(N) > cN\}\big) \ \ge\ \frac{\mu_c^2}{D}.$$

Proof. (a) Choose $\frac12 < \sigma < \Theta$ and apply Theorem 8(b); $N^{2\sigma} > cN$ for large N.

(b) Put $F(t) = e^{-t/2}U_{e^t}$ for $t \ge 0$, $F(t) = 0$ for $t < 0$, $g = \|F\|$ and $H(Y) = \int_0^Y g^2$. Then $H(\log X) = \int_1^X \|U_v\|^2 v^{-2}\,\mathrm{d}v$. By (17), $\int_N^{N+1}\|U_v\|^2v^{-2}\,\mathrm{d}v = Q_\lambda(N)N^{-2}(1+O(1/N))$, and $Q_\lambda(N) \le 5\max_{j\le N} L(j)^2 = o(N^2)$ by Corollary 1 and $L(x) = o(x)$. Hence $\limsup H(Y)/Y = D$. Fix $\varepsilon > 0$ and B with $H(Y) \le (D+\varepsilon)Y + B$ for all $Y \ge 0$.

Step 1. For $\delta > 0$, by the Cauchy–Schwarz inequality and $\int_0^\infty g^2e^{-\delta t}\,\mathrm{d}t = \delta\int_0^\infty H(y)e^{-\delta y}\,\mathrm{d}y$,

$$\int_0^\infty g(t)e^{-\delta t}\,\mathrm{d}t \le \delta^{-1/2}\big((D+\varepsilon)\delta^{-1} + B\big)^{1/2} < \infty. \tag{41}$$

So $\Phi(s) = \int_0^\infty F(t)e^{-(s-1/2)t}\,\mathrm{d}t$ converges absolutely and locally uniformly on $\operatorname{Re} s > \frac12$, and $\Phi(s) = M(s)w_s$ there, as in the proof of Theorem 7. Moreover $\int_n^{n+1} g \le H(n+1)^{1/2} \ll (n+1)^{1/2}$, so $\int P(y-t)g(t)\,\mathrm{d}t < \infty$, locally uniformly in y. The proof of Theorem 12 uses the bound $\|F\| \le B_0$ only to justify Fubini's theorem at fixed δ, which now follows from (41), and dominated convergence as $\delta \to 0$, which now follows from the domination by $P(y-t)g(t)$. The poles of $M(s)w_s$ on the line are simple because the zeros are simple. That proof therefore gives $\epsilon(y) := \|Z(y) - (P*F)(y)\| \to 0$ as $y \to \infty$.

Step 2. Let $A = \{t \ge 0 :\ g(t) > \sqrt{c}\}$. Since $P \ge 0$, $\int P = 1$ and $g \le \sqrt{c}$ outside A,

$$\|Z(y)\| \le \epsilon(y) + \int_{\mathbb{R}} P(y-t)g(t)\,\mathrm{d}t \le \epsilon(y) + \sqrt{c} + \int_A P(y-t)g(t)\,\mathrm{d}t,$$

hence $\varphi(y) := (\|Z(y)\| - \sqrt{c})_+ \le \epsilon(y) + \int_A P(y-t)g(t)\,\mathrm{d}t$.

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Step 3. Let $0<\eta<1$. Since P is even, $\int_0^Y P(y-t)\,dy=\int_{t-Y}^{t}P\le\min\bigl(1,\int_{t-Y}^{\infty}P\bigr)$, and $\int_u^\infty P\ll u^{-2}$. Integrating Step 2 over $[0,Y]$,

$$\int_0^Y\varphi\le o(Y)+\int_{A\cap[0,(1+\eta)Y]}g+O\Bigl(\sum_{j\ge0}(2^j\eta Y)^{-2}\int_{Y+2^j\eta Y}^{Y+2^{j+1}\eta Y}g\Bigr).$$

By the Cauchy–Schwarz inequality the j-th integral is at most $(2^j\eta Y)^{1/2}H(Y+2^{j+1}\eta Y)^{1/2}\ll_\eta 2^jY$, so the sum is $O_\eta(Y^{-1})$. Hence $\int_{A\cap[0,Y']}g\ge(\mu_c-\eta)Y'/(1+\eta)$ for $Y'=(1+\eta)Y$ large, and the Cauchy–Schwarz inequality gives

$$\frac{|A\cap[0,Y']|}{Y'}\ge\frac{(\int_{A\cap[0,Y']}g)^2}{Y'H(Y')}\ge\frac{(\mu_c-\eta)^2}{(1+\eta)^2(D+\varepsilon)}-o(1).$$

Step 4. Let $c'>c$ and let A' be the set of Step 2 for c'. For $t\in[\log N,\log(N+1))$, (17) gives $g(t)^2\le(N+2)Q_\lambda(N)/N^2$, so $t\in A'$ implies $Q_\lambda(N)>c'N^2/(N+2)>cN$ for $N>2c/(c'-c)$. These intervals have length $\log(1+1/N)\le1/N$, so $\sum_{N\le X,\,Q_\lambda(N)>cN}1/N\ge|A'\cap[0,\log X]|-O(1)$. By Step 3, $\underline{\delta}(\{N:Q_\lambda(N)>cN\})\ge\mu_{c'}^2/D$. Finally $|\mu_{c'}-\mu_c|\le\sqrt{c'}-\sqrt{c}$; let $c'\downarrow c$. □

Lemma 5. *In Theorem 15 assume $D<\infty$, so that $\Omega\subseteq\{0\}\cup\{\pm\gamma\}$ with $R(0)=-a$, $R(\pm\gamma)=A_\gamma,\overline{A_\gamma}$. Put $S_k=\sum_{\omega\in\Omega}|k(\omega)R(\omega)|^2K(\frac12+i\omega)$ and $B_k=\sum_{\omega\in\Omega}|k(\omega)R(\omega)|K(\frac12+i\omega)^{1/2}$.*

(i) $\mu_c\ge a\sqrt{2/3}-\sqrt{c}$.

(ii) $\mu_c\ge(S_k-c)/(2B_k)$.

(iii) *If $m=T$ and Γ_0 is q-independent below T, and V is the right side of (36), then* $\mu_c\ge(q+1)^{-|\Gamma_0|}(\sqrt{V}-\sqrt{c})$.

(iv) $\mu_c>0$ *if and only if* $c<\sup_y\|Z(y)\|^2$. *Moreover* $\sup_y\|Z(y)\|^2\ge V$ *in the situation of (iii).*

Proof. The mean value $\mathfrak{M}$ of a finite exponential sum exists, and $(x-\sqrt{c})_+\ge x-\sqrt{c}$. (i) Since $\|Z\|\ge\operatorname{Re}\langle Z,v\rangle$ with $v=-w_{1/2}/\|w_{1/2}\|$, we get $\mu_c\ge\mathfrak{M}[\operatorname{Re}\langle Z,v\rangle]-\sqrt{c}=a\|w_{1/2}\|-\sqrt{c}$, and $\|w_{1/2}\|^2=K(\frac12)=\frac23$. (ii) We have $\|Z\|\le B_k$. If $\|Z(y)\|>\sqrt{c}$, then $\|Z\|-\sqrt{c}=(\|Z\|^2-c)/(\|Z\|+\sqrt{c})\ge(\|Z\|^2-c)/(2B_k)$; otherwise the left side is 0. Hence $\varphi\ge(\|Z\|^2-c)/(2B_k)$ when $c\le B_k^2$, and the frequencies in Ω are distinct, so $\mathfrak{M}\|Z\|^2=S_k$. (iii) Let W be the weight in the proof of Theorem 13; then $0\le W\le\|W\|_\infty=(q+1)^{|\Gamma_0|}$, $\mathfrak{M}[W]=1$ and $\|\mathfrak{M}[WZ]\|=\sqrt{V}$. With $v=\mathfrak{M}[WZ]/\sqrt{V}$, $\varphi\ge\|W\|_\infty^{-1}W(\operatorname{Re}\langle Z,v\rangle-\sqrt{c})$, and the mean of the right side is $\|W\|_\infty^{-1}(\sqrt{V}-\sqrt{c})$. Also $\sqrt{V}\le\mathfrak{M}[W\|Z\|]\le\sup\|Z\|$. (iv) If $c\ge\sup\|Z\|^2$, then $\varphi=0$. Otherwise write $Z(y)=\sum_{j\le n}b_je^{i\omega_jy}$ and choose $\eta>0$ and y_0 with $\|Z(y_0)\|>\sqrt{c}+3\eta$. With $\Lambda=\sum\|b_j\||\omega_j|$ and $\ell=\eta/\Lambda$, $\|Z\|>\sqrt{c}+2\eta$ on $[y_0-\ell,y_0+\ell]$. Cover the torus $(\mathbb{R}/2\pi\mathbb{Z})^n$ by finitely many cells on which each $e^{i\theta_j}$ varies by at most $\eta/\sum\|b_j\|$. For each cell met by $\theta(t)=(\omega_jt)_j$, $t\ge0$, fix one such t, and let L_0 be the largest of these finitely many t. For $x\ge L_0$ the point $\theta(x)$ lies in some cell, with chosen point t_x; then $h=x-t_x\in[x-L_0,x]$ satisfies $\|Z(y+h)-Z(y)\|\le\eta$ for all y. So $\varphi\ge\eta$ on $[y_0+h-\ell,y_0+h+\ell]$. Taking $x=L_0+i(L_0+2\ell)$, $i=0,1,\dots$, gives disjoint intervals, one in each window of length $L_0+2\ell$, and $\mu_c\ge2\ell\eta/(L_0+2\ell)>0$. □

Corollary 16. *Unless the Riemann hypothesis holds and $D=\infty$, the set $\{N:Q_\lambda(N)>cN\}$ has positive lower logarithmic density for every $c<4.6242$. If $D<\infty$ and the positive ordinates satisfy no relation $\sum c_\gamma\gamma=0$ with $c_\gamma\in\{-2,\dots,2\}$ not all zero, the same holds for every $c>0$.*

Proof. If $\Theta>\frac12$, apply Theorem 15(a). If $D<\infty$, apply Theorem 15(b) and Lemma 5(iv) with the data of Corollary 13, for which $V>4.6242$. Under the second hypothesis the proof of Corollary 14 gives $\sup_y\|Z(y)\|\ge a/3+0.99\sum_{j\le J}k_T(\gamma_j)|A_{\gamma_j}|$, which is unbounded by (35). □

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For $c < S$, Lemma 5(i) and (ii) give usable constants. With the Jurkat–Peyerimhoff kernel and the computed residues one finds $\mu_c^2 \geq 0.0590,\ 0.0125,\ 0.0021,\ 1.5 \cdot 10^{-4}$ for $c = 0.1,\ 0.2,\ 0.3,\ 0.4$. For $c \geq S$, the only certified bound is (iii), which gives μ_c^2 of order 3161^{-700} for the data of Corollary 13.

The same argument applies to L itself, with F replaced by E, w_s by 1 (since $\int_0^\infty E(t)e^{-(s-1/2)t}\,\mathrm{d}t = M(s)$), Z by $z(y) = \sum_{\omega\in\Omega} k(\omega)R(\omega)e^{i\omega y}$, and D by D_L. Step 2 becomes $(-z(y)-c)_+ \leq \epsilon(y) + \int_{\{E<-c\}} P(y-t)|E(t)|\,\mathrm{d}t$, and its analogue with $(z-c)_+$ and $\{E > c\}$. For $t \in [\log N, \log(N+1))$, $E(t) < -c$ implies $L(N) < -c\sqrt{N}$. Since $\mathfrak{M}[z] = k(0)R(0) = 1/\zeta(\frac{1}{2})$, and the means of z and of E over $[0, Y]$ differ by $o(1)$ by Step 1 and the tail estimate of Step 3, this gives the following.

Theorem 16 (Frequency of large values of L). *Suppose that $\int_1^X L(x)^2 x^{-2}\,\mathrm{d}x \ll \log X$. Then*

$$\underline{\delta}\big(\{N : \pm L(N) > c\sqrt{N}\}\big) \;\geq\; \frac{1}{D_L}\Big(\liminf_{Y\to\infty} \frac{1}{Y}\int_0^Y (\pm z(y) - c)_+\,\mathrm{d}y\Big)^2 \qquad (c \geq 0).$$

In particular, the lower logarithmic density of $\{N : L(N) < -c\sqrt{N}\}$ is at least $(a-c)^2/D_L$ for $0 \leq c < a$, and it is positive for $c < 2.5257$. The lower logarithmic density of $\{N : L(N) > c\sqrt{N}\}$ is positive for $c < 1.1562$, and in particular for $c = 0$. Moreover $(\log X)^{-1}\int_1^X L(x)x^{-3/2}\,\mathrm{d}x \to 1/\zeta(\frac{1}{2})$.

The constants 2.5257 and 1.1562 are $\frac{2q}{q+1}\sum_{\gamma\in\Gamma_0} k_T(\gamma)|A_\gamma| \pm a$ for the data of Corollary 13, evaluated in ball arithmetic. Under the hypotheses of Theorem 14(b), Corollary 15(ii) gives $D_L = \zeta(\frac{1}{2})^{-2} + \sigma^2$ with $\sigma^2 = \sum_\gamma |A_\gamma|^2$, so

$$\underline{\delta}\big(\{N : L(N) < 0\}\big) \;\geq\; \frac{\zeta(\frac{1}{2})^{-2}}{\zeta(\frac{1}{2})^{-2} + \sigma^2},$$

without any independence hypothesis on the ordinates.

Proposition 8 (How often the energy is small). *For every $c > 0$, $\underline{\delta}(\{N : Q_\lambda(N) < cN\}) \geq 1 - D/c$. If the Riemann hypothesis holds and $J_{-1}(T) \leq c_1 T$ for $T \geq T_0 \geq 2$, then*

$$\sum_{|\gamma|>T_0} |A_\gamma|^2 \leq \frac{9c_1(\ell^2+\ell+1)}{4T_0}, \qquad \ell = \log(2T_0),$$

and $S \leq \frac{2}{3\zeta(1/2)^2} + \sum_{|\gamma|\leq T_0} K(\frac{1}{2}+i\gamma)|A_\gamma|^2 + \frac{9c_1(\ell^2+\ell+1)}{2T_0}$.

Proof. The first statement follows from $\sum_{N\leq X,\, Q_\lambda(N)\geq cN} 1/N \leq c^{-1}\sum_{N\leq X} Q_\lambda(N)/N^2$. For the second, $|A_\gamma|^2 \leq f(\gamma)|\zeta'(\rho)|^{-2}$ with $f(t) = \frac{9}{16}t^{-2}\log^2(2t)$, by $|\zeta(1+it)| \leq \frac{3}{4}\log t$ for $t \geq 3$ [58]; f decreases for $t \geq 2$. Partial summation gives

$$\sum_{\gamma>T_0} f(\gamma)|\zeta'(\rho)|^{-2} \leq -\int_{T_0}^\infty J_{-1}(t)f'(t)\,\mathrm{d}t \leq c_1\Big(T_0 f(T_0) + \int_{T_0}^\infty f\Big),$$

and $\int_{T_0}^\infty t^{-2}\log^2(2t)\,\mathrm{d}t = (\ell^2+2\ell+2)/T_0$. Double this for both signs of γ. The bound for S uses $K < 2$. □

With $T_0 = \gamma_{3800} = 4314.79\ldots$ and the computed partial sums, the proposition gives $S \leq 0.45798 + 0.0962\,c_1$. So the Riemann hypothesis together with $J_{-1}(T) \leq 5.6\,T$ for $T \geq \gamma_{3800}$, which gives $D = S$ by Corollary 15, implies that $\{N : Q_\lambda(N) < N\}$ has positive lower logarithmic density. No unconditional statement of this kind is possible: by Theorem 8, infinitely many N with $Q_\lambda(N) \leq N$ already imply the Riemann hypothesis.

5.5. Large Energy without a Mean-Square Hypothesis

Corollary 16 excludes the case that the Riemann hypothesis holds and $D = \infty$. This subsection removes the exclusion. We begin with an unconditional lower bound of the right order.

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Theorem 17 (Unconditional lower bound). *Let $0 < \alpha \le \frac{1}{2}$ and let $h \in C^2(\mathbb{R})$ be real, with support in $[\alpha, 1]$ and $\int h = 0$. Put $w(u) = \sum_{n\ge1} h(nu)$ for $u > 0$ and*

$$I_h = \int_0^1 \frac{h(z)}{z^{1/2}}\,\mathrm{d}z, \quad B_h = \int_0^1 \frac{|h(z)|}{z}\,\mathrm{d}z, \quad C_w = \int_0^1 \frac{w(u)^2}{u^2}\,\mathrm{d}u, \quad J_w = \int_0^1 \frac{w(u)}{u}\,\mathrm{d}u.$$

Then $w = 0$ on $(1,\infty)$ and $|w(u)| \le u\|h''\|_\infty/8$ for $0 < u \le \alpha$, so C_w, J_w are finite. For every integer $N \ge 1$ with $\sqrt{N}|I_h| > B_h$,

$$\sum_{j<N} L(j)^2 \ge \frac{N(\sqrt{N}|I_h| - B_h)^2}{C_w}, \qquad Q_\lambda(N) \ge \frac{2(\sqrt{N}|I_h| - B_h)^2}{C_w + J_w^2}.$$

In particular $\liminf_{N\to\infty} Q_\lambda(N)/N \ge 2I_h^2/(C_w + J_w^2) > 0$ whenever $I_h \ne 0$.

Proof. If $u > 1$ then $nu > 1$ for all $n \ge 1$, so $w(u) = 0$. Let $0 < u \le \alpha$ and $K = \lceil 1/u \rceil$. Since h and h' vanish at 0 and at $Ku \ge 1$, the composite trapezoidal rule with step u on $[0, Ku]$ equals $u\sum_{n\ge1} h(nu) = uw(u)$, and its error is at most $Ku \cdot u^2\|h''\|_\infty/12 \le (1+u)u^2\|h''\|_\infty/12$. As $\int h = 0$, $|w(u)| \le u\|h''\|_\infty/8$.

Every $x \ge 1$ satisfies $\sum_{n\le x} L(x/n) = \sum_{k\le x}(1*\lambda)(k) = \lfloor\sqrt{x}\rfloor$, since $1*\lambda$ is the indicator of the squares. In $\int_0^1 L(Nu)w(u)u^{-1}\,\mathrm{d}u$ only $n \le N$ contribute, because $L(Nu) = 0$ for $u < 1/N$ and $h(nu) = 0$ for $nu > 1$. Substituting $z = nu$ gives

$$\int_0^1 L(Nu)w(u)\frac{\mathrm{d}u}{u} = \int_\alpha^1 \Big(\sum_{n\le N} L(Nz/n)\Big)h(z)\frac{\mathrm{d}z}{z} = \int_0^1 \lfloor\sqrt{Nz}\rfloor h(z)\frac{\mathrm{d}z}{z},$$

and this differs from $\sqrt{N}\,I_h$ by at most B_h. By the Cauchy–Schwarz inequality its square is at most $C_w\int_0^1 L(Nu)^2\,\mathrm{d}u = C_w N^{-1}\sum_{j<N} L(j)^2$. This proves the first bound.

For the second, let $f = (f_1, f_2) \in \mathcal{H}$ with $f_1 = w/(2u) + J_w/2$ and $f_2 = J_w/2 - w/(2u)$. Then $\int f_2 = 0$ and $f_1 - f_2 = w/u$, so

$$\langle U_N, f\rangle = \int_0^1 L(Nu)(f_1 - f_2)\,\mathrm{d}u + L(N)\int_0^1 f_2\,\mathrm{d}u = \int_0^1 L(Nu)w(u)\frac{\mathrm{d}u}{u}.$$

Also $\|f\|^2 = \frac{1}{2}(C_w + J_w^2)$. By Lemma 2(a), $Q_\lambda(N) = \|U_N\|^2 \ge \langle U_N, f\rangle^2/\|f\|^2$. □

The best constants are found on the Mellin side. With $\widetilde{h}(s) = \int_0^1 h(z)z^{s-1}\,\mathrm{d}z$, the Mellin transform of w is $\zeta(s)\widetilde{h}(s)$, so $I_h = \widetilde{h}(\frac{1}{2})$, $J_w = \zeta(0)\widetilde{h}(0)$ and $C_w = \frac{1}{2\pi}\int|\zeta\widetilde{h}(-\frac{1}{2}+it)|^2\,\mathrm{d}t$. For $\alpha = \frac{1}{100}$ and h a polynomial of degree 20 times $((z-\alpha)(1-z))^4$, chosen by optimisation, one finds $I_h = 0.108757$, $B_h = 0.703105$, $C_w = 0.0673189$ and $J_w = -0.203564$. Hence $2I_h^2/(C_w + J_w^2) = 0.21751$, and $Q_\lambda(N) > 0.2N$ for $N \ge 10^5$. A direct computation shows that $Q_\lambda(N) \le 0.2N$ for $N \le 10^7$ only when $N \in \{6, 10, 11, 15, 17, 25, 27, 28, 42\}$. These values are computed in double precision and are not certified. Since $\zeta(\frac{1}{2})I_h = \int_0^1 (w/u)u^{1/2}\,\mathrm{d}u \le (C_w/2)^{1/2}$, the first bound of Theorem 17 never exceeds $\frac{1}{2}\zeta(\frac{1}{2})^{-2} = 0.2345\ldots$ times N^2.

From now on $F(t) = e^{-t/2}U_{e^t}$, $g = \|F\|$ and $E(t) = e^{-t/2}L(e^t)$ for $t \ge 0$. All three vanish for $t < 0$. Put $v(t) = e^{-2t}\int_0^{e^t} L(x)^2\,\mathrm{d}x$ for $t \ge 0$ and $v(t) = 0$ for $t < 0$.

Lemma 6. *For $t \ge 0$, $h \ge 0$ and integers $k \ge 0$:*

(a) $v(t) = \int_0^\infty E(t-r)^2 e^{-2r}\,\mathrm{d}r$;
(b) $v(t) \le g(t)^2 \le 2E(t)^2 + 3v(t)$;
(c) $v(t+h) \ge e^{-2h}v(t)$;
(d) $\int_k^{k+1} g^2 \le 5e^2 v(k+1)$;
(e) $v(t) \le e^t/3$.

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Proof. (a) Substitute $x = e^{t-r}$. (b) With $X = e^t$, $g(t)^2 = X^{-2}\int_0^X \big(L(x)^2 + (L(X) - L(x))^2\big)\,\mathrm{d}x$. Use $(L(X) - L)^2 \le 2L(X)^2 + 2L^2$. (c) The integral increases with its upper limit. (d) By (c), $v \le e^2 v(k+1)$ on $[k, k+1]$, and by (a), $\int_k^{k+1} E^2 \le e^2 v(k+1)$. Now apply (b). (e) Use $|L(x)| \le x$. □

Lemma 7 (Covering). *Let $g \ge 0$ be measurable on $\mathbb{R}$ and vanish on $(-\infty, 0)$. Let $v \ge 0$ satisfy, for some $C_1, C_2 > 0$, $t \ge 0$, $h \ge 0$ and integers $k \ge 0$,*

$$v(t) \le g(t)^2, \qquad v(t+h) \ge e^{-2h} v(t), \qquad \int_k^{k+1} g^2 \le C_1 v(k+1), \qquad v(t) \le C_2 e^t.$$

Let $\nu \ge 0$ be measurable with $\int \nu = 1$ and $\nu(r) \le C_\nu e^{-3|r|}$. Let $c', \eta > 0$. Suppose there are $0 < \ell \le L_0$ and Y_1 such that every interval of length L_0 in $[Y_1, \infty)$ contains a closed interval of length ℓ on which $\int \nu(y-t) g(t)\,\mathrm{d}t \ge \sqrt{c'} + \eta$. Then $A = \{t : g(t)^2 > c'\}$ satisfies $\liminf_{Y\to\infty} |A \cap [0, Y]|/Y > 0$.

Proof. Let R^* be the union of one such interval for each window $[Y_1 + jL_0, Y_1 + (j+1)L_0]$, $j \ge 0$. For $y \in \mathbb{R}$ put $I_k = [k, k+1]$ and $d_k = \operatorname{dist}(y, I_k)$. Then $\sum_{k\ge 0} e^{-3d_k/2} \le 1 + 2(1 - e^{-3/2})^{-1} < 4$. Put

$$\beta = \frac{\eta}{8C_\nu e\,(C_1 c')^{1/2}}, \qquad \kappa = \frac{\eta}{8C_\nu (C_1 c')^{1/2}}.$$

Fix $y \in R^*$. Since $g \le \sqrt{c'}$ off A and $\int \nu = 1$, we have $\int_A \nu(y-t) g(t)\,\mathrm{d}t \ge \eta$. On I_k, $\nu(y - \cdot) \le C_\nu e^{-3d_k}$. The Cauchy–Schwarz inequality gives

$$\eta \le C_\nu C_1^{1/2} \sum_{k\ge 0} e^{-3d_k} v(k+1)^{1/2} |A \cap I_k|^{1/2}.$$

Call k low if $v(k+1) \le e^2 c'$ and high otherwise. Either the low terms or the high terms contribute at least $\eta/2$.

(a) Suppose the low terms contribute at least $\eta/2$. Then some low k has $e^{-3d_k/2}|A \cap I_k|^{1/2} \ge \beta$. As $|A \cap I_k| \le 1$, this forces $\beta \le 1$ and $d_k \le d_* := \frac{2}{3}\log(1/\beta)$. So $B_y = [y - d_* - 1, y + d_* + 1]$ satisfies $|A \cap B_y| \ge \beta^2 \ge \theta_1 |B_y|$, with $\theta_1 = \beta^2/(2d_* + 2)$.

(b) Suppose the high terms contribute at least $\eta/2$. Then some high k has $e^{-3d_k/2} v(k+1)^{1/2} \ge \kappa\sqrt{c'}$. Put $\rho = \frac{1}{2}\log(v(k+1)/c') > 1$. Then $\rho \ge \frac{3}{2}d_k + \log\kappa$. For $t \in J = [k+1, k+1+\rho)$, $g(t)^2 \ge v(t) \ge e^{-2(t-k-1)} v(k+1) > c'$, so $J \subseteq A$. Since $|y - (k+1)| \le d_k + 1$, $J \subseteq B_y = [y - r, y + r]$ with $r = d_k + 1 + \rho \le C_3\rho$, where $C_3 = \frac{8}{3} + \frac{2}{3}|\log\kappa|$. Hence $|A \cap B_y| \ge \rho \ge \theta_2 |B_y|$, with $\theta_2 = 1/(2C_3)$. Moreover $e^{2\rho} \le C_2 e^{k+1}/c'$, so $\rho \le \frac{k+1}{2} + C_4$ with $C_4 = \frac{1}{2}\log^+(C_2/c')$. If $k > y$, then $\frac{3}{2}(k - y) + \log\kappa \le \rho$ forces $k \le \frac{3}{2}y + C_4 + \frac{1}{2} + |\log\kappa|$. In both cases $\rho \le \frac{3}{4}y + C_5$, so $B_y \subseteq (-\infty, C_6 y + C_7]$ for constants independent of y. We take $C_7 \ge d_* + 1$ to cover case (a) as well.

Let $\theta = \min(\theta_1, \theta_2)$, let $Y > Y_1$ and let $R_Y = R^* \cap [Y_1, Y]$, a compact set. Finitely many interiors of the B_y, $y \in R_Y$, cover R_Y. The finite Vitali lemma gives disjoint $B_{y_1}, \dots, B_{y_n}$ with $R_Y \subseteq \bigcup 3B_{y_i}$. As $A \subseteq [0, \infty)$,

$$|A \cap [0, C_6 Y + C_7]| \ge \sum_i |A \cap B_{y_i}| \ge \theta \sum_i |B_{y_i}| \ge \frac{\theta}{3}|R_Y| \ge \frac{\theta\ell}{3}\Big\lfloor \frac{Y - Y_1}{L_0} \Big\rfloor.$$

Hence $\liminf_{Z\to\infty} |A \cap [0, Z]|/Z \ge \theta\ell/(3L_0C_6) > 0$. □

Lemma 8 (A positive kernel). *Assume the Riemann hypothesis. Let $T_0 > 0$, and let $\gamma_1 \le \gamma_2 \le \cdots$ be the ordinates $\ge T_0$, each repeated according to multiplicity. Put $a_j = \pi/(2\gamma_j)$ and $\sigma^2 = \sum a_j^2 < \infty$. Let ε_j be independent random signs, $S = \sum a_j \varepsilon_j$, and μ_B the law of S. Let $\psi = c_\varepsilon^{-1} P(r) e^{-\varepsilon r^2}$, where $\varepsilon > 0$, $P \ge 0$ is smooth with all derivatives bounded, and $c_\varepsilon = \int P e^{-\varepsilon r^2} > 0$. Put $\nu(r) = \mathbb{E}\,\psi(r - S)$, $\Psi(s) = \int \psi(r) e^{-(s-1/2)r}\,\mathrm{d}r$ and $\widehat{\nu}(s) = \int \nu(r) e^{-(s-1/2)r}\,\mathrm{d}r$.*

(a) $\nu \ge 0$, $\int \nu = 1$, *and* $\nu(r) \le C_\nu e^{-3|r|}$.

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(b) $\widehat{\nu}(s) = \Psi(s)\prod_j \cosh\bigl(a_j(s-\frac12)\bigr)$ *is entire. Uniformly for* $\frac14 \le \operatorname{Re} s \le 2$, $\widehat{\nu}(s) \ll_n (1+|\operatorname{Im} s|)^{-n}$ *for every* n. *It vanishes at* $\frac12 \pm i\gamma_j$ *to at least the multiplicity of the zero. Moreover* $\widehat{\nu}(\frac12+i\omega) = \Psi(\frac12+i\omega)\Pi_{T_0}(\omega)$, *where* $\Pi_{T_0}(\omega) = \prod_j \cos(\pi\omega/2\gamma_j)$ *lies in* $(0,1]$ *for* $|\omega| < T_0$ *and tends to* 1 *as* $T_0 \to \infty$ *for each fixed* ω.

(c) *For* $y \in \mathbb{R}$, *with the sum over the zeros* $\rho = \frac12 + i\gamma$, $|\gamma| < T_0$, *taken without multiplicity,*

$$(\nu * F)(y) = -a\, w_{1/2} + \sum_{|\gamma|<T_0} \operatorname*{Res}_{s=\rho} \bigl(\widehat{\nu}(s) M(s) w_s e^{(s-1/2)y}\bigr) + O_\nu(e^{-y/4}).$$

Proof. (a) Write S_n for the partial sums. Since $\cosh x \le e^{x^2/2}$, $\mathbb{E}e^{\lambda S_n} \le e^{\lambda^2\sigma^2/2}$. Hence $S_n \to S$ almost surely, $\sup_n \mathbb{E}e^{\lambda|S_n|} < \infty$ for every λ, and $\Pr(|S| > x) \le 2e^{-x^2/2\sigma^2}$. Then $\nu(r) \le \sup_{|u|\ge|r|/2} \psi(u) + \|\psi\|_\infty \Pr(|S| \ge |r|/2)$, and Fubini's theorem gives $\int \nu = 1$.

(b) By uniform integrability, $\mathbb{E}e^{zS} = \lim \mathbb{E}e^{zS_n} = \prod_j \cosh(a_j z)$ for complex z. The product converges locally uniformly because $\sum a_j^2 < \infty$. Fubini's theorem gives the product formula, and cosh is even. Since $|\cosh(a(x+i\tau))| \le \cosh(ax)$, the product is at most $e^{9\sigma^2/8}$ on the strip. Moreover $\Psi(\sigma+i\tau)$ is the Fourier transform at τ of $c_\varepsilon^{-1}P(r)e^{-\varepsilon r^2-(\sigma-1/2)r}$, a Schwartz function with seminorms bounded uniformly for $\frac14 \le \sigma \le 2$. The factor $\cosh(a_j(s-\frac12))$ vanishes at $\frac12 \pm i\gamma_j$, and it occurs once for each repetition of γ_j. For $|\omega| < T_0$, every $|\pi\omega/2\gamma_j| < \pi/2$, so every factor is positive and the absolutely convergent product is positive. If $T_0 \ge 2|\omega|$, then $\cos x \ge e^{-x^2}$ for $|x| \le \pi/4$ gives $\Pi_{T_0}(\omega) \ge \exp(-\frac{\pi^2\omega^2}{4}\sum_{\gamma_j\ge T_0}\gamma_j^{-2}) \to 1$.

(c) We have $\|F(t)\| \le \sqrt{2}e^{t/2}$. With (a) and Lemma 2(c), Fubini's theorem gives $\int (\nu * F)(y)e^{-(s-1/2)y}\,dy = \widehat{\nu}(s)M(s)w_s$ on $\operatorname{Re} s = 2$. The function $(\nu*F)(y)e^{-3y/2}$ is continuous and integrable, and by (b) its Fourier transform is integrable. So $(\nu*F)(y) = \frac{1}{2\pi i}\int_{(2)} \widehat{\nu}Mw_s e^{(s-1/2)y}\,ds$ for every y. Shift to $\operatorname{Re} s = \frac14$ through the horizontal segments at height $\pm T_n$ of the proof of Lemma 4. The bounds for $1/\zeta$ used there need only the Riemann hypothesis, and they give $M(s) \ll |s|$ on these segments and on $\operatorname{Re} s = \frac14$. By (b) the horizontal integrals tend to 0, and the vertical one is $O_\nu(e^{-y/4})$. Inside the strip, M has the simple pole $\frac12$ with residue $1/\zeta(\frac12) = -a$, where $\widehat{\nu}(\frac12) = 1$, and poles at the zeros. At a zero of multiplicity m_ρ, the pole of M has order exactly m_ρ, because $\zeta(2\rho) \ne 0$. For $|\gamma| \ge T_0$ it is cancelled by (b). □

Theorem 18 (Frequency of large energy, unconditionally)**.** *Let* k, m, P *be as in Theorem 12. If every pole of* M *on* $\{\frac12+i\omega : |\omega| < m\}$ *is simple, let* Z *be as there and put* $c_* = \sup_y \|Z(y)\|^2$; *otherwise put* $c_* = \infty$. *Then* $\underline{\delta}(\{N : Q_\lambda(N) > cN\}) > 0$ *for every* $c < c_*$. *If* ζ *has a multiple zero, or* $\Theta > \frac12$, *this holds for every* $c \ge 0$.

Proof. If $\Theta > \frac12$, apply Theorem 15(a). Assume the Riemann hypothesis, and fix $c < c' < c_*$. By Step 4 of the proof of Theorem 15, it suffices to show that $A' = \{t \ge 0 : g(t)^2 > c'\}$ has positive lower density. By Lemma 6, the hypotheses of Lemma 7 on g and v hold with $C_1 = 5e^2$ and $C_2 = \frac13$. Since $\nu \ge 0$ and $\int \nu = 1$, $\nu * g \ge \|\nu * F\|$. Choose T_0 as follows. If ζ has a multiple zero, take T_0 larger than its ordinate. Otherwise $c_* < \infty$; take $T_0 > 2m$ so large that $Z^{(T_0)}(y) = \sum_{\omega\in\Omega} k(\omega)\Pi_{T_0}(\omega)R(\omega)e^{i\omega y}w_{1/2+i\omega}$ satisfies $\sup_y \|Z^{(T_0)}(y)\|^2 > c'$. This is possible by Lemma 8(b), because Ω is finite.

Case 1: some zero with $|\gamma| < T_0$ *is multiple.* Take $\psi(r) = (2\pi)^{-1/2}e^{-r^2/2}$, so $\Psi(s) = e^{(s-1/2)^2/2} \ne 0$, and let $D \ge 2$ be the largest multiplicity below T_0. By Lemma 8(c), $(\nu*F)(y) = y^{D-1}p(y) + O(y^{D-2})$ with

$$p(y) = \sum_{|\gamma|<T_0,\ m_\rho=D} b_\rho e^{i\gamma y}, \qquad b_\rho = \frac{D\,\zeta(2\rho)\widehat{\nu}(\rho)}{\rho\,\zeta^{(D)}(\rho)}\, w_\rho \ne 0,$$

because $\widehat{\nu}(\rho) = e^{-\gamma^2/2}\Pi_{T_0}(\gamma) > 0$. The mean value of $\|p\|^2$ is $\sum\|b_\rho\|^2 > 0$. The argument in the proof of Lemma 5(iv) then gives $\beta, \ell, L_0 > 0$ such that every interval of length L_0 in $[0,\infty)$ contains an interval of length ℓ on which $\|p\| \ge \beta$. There $\nu * g \ge \|\nu * F\| \ge \sqrt{c'} + 1$ once y is large. Lemma 7 applies, for every c'.

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Case 2: every zero with $|\gamma| < T_0$ is simple. Then $c_* < \infty$, and $\Omega = \{0\} \cup \{\gamma : |\gamma| < m\}$. Take $\psi_\varepsilon = c_\varepsilon^{-1} P e^{-\varepsilon r^2}$; its derivatives are bounded since k has compact support. Write $\widehat{\psi}_\varepsilon(\omega) = \int \psi_\varepsilon(r) e^{-i\omega r}\,\mathrm{d}r$, $R(0) = -a$ and $R(\gamma) = A_\gamma$. By Lemma 8(c), $(\nu_\varepsilon * F)(y) = Z_\varepsilon(y) + O_\varepsilon(e^{-y/4})$ with

$$Z_\varepsilon(y) = \sum_{\omega \in \{0\} \cup \{\gamma : |\gamma| < T_0\}} \widehat{\psi}_\varepsilon(\omega) \Pi_{T_0}(\omega) R(\omega) e^{i\omega y} w_{1/2+i\omega}.$$

By dominated convergence, $c_\varepsilon \to \int P = k(0) = 1$ and $\widehat{\psi}_\varepsilon(\omega) \to \int P e^{-i\omega r} = k(\omega)$ for each ω. Since $k(\omega) = 0$ for $|\omega| \geq m$ and the sum is finite, $\sup_y \|Z_\varepsilon - Z^{(T_0)}\| \to 0$. Fix ε with $\|Z_\varepsilon(y_0)\| > \sqrt{c'} + 3\eta$ for some y_0 and some $\eta > 0$. The argument in the proof of Lemma 5(iv) gives ℓ, L_0 such that every interval of length L_0 in $[0, \infty)$ contains an interval of length ℓ on which $\|Z_\varepsilon\| \geq \sqrt{c'} + \eta$. For large y the error is less than $\eta/2$, so $\nu_\varepsilon * g \geq \sqrt{c'} + \eta/2$ there. Lemma 7 completes the proof. □

Corollary 17. *The set $\{N : Q_\lambda(N) > cN\}$ has positive lower logarithmic density for every $c < 4.6242$. If the distinct positive ordinates satisfy no relation $\sum c_\gamma \gamma = 0$ with $c_\gamma \in \{-2, \ldots, 2\}$ not all zero, the same holds for every $c > 0$.*

Proof. For the data of Corollary 13, either some pole on the segment is multiple and $c_* = \infty$, or Lemma 5(iv) gives $c_* \geq V > 4.6242$. Its second statement uses only the exponential sum Z. For the second assertion we may assume the Riemann hypothesis and that all zeros are simple, by Theorem 18. Then the proof of Corollary 14 and (35) give kernels with c_* arbitrarily large. □

The proof uses that v persists forward and dominates g^2. No such quantity dominates $(-E)_+$, so Theorem 16 still needs $\int_1^X L(x)^2 x^{-2}\,\mathrm{d}x \ll \log X$.

5.6. A Weighted Mellin Energy

The same mechanism applies to the smoothed sum

$$B(X) = \sum_{n \geq 1} \lambda(n) e^{-n/X} \qquad (X > 0),$$

whose Mellin transform carries a Gamma factor. For $\operatorname{Re} s > 1$, termwise integration with $\int_0^\infty e^{-n/X} X^{-s-1}\,\mathrm{d}X = \Gamma(s) n^{-s}$ gives

$$\int_0^\infty B(X) X^{-s-1}\,\mathrm{d}X = \Gamma(s) \frac{\zeta(2s)}{\zeta(s)}. \tag{42}$$

Define

$$J(\sigma) = \int_1^\infty |B(X)|^2 X^{-2\sigma-1}\,\mathrm{d}X \in [0, \infty], \qquad H(s) = \int_0^1 B(X) X^{-s-1}\,\mathrm{d}X.$$

Since $|B(X)| \leq \sum_n e^{-n/X} \leq 2e^{-1/X}$ for $0 < X \leq 1$, the function H is entire.

Theorem 19 (Finite weighted energy excludes zeros). *Let $\sigma > \frac{1}{2}$. If $J(\sigma) < \infty$, then $\zeta(s) \neq 0$ for $\operatorname{Re} s > \sigma$, and on that half-plane*

$$\left| \Gamma(s) \frac{\zeta(2s)}{\zeta(s)} - H(s) \right|^2 \leq \frac{J(\sigma)}{2(\operatorname{Re} s - \sigma)}. \tag{43}$$

Conversely $J(\sigma) < \infty$ for every $\sigma > \max(\Theta, \frac{1}{2})$. Consequently the Riemann hypothesis is equivalent to $J(\sigma) < \infty$ for every $\sigma > \frac{1}{2}$.

Proof. Let $T(s) = \int_1^\infty B(X) X^{-s-1}\,\mathrm{d}X$. For $\operatorname{Re} s > \sigma$ the Cauchy–Schwarz inequality gives

$$|T(s)|^2 \leq J(\sigma) \int_1^\infty X^{-2(\operatorname{Re} s - \sigma) - 1}\,\mathrm{d}X = \frac{J(\sigma)}{2(\operatorname{Re} s - \sigma)},$$

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and the same estimate, applied locally uniformly, shows that T is holomorphic on $\operatorname{Re} s > \sigma$. By (42), $\zeta(s)\big(T(s)+H(s)\big) = \Gamma(s)\zeta(2s)$ for $\operatorname{Re} s > \max(1,\sigma)$, and the identity extends to $\operatorname{Re} s > \sigma$. The right side is holomorphic and has no zero on $\operatorname{Re} s > \sigma \geq \frac{1}{2}$, so ζ has no zero there, and $T = \Gamma\zeta(2\cdot)/\zeta - H$. This gives (43).

For the converse, partial summation gives $B(X) = X^{-1}\int_1^\infty L(x)e^{-x/X}\,\mathrm{d}x$. If $\theta > \Theta$, then $L(x) \ll x^\theta$ by Theorem 6, so

$$|B(X)| \ll \frac{1}{X}\int_0^\infty x^\theta e^{-x/X}\,\mathrm{d}x = \Gamma(1+\theta)\,X^\theta,$$

and $J(\sigma) < \infty$ for $\sigma > \theta$.[11] If $J(\sigma) < \infty$ for every $\sigma > \frac{1}{2}$, then ζ has no zero in $\operatorname{Re} s > \frac{1}{2}$, and the functional equation gives the Riemann hypothesis. □

Proposition 9 (The cost of a pole). *Let ρ be a zero of ζ of order m with $\beta = \operatorname{Re}\rho > \frac{1}{2}$, and put*

$$a_\rho = \frac{m!\,\Gamma(\rho)\,\zeta(2\rho)}{\zeta^{(m)}(\rho)} \neq 0.$$

Then, with the value $+\infty$ allowed for J,

$$\liminf_{u\downarrow 0}\; u^{2m-1}J(\beta+u/2) \geq |a_\rho|^2. \tag{44}$$

Independently of the Riemann hypothesis,

$$\liminf_{u\downarrow 0}\; u\,J(\tfrac{1}{2}+u/2) \geq \Big|\frac{\Gamma(1/2)}{2\zeta(1/2)}\Big|^2 = \frac{\pi}{4\,\zeta(1/2)^2} > 0. \tag{45}$$

Proof. Let $u > 0$. If $J(\beta+u/2) = \infty$ the inequality is trivial. Otherwise apply (43) with $\sigma = \beta + u/2$ at the point $s = \rho + u$, where $2(\operatorname{Re} s - \sigma) = u$:

$$u^{2m-1}J(\beta+u/2) \geq \Big|u^m\Big(\Gamma(s)\frac{\zeta(2s)}{\zeta(s)} - H(s)\Big)\Big|^2_{s=\rho+u}.$$

Near ρ, $\Gamma(s)\zeta(2s)/\zeta(s) = a_\rho(s-\rho)^{-m} + O(|s-\rho|^{-m+1})$, with $a_\rho \neq 0$ because $\operatorname{Re}(2\rho) > 1$, and H is entire. So the right side tends to $|a_\rho|^2$. If $\beta < \Theta$, then $J(\beta+u/2) = \infty$ for all small u by Theorem 19; the inequality has content when $\beta = \Theta$. For (45) the same argument applies at $s = \frac{1}{2}+u$ with $\sigma = \frac{1}{2}+u/2$, using the simple pole of $\zeta(2s)$ at $s = \frac{1}{2}$, where $\Gamma(s)\zeta(2s)/\zeta(s)$ has residue $\Gamma(\frac{1}{2})/(2\zeta(\frac{1}{2}))$. □

The pole at $s = \frac{1}{2}$ is present whether or not the Riemann hypothesis holds. By (45) the energy $J(\sigma)$ grows at least like $0.368/(2\sigma-1)$ as $\sigma \downarrow \frac{1}{2}$, so a proof of the Riemann hypothesis through Theorem 19 cannot bound $J(\sigma)$ uniformly in $\sigma > \frac{1}{2}$. Table 5 compares the three energies of this part.

Table 5. Three energy criteria. In each row the finiteness or growth condition, for every $\sigma > \frac{1}{2}$, is equivalent to the Riemann hypothesis; for a fixed σ it lies between $\Theta < \sigma$ and $\Theta \leq \sigma$.

energy	condition at σ	transform	reference
$Q_\lambda(N)$	$Q_\lambda(N) \ll N^{2\sigma}$	$M(s)\,w_s$	Theorem 6
$\sum_N Q_\lambda(N)N^{-1-2\sigma}$	finite	$\frac{1}{2\pi}\int \lvert M\rvert^2 K\,\mathrm{d}\tau$	Theorem 7
$J(\sigma)$	finite	$\Gamma(s)\zeta(2s)/\zeta(s) - H(s)$	Theorem 19

6. Sensitivity to Sparse Sets of Primes

Theorem 6 identifies the growth exponent of $Q_\lambda(N)$ with 2Θ. A proof that this exponent is 1 must therefore use some property of λ that fails for every sequence whose energy grows faster. This section measures how fine that property has to be. We change the sign of λ on a set of primes and ask how

[11] Bounds for $B(X)$ under a zero-free hypothesis are treated in Proposition 22.

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thin the set can be while the exponent still moves. The answer is a set with about x^β elements up to x, for any $\beta > 1/2$, and no thinner.

For a real sequence a write $L_a(x) = \sum_{n\le x} a_n$, so that $L = L_\lambda$, and recall from Proposition 1 and (6) that

$$\tfrac{1}{2}|L_a(N)|^2 \le Q_a(N) \le 5 \max_{0\le j\le N} |L_a(j)|^2 \qquad (N \ge 1). \tag{46}$$

Definition 2. *Let S be a set of primes. The* modified Liouville function f_S *is the completely multiplicative function with $f_S(p) = +1$ for $p \in S$ and $f_S(p) = -1$ for every other prime.*[12] *We write $\pi_S(x) = \#\{p \in S : p \le x\}$ and*

$$G_S(s) = \prod_{p\in S} \frac{1+p^{-s}}{1-p^{-s}} = \sum_{d\ge 1} \frac{g_S(d)}{d^s},$$

where g_S is the multiplicative function with $g_S(p^j) = 2$ for $p \in S$, $j \ge 1$, and $g_S(p^j) = 0$ at the other prime powers.

Thus $f_\emptyset = \lambda$, and $g_S(d) = 2^{\omega(d)}$ when every prime factor of d lies in S.

Lemma 9 (Two convolutions)**.** *Let S be a set of primes with $\sum_{p\in S} p^{-\sigma_0} < \infty$ for some $\sigma_0 > 0$. The Dirichlet inverse g_S^{-1} is the multiplicative function with $g_S^{-1}(p^j) = 2(-1)^j$ for $p \in S$, $j \ge 1$, and zero at the other prime powers. Both series $\sum_d g_S(d)d^{-s}$ and $\sum_d g_S^{-1}(d)d^{-s}$ converge absolutely for $\operatorname{Re} s \ge \sigma_0$, with $\sum_d |g_S^{\pm 1}(d)|d^{-\sigma} = G_S(\sigma)$ for real $\sigma \ge \sigma_0$. Moreover $f_S = \lambda * g_S$ and $\lambda = f_S * g_S^{-1}$, so that for every $x \ge 1$*

$$L_{f_S}(x) = \sum_{d\le x} g_S(d) L(x/d), \qquad L(x) = \sum_{d\le x} g_S^{-1}(d) L_{f_S}(x/d). \tag{47}$$

Proof. For $|u| < 1$ one has $(1+u)/(1-u) = 1 + 2\sum_{j\ge1} u^j$ and $(1-u)/(1+u) = 1 + 2\sum_{j\ge1}(-1)^j u^j$. These are the local factors of g_S and of its inverse at $p \in S$, with $u = p^{-s}$. Since $p^{-\sigma_0} \le 2^{-\sigma_0} < 1$ and $\sum_{p\in S} p^{-\sigma_0} < \infty$, the product $G_S(\sigma)$ converges for $\sigma \ge \sigma_0$, and it equals the sum of the absolute values of either series. Both sides of $f_S = \lambda * g_S$ are multiplicative. At p^j with $p \notin S$ only $d = 1$ contributes, giving $\lambda(p^j) = f_S(p^j)$. At p^j with $p \in S$ the convolution equals $(-1)^j + 2\sum_{i=1}^{j}(-1)^{j-i} = 1$. Convolving with g_S^{-1} gives the second identity, and summing over $n \le x$ gives (47). □

Theorem 20 (Sensitivity to a sparse set of primes)**.** *Let $1/2 < \beta < 1$.*

(i) *Let S be a set of primes for which $\sum_{p\in S} p^{-\sigma}$ is finite for every $\sigma > \beta$ and infinite at $\sigma = \beta$. Then, for every $\alpha < \beta$,*

$$L_{f_S}(x) \ne O(x^\alpha), \qquad Q_{f_S}(N) \ne O(N^{2\alpha}).$$

(ii) *For the same S and every $\varepsilon > 0$,*

$$L_{f_S}(x) \ll_{\varepsilon,S} x^{\max(\beta,\Theta)+\varepsilon}, \qquad Q_{f_S}(N) \ll_{\varepsilon,S} N^{2\max(\beta,\Theta)+\varepsilon}.$$

(iii) *Such sets exist with $\pi_S(x) \asymp x^\beta$ for $x \ge x_0$. In particular $\sum_{p\in S} 1/p < \infty$.*

Proof. (iii) By the prime number theorem [49] the interval $(2^j, 2^{j+1}]$ contains more than $2^{j\beta}$ primes once $j \ge j_0$. For each such j let S_j consist of exactly $\lfloor 2^{j\beta} \rfloor$ of them, and put $S = \bigcup_{j\ge j_0} S_j$. For $2^J \le x < 2^{J+1}$ with $J > j_0$,

$$\lfloor 2^{(J-1)\beta} \rfloor \le \pi_S(x) \le \sum_{j\le J} 2^{j\beta} \le \frac{2^{J\beta}}{1-2^{-\beta}},$$

[12] This is the class studied by Borwein, Choi and Coons [28], who call its members Liouville-like functions and determine, among other things, which of them have small partial sums when S is defined by a congruence condition.

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so $\pi_S(x) \asymp x^\beta$. Moreover

$$\sum_{j\ge j_0} \lfloor 2^{j\beta}\rfloor 2^{-(j+1)\sigma} \le \sum_{p\in S} p^{-\sigma} \le \sum_{j\ge j_0} 2^{j(\beta-\sigma)},$$

and both outer series converge exactly when $\sigma > \beta$.

(i) Write $f = f_S$ and suppose $L_f(x) \ll x^\alpha$ with $\alpha < \beta$; we may take $\alpha > 1/2$. By partial summation

$$F(s) = s\int_1^\infty L_f(x) x^{-s-1}\,\mathrm{d}x$$

is holomorphic on $\operatorname{Re} s > \alpha$ and equals $\sum_n f(n)n^{-s}$ for $\operatorname{Re} s > 1$. Comparing Euler factors, $(1-p^{-s})^{-1}$ for $p \in S$ against $(1+p^{-s})^{-1}$, gives

$$F(s) = \frac{\zeta(2s)}{\zeta(s)}\, G_S(s) \qquad (\operatorname{Re} s > 1). \tag{48}$$

By Lemma 9 the function G_S is holomorphic on $\operatorname{Re} s > \beta$, where $\zeta(2s)/\zeta(s)$ is meromorphic. Hence (48) persists on $\operatorname{Re} s > \beta$ as an identity of meromorphic functions. On the real segment $\beta < \sigma < 1$ the factor $\zeta(2\sigma)/\zeta(\sigma)$ is finite and nonzero: $2\sigma > 1$, and $\zeta(\sigma) < 0$ on $(0,1)$ because $(1-2^{1-\sigma})\zeta(\sigma) = \sum_{n\ge1}(-1)^{n-1}n^{-\sigma} > 0$. As $\sigma \downarrow \beta$ the left side of (48) tends to the finite value $F(\beta)$ and $\zeta(2\sigma)/\zeta(\sigma)$ tends to the finite nonzero value $\zeta(2\beta)/\zeta(\beta)$. So $G_S(\sigma)$ stays bounded. But $g_S(p) = 2$ and $g_S \ge 0$ give

$$G_S(\sigma) \ge 1 + 2\sum_{p\in S} p^{-\sigma} \longrightarrow \infty \qquad (\sigma \downarrow \beta)$$

by monotone convergence. This contradiction proves the first assertion. If $Q_f(N) \ll N^{2\alpha}$, then (46) gives $L_f(N) \ll N^\alpha$ at the integers, hence at all real x, which has just been excluded.

(ii) If $\Theta = 1$ the bounds are trivial. Otherwise let $\theta' = \max(\beta, \Theta) + \varepsilon < 1$. Theorem 6 gives $|L(y)| \le C y^{\theta'}$ for $y \ge 1$, and $G_S(\theta') < \infty$. By (47),

$$|L_f(x)| \le C\sum_{d\le x} g_S(d)\left(\frac{x}{d}\right)^{\theta'} \le C\, G_S(\theta')\, x^{\theta'}.$$

The bound for Q_f follows from the upper inequality in (46). □

Remark 7 (Attribution). *Part (i) is not new. Aymone [18] proved that if a completely multiplicative f with values in $[-1,1]$ has $\sum_{n\le x} f(n) \ll x^{1-\delta}$ and its Dirichlet series vanishes at $s = 1$, then $\sum_{p\le x}(1+f(p))\log p \ll x^{1-\delta+\varepsilon}$. For $f = f_S$ the last sum is $2\sum_{p\le x,\, p\in S}\log p$, so a bound $L_{f_S}(x) \ll x^\alpha$ forces $\pi_S(x) \ll x^{\alpha+\varepsilon}$, and part (i) is the contrapositive within the class of Definition 2. The argument above is a short self-contained proof of this case. Section 10 gives a refinement in which the Dirichlet series of f_S is computed in closed form up to a nonvanishing factor.*

The content of the theorem for the energy problem is the following two-sided threshold.

Corollary 18 (The threshold $1/2$). *Let $1/2 \le \theta < 1$.*

(a) *For every β with $\theta < \beta < 1$, some set S of primes with $\pi_S(x) \asymp x^\beta$ breaks the exponent: there is a completely multiplicative $f : \mathbb{N} \to \{\pm1\}$ that agrees with λ at every prime outside S and for which $Q_f(N) \ll_\varepsilon N^{2\theta+\varepsilon}$ fails for every $\varepsilon < 2(\beta-\theta)$. This holds whether or not ζ has zeros in $\operatorname{Re} s > \theta$.*

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(b) *No set of size $x^{1/2+o(1)}$ does. If $\sum_{p\in S} p^{-\sigma} < \infty$ for every $\sigma > 1/2$, in particular if $\pi_S(x) \ll x^{1/2+o(1)}$, then for every $\theta \in [1/2, 1]$ the three statements*

$$Q_{f_S}(N) \ll_\varepsilon N^{2\theta+\varepsilon} \text{ for every } \varepsilon > 0, \qquad Q_\lambda(N) \ll_\varepsilon N^{2\theta+\varepsilon} \text{ for every } \varepsilon > 0,$$
$$\zeta(s) \neq 0 \text{ for } \operatorname{Re} s > \theta$$

are equivalent, and $\limsup_N \log(1 + Q_{f_S}(N))/\log N = 2\Theta$.

Consequently, any argument that proves $Q_a(N) \ll_\varepsilon N^{2\theta+\varepsilon}$ from hypotheses on a which hold for every completely multiplicative sign sequence agreeing with λ outside a set of $O(x^\beta)$ primes, for some $\beta > \theta$, is invalid. A proof of the exponent 1 must detect a change of λ on $O(x^{1/2+\delta})$ primes up to x, for every $\delta > 0$.

Proof. (a) Take S from Theorem 20(iii) and $f = f_S$. If $Q_f(N) \ll N^{2\theta+\varepsilon}$ with $\theta + \varepsilon/2 < \beta$, then part (i) with $\alpha = \theta + \varepsilon/2$ is contradicted.

(b) Partial summation shows that $\pi_S(x) \ll x^{1/2+o(1)}$ implies the convergence of $\sum_{p\in S} p^{-\sigma}$ for $\sigma > 1/2$. Under that convergence Lemma 9 applies with every $\sigma_0 > 1/2$. Let $\theta \geq 1/2$ and $\varepsilon > 0$. If $|L(y)| \leq Cy^{\theta+\varepsilon}$, the first identity in (47) gives $|L_{f_S}(x)| \leq CG_S(\theta+\varepsilon)x^{\theta+\varepsilon}$. If $|L_{f_S}(y)| \leq Cy^{\theta+\varepsilon}$, the second gives the same bound for L, because $\sum_d |g_S^{-1}(d)|d^{-\theta-\varepsilon} = G_S(\theta+\varepsilon)$. By (46), bounds $O(x^{\theta+\varepsilon})$ for the partial sums for all ε are equivalent to bounds $O(N^{2\theta+\varepsilon})$ for the energy for all ε, for either function. Theorem 6 supplies the third statement.

For the exponent, write $\kappa = \limsup_N \log(1 + Q_{f_S}(N))/\log N$. Since $\Theta \geq 1/2$ and $\zeta \neq 0$ on $\operatorname{Re} s > \Theta$, the equivalence with $\theta = \Theta$ gives $\kappa \leq 2\Theta$. If $\kappa < 2\Theta$ and $\Theta > 1/2$, choose $\theta \in [1/2, \Theta)$ with $2\theta > \kappa$; then $Q_{f_S}(N) \ll N^{2\theta+\varepsilon}$ for every $\varepsilon > 0$, and the equivalence gives $\zeta \neq 0$ on $\operatorname{Re} s > \theta$, contradicting the definition of Θ. If $\kappa < 1 = 2\Theta$, choose $\kappa < 2\kappa' < 1$. By (46), $L_{f_S}(x)^2 \leq 2Q_{f_S}(\lfloor x \rfloor) \ll x^{2\kappa'}$, so the Dirichlet series $F_S(s) = \sum_n f_S(n)n^{-s}$ converges and is holomorphic on $\operatorname{Re} s > \kappa'$, in particular bounded near $s = 1/2$. For real $\sigma > 1/2$, however, comparing Euler factors gives

$$F_S(\sigma) = \frac{\zeta(2\sigma)}{\zeta(\sigma)} G_S(\sigma), \qquad G_S(\sigma) = \prod_{p\in S} \frac{1+p^{-\sigma}}{1-p^{-\sigma}} \geq 1,$$

and as $\sigma \downarrow 1/2$ one has $\zeta(2\sigma) \to +\infty$ and $\zeta(\sigma) \to \zeta(1/2) < 0$, so $F_S(\sigma) \to -\infty$. This contradiction gives $\kappa = 2\Theta$. □

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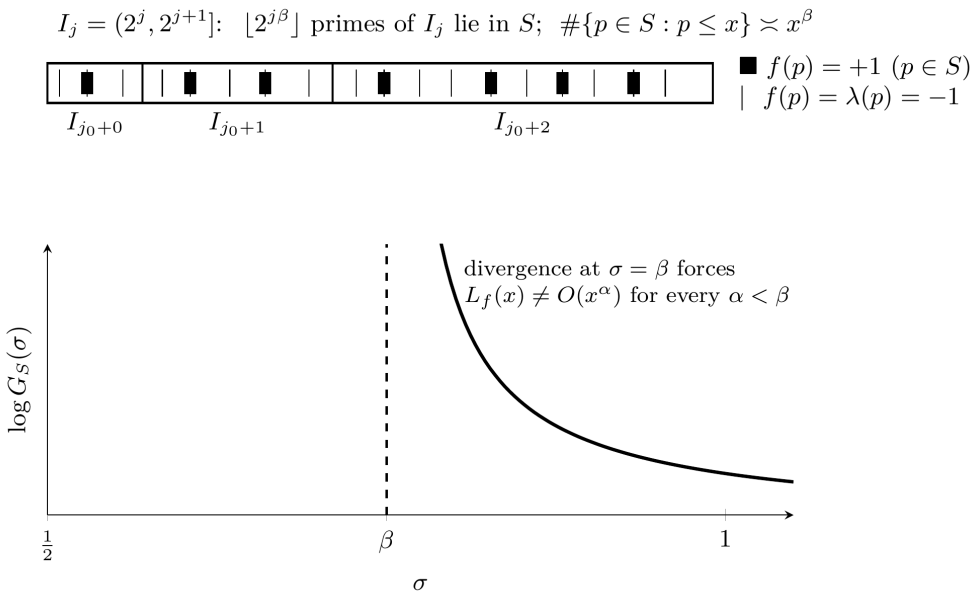

Figure 6. The sparse flip of Corollary 18. Top: in each dyadic interval $I_j = (2^j, 2^{j+1}]$ about $2^{j\beta}$ primes are placed in S (filled), and f is changed from $\lambda(p) = -1$ to $+1$ on them; all other primes keep $f(p) = \lambda(p)$. Bottom: the factor $G_S(\sigma)$ of the Dirichlet series diverges as $\sigma \downarrow \beta$, so the series of f has a singularity at β, and $L_f(x)$ cannot be $O(x^\alpha)$ for any $\alpha < \beta$.

The threshold is sharp in both directions: above $1/2$ some sparse set changes the exponent, and at $1/2$ none does.

Remark 8 (Every set with abscissa β)**.** *The divergence at $\sigma = \beta$ in Theorem 20(i) can be replaced by the weaker hypothesis that $\beta \in (1/2, 1)$ is the abscissa of convergence of $\sum_{p \in S} p^{-s}$, which holds for instance whenever $\pi_S(x) \asymp x^\beta$. Indeed G_S has nonnegative coefficients and the same abscissa of convergence, so by Landau's theorem [3] [Ch. 11] the point $s = \beta$ is a singularity of G_S. If $L_{f_S}(x) \ll x^\alpha$ with $1/2 < \alpha < \beta$, then $G_S(s) = F(s)\zeta(s)/\zeta(2s)$ would be holomorphic near every real point of $(\alpha, 1)$, since $\zeta(2\sigma) > 1$ there. This agrees with Aymone's theorem quoted in Remark 7.*

The next proposition lists what the functions f_S share with λ. For multiplicative f, g with values in the closed unit disc, the pretentious distance up to x is

$$\mathbb{D}(f, g; x) = \Big(\sum_{p \le x} \frac{1 - \operatorname{Re} f(p)\overline{g(p)}}{p} \Big)^{1/2}.$$

Proposition 10 (Hypotheses invariant under the flip)**.** *Let S be a set of primes with $\sum_{p \in S} 1/p < \infty$ and $f = f_S$.*

(a) *f is completely multiplicative with values ± 1, and the primes with $f(p) \neq \lambda(p)$ have relative logarithmic density zero among the primes. If $\pi_S(x) \ll x^\beta$ with $\beta < 1$, they have relative natural density zero.*

(b) *$\mathbb{D}(f, \lambda; x)^2 \le 2 \sum_{p \in S} 1/p$ for every x. For every Dirichlet character χ, real t and $x \ge 2$,*

$$\big| \mathbb{D}(f, \chi n^{it}; x) - \mathbb{D}(\lambda, \chi n^{it}; x) \big| \le \Big(2 \sum_{p \in S} \frac{1}{p} \Big)^{1/2}.$$

(c) *f satisfies the hypothesis of Tao's theorem [56], and therefore, for each fixed $d \ge 1$,*

$$\sum_{t \le T} \frac{f(t) f(t+d)}{t} = o_{d,S}(\log T).$$

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(d) *If moreover* $\sum_{p\in S} p^{-\sigma_0} < \infty$ *for some* $\sigma_0 < 1$, *then for every* $A > 0$, *uniformly in real* α,

$$\sum_{n\le x} f(n)\mathrm{e}(n\alpha) \ll_{A,S} \frac{x}{(\log x)^A} \qquad (x \ge 2).$$

(e) *For any* $P \ge 2$, *removing from* S *the primes up to* P *preserves the hypotheses and conclusions of Theorem 20 and gives* $f(n) = \lambda(n)$ *for all* $n \le P$.

Proof. (a) This is the definition, and $\sum_{p\in S,\, p\le x} 1/p = O(1)$ while $\sum_{p\le x} 1/p \to \infty$.

(b) At a prime $p \notin S$ the two functions agree, and at $p \in S$ one has $1 - f(p)\lambda(p) = 2$. For the second claim, let q be the modulus of χ and $g(n) = \chi(n)n^{it}$. At primes $p \nmid q$ all three functions are unimodular and $1 - \operatorname{Re} u\bar{v} = \frac{1}{2}|u - v|^2$. Hence

$$\mathbb{D}(f, g; x)^2 = \sum_{\substack{p\le x\\ p\nmid q}} \frac{|f(p) - g(p)|^2}{2p} + \sum_{\substack{p\le x\\ p\mid q}} \frac{1}{p},$$

and likewise for λ. The first sums are squared ℓ^2 norms, to which Minkowski's inequality applies, and the second sums coincide. The claim follows from the inequality $|\sqrt{a^2 + c^2} - \sqrt{b^2 + c^2}| \le |a - b|$.

(c) Tao's theorem applies to a multiplicative function g_1 bounded by 1 such that, for every A, one has $\inf \mathbb{D}(g_1, \chi n^{it}; x)^2 \ge A$ for all large x, the infimum being over characters of period at most A and real $|t| \le Ax$. It is shown in [56], using [47], that λ has this property. By (b) it passes to f, with A replaced by $(\sqrt{A} + (2\sum_{p\in S} 1/p)^{1/2})^2$. The conclusion of the theorem with $g_1 = g_2 = f$ is the displayed estimate.

(d) By Lemma 9, $\sum_{n\le x} f(n)\mathrm{e}(n\alpha) = \sum_{d\le x} g_S(d) \sum_{m\le x/d} \lambda(m)\mathrm{e}(md\alpha)$. Davenport's bound $\sup_\alpha |\sum_{m\le y} \lambda(m)\mathrm{e}(m\alpha)| \le C_A y(\log 2y)^{-A}$, [41, Ch. 13], [47], applies to the inner sum for $d \le \sqrt{x}$, where $\log(2x/d) \ge \frac{1}{2}\log x$. For $d > \sqrt{x}$ use the trivial bound x/d and $\sum_{d>\sqrt{x}} g_S(d)/d \le x^{(\sigma_0-1)/2} G_S(\sigma_0)$. Altogether

$$\Big|\sum_{n\le x} f(n)\mathrm{e}(n\alpha)\Big| \le 2^A C_A G_S(1) \frac{x}{(\log x)^A} + G_S(\sigma_0) x^{(1+\sigma_0)/2}.$$

(e) Removing finitely many primes changes neither the convergence nor the divergence of $\sum_{p\in S} p^{-\sigma}$, and an integer $n \le P$ has all its prime factors at most P. □

Table 6 summarizes the comparison. Every hypothesis in its upper block holds for λ and for each f_S of Theorem 20, so none of them, alone or in combination, can yield an energy exponent below 2β; and β may be taken arbitrarily close to 1. In particular the logarithmically averaged Chowla theorem cannot give any exponent smaller than 2. What separates λ from these functions is the location of the singularities of $\sum_n f(n)n^{-s}$ in the strip $1/2 < \operatorname{Re} s < 1$, which for λ are exactly the zeros of ζ.

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Table 6. Properties of λ and of a modified Liouville function f_S with $\sum_{p\in S} p^{-\sigma}$ convergent exactly for $\sigma > \beta$, where $1/2 < \beta < 1$. The upper block is invariant under the flip; the lower block is not.

Property	λ	f_S	Reference
completely multiplicative, values ±1	yes	yes	Definition 2
value -1 at a density-one set of primes	yes	yes	Proposition 10(a)
bounded pretentious distance from λ	yes	yes	Proposition 10(b)
non-pretentious in the sense of [56]	yes	yes	Proposition 10(c)
log-averaged two-point Chowla	yes	yes	Proposition 10(c)
saving $(\log x)^{-A}$ in every additive twist	yes	yes	Proposition 10(d)
agreement with λ on $[1, P]$, any fixed P	yes	can be imposed	Proposition 10(e)
simple zero of $\sum f(n)n^{-s}$ at $s = 1$	yes	yes	Proposition 17
derivative of $\sum f(n)n^{-s}$ at $s = 1$ equal to $\zeta(2)$	yes	no	Proposition 17
singularities in $1/2 < \mathrm{Re}\, s < 1$	zeros of ζ	also $s = \beta$	Theorem 20
exponent of $Q_f(N)$	2Θ	$2\max(\beta, \Theta)$	Theorem 27

7. Shift Correlations and the Maximal Collar

The expansion (50) below writes $|\sum_{n\le N} a_n \mathrm{e}(n\alpha)|^2$ as a sum over shifts of two-point correlations. A natural way to bound it is to estimate each correlation separately, in absolute value and uniformly in its endpoint, and to add the results. This section examines that approach. The hypothesis it needs would imply the generalized Riemann hypothesis for every Dirichlet character; no estimate that is uniform in the coefficients can supply it; the general maximal inequality that is available is a full power of N away; and Sections 8 and 9 show that the hypothesis fails for natural model sequences and, at its endpoint, for λ itself.

7.1. Definitions and Two Identities

Write $\mathrm{e}(t) = e^{2\pi i t}$. A sequence $a = (a_n)_{n\ge1}$ is real with $|a_n| \le 1$ unless stated otherwise. Put

$$A_N(\alpha) = \sum_{n=1}^{N} a_n \mathrm{e}(n\alpha), \qquad C_{a,h}(Y) = \sum_{n=1}^{Y} a_n a_{n+h} \qquad (h \ge 1,\ Y \ge 0),$$

so that $C_{a,h}(0) = 0$. For $1 \le h < N$ let

$$C^*_{a,h}(N) = \max_{0\le Y\le N-h} |C_{a,h}(Y)|,$$

and put $C^*_{a,h}(N) = 0$ for $h \ge N$. The *collar energy* on a dyadic range of shifts is

$$\mathbb{E}_a(N, H) = \sum_{H<h\le 2H} C^*_{a,h}(N)^2 \qquad (H = 1, 2, 4, \dots).$$

The dyadic ranges cover every shift $h \ge 2$ once. The *maximal collar hypothesis* for a is the family of bounds

$$\mathbb{E}_a(N, H) \ll_\varepsilon \frac{N^{2+\varepsilon}}{H} \qquad (1 \le H < N,\ H \text{ dyadic},\ \varepsilon > 0). \tag{49}$$

It asserts square-root cancellation, on average over $h \in (H, 2H]$, in every prefix of every shift correlation.

Lemma 10 (Square and twist identities)**.** *For every real α,*

$$|A_N(\alpha)|^2 = \sum_{n\le N} a_n^2 + 2\,\mathrm{Re} \sum_{h=1}^{N-1} \mathrm{e}(h\alpha)\, C_{a,h}(N-h). \tag{50}$$

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For every finite set J of shifts,

$$\Big(\sum_{h\in J} C^*_{a,h}(N)^2\Big)^{1/2} = \max_{(Y_h)} \sup_{\sum_{h\in J}|\beta_h|^2\le 1} \Big|\sum_{h\in J}\beta_h C_{a,h}(Y_h)\Big|, \tag{51}$$

where $0 \le Y_h \le N-h$ for $h < N$ and the terms with $h \ge N$ are omitted.

Proof. Group the terms of $A_N(\alpha)\overline{A_N(\alpha)}$ by the difference of their indices; the pairs $(n+h,n)$ and $(n,n+h)$ with $1 \le n \le N-h$ contribute $\mathrm{e}(h\alpha)a_n a_{n+h}$ and its conjugate. For fixed endpoints (Y_h), the Cauchy–Schwarz inequality gives the dual norm in (51), with equality at $\beta_h = \overline{C_{a,h}(Y_h)}/\|C\|_2$. The maxima over the Y_h are attained and are independent of one another. □

The second identity explains the name: $\mathbb{E}_a(N,H)^{1/2}$ is the largest value of a linear form in the correlations in which the coefficients β_h and the endpoints Y_h may be chosen separately for every shift.

Lemma 11 (Square divisors). *For every $n \ge 1$, $\lambda(n) = \sum_{r^2\mid n}\mu(n/r^2)$. Consequently, for $Y, h \ge 1$,*

$$C_{\lambda,h}(Y) = \sum_{r\le\sqrt{Y}}\sum_{s\le\sqrt{Y+h}}\sum_{\substack{1\le n\le Y\\ r^2\mid n,\ s^2\mid n+h}} \mu(n/r^2)\,\mu((n+h)/s^2),$$

and the inner sum is empty unless $\gcd(r,s)^2 \mid h$.

Proof. Both sides of the first identity are multiplicative, and at p^e only the term with $e-2j \in \{0,1\}$ survives in $\sum_{2j\le e}\mu(p^{e-2j})$, giving $(-1)^e$. Insert the identity in both factors of the correlation. The congruences $n \equiv 0 \pmod{r^2}$ and $n \equiv -h \pmod{s^2}$ are compatible exactly when $\gcd(r^2,s^2) = \gcd(r,s)^2$ divides h. □

The stratum $r = s = 1$ is the Möbius correlation $\sum_{n\le Y}\mu(n)\mu(n+h)$. The identity keeps all signs. Replacing the Möbius values by their absolute values gives a quantity of order Y, that is, the trivial bound behind the coefficient-uniform maximum of Proposition 12.

7.2. What the Hypothesis Would Give

Theorem 21 (Consequence of the full maximal collar). *Suppose (49) holds for $a = \lambda$. Then*

$$\sup_{\alpha\in\mathbb{R}}\Big|\sum_{n\le N}\lambda(n)\mathrm{e}(n\alpha)\Big| \ll_\varepsilon N^{1/2+\varepsilon}$$

for every $\varepsilon > 0$, and every Dirichlet L-function $L(s,\chi)$, as well as ζ, is free of zeros in $\operatorname{Re} s > 1/2$.

Proof. By (50), $|A_N(\alpha)|^2 \le N + 2\sum_{h<N} C^*_{\lambda,h}(N)$. The shift $h = 1$ contributes at most N. On a dyadic block the Cauchy–Schwarz inequality and (49) give

$$\sum_{H<h\le 2H} C^*_{\lambda,h}(N) \le \sqrt{H}\,\mathbb{E}_\lambda(N,H)^{1/2} \ll_\varepsilon N^{1+\varepsilon/2},$$

and there are at most $1+\log_2 N$ blocks. This proves the first assertion after renaming ε.

Let χ be a character modulo q and $\widehat{\chi}(r) = q^{-1}\sum_{b \bmod q}\chi(b)\mathrm{e}(-rb/q)$, so that $\chi(n) = \sum_{r \bmod q}\widehat{\chi}(r)\mathrm{e}(rn/q)$ and $\sum_r|\widehat{\chi}(r)|^2 \le 1$ by Parseval's identity. Then $\sum_r|\widehat{\chi}(r)| \le \sqrt{q}$ and

$$\sum_{n\le x}\lambda(n)\chi(n) \ll_\varepsilon \sqrt{q}\,x^{1/2+\varepsilon}.$$

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By partial summation, $F_\chi(s) = s\int_1^\infty \big(\sum_{n\le x}\lambda(n)\chi(n)\big)x^{-s-1}\,\mathrm{d}x$ is holomorphic on $\operatorname{Re} s > 1/2$. For $\operatorname{Re} s > 1$ the Euler factors $(1+\chi(p)p^{-s})^{-1} = (1-\chi(p)p^{-s})/(1-\chi(p)^2p^{-2s})$ give

$$L(s,\chi)\,F_\chi(s) = L(2s,\chi^2).$$

The identity extends to $\operatorname{Re} s > 1/2$, where the right side is a convergent Euler product without zeros. A zero of $L(s,\chi)$ there would make the left side vanish. The trivial character modulo 1 gives the statement for ζ. □

For primitive χ the functional equation then confines the nontrivial zeros of $L(s,\chi)$ to the critical line. The hypothesis (49) for λ is thus at least as strong as the Riemann hypothesis for all Dirichlet L-functions simultaneously, with the uniformity in q of the displayed character-sum bound. Only the short and middle ranges of shifts need the collar form:

Proposition 11 (Terminal-range form). *Fix $0 < \eta < 1$. Suppose that, for every $\delta > 0$,*

$$\mathbb{E}_a(N,H) \ll_{\eta,\delta} \frac{N^{2+\delta}}{H} \quad (H \le N^{1-\eta},\ H \text{ dyadic}),$$
$$\sum_{N^{1-\eta}<h<N} C^*_{a,h}(N) \ll_{\eta,\delta} N^{1+\delta}.$$

Then $\sup_\alpha |A_N(\alpha)| \ll_{\eta,\varepsilon} N^{1/2+\varepsilon}$ for every $\varepsilon > 0$. For $a = \lambda$ this implies RH.

Proof. The term $h = 1$ is at most N. Each dyadic block beginning at $H \le N^{1-\eta}$ contributes $O_{\eta,\delta}(N^{1+\delta/2})$ to $\sum_h C^*_{a,h}(N)$, as in the preceding proof, and there are $O(\log 2N)$ of them. Every shift not covered by these blocks or by $h = 1$ exceeds $N^{1-\eta}$, and the sum over those is bounded by hypothesis. Now use (50). For $a = \lambda$ take $\alpha = 0$ and apply Theorem 6. □

7.3. The Coefficient-Uniform Maximum

Proposition 12 (Exact coefficient-uniform maximum). *For all N and all dyadic H,*

$$\sup_{|a_n|\le 1} \mathbb{E}_a(N,H) = \sum_{H<h\le \min(2H,N-1)} (N-h)^2.$$

The supremum is attained by $a_n = 1$ and by $a_n = (-1)^n$. If $2H \le N/2$ it lies between $HN^2/4$ and HN^2.

Proof. One has $|C_{a,h}(Y)| \le Y \le N-h$. For the two stated sequences $a_n a_{n+h}$ is constant in n and of modulus one, so $|C_{a,h}(N-h)| = N-h$. If $2H \le N/2$, each of the H terms lies between $N^2/4$ and N^2. □

The two extremal sequences behave in opposite ways in the signed energy of Proposition 1: $Q_{\mathbf{1}}(N) = (2N^2+1)/3$ while $Q_a(N) = 1$ for $a_n = (-1)^n$. The collar energy cannot tell them apart, because the absolute value in C^* removes the sign $(-1)^h$ that produces the cancellation between shifts. Table 7 compares the uniform maximum with the scale in (49).

Table 7. Exponents of N at $H \asymp N^u$, $0 < u < 1$. The last column is the ratio of the uniform maximum to the collar scale with $\varepsilon = 0$.

	general u	$u = 1/4$	$u = 1/2$	$u = 3/4$
$\sup_{\lvert a_n\rvert\le 1} \mathbb{E}_a(N,H)$	$2+u$	$9/4$	$5/2$	$11/4$
collar scale N^2/H	$2-u$	$7/4$	$3/2$	$5/4$
ratio	$2u$	$1/2$	1	$3/2$

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Corollary 19 (No coefficient-uniform proof). *Let $0 < u < 1$ and let H be the largest dyadic integer not exceeding N^u. For all sufficiently large N,*

$$\mathbb{E}_{\mathbf{1}}(N,H) \geq \tfrac{1}{8}N^{2+u}, \qquad \frac{N^{2+\varepsilon}}{H} \leq 2N^{2-u+\varepsilon}.$$

Hence no bound of the form (49) with $\varepsilon < 2u$ holds with a constant independent of the bounded sequence a. In particular no argument that uses only $|a_n| \leq 1$ proves (49) for any $\varepsilon < 2$.

Proof. Here $N^u/2 < H \leq N^u$, and $2H \leq N/2$ for large N. Proposition 12 gives $\mathbb{E}_{\mathbf{1}}(N,H) \geq HN^2/4 > N^{2+u}/8$. □

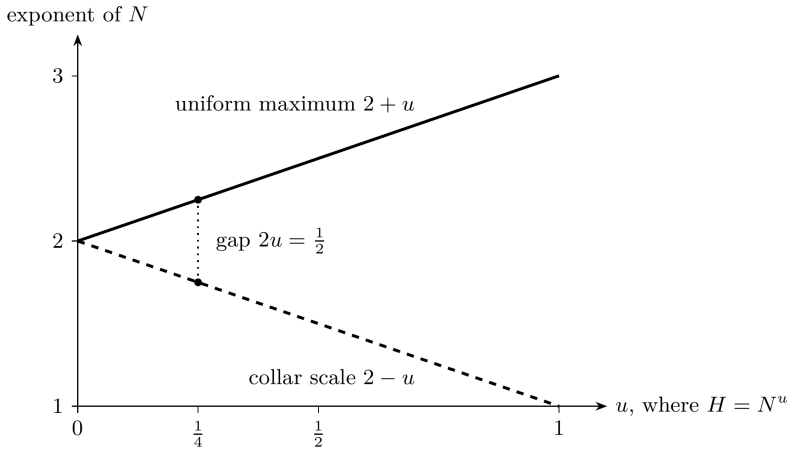

Figure 7. The exact coefficient-uniform maximum of the collar energy, of size N^{2+u} when $H = N^u$, against the collar scale N^{2-u}. The two lines separate at once; at $u = \frac{1}{4}$ the gap is already $N^{1/2}$. No bound that treats the shift correlations uniformly in the coefficients can close this gap.

7.4. The Maximal Bessel Bound

A maximal inequality over prefixes is available whenever a Bessel inequality holds on a common index set. It is the Rademacher–Menshov device, which costs only a squared logarithm.[13]

Theorem 22 (Maximal Bessel bound). *Let (Ω,μ) be a measure space and let $\rho_1,\dots,\rho_M \in L^2(\Omega,\mu)$ satisfy*

$$\int_\Omega \Big|\sum_{n=1}^{M} b_n\rho_n\Big|^2 \,\mathrm{d}\mu \leq B\sum_{n=1}^{M}|b_n|^2 \qquad \text{for every } b\in\mathbb{C}^M.$$

Then, with $K = \lceil \log_2 M\rceil$,

$$\int_\Omega \max_{0\leq Y\leq M}\Big|\sum_{n\leq Y} b_n\rho_n\Big|^2 \,\mathrm{d}\mu \leq (K+1)^2 B\sum_{n=1}^{M}|b_n|^2.$$

Proof. Extend b by zero to length 2^K. Call an interval $(i2^k,(i+1)2^k]$ with $0\leq k\leq K$ aligned. Reading the binary expansion of Y from the top shows that $[1,Y]$ is a disjoint union of at most one aligned interval of each length, hence of at most $K+1$ aligned intervals. By the Cauchy–Schwarz inequality the squared prefix sum is at most $K+1$ times the sum of the squares of the corresponding block sums, and therefore at most $K+1$ times the sum over all aligned intervals I of $|\sum_{n\in I} b_n\rho_n|^2$, which no

[13] The dyadic decomposition of a prefix into aligned blocks goes back to Rademacher and Menshov in their work on orthogonal series. The form below is the one that combines with spectral large sieve inequalities of Deshouillers–Iwaniec type [33]: if the fixed-endpoint inequality holds with constant $B \ll_\varepsilon (T^2+N)(TN)^\varepsilon$ for coefficients supported on $N < n \leq 2N$, then the maximal version holds with the same exponents.

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longer depends on Y. Integrate. For each k the aligned intervals of length 2^k partition $[1, 2^K]$, and the hypothesis applied to each restricted vector bounds the integral of that level by $B\sum_n |b_n|^2$. There are $K+1$ levels. □

Proposition 13 (The additive benchmark)**.** *For every real sequence a with $|a_n| \le 1$, every $N \ge 2$ and every dyadic H,*

$$\mathbb{E}_a(N,H) \le (1+\lceil \log_2 N\rceil)^2 \, \|A_N\|^2_{L^\infty(\mathbb{R}/\mathbb{Z})} \sum_{n\le N} a_n^2.$$

In particular $\mathbb{E}_\lambda(N,H) \ll_A N^3(\log N)^{-A}$ for every $A>0$, uniformly in H.

Proof. Let $a^{(N)} = a\mathbf{1}_{[1,N]}$, extended by zero to $\mathbb{Z}$, and $K = \lceil \log_2 N\rceil$. For an aligned interval $I \subset [1,2^K]$ put

$$c_I(h) = \sum_{n\in I} a_n^{(N)} a_{n+h}^{(N)} \qquad (h\in\mathbb{Z}).$$

If $0 \le Y \le N-h$, then $n+h \le N$ for every $n \le Y$, so $C_{a,h}(Y)$ is the sum of $c_I(h)$ over the at most $K+1$ aligned intervals that make up $[1,Y]$. As in the proof of Theorem 22,

$$C^*_{a,h}(N)^2 \le (K+1)\sum_I |c_I(h)|^2.$$

Let $A_I(\alpha) = \sum_{n\in I} a_n^{(N)} \mathrm{e}(n\alpha)$. The Fourier series of $h \mapsto c_I(h)$ is $\overline{A_I(\alpha)}A_N(\alpha)$, and by Parseval's identity

$$\sum_{h\in\mathbb{Z}} |c_I(h)|^2 = \int_0^1 |A_I(\alpha)|^2 |A_N(\alpha)|^2 \,\mathrm{d}\alpha \le \|A_N\|_\infty^2 \sum_{n\in I} |a_n^{(N)}|^2.$$

Summing over the intervals of one length gives at most $\|A_N\|_\infty^2 \sum_{n\le N} a_n^2$, and there are $K+1$ lengths. For λ insert Davenport's bound $\|A_N\|_\infty \ll_A N(\log N)^{-A}$, in the Liouville form quoted in [47]; see also [41, Ch. 13]. □

Proposition 13 is essentially all that the maximal method yields from additive information: a bound $\|A_N\|_\infty \ll N^{1/2+\varepsilon}$, which is the conclusion of Theorem 21, returns only $\mathbb{E}_a(N,H) \ll N^{2+\varepsilon}$, without the factor $1/H$ of (49). The loss is real. Theorem 24 exhibits a sign sequence with $\|A_N\|_\infty \ll \sqrt{N}$ and $\mathbb{E}(3H,H) \ge H^2$. Table 8 collects the scales.

Table 8. The collar energy $\mathbb{E}_a(N,H)$ at $N=3H$: hypothesis, benchmarks and lower bounds.

Statement	Size at $N=3H$	Source
maximal collar hypothesis (49)	$\ll H^{1+\varepsilon}$	would imply GRH
uniform maximum over $\|a_n\|\le 1$	$\asymp H^3$	Proposition 12
additive benchmark for λ	$\ll_A H^3(\log H)^{-A}$	Proposition 13
Rudin–Shapiro signs	$\ge H^2$	Theorem 24
independent signs, almost surely	$\ge H^2/2$	Proposition 14
random multiplicative signs, infinitely often	$\gg H^2/\log H$	Proposition 15
every sign sequence	$\mathbb{E}/H \to \infty$; $\gg H\log H$ for a positive proportion of scales	Theorem 25
λ, fixed endpoint M	mean of $\mathbb{E}/H$ at least $M\log 2$	Theorem 26
λ, under power-saving Chowla	$\gg H^{1+\kappa}$	Proposition 16

8. Model Sequences

Five models show that (49) fails for sequences that share with λ the properties one might hope to use. The first keeps the arithmetic data at any finite set of primes. The second has square-root cancellation in every additive twist, which is the conclusion of Theorem 21. The third and fourth are random. The last is a statement about every sign sequence.

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8.1. Finite Prime Data

Theorem 23 (Finite prime data)**.** *Let P be a finite set of primes and $(\epsilon_p)_{p\in P}$ signs. There is a completely multiplicative $f:\mathbb{N}\to\{\pm1\}$ with $f(p)=\epsilon_p$ for $p\in P$ and*

$$\Big|\sum_{n\le N} f(n)\Big| \ll_P (1+\log N)^{|P\cup\{3\}|}.$$

There is an integer $D\ge 1$ depending only on P such that $\mathbb{E}_f(N,H)\ge HN^2/(32D)$ whenever $2D\le H\le N/4$.

Proof. Let $S=P\cup\{3\}$, $W=\prod_{p\in S}p$, and let χ_3 be the nonprincipal character modulo 3. Put $f(p)=\epsilon_p$ on P, $f(3)=1$ if $3\notin P$, and $f(p)=\chi_3(p)$ for $p\notin S$. Write $n=bm$ with b composed of primes in S and $(m,W)=1$; then $f(n)=f(b)\chi_3(m)$. The function $m\mapsto\chi_3(m)\mathbf{1}_{(m,W)=1}$ has period W, and its sum over a period vanishes: by the Chinese remainder theorem the reduced residues modulo W that are 1 modulo 3 and those that are 2 modulo 3 are equinumerous. Each inner sum over $m\le N/b$ is therefore at most W in absolute value, and the number of admissible $b\le N$ is at most $\prod_{p\in S}(1+\log N/\log p)$.

Choose k with $\sum_{p\in S}p^{-k}\le 1/4$ and put $D=\prod_{p\in S}p^k$. Let $D\mid h$ and let n satisfy $p^k\nmid n$ for every $p\in S$. Then $v_p(n+h)=v_p(n)$ for $p\in S$, so n and $n+h$ have the same factor b; moreover $3\mid h/b$, so the cofactors agree modulo 3. Hence $f(n+h)=f(n)$. At most $Y/4$ integers $n\le Y$ are exceptional, so $C_{f,h}(Y)\ge Y/2$ whenever $D\mid h$. For $2D\le H\le N/4$ the range $(H,2H]$ contains at least $H/(2D)$ multiples of D, and for each of them $Y=N-h\ge N/2$ is admissible. Summing $(N/4)^2$ over these shifts gives the bound. □

So the collar energy of f is within a constant of the uniform maximum, although f agrees with λ, or with any other prescription, on P, and its partial sums are bounded by a power of $\log N$.

8.2. Rudin–Shapiro Signs

The Rudin–Shapiro polynomials are defined by $P_0=Q_0=1$ and

$$P_{m+1}(z)=P_m(z)+z^{2^m}Q_m(z),\qquad Q_{m+1}(z)=P_m(z)-z^{2^m}Q_m(z). \tag{52}$$

The coefficients of P_{m+1} begin with those of P_m, so there is a sign sequence $r=(r_n)_{n\ge1}$ with $r_n=[z^{n-1}]P_m$ for $n\le 2^m$.

Theorem 24 (Additive cancellation without the collar)**.** *The Rudin–Shapiro sequence satisfies*

$$\sup_\alpha\Big|\sum_{n\le N} r_n\mathrm{e}(n\alpha)\Big|\le(2+2\sqrt{2})\sqrt{N},\qquad |C_{r,h}(Y)|\le 5\sqrt{hY}\quad(h,Y\ge1).$$

Nevertheless $C_{r,2H}(H)=H$ and hence $\mathbb{E}_r(3H,H)\ge H^2$ for every dyadic H.

Proof. The parallelogram law applied to (52) gives $|P_{m+1}|^2+|Q_{m+1}|^2=2(|P_m|^2+|Q_m|^2)$ on $|z|=1$, hence

$$|P_m(z)|^2+|Q_m(z)|^2=2^{m+1}\qquad(|z|=1). \tag{53}$$

By (52) the coefficient vector of P_{m+1} is that of P_m followed by that of Q_m, and the coefficient vector of Q_{m+1} is that of P_m followed by that of $-Q_m$. By induction every aligned block $(i2^j,(i+1)2^j]$ of r is the coefficient vector of one of $\pm P_j,\pm Q_j$. The three assertions follow from this block structure; the computations are carried out in Appendix C.4. In outline: a prefix of length N is a union of at most one aligned block of each length $2^j\le N$, each of sup norm at most $2^{(j+1)/2}$ by (53), which gives the first bound. For the second, if $h>Y$ the trivial bound suffices. Otherwise cut $[1,Y]$ into aligned blocks of length 2^{k+1}, where 2^k is the power of two in $[\sqrt{hY},2\sqrt{hY})$. Inside a block the pairs $(n,n+h)$ lying in the same half contribute the lag-h autocorrelations of P_k and of Q_k, whose sum is the h-th Fourier

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coefficient of (53) and vanishes. The remaining pairs number at most $2h$ per block. Finally P_{m+2} consists of the four blocks P_m, Q_m, P_m, $-Q_m$, so for $H = 2^m$ the first and third blocks agree term by term and $C_{r,2H}(H) = \sum_{n \le H} r_n^2 = H$. The endpoint $Y = H$ is admissible for $N = 3H$ and $h = 2H$. □

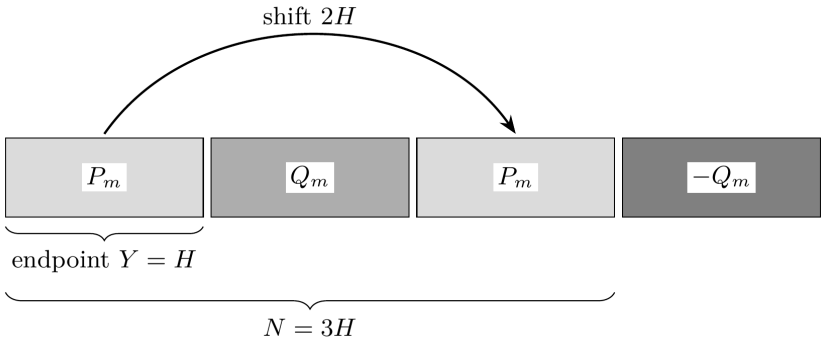

Figure 8. Rudin–Shapiro blocks of length H. The first and third blocks agree entry by entry, so the correlation at shift $2H$ is as large as possible. The endpoint $Y = H$ sees this agreement before the fourth block, with the opposite sign, contributes.

The prefix maximum in C^* is essential here. With the full endpoint $Y = N - h$ the fourth block would enter and the second estimate of the theorem would apply. The sequence r satisfies the conclusion of Theorem 21 with $\varepsilon = 0$, and every one of its correlations has square-root size uniformly in h and Y; the collar hypothesis at $N = 3H$ would require $\mathbb{E}_r(3H, H) \ll H^{1+\varepsilon}$.

8.3. Random Signs

Proposition 14 (Independent signs). *Let (ϵ_n) be independent fair signs. Almost surely $\mathbb{E}_\epsilon(3H, H) \ge H^2/2$ for all sufficiently large dyadic H.*

Proof. Put $T_H = \sum_{H<h\le 2H} |C_{\epsilon,h}(H)|^2$, so that $\mathbb{E}_\epsilon(3H, H) \ge T_H$. Expanding the squares, the diagonal contributes H for each h, in total H^2. The other terms are $2\epsilon_n\epsilon_m\epsilon_{n+h}\epsilon_{m+h}$ with $1 \le n < m \le H < n + h < m + h$. The four indices are distinct and determine (n, m, h), so these monomials are orthonormal. Hence $\mathbf{E}\, T_H = H^2$ and $\operatorname{Var} T_H = 4H\binom{H}{2} \le 2H^3$. Chebyshev's inequality gives $\mathbf{P}(T_H < H^2/2) \le 8/H$, which is summable over dyadic H. Apply the Borel–Cantelli lemma. □

Proposition 15 (Random completely multiplicative signs). *Give the primes independent fair signs and extend completely multiplicatively to f. Almost surely $\mathbb{E}_f(3H, H) \gg H^2/\log H$ for infinitely many dyadic H, with an absolute implied constant.*

Proof. Fix a dyadic $H \ge 2$ and condition on the signs of all primes up to $2H$. A composite integer at most $3H$ has all its prime factors at most $3H/2$. So in the vector $(C_{f,h}(H))_{H<h\le 2H}$ the only signs not yet determined are $\epsilon_p = f(p)$ for primes $p \in (2H, 3H]$ occurring as $n + h$. Write the vector as $u + \sum_p \epsilon_p A_p$, where u is determined by the conditioning and $A_p(h) = f(p - h)$ if $1 \le p - h \le H$, and $A_p(h) = 0$ otherwise. Let $T = \sum_p \|A_p\|_2^2$. A prime $p \in (2H, 5H/2]$ has $\|A_p\|_2^2 = 3H - p + 1 \ge H/2$, and the prime number theorem gives $T \gg H^2/\log H$.

Let $Z = \|u + \sum_p \epsilon_p A_p\|_2^2$. Its conditional mean is $m = \|u\|_2^2 + T$, and by orthogonality of the monomials ϵ_p and $\epsilon_p\epsilon_q$,

$$\operatorname{Var}(Z) = 4\sum_p \langle u, A_p\rangle^2 + 4\sum_{p<q} \langle A_p, A_q\rangle^2 \le 4\|u\|_2^2 T + 2T^2 \le 2m^2.$$

Thus $\mathbf{E}[Z^2] \le 3m^2$. Since $\mathbf{E}[Z\mathbf{1}_{Z\ge m/2}] \ge m/2$, the Cauchy–Schwarz inequality gives $\mathbf{P}(Z \ge m/2) \ge 1/12$ conditionally. The events $\{Z \ge T/2\}$ at smaller dyadic scales depend only on primes up to $3H/2$ and are measurable with respect to the conditioning at scale H. Iterating, the probability that none of k

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given dyadic scales succeeds is at most $(11/12)^k$. Let $k \to \infty$ and take a countable union over starting scales. Finally $\mathbb{E}_f(3H, H) \geq Z$. □

8.4. A Deterministic Logarithmic Obstruction

Theorem 25 (Gram matrices)**.** *There is an absolute constant C_0 with the following property. For every sign sequence a and every $B \geq 0$, the number of dyadic H with*

$$\mathbb{E}_a(3H, H) \leq BH$$

is at most $C_0(B+1)$. Consequently $\mathbb{E}_a(3H, H)/H \to \infty$ along the dyadic integers, and for all large J at least a quarter of the scales $H = 2^j$, $J/2 \leq j \leq J$, satisfy $\mathbb{E}_a(3H, H) \geq c_0 H \log H$ with an absolute $c_0 > 0$.

Proof. Restrict to exponents j in one residue class modulo 3, so that distinct scales differ by a factor at least 8. Let $M = \lceil 64(B+1) \rceil$ and suppose that there are $K = 2M$ scales $H_1 < \cdots < H_K$ in the class, all at least M, with $\mathbb{E}_a(3H_i, H_i) \leq BH_i$. Since $M \geq 64$, each H_i is divisible by 4; let $I_i = (3H_i/2, 7H_i/4] \cap \mathbb{Z}$, a set of $H_i/4$ integers. For $n \geq 1$ let $v_n = (a_{n+1}, \ldots, a_{n+M})^{\mathsf{T}}$ and

$$B_i = \frac{1}{|I_i|} \sum_{n \in I_i} v_n v_n^{\mathsf{T}}.$$

Each B_i is a positive semidefinite $M \times M$ matrix of trace M, so $\operatorname{tr}(B_i^2) \leq M^2$. Let $i < j$, $n \in I_i$, $m \in I_j$ and $h = m - n$. Since $H_i \leq H_j/8$, one has $H_j < h < 2H_j$ and $m + M \leq 3H_j$. Therefore both n and $n + M$ are admissible endpoints at $N = 3H_j$, and

$$|v_n^{\mathsf{T}} v_m| = |C_{a,h}(n+M) - C_{a,h}(n)| \leq 2C^*_{a,h}(3H_j).$$

For a fixed h there are at most $|I_i|$ pairs (n, m) with $m - n = h$. Hence

$$\operatorname{tr}(B_i B_j) = \frac{1}{|I_i||I_j|} \sum_{n \in I_i} \sum_{m \in I_j} (v_n^{\mathsf{T}} v_m)^2 \leq \frac{4\,\mathbb{E}_a(3H_j, H_j)}{|I_j|} \leq 16B.$$

On the other hand $\sum_i B_i$ is positive semidefinite with trace KM and M eigenvalues, so $\operatorname{tr}((\sum_i B_i)^2) \geq K^2 M$ by the Cauchy–Schwarz inequality. Combining,

$$K^2 M \leq KM^2 + 16BK(K-1).$$

With $K = 2M$ this gives $2M^3 < 64BM^2$, that is $M < 32B$, contrary to the choice of M.

So each residue class contains fewer than $2M$ such scales above M, and there are at most $1 + \log_2 M$ dyadic scales below M. This proves the first claim. For the last, apply it with $B = c_0 J$: for c_0 small and J large the exceptional scales number at most $J/4$ among the $J/2 + O(1)$ scales of the range, and on the others $\mathbb{E}_a(3H, H) > c_0 JH \geq c_0 H \log_2 H$. □

The theorem applies to λ without any information on its correlations. It shows that (49) with $\varepsilon = 0$ fails for every sign sequence, along every infinite set of dyadic scales. The mechanism is linear algebra: a sign sequence has too many windows of length M for them to be pairwise almost orthogonal in $\mathbb{R}^M$.

Table 9 lists the values of $\mathbb{E}_a(3H, H)/H$ for three sequences, as a numerical illustration. For λ the ratio grows linearly in H in this range, as it does for random signs. The theorems of this paper do not prove that growth for λ; Section 9 proves what the available correlation estimates give.

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Table 9. Computed values of $\mathbb{E}_a(3H, H)/H$. The last column is the uniform maximum of Proposition 12 divided by H. The random column is one sample.

H	$a = \lambda$	Rudin–Shapiro	random signs	maximum
16	35.4	41.0	34.1	573.5
64	155.4	162.8	159.7	9461.5
256	638.4	673.4	672.0	152533.5
1024	2630.6	2684.0	2622.6	2445141.5
2048	5399.2	5366.1	5656.7	9783637.5

9. The Endpoint for Liouville Itself

The models of Section 8 are other sequences, and Theorem 25 gives for λ only a logarithmic lower bound at most scales. Tao's logarithmically averaged two-point Chowla theorem [56] gives an asymptotic for a second moment of the Liouville correlations themselves. We use it in the form

$$\sum_{t\le T} \frac{\lambda(t)\lambda(t+d)}{t} = o_d(\log T) \qquad (d \ge 1 \text{ fixed}). \tag{54}$$

Theorem 26 (The endpoint for Liouville)**.** *For every fixed integer $M \ge 1$,*

$$\sum_{h\le T} \frac{|C_{\lambda,h}(M)|^2}{h} = M \log T + o_M(\log T) \qquad (T \to \infty). \tag{55}$$

Consequently

$$\liminf_{J\to\infty} \frac{1}{J} \sum_{j=0}^{J-1} \frac{\mathbb{E}_\lambda(3\cdot 2^j, 2^j)}{2^j} \ge M \log 2 \quad \textit{for every } M, \tag{56}$$

so the left side is $+\infty$, and the bound (49) with $\varepsilon = 0$ fails for λ at $N = 3H$.

Proof. Expand

$$|C_{\lambda,h}(M)|^2 = \sum_{1\le n,m\le M} \lambda(n)\lambda(m)\lambda(n+h)\lambda(m+h).$$

The diagonal $n = m$ contributes M, hence $M \log T + O(M)$ to (55). Fix $n < m$ and put $d = m - n$, $t = n + h$. Then

$$\begin{aligned}\sum_{h\le T} \frac{\lambda(n+h)\lambda(m+h)}{h} &= \sum_{n<t\le T+n} \frac{\lambda(t)\lambda(t+d)}{t} + O\Big(\sum_{t>n} \frac{n}{t(t-n)}\Big) \\ &= o_d(\log T) + O_M(1)\end{aligned}$$

by (54); the change of the range of summation costs $O_M(1)$. The pairs with $m < n$ are treated in the same way. There are fewer than M^2 pairs, which proves (55).

Let $H \ge M$ be dyadic and $H < h \le 2H$. The endpoint M is admissible at $N = 3H$, so $|C_{\lambda,h}(M)| \le C^*_{\lambda,h}(3H)$, and $1/h < 1/H$. Hence the part of the sum (55) with $H < h \le 2H$ is at most $\mathbb{E}_\lambda(3H, H)/H$. Summing over the dyadic H with $M \le H < 2^J$ and using (55) with $T = 2^J$ gives

$$\sum_{j<J} \frac{\mathbb{E}_\lambda(3\cdot 2^j, 2^j)}{2^j} \ge MJ \log 2 - o_M(J) - O_M(1),$$

where the last term accounts for the shifts $h \le 2M$. Divide by J. □

The divergence in (56) alone already follows from Theorem 25, for every sign sequence. What the correlation theorem adds is the exact constant in (55): on logarithmic average over the shift, the correlations of λ with a fixed endpoint M have the mean square M of independent signs. By

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Proposition 10(c) the same asymptotic holds for every modified Liouville function f_S with $\sum_{p\in S} 1/p < \infty$. The argument takes M fixed before $T \to \infty$ and gives no positive power of H. The next proposition shows what uniformity in M would give.

Proposition 16 (A power saving in ordinary correlations contradicts the collar)**.** *Suppose there are $\delta, \eta > 0$ and $K < \infty$ such that, for all sufficiently large dyadic H and all $1 \le n < m \le H^\eta$,*

$$\Big| \sum_{H<h\le 2H} \lambda(h+n)\lambda(h+m) \Big| \le KH^{1-\delta}.$$

Then $\mathbb{E}_\lambda(3H, H) \gg H^{1+\kappa}$ for every $0 < \kappa < \min(\delta, \eta, 1)$ and all large dyadic H. In particular (49) fails for λ for every $\varepsilon < \kappa$.

Proof. Let $M = \lfloor H^\kappa \rfloor \le H$ and expand $\sum_{H<h\le 2H} |C_{\lambda,h}(M)|^2$. The diagonal equals HM. The off-diagonal terms sum to at most $M(M-1)KH^{1-\delta}$ in absolute value, and the ratio of this to HM is at most $KH^{\kappa-\delta} \to 0$. Since M is an admissible endpoint at $N = 3H$, the collar energy is at least this second moment. □

The hypothesis of Proposition 16 is a uniform form of the two-point Chowla conjecture with a power saving, which is the expected truth. Under it the maximal collar hypothesis is false, and Theorem 21 would be an implication with a false premise. The reason is a matter of scale. If every correlation $C_{\lambda,h}(Y)$ had square-root size and no smaller, then $C^*_{\lambda,h}(N)^2$ would be of order N and $\mathbb{E}_\lambda(N, H)$ of order HN, which exceeds the collar scale N^2/H as soon as $H > \sqrt{N}$. The best cancellation one expects in each term is therefore incompatible with (49) in the upper half of the range of shifts, and this is the precise sense in which the termwise approach cannot work. For comparison, the two-point Chowla conjecture at each fixed shift would give, for fixed M,

$$\frac{1}{H} \sum_{H<h\le 2H} |C_{\lambda,h}(M)|^2 \longrightarrow M \qquad (H \to \infty),$$

by the same expansion, and hence $\mathbb{E}_\lambda(3H, H)/H \to \infty$ with explicit dependence on M and without averaging over scales. Only the logarithmically averaged statement (54) is known, and Theorem 26 uses nothing more.

The signed energy $Q_\lambda(N)$ of Proposition 1 avoids this defect. In its shift expansion the correlations enter with the Fejér weights $1 - h/N$, at the single endpoint $N - h$, and without absolute values.

10. Sparse Primes and Abel Data

Theorem 20 produces a singularity at $s = \beta$ without describing it. With a more careful choice of the set of primes the Dirichlet series can be computed up to a nonvanishing factor, and the resulting functions can be made to imitate λ to any finite precision in a neighbourhood of $s = 1$. The local behaviour at $s = 1$ is what Abel summation $\sum_n f(n)n^{-1-\delta}$, $\delta \downarrow 0$, detects, and we refer to the Taylor coefficients there as Abel data.

10.1. A Rounded Construction with an Explicit Factorization

Fix $1/2 < \beta < 1$ and an integer $P \ge 2$. For primes $p > P$ put

$$q_p = \tfrac{1}{2}p^{\beta-1}, \qquad S(t) = \sum_{P<p\le t} q_p, \qquad I_p = \lfloor S(p) \rfloor - \lfloor S(p) - q_p \rfloor \in \{0, 1\}. \tag{57}$$

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Let $\mathcal{S}_P = \{p > P : I_p = 1\}$, let $f_P = f_{\mathcal{S}_P}$ be the corresponding modified Liouville function, and $G_P = G_{\mathcal{S}_P}$. Thus $f_P(p) = 2I_p - 1$ for $p > P$ and $f_P(n) = \lambda(n)$ for all $n \le P$. The set $\mathcal{S}_P$ contains a prime each time the sum S passes an integer. Telescoping the integer parts gives

$$E(t) := \sum_{P<p\le t} (I_p - q_p) = \lfloor S(t) \rfloor - S(t) \in (-1, 0] \qquad (t \ge P). \tag{58}$$

Theorem 27 (Explicit factorization and exact exponent)**.** *The function $f = f_P$ has the following properties.*

(i) *One has $\pi_{\mathcal{S}_P}(x) = \frac{1}{2}\sum_{P<p\le x} p^{\beta-1} + O(1)$, which is asymptotic to $x^\beta/(2\beta \log x)$. The series $\sum_{p\in\mathcal{S}_P} p^{-\sigma}$ converges for $\sigma > \beta$ and diverges at $\sigma = \beta$.*

(ii) *The Dirichlet series $F_f(s) = \sum_n f(n)n^{-s}$ extends meromorphically to $\operatorname{Re} s > 1/2$ as*

$$F_f(s) = \frac{\zeta(2s)\,\zeta(s+1-\beta)}{\zeta(s)}\, R(s), \tag{59}$$

where R is holomorphic and zero-free on $\operatorname{Re} s > 1/2$ and positive on the real axis. It has a simple zero at $s = 1$ and a simple pole at $s = \beta$ with residue $\zeta(2\beta)R(\beta)/\zeta(\beta) \ne 0$.

(iii) *$L_f(x) \ll_{A,f} x(\log x)^{-A}$ and $Q_f(N) \ll_{A,f} N^2(\log N)^{-A}$ for every $A > 0$.*

(iv) *The energy exponent is exact:*

$$\limsup_{N\to\infty} \frac{\log(1 + Q_f(N))}{\log N} = 2\max(\Theta, \beta).$$

Proof. (i) By (58), $\pi_{\mathcal{S}_P}(x) = S(x) + E(x)$, and the asymptotic follows from the prime number theorem by partial summation. The centred series $A(s) = \sum_{p>P}(I_p - q_p)p^{-s}$ satisfies, by partial summation and the boundedness of E,

$$A(s) = s\int_P^\infty E(t)\, t^{-s-1}\,\mathrm{d}t \qquad (\operatorname{Re} s > 0), \tag{60}$$

and is holomorphic there. For real $\sigma > \beta$,

$$\sum_{p\in\mathcal{S}_P} p^{-\sigma} = \frac{1}{2}\sum_{p>P} p^{-(\sigma+1-\beta)} + A(\sigma) < \infty.$$

As $\sigma \downarrow \beta$ the first term tends to $\frac{1}{2}\sum_{p>P} 1/p = \infty$ while $A(\sigma)$ stays bounded, so the series diverges at β by monotone convergence.

(ii) For $\operatorname{Re} s > 1$, $F_f = \zeta(2s)G_P(s)/\zeta(s)$ as in (48). Subtracting the logarithm of the Euler product of $\zeta(s+1-\beta)$ from that of G_P gives, for $\operatorname{Re} s > 1$,

$$\begin{aligned}\log G_P(s) - \log\zeta(s+1-\beta) = 2A(s) &- \sum_{p\le P} p^{-(s+1-\beta)} \\ &+ 2\sum_{p\in\mathcal{S}_P}\sum_{\substack{k\ge 3\\ k\ \text{odd}}} \frac{p^{-ks}}{k} - \sum_p\sum_{k\ge 2}\frac{p^{-k(s+1-\beta)}}{k}.\end{aligned} \tag{61}$$

The details are in Appendix C.1. The first line is holomorphic on $\operatorname{Re} s > 0$. The two double series converge absolutely and locally uniformly on $\operatorname{Re} s > 1/2$, being dominated there by $\sum_p p^{-3\sigma}$ and $\sum_p p^{-2(\sigma+1-\beta)}$, whose exponents exceed 1. Let $R(s)$ be the exponential of the right side of (61). It is holomorphic and zero-free on $\operatorname{Re} s > 1/2$, positive at real points, and $G_P(s) = \zeta(s+1-\beta)R(s)$; this proves (59). At $s = 1$ the factors $\zeta(2s)$, $\zeta(s+1-\beta)$ and R are finite and positive and $1/\zeta$ has a simple zero. At $s = \beta$ the factor $\zeta(s+1-\beta)$ has a simple pole with residue 1, while $\zeta(2\beta)$ is finite and positive, $\zeta(\beta) < 0$ and $R(\beta) > 0$.

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(iii) This is Proposition 10(d) at $\alpha = 0$, applicable by (i) with any $\sigma_0 \in (\beta, 1)$. For the energy use (46), bounding $|L_f(j)|$ by $\sqrt{N}$ for $j \le \sqrt{N}$.

(iv) The upper bound is Theorem 20(ii). For the lower bound let ρ be a pole of F_f in $\operatorname{Re} s > 1/2$. If the limit superior were less than $2\operatorname{Re}\rho$, choose α with $\alpha < \operatorname{Re}\rho$ and $Q_f(N) \ll N^{2\alpha}$. Then $L_f(x) \ll x^\alpha$ by (46), and $s\int_1^\infty L_f(x)x^{-s-1}\,\mathrm{d}x$ would continue F_f holomorphically to $\operatorname{Re} s > \alpha$, a contradiction. The point β is a pole. So is every zero ρ of ζ with $\operatorname{Re}\rho > \beta$: there $\operatorname{Re} 2\rho > 1$ and $\operatorname{Re}(\rho + 1 - \beta) > 1$, so neither zeta factor in the numerator of (59) vanishes, and $R(\rho) \neq 0$. If $\Theta > \beta$ such zeros have real parts arbitrarily close to Θ. □

The same function f_P serves for all A in (iii); it is one completely multiplicative function, fixed once β and P are chosen. Table A3 in Appendix C.1 lists the first steps of the rounding for $\beta = 3/4$, $P = 2$.

10.2. The Zero at $S = 1$ and Its Leading Coefficient

Proposition 17 (Rigidity of the leading coefficient)**.** *Let $g : \mathbb{N} \to \{\pm 1\}$ be completely multiplicative, and suppose its Dirichlet series F_g continues holomorphically to a neighbourhood of $s = 1$ with $F_g(1) = 0$. Then the zero is simple, $F_g'(1) \ge \zeta(2)$, and equality holds if and only if $g = \lambda$.*

Proof. For real $\sigma > 1$ the Euler products give

$$\frac{F_g(\sigma)}{F_\lambda(\sigma)} = \prod_{p:\, g(p)=1} \frac{1+p^{-\sigma}}{1-p^{-\sigma}} \ge 1, \qquad F_\lambda(1+\delta) = \frac{\zeta(2+2\delta)}{\zeta(1+\delta)} \sim \zeta(2)\,\delta \quad (\delta \downarrow 0).$$

Hence $F_g(1+\delta) \ge (1+o(1))\zeta(2)\delta$. A zero of order at least two would give $F_g(1+\delta) = O(\delta^2)$. Dividing by δ and letting $\delta \downarrow 0$ gives $F_g'(1) \ge \zeta(2)$. If $g(p_0) = 1$ for some prime p_0, the ratio is at least $(1+p_0^{-\sigma})/(1-p_0^{-\sigma})$, which tends to $(p_0+1)/(p_0-1) > 1$, so $F_g'(1) > \zeta(2)$. Otherwise $g(p) = -1$ for all p and $g = \lambda$. □

For $g = f_P$ the proposition applies, and $F_{f_P}'(1) = \zeta(2)G_P(1) > \zeta(2)$ because $\mathcal{S}_P$ is nonempty: $S(t) \to \infty$ since $\sum_p p^{\beta-1}$ diverges. Thus complete multiplicativity, agreement with λ up to P, every logarithmic saving and a simple zero at $s = 1$ together still allow energy growth $N^{2\beta}$. Within the class of completely multiplicative sign functions the exact value $F'(1) = \zeta(2)$ singles out λ. The next subsection shows that no approximate form of this equality does.

10.3. Approximation of Finite Abel Jets

Normalize the zero at $s = 1$ by putting

$$\mathcal{J}_\lambda(s) = \frac{\zeta(2s)}{\big((s-1)\zeta(s)\big)^2}, \qquad \mathcal{J}_P(s) = \mathcal{J}_\lambda(s)\,G_P(s), \tag{62}$$

so that $\mathcal{J}_f(s) = F_f(s)/((s-1)^2\zeta(s))$ for $f = \lambda$ and $f = f_P$, with the removable value at $s = 1$ assigned. The derivatives $\mathcal{J}^{(k)}(1)$, $0 \le k \le K$, form the Abel jet of order K; $\mathcal{J}_f(1) = F_f'(1)$.

Lemma 12 (Euler-factor bound)**.** *For real $\sigma > \beta$,*

$$\sup_{\operatorname{Re} s \ge \sigma} |\log G_P(s)| \le \Delta(P,\sigma) := \frac{P^{\beta-\sigma}}{(\sigma-\beta)(1-P^{-2\sigma})},$$

where $\log G_P$ is defined by its Euler series. Consequently $|G_P(s) - 1| \le e^{\Delta(P,\sigma)} - 1$ for $\operatorname{Re} s \ge \sigma$.

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Proof. For $T > P$, partial summation and $E \le 0$ give

$$\sum_{P<p\le T} (I_p - q_p)p^{-\sigma} = E(T)T^{-\sigma} + \sigma \int_P^T E(t)t^{-\sigma-1}\,\mathrm{d}t \le 0.$$

Therefore

$$\sum_{p\in\mathcal{S}_P} p^{-\sigma} \le \frac{1}{2}\sum_{p>P} p^{-(1+\sigma-\beta)} \le \frac{1}{2}\sum_{n>P} n^{-(1+\sigma-\beta)} \le \frac{P^{\beta-\sigma}}{2(\sigma-\beta)},$$

the last step comparing the decreasing sum with $\int_P^\infty t^{-(1+\sigma-\beta)}\,\mathrm{d}t$. For $\operatorname{Re} s \ge \sigma$,

$$|\log G_P(s)| = \Big|2\sum_{p\in\mathcal{S}_P}\sum_{j\ \text{odd}} \frac{p^{-js}}{j}\Big| \le 2\sum_{p\in\mathcal{S}_P} \frac{p^{-\sigma}}{1-p^{-2\sigma}} \le \frac{2}{1-P^{-2\sigma}}\sum_{p\in\mathcal{S}_P} p^{-\sigma}.$$

The last assertion follows from $|e^w - 1| \le e^{|w|} - 1$. □

Theorem 28 (Approximation of every finite Abel jet). *Let $0 < r < 1-\beta$ and $C_r = \zeta(2-2r)/(1-2r)^2$. The functions (62) are holomorphic on a neighbourhood of $|s-1| \le r$, and for every integer $k \ge 0$,*

$$\big|\mathcal{J}_P^{(k)}(1) - \mathcal{J}_\lambda^{(k)}(1)\big| \le k!\, r^{-k}\, C_r\big(e^{\Delta(P,1-r)} - 1\big). \tag{63}$$

For fixed r and K the right side tends to zero as $P \to \infty$, uniformly in $0 \le k \le K$. Explicitly, if $\eta > 0$, $d = 1-r-\beta$, $M = C_r \max_{k\le K} k!\, r^{-k}$ and

$$P \ge \max\Big\{2,\ \Big[\frac{2}{d\,\log(1+\eta/M)}\Big]^{1/d}\Big\}, \tag{64}$$

then $|\mathcal{J}_P^{(k)}(1) - \mathcal{J}_\lambda^{(k)}(1)| \le \eta$ for all $0 \le k \le K$.

Proof. Euler summation gives, for $\operatorname{Re} s > 0$,

$$\zeta(s) = \frac{s}{s-1} - s\int_1^\infty \{x\}x^{-s-1}\,\mathrm{d}x,$$

and the integral has modulus at most $1/\operatorname{Re} s$ [57, §2.1]. Since $r < 1/2$, on $|s-1| \le r$ this yields

$$|(s-1)\zeta(s)| \ge |s|\Big(1 - \frac{|s-1|}{\operatorname{Re} s}\Big) \ge (1-r)\Big(1 - \frac{r}{1-r}\Big) = 1-2r > 0.$$

Also $|\zeta(2s)| \le \zeta(2-2r)$ there. Hence $|\mathcal{J}_\lambda| \le C_r$ on the disc. Since $1-r > \beta$, Lemma 12 bounds $|G_P - 1|$ by $e^{\Delta(P,1-r)} - 1$ on the disc, and Cauchy's inequalities for $\mathcal{J}_P - \mathcal{J}_\lambda = \mathcal{J}_\lambda(G_P - 1)$ on $|s-1| = r$ give (63). For $P \ge 2$ and $\sigma = 1-r > 1/2$ one has $1 - P^{-2\sigma} \ge 1/2$, so $\Delta(P, 1-r) \le 2P^{-d}/d$, and (64) makes this at most $\log(1+\eta/M)$. □

Theorem 29 (The local energy still fails). *Let $B_P(X) = \sum_{n\ge1} f_P(n)e^{-n/X}$ and $\mathcal{H}_{B_P}(X) = \int_{X/2}^{2X} |B_P(Y)|^2\,\mathrm{d}Y/Y$. For every P and every $0 < \varepsilon < 2\beta - 1$,*

$$\mathcal{H}_{B_P}(X) \ne O(X^{1+\varepsilon}), \qquad B_P(X) \ne O(X^{(1+\varepsilon)/2}).$$

Proof. The second statement follows from the first. Suppose $\mathcal{H}_{B_P}(X) \ll X^{1+\varepsilon}$. For real $\sigma > (1+\varepsilon)/2$ the Cauchy–Schwarz inequality on $[2^j, 2^{j+1}] \subset [2^j/2, 2\cdot 2^j]$ gives

$$\int_{2^j}^{2^{j+1}} |B_P(Y)|\, Y^{-\sigma}\,\frac{\mathrm{d}Y}{Y} \le (\log 2)^{1/2} 2^{-j\sigma}\mathcal{H}_{B_P}(2^j)^{1/2} \ll 2^{j((1+\varepsilon)/2-\sigma)}.$$

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The sum over $j \geq 0$ converges, locally uniformly in σ. On $0 < Y \leq 1$ one has $|B_P(Y)| \leq (e^{1/Y} - 1)^{-1}$, which decays faster than any power. Hence $\int_0^\infty B_P(Y) Y^{-s} \, \mathrm{d}Y/Y$ is holomorphic on $\operatorname{Re} s > (1 + \varepsilon)/2$. For $\operatorname{Re} s > 1$ it equals $\Gamma(s) F_{f_P}(s)$, by absolute convergence and $\int_0^\infty e^{-n/Y} Y^{-s} \, \mathrm{d}Y/Y = \Gamma(s) n^{-s}$. This would continue $\Gamma(s) F_{f_P}(s)$ holomorphically across $s = \beta > (1 + \varepsilon)/2$, where it has a pole by Theorem 27(ii) and $\Gamma(\beta) \neq 0$. □

Theorems 28 and 29 concern the same function. Table 10 gives certified instances of (63); the arithmetic is in Appendix C.2. In the first row every value $f_P(n)$ with $n \leq 2^{320}$ equals $\lambda(n)$, the first four Abel derivatives agree with those of λ to within $2 \cdot 10^{-7}$, and nevertheless $\mathcal{H}_{B_P}(X) \neq O(X^{1+\varepsilon})$ for every $\varepsilon < 1/2$. Exact agreement is different: by Proposition 17, $\mathcal{J}_P(1) = \zeta(2) G_P(1) > \zeta(2) = \mathcal{J}_\lambda(1)$ for every P, although the difference tends to zero.

Table 10. Upper bounds for $\max_{k \leq K} |\mathcal{J}_P^{(k)}(1) - \mathcal{J}_\lambda^{(k)}(1)|$ from (63), with C_r replaced by the rational bound $(2 - 2r)/(1 - 2r)^3$.

β	r	K	P	$\Delta(P, 1 - r) \leq$	bound
3/4	1/8	3	2^{320}	2^{-36}	$1.9 \cdot 10^{-7}$
3/4	1/8	0	2^{160}	2^{-16}	$6.4 \cdot 10^{-5}$
3/4	1/8	6	2^{480}	2^{-56}	$1.1 \cdot 10^{-8}$
5/8	1/4	3	2^{320}	2^{-36}	$6.8 \cdot 10^{-8}$
7/8	1/16	3	2^{960}	2^{-55}	$2.0 \cdot 10^{-12}$

10.4. Bounded Coefficients with Prescribed Vanishing Moments

The preceding results stay within completely multiplicative sign functions and match Abel data only approximately. If multiplicativity is dropped, any finite number of Taylor coefficients at $s = 1$ can be made to vanish exactly while the smoothed sum is as large as X^β.

Theorem 30 (Finitely many vanishing moments)**.** *Let $k \geq 1$, $1/2 < \beta < 1$ and $\gamma > 0$. There are a real sequence (a_n) with $|a_n| \leq 1$ and a number $\varepsilon_0 > 0$ such that:*

(i) *$D_a(s) = \sum_n a_n n^{-s}$ converges absolutely for $\operatorname{Re} s > \beta$ and $D_a^{(j)}(1) = 0$ for $0 \leq j < k$;*
(ii) *D_a is meromorphic on $\mathbb{C}$, with simple poles at $\beta \pm i\gamma$ of residue $\varepsilon_0/2$ and no others;*
(iii) *with $z = \beta + i\gamma$, for $X \geq 1$,*

$$\sum_{n \geq 1} a_n e^{-n/X} = \varepsilon_0 \operatorname{Re}\bigl(\Gamma(z) X^z\bigr) + O_{k,\beta,\gamma}(1).$$

In particular $\limsup_{X \to \infty} X^{-\beta} \sum_n a_n e^{-n/X} = \varepsilon_0 |\Gamma(z)|$, the limit inferior is $-\varepsilon_0 |\Gamma(z)|$, and the smoothed sum is not $O(X^{1/2+\eta})$ for any $\eta < \beta - 1/2$.

Proof. Let $r_n = n^{\beta - 1} \cos(\gamma \log n) = \operatorname{Re} n^{z-1}$ and $\mathcal{T} = \{2^m : 0 \leq m < k\}$. For $0 \leq j < k$ the moments

$$t_j = \sum_{n \notin \mathcal{T}} \frac{r_n (-\log n)^j}{n}$$

converge absolutely, since $|r_n| \leq n^{\beta - 1}$. Choose real numbers $b_0, \ldots, b_{k-1}$ with

$$\sum_{m=0}^{k-1} \frac{b_m (-m \log 2)^j}{2^m} = -t_j \qquad (0 \leq j < k). \tag{65}$$

After scaling rows and columns the matrix of this system is the Vandermonde matrix $(m^j)_{0 \leq j, m < k}$, which is invertible; Appendix C.3 gives the solution in closed form. Put

$$\varepsilon_0 = \frac{1}{1 + \max_m |b_m|}, \qquad a_n = \varepsilon_0 b_m \ \ (n = 2^m \in \mathcal{T}), \qquad a_n = \varepsilon_0 r_n \ \ (n \notin \mathcal{T}).$$

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Then $|a_n| \le 1$, and (65) is the statement $D_a^{(j)}(1) = 0$, the series being differentiable term by term. For $\operatorname{Re} s > \beta$,

$$D_a(s) = \frac{\varepsilon_0}{2}\big(\zeta(s+1-z) + \zeta(s+1-\bar{z})\big) + \varepsilon_0 \sum_{m=0}^{k-1} (b_m - r_{2^m}) 2^{-ms}. \tag{66}$$

The finite sum is entire, and the two shifted zeta functions have their only poles at $s = z$ and $s = \bar{z}$. This proves (i) and (ii).

For (iii) let $\phi_X(t) = t^{z-1}e^{-t/X}$. For a continuously differentiable integrable ϕ on $[1,\infty)$ with integrable derivative, $|\sum_{n\ge1}\phi(n) - \int_1^\infty \phi| \le \int_1^\infty |\phi'|$. Here, for $X \ge 1$,

$$\int_1^\infty |\phi_X'(t)|\,\mathrm{d}t \le |z-1|\int_1^\infty t^{\beta-2}\,\mathrm{d}t + \frac{1}{X}\int_0^\infty t^{\beta-1}e^{-t/X}\,\mathrm{d}t \le \frac{|z-1|}{1-\beta} + \Gamma(\beta),$$

and $\int_0^1 |\phi_X| \le 1/\beta$. Since $\int_0^\infty t^{z-1}e^{-t/X}\,\mathrm{d}t = \Gamma(z)X^z$,

$$\sum_{n\ge1} n^{z-1}e^{-n/X} = \Gamma(z)X^z + O_{\beta,\gamma}(1).$$

Take real parts, multiply by ε_0, and note that changing the k coefficients on $\mathcal{T}$ alters the sum by $O_k(1)$. The limits follow by choosing $\gamma \log X \equiv -\arg\Gamma(z)$ and $\gamma \log X \equiv \pi - \arg\Gamma(z)$ modulo 2π. □

The poles in (66) are built into the coefficients; they are not zeros of ζ. The sequence depends on k, and ε_0 may tend to zero as k grows, so no single sequence with all moments vanishing is obtained. The theorem excludes a general inference from boundedness and finitely many exact cancellations at $s = 1$ to a square-root bound for the smoothed sum. Theorem 28 shows that the multiplicative structure does not restore that inference when the cancellations are matched only to finite precision.

11. Local Energy and the Prime Tail

By Proposition 22, RH is equivalent to $B(X) \ll_\varepsilon X^{1/2+\varepsilon}$ for the smoothed sum $B(X) = \sum_n \lambda(n)e^{-n/X}$. This section records four exact reformulations of that bound: a mean square over neighbouring scales, a one-sided bound for a primitive correlation, a signed dilation identity over prime powers, and a cancellation in the tail over large primes. In each case the reformulation is proved and the place where an arithmetic estimate is missing is identified. Table 13 at the end of the section summarizes them.

11.1. Reconstruction of a Point Value from a Local Mean Square

For a complex sequence with $|a_n| \le 1$ put

$$F(X) = \sum_{n\ge1} a_n e^{-n/X}, \qquad \mathcal{H}_F(X) = \int_{X/2}^{2X} |F(Y)|^2 \frac{\mathrm{d}Y}{Y} \qquad (X > 0).$$

Theorem 31 (Analytic localization). *Let $X \ge 1$ and $m = \lceil 8\log_2(2X)\rceil$. Then*

$$|F(X)|^2 \le 10(m+2)\,\mathcal{H}_F(X) + X^{-13}. \tag{67}$$

Moreover there is a polynomial $K_m(0,u)$ of degree at most m, independent of the sequence, with

$$F(X) = \int_{-1/4}^{1/4} K_m(0,u)\,F\Big(\frac{X}{1+u}\Big)\,\mathrm{d}u + \mathcal{R}_X, \qquad |\mathcal{R}_X| \le \frac{\sqrt{2(m+2)}}{128X^7}.$$

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Proof. The function $G_X(z) = \sum_n a_n e^{-n(1+z)/X}$ is holomorphic on $\operatorname{Re} z > -1$, $G_X(u) = F(X/(1+u))$ for real u, and on $|z| \le 1/2$

$$|G_X(z)| \le \sum_{n\ge1} e^{-n/(2X)} = \frac{1}{e^{1/(2X)}-1} \le 2X.$$

Let p_m be its Taylor polynomial of degree m at 0. Cauchy's inequalities on $|z| = 1/2$ give

$$\sup_{|u|\le1/4} |G_X(u) - p_m(u)| \le 2X \sum_{j>m} 2^j 4^{-j} = 2X\, 2^{-m} =: \delta \le \frac{1}{128X^7},$$

and $p_m(0) = F(X)$. Let Λ_j be the Legendre polynomial of degree j, normalized by $\Lambda_j(1) = 1$, so that $\int_{-1}^{1} \Lambda_i\Lambda_j = 2\delta_{ij}/(2j+1)$. The polynomials $\sqrt{2(2j+1)}\,\Lambda_j(4u)$, $0 \le j \le m$, are orthonormal on $[-1/4, 1/4]$, and

$$K_m(v,u) = 2\sum_{j=0}^{m} (2j+1)\Lambda_j(4v)\Lambda_j(4u)$$

is the reproducing kernel of the space of polynomials of degree at most m in $L^2[-1/4,1/4]$. Hence, for such a polynomial p,

$$p(0) = \int_{-1/4}^{1/4} K_m(0,u)p(u)\,\mathrm{d}u, \qquad |p(0)|^2 \le K_m(0,0)\int_{-1/4}^{1/4} |p(u)|^2\,\mathrm{d}u. \tag{68}$$

Appendix C.5 proves the facts about Λ_j used here and the bound

$$K_m(0,0) = 2\sum_{i\le m/2} (4i+1)\left(\binom{2i}{i}4^{-i}\right)^2 \le 4(m+2). \tag{69}$$

Apply (68) to p_m with $|p_m|^2 \le 2|G_X|^2 + 2\delta^2$:

$$|F(X)|^2 \le 8(m+2)\int_{-1/4}^{1/4} |G_X(u)|^2\,\mathrm{d}u + 4(m+2)\delta^2.$$

The substitution $Y = X/(1+u)$ gives

$$\int_{-1/4}^{1/4} |G_X(u)|^2\,\mathrm{d}u = \int_{4X/5}^{4X/3} |F(Y)|^2 \frac{X}{Y}\frac{\mathrm{d}Y}{Y} \le \frac{5}{4}\,\mathcal{H}_F(X).$$

Since $m + 2 \le 8\log_2 X + 11 \le 16X$, the last term is at most $64X/(128X^7)^2 \le X^{-13}$. This proves (67). Finally, by (68) for p_m,

$$\mathcal{R}_X = \int_{-1/4}^{1/4} K_m(0,u)\big(p_m(u) - G_X(u)\big)\,\mathrm{d}u,$$
$$|\mathcal{R}_X| \le \delta\Big(\tfrac12\int_{-1/4}^{1/4} K_m(0,u)^2\,\mathrm{d}u\Big)^{1/2} = \delta\sqrt{K_m(0,0)/2}.$$

□

The polynomial approximation acts on the analytic dependence on the scale, not on the coefficients, and uses no cancellation. For $F = B$ it shows that

$$\mathcal{H}_B(X) \ll_\varepsilon X^{1+\varepsilon} \qquad (\varepsilon > 0,\ X \ge 1) \tag{70}$$

implies $B(X) \ll_\varepsilon X^{1/2+\varepsilon}$, and the converse follows by integration. So (70) is equivalent to RH. The proof of Theorem 29 gives the forward implication in another way: (70) makes the Mellin transform of B, which is $\Gamma(s)\zeta(2s)/\zeta(s)$, holomorphic on $\operatorname{Re} s > 1/2$.

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11.2. Diagonal and Signed Parts

Let $K(v) = \int_{1/2}^{2} e^{-v/t}\,\mathrm{d}t/t$ for $v > 0$. Expanding $|B(Y)|^2$ and integrating term by term, which is justified by absolute convergence, gives

$$\begin{aligned} \mathcal{H}_B(X) &= \mathcal{D}(X) + \mathcal{N}(X), \qquad \mathcal{D}(X) = \int_{X/2}^{2X} \frac{1}{e^{2/Y}-1}\frac{\mathrm{d}Y}{Y}, \\ \mathcal{N}(X) &= 2\sum_{h\ge 1}\sum_{n\ge 1} \lambda(n)\lambda(n+h)K\Big(\frac{2n+h}{X}\Big). \end{aligned} \tag{71}$$

Proposition 18 (The diagonal). *For every $X > 0$,*

$$0 < \mathcal{D}(X) - \Big(\frac{3X}{4} - \log 2\Big) < \frac{1}{4X}.$$

Proof. For $t > 0$,

$$0 < \frac{1}{e^t - 1} - \frac{1}{t} + \frac{1}{2} < \frac{t}{12}. \tag{72}$$

The lower inequality is equivalent to $\varphi(t) = (t-2)e^t + t + 2 > 0$, which holds because $\varphi(0) = \varphi'(0) = 0$ and $\varphi''(t) = te^t > 0$. After multiplication by $12t(e^t - 1)$ the upper inequality becomes

$$(t^2 - 6t + 12)(e^t - 1) - 12t = \sum_{j\ge 5} \frac{(j-3)(j-4)}{j!}\,t^j > 0.$$

Put $t = 2/Y$ and integrate against $\mathrm{d}Y/Y$ over $[X/2, 2X]$. The terms $Y/2 - 1/2$ give $3X/4 - \log 2$, and the error is less than $\frac{1}{6}\int_{X/2}^{2X} Y^{-2}\,\mathrm{d}Y = 1/(4X)$. □

Since $\mathcal{D}(X) \asymp X$ for $X \ge 1$, the bound (70), and with it RH, is equivalent to the upper bound $\mathcal{N}(X) \le K_\varepsilon X^{1+\varepsilon}$ for the signed part. Bounding the terms of $\mathcal{N}$ in absolute value gives order X^2. The positivity of $\mathcal{H}_B$ gives the lower bound $\mathcal{N}(X) \ge -\mathcal{D}(X)$ and nothing from above. Table 11 checks the proposition numerically.

Table 11. Numerical values of the diagonal $\mathcal{D}(X)$ against Proposition 18.

X	$\mathcal{D}(X)$	$\mathcal{D}(X) - (3X/4 - \log 2)$	$1/(4X)$
1	0.282975	0.226122	0.250000
10	6.831824	0.024971	0.025000
100	74.309353	0.002500	0.002500
1000	749.307103	0.000250	0.000250

11.3. The Primitive Correlation

Complete multiplicativity removes common factors from the pairs in $|B|^2$. Put

$$\begin{aligned} c(k) &= \sum_{\substack{1\le a<k \\ (a,k)=1}} \lambda(a)\lambda(k-a), \\ \mathcal{C}_\lambda(X) &= \sum_{\substack{a,b\ge 1 \\ (a,b)=1}} \lambda(a)\lambda(b)e^{-(a+b)/X} = \sum_{k\ge 2} c(k)e^{-k/X}. \end{aligned}$$

Proposition 19 (Primitive decomposition). *For every $X > 0$, with absolutely convergent series,*

$$B(X)^2 = \sum_{d\ge 1} \mathcal{C}_\lambda(X/d), \qquad \mathcal{C}_\lambda(X) = \sum_{d\ge 1} \mu(d)B(X/d)^2, \tag{73}$$

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and

$$\mathcal{H}_B(X) = \sum_{k\geq 2} c(k) \int_{X/2}^{2X} \frac{1}{e^{k/Y}-1} \frac{\mathrm{d}Y}{Y}.$$

The term $k = 2$, where $c(2) = 1$, is the diagonal $\mathcal{D}(X)$.

Proof. Sort the pairs (m,n) in $B(X)^2$ by $d = \gcd(m,n)$ and write $m = da$, $n = db$ with $(a,b) = 1$. Since $\lambda(d)^2 = 1$, $\lambda(da)\lambda(db) = \lambda(a)\lambda(b)$; this gives the first identity. For the second insert $\mathbf{1}_{(m,n)=1} = \sum_{d|m,\, d|n} \mu(d)$ in the definition of $\mathcal{C}_\lambda$. The rearrangements are justified by the majorant $\sum_d (\sum_a e^{-da/X})^2 = \sum_d (e^{d/X}-1)^{-2} < \infty$. For the third, sum $e^{-d(a+b)/Y}$ over d first, put $k = a+b$, note $(a,b) = 1$ if and only if $(a,k) = 1$, and integrate. The pair $a = b = 1$ collects exactly the pairs $m = n$. □

Theorem 32 (A one-sided primitive criterion)**.** *The following are equivalent.*

(i) RH.
(ii) $B(X) \ll_\varepsilon X^{1/2+\varepsilon}$ *for every* $\varepsilon > 0$ *and* $X \geq 1$.
(iii) *For every* $\varepsilon \in (0,1]$ *there is* K_ε *with* $\mathcal{C}_\lambda(X) \leq K_\varepsilon X^{1+\varepsilon}$ *for all* $X \geq 1$.

In (iii) *only an upper bound is required, and $\mathcal{C}_\lambda$ is not assumed to have a sign.*

Proof. The equivalence of (i) and (ii) is Proposition 22. Assume (iii). For $0 < X < 1$, $|\mathcal{C}_\lambda(X)| \leq (e^{1/X}-1)^{-2} \leq X^2 \leq X^{1+\varepsilon}$, so after enlarging K_ε to at least 1 the bound holds for all $X > 0$. The first identity in (73) gives

$$0 \leq B(X)^2 \leq K_\varepsilon \sum_{d\geq 1} (X/d)^{1+\varepsilon} = K_\varepsilon\, \zeta(1+\varepsilon)\, X^{1+\varepsilon},$$

which is (ii). Conversely, (ii) and $|B(X)| \leq (e^{1/X}-1)^{-1} \leq X$ for $X < 1$ give $B(X)^2 \leq A_\varepsilon X^{1+\varepsilon}$ for all $X > 0$, and the second identity in (73) gives

$$|\mathcal{C}_\lambda(X)| \leq A_\varepsilon X^{1+\varepsilon} \sum_{d\geq 1} \frac{\mu(d)^2}{d^{1+\varepsilon}} = A_\varepsilon \frac{\zeta(1+\varepsilon)}{\zeta(2+2\varepsilon)} X^{1+\varepsilon}.$$

□

The criterion involves one weighted sum of the coefficients $c(k)$ and no bound for an individual $c(k)$ or for a fixed additive shift. The upper bound in (iii) is not proved here. The identities (73) do not supply it, and $B(X)^2 \geq 0$ bounds a sum of values of $\mathcal{C}_\lambda$ from below only. At the level of coefficients the same proof gives $A(k) = \sum_{d|k} c(k/d)$ and $c(k) = \sum_{d|k} \mu(d) A(k/d)$, where $A(k) = \sum_{a=1}^{k-1} \lambda(a)\lambda(k-a)$.

11.4. Prime Dilations

For a completely multiplicative f one has $f \log = f * (f\Lambda)$, and $\lambda(p^k)\Lambda(p^k) = (-1)^k \log p$. Smoothing gives the exact identity

$$\sum_{n\geq 1} \lambda(n) \log n\, e^{-n/X} = \sum_{p^k} (-1)^k \log p\; B(X/p^k), \tag{74}$$

with absolute convergence for each X. It expresses a logarithmic derivative of B through signed dilations of B itself. One might hope to treat the dilations as nearly orthogonal, or the right side as a contraction. Neither is possible in general. For distinct primes p, q,

$$\sum_{n\leq N} \lambda(pn)\lambda(qn) = N,$$

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so prime dilates of λ are perfectly correlated.[14] The following theorem computes the norm of the dilation operator exactly. Let

$$\mathcal{K}_\sigma = L^2\big((0,\infty), X^{-2\sigma}\,\mathrm{d}X/X\big), \qquad T_R g(X) = \sum_{p^k \le R} (-1)^k \log p \; g(X/p^k).$$

Theorem 33 (Exact dilation norm). *For every real σ and $R \ge 2$,*

$$\|T_R\|_{\mathcal{K}_\sigma \to \mathcal{K}_\sigma} = \sum_{p^k \le R} (\log p)\, p^{-k\sigma}. \tag{75}$$

For fixed $1/2 < \sigma < 1$, $\|T_R\| \sim R^{1-\sigma}/(1-\sigma)$ as $R \to \infty$.

Proof. The map $Ug(t) = e^{-\sigma t} g(e^t)$ is an isometry of $\mathcal{K}_\sigma$ onto $L^2(\mathbb{R})$, and it carries $g(X/d)$ to $d^{-\sigma}(Ug)(t - \log d)$. So UT_RU^{-1} is a finite combination of translations, with Fourier multiplier

$$m_{R,\sigma}(\tau) = \sum_{p^k \le R} (-1)^k (\log p)\, p^{-k\sigma} e^{-ik\tau \log p}.$$

By Plancherel's theorem the operator norm is $\sup_\tau |m_{R,\sigma}(\tau)|$, which is at most the right side of (75). For the reverse inequality it suffices to find τ with $e^{-i\tau \log p}$ arbitrarily close to -1 for all primes $p \le R$ simultaneously, since then every term $(-1)^k e^{-ik\tau \log p}$ is close to 1. The numbers $\log p$, $p \le R$, are linearly independent over $\mathbb{Q}$ by unique factorization, so this is Kronecker's theorem [57, Ch. VIII]. Directly: for integers (n_p) not all zero the frequency $\omega = \sum_p n_p \log p$ is nonzero and $(2T)^{-1} \int_{-T}^{T} e^{-i\tau\omega}\,\mathrm{d}\tau \to 0$; by approximation with trigonometric polynomials, the averages over $\tau \in [-T, T]$ of any continuous function on the torus evaluated along $(e^{-i\tau \log p})_{p \le R}$ converge to its Haar integral, and a nonnegative bump supported near $(-1, \dots, -1)$ has positive integral.

For the asymptotic, the prime number theorem $\sum_{p \le u} \log p \sim u$ [49] and partial summation give $\sum_{p \le R} (\log p) p^{-\sigma} \sim R^{1-\sigma}/(1-\sigma)$. The terms with $k \ge 2$ contribute at most $(1 - 2^{-\sigma})^{-1} \sum_p (\log p) p^{-2\sigma} < \infty$. □

Proposition 20 (A sign change at low frequency). *Fix $1/2 < \sigma < 1$ and let $P_R = \sum_{p \le R} (\log p) p^{-\sigma}$, $\tau_R = \pi/\log R$. Then*

$$m_{R,\sigma}(0) = -P_R + O_\sigma(1), \qquad m_{R,\sigma}(\tau_R) = P_R + O_\sigma\Big(\frac{R^{1-\sigma}}{\log R}\Big).$$

Proof. The terms with $k \ge 2$ are $O_\sigma(1)$ at every frequency, which gives the first formula. The terms with $k = 1$ at τ_R are

$$-\sum_{p \le R} (\log p) p^{-\sigma} e^{-i\pi \log p/\log R} = \sum_{p \le R} (\log p) p^{-\sigma} e^{i\pi \log(R/p)/\log R},$$

and $|e^{i\vartheta} - 1| \le |\vartheta|$ bounds the difference from P_R by

$$\frac{\pi}{\log R} \sum_{p \le R} (\log p) p^{-\sigma} \log \frac{R}{p} = \frac{\pi}{\log R} \int_2^R P_u \frac{\mathrm{d}u}{u} \ll_\sigma \frac{R^{1-\sigma}}{\log R},$$

using $P_u \ll_\sigma u^{1-\sigma}$. □

So the operator in (74), truncated at R, has norm of size $R^{1-\sigma}$ on every space $\mathcal{K}_\sigma$ with $\sigma < 1$, and its multiplier changes sign between frequency 0 and frequency $\pi/\log R$ at nearly full size. No

[14] Results on multiplicative functions in almost all short intervals [46] compare short averages with long ones. They do not give the square-root size of the long average, which is what is needed here.

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contraction estimate and no fixed sign are available, and smoothing in frequency near zero does not help. These are statements about the operator; they do not assert that B itself is extremal, nor that $B \in \mathcal{K}_\sigma$, which for every $\sigma > 1/2$ is again equivalent to RH. Table 12 illustrates (75).

Table 12. The norm (75) at $\sigma = 3/4$, compared with its asymptotic value and with $\log R$.

R	$\|T_R\|$	$R^{1/4}/(1/4)$	$\|T_R\|/\log R$
10^2	7.89	12.65	1.71
10^3	17.86	22.49	2.58
10^4	35.39	40.00	3.84
10^5	66.51	71.13	5.78
10^6	121.85	126.49	8.82

11.5. A Finite-Prime Model and the Prime Tail

For $y \geq 2$ let f_y be the completely multiplicative function with $f_y(p) = -1$ for $p \leq y$ and $f_y(p) = +1$ for $p > y$, and put

$$B_y(X) = \sum_{n\geq 1} f_y(n) e^{-n/X}, \qquad c_y(s) = \prod_{p\leq y} \frac{1-p^{-s}}{1+p^{-s}}, \qquad c_y = c_y(1).$$

Thus $c_y = \prod_{p\leq y}(p-1)/(p+1)$.

Theorem 34 (Finite-prime main term). *For $X > 0$, $y \geq 2$ and $0 < \sigma < 1$,*

$$|B_y(X) - c_y X| \leq \frac{X^\sigma}{c_y(\sigma)}. \tag{76}$$

For $y = (\log X)^2$ and $X \to \infty$,

$$B_y(X) = c_y X + O(X^{1/2+o(1)}), \qquad c_y \geq (\log X)^{-4},$$

and hence $B_y(X) \geq X/(2(\log X)^4)$ for $X \geq X_0$.

Proof. Let g_y be multiplicative with $g_y(p^j) = 2(-1)^j$ for $p \leq y$, $j \geq 1$, and zero at powers of larger primes. Comparing Euler factors, $f_y = \mathbf{1} * g_y$, $\sum_d g_y(d) d^{-s} = c_y(s)$ and $\sum_d |g_y(d)| d^{-\sigma} = 1/c_y(\sigma)$, all series being finite products of geometric series for $\sigma > 0$. Hence

$$B_y(X) = \sum_{d\geq 1} \frac{g_y(d)}{e^{d/X} - 1}, \qquad c_y X = \sum_{d\geq 1} g_y(d) \frac{X}{d}.$$

By (72), $0 < 1/t - 1/(e^t - 1) < \min(1/2, 1/t) \leq t^{-\sigma}$ for $t > 0$. Applying this with $t = d/X$ term by term gives (76).

For $u = p^{-1/2} \leq 2^{-1/2}$ one has $\log((1+u)/(1-u)) \leq 2u/(1-u) < 7u$, so $-\log c_y(1/2) \leq 7\sum_{p\leq y} p^{-1/2} \ll \sqrt{y}/\log y$ by Chebyshev's bound; the computation is in Appendix C.6. For $y = (\log X)^2$ this is $O(\log X/\log\log X)$, so $1/c_y(1/2) = X^{o(1)}$. With $r = \lfloor y \rfloor$,

$$c_y \geq \prod_{n=2}^{r} \frac{n-1}{n+1} = \frac{2}{r(r+1)} \geq y^{-2}.$$

□

So complete multiplicativity together with agreement with λ at all primes up to $(\log X)^2$ is compatible with a smoothed sum of size $X/(\log X)^4$ at scale X. To pass from f_y to λ, let $h_y(d) =$

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$2^{\omega(d)}\lambda(d)$ if every prime factor of d exceeds y and $h_y(d) = 0$ otherwise, with $h_y(1) = 1$. Its local factor at $p > y$ is $(1 - p^{-s})/(1 + p^{-s})$, so $\lambda = f_y * h_y$ and

$$B(X) = B_y(X) + T_y(X), \qquad T_y(X) = \sum_{\substack{d>1 \\ P^-(d)>y}} h_y(d)\, B_y(X/d), \tag{77}$$

where $P^-(d)$ is the least prime factor. The series converges absolutely for each X.

Proposition 21 (The prime tail)**.** *Let* $y = (\log X)^2$.

(i) *For every* $A > 0$, $T_y(X) = -c_y X + O_A(X(\log X)^{-A})$. *In particular* $T_y(X)/(c_y X) \to -1$.
(ii) *The estimate* $T_y(X) = -c_y X + O_\varepsilon(X^{1/2+\varepsilon})$ *for every* $\varepsilon > 0$ *is equivalent to* RH.

Proof. By (77) and Theorem 34,

$$T_y(X) + c_y X = B(X) - \big(B_y(X) - c_y X\big) = B(X) + O(X^{1/2+o(1)}).$$

Part (ii) follows from Proposition 22. For (i), partial summation gives $B(X) = X^{-1}\int_1^\infty L(t)e^{-t/X}\,\mathrm{d}t$. The prime number theorem with the classical error term gives $|L(t)| \le C_A t(\log 2t)^{-A}$ for $t \ge 1$ and every $A > 0$ [49]. For $X \ge 4$ the range $t \le \sqrt{X}$ contributes at most $1/2$, and on $t \ge \sqrt{X}$ one has $\log 2t \ge \frac{1}{2}\log X$, so

$$|B(X)| \le \frac{1}{2} + \frac{2^A C_A}{X(\log X)^A}\int_0^\infty te^{-t/X}\,\mathrm{d}t = \frac{1}{2} + \frac{2^A C_A X}{(\log X)^A}.$$

Since $c_y \ge (\log X)^{-4}$, taking $A > 4$ gives the last statement. □

Part (i) is a genuine cancellation over all large primes in the exponential kernel: the tail removes the main term $c_y X$ of the model. It follows from classical additive cancellation and is far from the square-root error of part (ii). There is a companion cancellation in the Abel weight, where it is exact. For real $s > 1$,

$$H_y(s) := \sum_{d\ge 1}\frac{h_y(d)}{d^s} = \frac{\zeta(2s)}{\zeta(s)^2\, c_y(s)}, \tag{78}$$

and, uniformly for $0 < \delta \le 1$ with $\delta \log y \le 1$,

$$H_y(1+\delta) = \frac{\zeta(2)\,\delta^2}{c_y}\big(1 + O(\delta(1+\log y))\big). \tag{79}$$

Indeed $\zeta(2+2\delta)/\zeta(1+\delta)^2 = \zeta(2)\delta^2(1+O(\delta))$, and

$$\frac{\mathrm{d}}{\mathrm{d}s}\log c_y(s) = 2\sum_{p\le y}\frac{(\log p)\,p^{-s}}{1-p^{-2s}} \ll 1 + \log y \qquad (s \ge 1)$$

by Mertens' estimate $\sum_{p\le y}(\log p)/p \ll 1 + \log y$ [3]. So $\sum_{d>1} h_y(d)d^{-1-\delta} \to -1$ as $\delta \downarrow 0$ for fixed y: the tail cancels the term $d = 1$ to second order. The two cancellations are not interchangeable. The series of absolute values is $\sum_d |h_y(d)|d^{-s} = \zeta(s)^2 c_y(s)/\zeta(2s)$, with a double pole at $s = 1$, so no comparison in absolute value between the weight $d^{-1-\delta}$ of (79) and the kernel $B_y(X/d)$ of (77) preserves the small signed value. Nor does the double zero at $s = 1$ constrain zeros of ζ elsewhere: if ρ is a zero of order r with $\mathrm{Re}\,\rho > 1/2$, then by (78)

$$H_y(s) = \frac{(r!)^2\,\zeta(2\rho)}{c_y(\rho)\,\zeta^{(r)}(\rho)^2}\,(s-\rho)^{-2r} + O\big(|s-\rho|^{1-2r}\big),$$

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with a nonzero leading coefficient. Theorems 28 and 30 show in two different classes that local information at $s = 1$ of this kind cannot be converted into the bound of part (ii).

Table 13. Reformulations of the square-root bound for $B(X)$. In each row the stated estimate, for every $\varepsilon > 0$, is equivalent to RH; the last column records what is proved unconditionally.

Object	Estimate equivalent to RH	Proved here
local mean square	$\mathcal{H}_B(X) \ll X^{1+\varepsilon}$	point value bounded by $\mathcal{H}_B$ with loss $O(\log X)$ (Theorem 31)
signed part	$\mathcal{N}(X) \le K_\varepsilon X^{1+\varepsilon}$	$\mathcal{D}(X) = \frac{3}{4}X - \log 2 + O(1/X)$ (Proposition 18)
primitive correlation	$\mathcal{C}_\lambda(X) \le K_\varepsilon X^{1+\varepsilon}$, one-sided	exact decomposition (Proposition 19)
prime dilations	$B \in \mathcal{K}_\sigma$ for all $\sigma > 1/2$	exact norm $\|T_R\| \asymp R^{1-\sigma}$ (Theorem 33)
prime tail, $y = (\log X)^2$	$T_y(X) = -c_y X + O(X^{1/2+\varepsilon})$	error $O_A(X(\log X)^{-A})$ (Proposition 21)

12. The Smoothed Sum

The sharp cut-off in $L(x)$ can be replaced by an exponential weight without changing what the sum detects. For $X > 0$ put

$$B(X) = \sum_{n\ge1} \lambda(n)\mathrm{e}^{-n/X}.$$

The series converges absolutely, and

$$|B(X)| \le \sum_{n\ge1} \mathrm{e}^{-n/X} = \frac{1}{\mathrm{e}^{1/X}-1} \le \min\bigl(X,\, 2\mathrm{e}^{-1/X}\bigr), \tag{80}$$

where the second bound is used for $X \le 1$. Since $\int_0^\infty \mathrm{e}^{-n/X} X^{-s}\,\mathrm{d}X/X = \Gamma(s)n^{-s}$ for $\operatorname{Re} s > 0$, summation over n gives, for $\operatorname{Re} s > 1$,

$$\int_0^\infty B(X)X^{-s}\frac{\mathrm{d}X}{X} = \Phi(s), \qquad \Phi(s) = \Gamma(s)\frac{\zeta(2s)}{\zeta(s)}. \tag{81}$$

The part of the integral over $0 < X \le 1$ is an entire function of s by (80). The function Φ is meromorphic in the plane. In the half-plane $\operatorname{Re} s > 1/2$ the factors $\Gamma(s)$ and $\zeta(2s)$ are holomorphic and have no zeros, so the poles of Φ there are exactly the zeros of ζ, with the same orders; at $s = 1$ the pole of ζ makes Φ vanish. At $s = 1/2$ the factor $\zeta(2s)$ has a simple pole with residue $1/2$, and $\zeta(1/2) < 0$,[15] so Φ has a simple pole there with residue

$$\operatorname*{Res}_{s=1/2} \Phi(s) = \frac{\sqrt{\pi}}{2\zeta(1/2)} = -c_0, \qquad c_0 = -\frac{\sqrt{\pi}}{2\zeta(1/2)} = 0.60685\ldots \tag{82}$$

12.1. The Equivalence

Proposition 22 (Smoothed form of the criterion)**.** *Let $1/2 \le \theta \le 1$. The following are equivalent.*

1. *$\zeta(s) \ne 0$ for $\operatorname{Re} s > \theta$.*
2. *$B(X) \ll_{\theta,\varepsilon} X^{\theta+\varepsilon}$ for $X \ge 1$, for every $\varepsilon > 0$.*

Consequently

$$\limsup_{X\to\infty} \frac{\log(1+|B(X)|)}{\log X} = \Theta.$$

Proof. Assume the second condition. With (80) for $X \le 1$, the integral in (81) converges absolutely and locally uniformly in $\operatorname{Re} s > \theta$, so it is holomorphic there. It agrees with Φ for $\operatorname{Re} s > 1$, hence Φ is holomorphic in $\operatorname{Re} s > \theta$, and ζ has no zero in that half-plane.

[15] For $0 < s < 1$ one has $(1-2^{1-s})\zeta(s) = \sum_{n\ge1}(-1)^{n-1}n^{-s} > 0$ and $1-2^{1-s} < 0$, so $\zeta(s) < 0$ on $(0,1)$ [57, §2.2].

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Assume the first condition. Theorem 6 gives $L(u) \ll_{\theta,\varepsilon} u^{\theta+\varepsilon}$ for $u \geq 1$. Partial summation gives

$$B(X) = \int_{1^-}^{\infty} \mathrm{e}^{-u/X}\,\mathrm{d}L(u) = \frac{1}{X}\int_1^{\infty} L(u)\mathrm{e}^{-u/X}\,\mathrm{d}u,$$

and therefore

$$|B(X)| \ll_{\theta,\varepsilon} \frac{1}{X}\int_0^{\infty} u^{\theta+\varepsilon}\mathrm{e}^{-u/X}\,\mathrm{d}u = \Gamma(1+\theta+\varepsilon)X^{\theta+\varepsilon}.$$

Let r denote the limit superior. The equivalence at $\theta = \Theta$ gives $r \leq \Theta$. If $r < \Theta$, the Mellin integral is holomorphic in $\operatorname{Re} s > r + \varepsilon$ for every $\varepsilon > 0$. When $\Theta > 1/2$ this half-plane contains a zero of ζ with real part greater than $1/2$, which is a pole of Φ; when $\Theta = 1/2$ it contains the pole of Φ at $s = 1/2$. Both are impossible. □

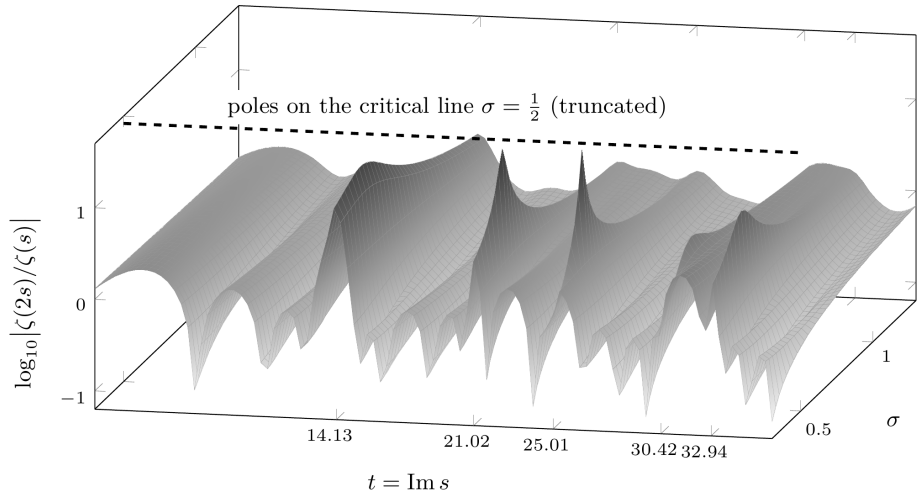

Figure 9. The surface $\log_{10}|\zeta(2s)/\zeta(s)|$ over $0.3 \leq \sigma \leq 1.3$, $2 \leq t \leq 36$. The peaks on the line $\sigma = \frac{1}{2}$ are the poles at the zeros of ζ (truncated for display). Any zero of ζ with $\sigma > \frac{1}{2}$ would appear as a peak away from that line, and by Proposition 22 it would force $|B(X)|$ to exceed $X^{\beta-\varepsilon}$ infinitely often.

The proof uses only that the Mellin transform $\Gamma(s)$ of the weight has no zeros. This is a real restriction on the weight, as the next remark shows.

Remark 9 (Weights that cancel the poles)**.** *Let V be smooth and compactly supported in $(0,\infty)$, write $\widehat{V}(s) = \int_0^{\infty} V(x)x^{s-1}\,\mathrm{d}x$, which is entire, and form the divisor-convolution weight $W(x) = \sum_{m\geq 1} V(mx)$. For each $X > 0$ the sums below are finite, and*

$$\sum_{n\geq 1}\lambda(n)W(n/X) = \sum_{r\geq 1} V(r/X)\sum_{n\mid r}\lambda(n) = \sum_{j\geq 1} V(j^2/X),$$

because $\sum_{n\mid r}\lambda(n)$ equals 1 when r is a square and 0 otherwise: at a prime power p^{ℓ} the sum is $\sum_{k=0}^{\ell}(-1)^k$. If V is supported in $[a,b]$, the last sum has at most $\sqrt{bX}$ terms, so the weighted sum is $O_V(\sqrt{X})$ for $X \geq 1$, with no hypothesis on the zeros. There is no contradiction with Proposition 22. For $\operatorname{Re} s > 1$ the Mellin transform of W is $\zeta(s)\widehat{V}(s)$, and so

$$\int_0^{\infty}\Big(\sum_{n\geq 1}\lambda(n)W(n/X)\Big)X^{-s}\frac{\mathrm{d}X}{X} = \frac{\zeta(2s)}{\zeta(s)}\cdot\zeta(s)\widehat{V}(s) = \zeta(2s)\widehat{V}(s).$$

The factor $\zeta(s)$ contributed by the weight cancels every pole that a zero of ζ would produce; only the pole at $s = 1/2$ survives, and it accounts for the size $\sqrt{X}$. The argument of Proposition 22 works for any weight whose Mellin transform has no zeros in $\operatorname{Re} s > 1/2$; the exponential weight, with transform $\Gamma(s)$, is one such weight, and W is not.

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12.2. The Unconditional Bound

Write

$$\ell(t) = (\log t)^{2/3}(\log\log t)^{1/3} \qquad (t \geq 3),$$

an increasing function. The strongest known zero-free region is that of Vinogradov and Korobov.

Lemma 13 (Vinogradov–Korobov)**.** *There are absolute constants $c_1 > 0$ and C_1 such that, for $|t| \geq 3$ and $\sigma \geq 1 - c_1/\ell(|t|)$,*

$$\zeta(\sigma + it) \neq 0 \qquad \text{and} \qquad \left|\frac{1}{\zeta(\sigma+it)}\right| \leq C_1\,\ell(|t|).$$

This is classical: see [57, Notes to Ch. VI] and [40, Ch. 6]; Ford [35] gives the zero-free region with the explicit constant $c_1 = 1/57.54$.[16]

Theorem 35 (The smoothed sum, unconditionally)**.** *There are absolute constants $c > 0$ and X_0 such that, for $X \geq X_0$,*

$$|B(X)| \leq X \exp\left(-\frac{c\,\log X}{(\log\log X)^{2/3}(\log\log\log X)^{1/3}}\right).$$

For comparison, partial summation from the unconditional bound for L in Theorem 48 gives only

$$B(X) \ll X\exp\bigl(-c\,\eta(X)\bigr), \qquad \eta(X) = (\log X)^{3/5}(\log\log X)^{-1/5}.$$

The smoothing gains because $\Gamma(s)$ decays exponentially on vertical lines: at scale X the sum $B(X)$ is sensitive to the zeros of height at most a constant multiple of $\log X$, where the Vinogradov–Korobov region is much wider than at height X.

Proof. For $y > 0$, Mellin inversion gives $\mathrm{e}^{-y} = \frac{1}{2\pi i}\int_{\mathrm{Re}\,s=2}\Gamma(s)y^{-s}\,\mathrm{d}s$ [49, §5.1]. Taking $y = n/X$, multiplying by $\lambda(n)$ and summing, which is permitted by absolute convergence, gives

$$B(X) = \frac{1}{2\pi i}\int_{\mathrm{Re}\,s=2}\Phi(s)X^s\,\mathrm{d}s \qquad (X > 0). \tag{83}$$

Choose $\omega_0 \in (0, 1/4]$ such that ζ has no zero in the rectangle $1 - \omega_0 \leq \sigma \leq 2$, $|t| \leq 3$. This is possible because ζ has finitely many zeros in $1/2 \leq \sigma \leq 1$, $|t| \leq 3$, and none of them has real part 1.[17] Let X_0 be so large that $T = \log X \geq 3$ and

$$\omega = \frac{c_1}{\ell(T)} \leq \omega_0 \qquad \text{for } X \geq X_0.$$

Let $\mathcal{C}_X$ be the path made of the half-line $\mathrm{Re}\,s = 2$, $\mathrm{Im}\,s \leq -T$; the segment from $2 - iT$ to $1 - \omega - iT$; the segment from $1 - \omega - iT$ to $1 - \omega + iT$; the segment from $1 - \omega + iT$ to $2 + iT$; and the half-line $\mathrm{Re}\,s = 2$, $\mathrm{Im}\,s \geq T$.

[16] The sources state these facts for $|t| \geq t_0$. On $3 \leq |t| \leq t_0$ the function ζ has finitely many zeros in $1/2 \leq \sigma \leq 1$ and none on $\sigma = 1$, so after decreasing c_1 and increasing C_1 the statement holds from $|t| = 3$ on.

[17] In fact ζ has no zero at all with $|t| < 14.13$, so any $\omega_0 \leq 1/4$ would do. The argument does not need this numerical fact.

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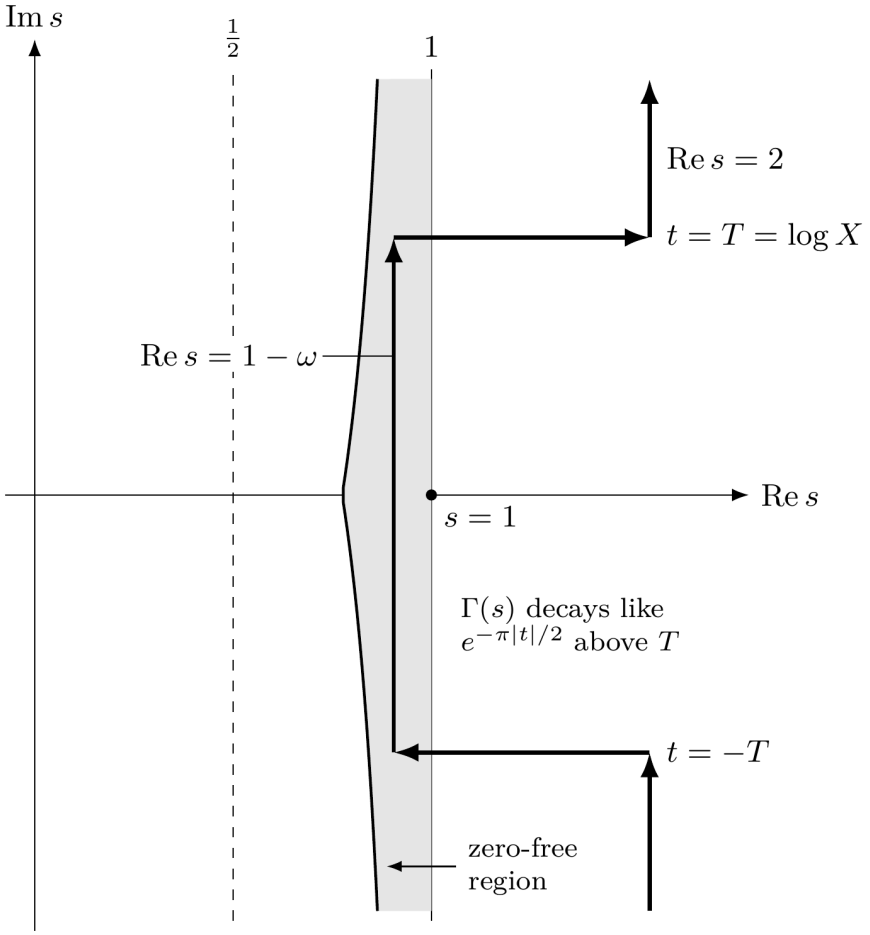

Figure 10. The contour $\mathcal{C}_X$ in the proof of Theorem 35, with $T = \log X$ and $\omega = c_1/\ell(T)$. The integral starts on the line $\operatorname{Re} s = 2$ (dashed) and is moved left to the solid path. On the vertical segment $\operatorname{Re} s = 1 - \omega$ the integrand is $O(X^{1-\omega})$ times logarithmic factors; on the horizontal segments and on the half-lines the factor $\Gamma(s)$ gives $O(X^2 e^{-\pi T/2}) = O(X^{2-\pi/2})$. The shaded region is the Vinogradov–Korobov zero-free region.

The closed rectangle $R_X = \{1 - \omega \le \sigma \le 2,\ |t| \le T\}$ lies between the line $\operatorname{Re} s = 2$ and $\mathcal{C}_X$. The function Φ is holomorphic on a neighbourhood of R_X: $\Gamma(s)$ and $\zeta(2s)$ are holomorphic in $\operatorname{Re} s > 1/2$; the reciprocal $1/\zeta$ is holomorphic wherever ζ has no zero, with a zero at $s = 1$; for $|t| \le 3$ there is no zero in R_X by the choice of ω_0; and for $3 \le |t| \le T$ the monotonicity of ℓ gives $1 - \omega \ge 1 - c_1/\ell(|t|)$, so Lemma 13 applies. Cauchy's theorem on R_X therefore replaces the segment of $\operatorname{Re} s = 2$ with $|t| \le T$ by the other three sides, and

$$B(X) = \frac{1}{2\pi i} \int_{\mathcal{C}_X} \Phi(s) X^s \, ds.$$

The three kinds of piece are estimated in Appendix E, with absolute implied constants:

$$\int_{\operatorname{Re} s = 2,\ |t| \ge T} |\Phi(s) X^s| \, |ds| \ll X^{2-\pi/2} (\log X)^{3/2}, \tag{84}$$

$$\int_{1-\omega}^{2} |\Phi(\sigma \pm iT)| X^{\sigma} \, d\sigma \ll X^{2-\pi/2} (\log X)^2, \tag{85}$$

$$\int_{-T}^{T} |\Phi(1 - \omega + it)| X^{1-\omega} \, dt \ll X^{1-\omega}. \tag{86}$$

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The first two come from $|\Gamma(\sigma+it)| \ll (1+|t|)^{3/2}e^{-\pi|t|/2}$ for $1/2 \le \sigma \le 2$ and from $e^{-\pi T/2} = X^{-\pi/2}$; the third from the same bound, which makes the integral over t converge, and from Lemma 13. Since $2 - \pi/2 = 0.4292\ldots < 1/2$, we obtain

$$B(X) \ll X^{1-\omega} + X^{0.43} \ll X \exp\left(-\frac{c_1 \log X}{(\log\log X)^{2/3}(\log\log\log X)^{1/3}}\right).$$

Replacing c_1 by $c = c_1/2$ and enlarging X_0 absorbs the implied constant. □

12.3. Why No Power Saving Is Possible

By Proposition 22, a bound $B(X) \ll X^{1-\delta}$ with a fixed $\delta > 0$ is the statement that ζ has no zero in $\operatorname{Re} s > 1-\delta$. No such half-plane is known, so Theorem 35 cannot be improved to a power saving by any argument that does not also produce a zero-free half-plane.

The proof shows the mechanism. A zero $\rho = \beta + i\gamma$ enters $B(X)$ with the weight $X^\beta|\Gamma(\rho)|$, and $|\Gamma(\rho)|$ is of order $|\gamma|^{\beta-1/2}e^{-\pi|\gamma|/2}$. Against a target $X^{1-\delta}$, the zero is negligible when $\frac{\pi}{2}|\gamma|$ exceeds $(\beta - 1 + \delta)\log X$ by a margin, and not otherwise. A saving of $X^{-\delta}$ at the single scale X therefore needs a rectangle

$$\operatorname{Re} s > 1-\delta, \qquad |\operatorname{Im} s| \le \frac{2\delta}{\pi}\log X$$

free of zeros, together with a bound for $1/\zeta$ inside it; as $X \to \infty$ these rectangles exhaust the half-plane $\operatorname{Re} s > 1-\delta$. The Vinogradov–Korobov region has width $c_1/\ell(\log X)$ at height $\log X$, and this width is what appears in the exponent of Theorem 35. Conversely, a single zero to the right of the critical line forces oscillations of exact order X^β in both directions; this is Theorem 36 below.

Remark 10 (A finite range). *Platt and Trudgian [54] verified that every zero of ζ with $0 < \operatorname{Im} s \le 3\cdot 10^{12}$ lies on the critical line. For $1 \le X \le \exp(3\cdot 10^{12})$ and any fixed $\sigma_1 \in (1/2, 1)$, the contour above may therefore be drawn with left side $\operatorname{Re} s = \sigma_1$, and the same estimates give*

$$|B(X)| \le K(\sigma_1)X^{\sigma_1} \qquad (1 \le X \le \exp(3\cdot 10^{12})),$$

where $K(\sigma_1)$ is a finite constant depending on the maximum of $|1/\zeta|$ on the compact set $\sigma_1 \le \sigma \le 2$, $|t| \le 3\cdot 10^{12}$. We have not evaluated $K(\sigma_1)$. The statement concerns a bounded range of X and carries no information about Θ.

Table 14 records values of $B(X)/\sqrt{X}$. They are consistent with the negative bias $-c_0$ produced by the pole at $s = 1/2$ and prove nothing about large X.

Table 14. The smoothed sum at small scales, computed from the definition. The last column is the constant of (82).

X	1	3	10^2	10^3	10^4	10^5	$-c_0$
$B(X)/\sqrt{X}$	0.1957	−0.0457	−0.5069	−0.5752	−0.5969	−0.6037	−0.6069

12.4. The Positivity Principle

The pole at $s = 1/2$ suggests that $B(X)$ might eventually be negative. The following classical principle, due to Landau for Dirichlet series [3, Ch. 11], turns a sign condition into information about singularities. The integral form is [49, Lemma 15.1]; we include the short proof.

Lemma 14 (Positivity principle). *Let $f : [0,\infty) \to [0,\infty)$ be locally integrable, and suppose that*

$$\mathcal{L}(s) = \int_0^\infty f(t)e^{-st}\,dt$$

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converges for some real s. Let a be the infimum of the real s for which it converges. Then the integral converges absolutely and is holomorphic in $\operatorname{Re} s > a$. If $a > -\infty$, then $\mathcal{L}$ has no holomorphic continuation to any neighbourhood of the point $s = a$.

Proof. Since $f \geq 0$, convergence at a real point s_1 gives absolute and locally uniform convergence in $\operatorname{Re} s > s_1$; this proves the first assertion. Suppose that a is finite and that $\mathcal{L}$ continues holomorphically to a disc of radius r about a. Put $b = a + r/4$. The disc of radius $r/2$ about b lies in the disc of radius r about a, so the Taylor series of $\mathcal{L}$ at b converges at $b - z$ for every real $z \in (r/4, r/2)$. Differentiating under the integral sign, which is justified by convergence at a real point between a and b, gives

$$\frac{(-1)^j \mathcal{L}^{(j)}(b)}{j!} = \frac{1}{j!} \int_0^\infty t^j f(t) \mathrm{e}^{-bt} \, \mathrm{d}t \geq 0.$$

Hence, by Tonelli's theorem,

$$\mathcal{L}(b - z) = \sum_{j \geq 0} \frac{z^j}{j!} \int_0^\infty t^j f(t) \mathrm{e}^{-bt} \, \mathrm{d}t = \int_0^\infty f(t) \mathrm{e}^{-(b-z)t} \, \mathrm{d}t < \infty.$$

But $b - z < a$, which contradicts the definition of a. □

With $X = X_0 \mathrm{e}^t$ the lemma applies to integrals $\int_{X_0}^\infty A(X) X^{-s} \, \mathrm{d}X/X$ with $A \geq 0$ on $[X_0, \infty)$. We call a the abscissa of the integral.

Theorem 36 (Both signs forced by a right-hand zero)**.** *Let $\rho = \beta + i\gamma$ be a zero of ζ of order r with $\beta > 1/2$, and put*

$$C_\rho = \frac{r \, |\Gamma(\rho) \zeta(2\rho)|}{|\zeta^{(r)}(\rho)|} > 0.$$

Then

$$\limsup_{X \to \infty} \frac{B(X)}{X^\beta (\log X)^{r-1}} \geq C_\rho, \qquad \liminf_{X \to \infty} \frac{B(X)}{X^\beta (\log X)^{r-1}} \leq -C_\rho.$$

Proof. Near ρ we have $\Phi(s) = A_\rho (s - \rho)^{-r} + O(|s - \rho|^{1-r})$ with

$$A_\rho = \frac{r! \, \Gamma(\rho) \zeta(2\rho)}{\zeta^{(r)}(\rho)}, \qquad |A_\rho| = (r-1)! \, C_\rho,$$

and $A_\rho \neq 0$ because $\operatorname{Re}(2\rho) > 1$. Also $\gamma \neq 0$ and $\beta < 1$, and Φ is holomorphic at every real point of $(1/2, \infty)$, since ζ has no zero on $(1/2, 1)$ and its pole at 1 is a zero of Φ.

Suppose the first inequality fails. Then there are $c \in (0, C_\rho)$ and $X_1 > 1$ with

$$A(X) = c X^\beta (\log X)^{r-1} - B(X) \geq 0 \qquad (X \geq X_1).$$

Let $I(s) = \int_{X_1}^\infty A(X) X^{-s} \, \mathrm{d}X/X$. It converges for real $s > 1$ by (80). Using $\int_1^\infty X^{\beta - s} (\log X)^{r-1} \, \mathrm{d}X/X = (r-1)!/(s-\beta)^r$ and (81), we find for $\operatorname{Re} s > 1$

$$I(s) = J(s), \qquad J(s) = \frac{c \, (r-1)!}{(s-\beta)^r} - \Phi(s) + E(s), \tag{87}$$

where E is entire and accounts for the integrals over the bounded ranges $[1, X_1]$ and $(0, X_1]$. The function J is meromorphic in the plane.

Let $a \leq 1$ be the abscissa of I. By Lemma 14, I is holomorphic in $\operatorname{Re} s > a$, and by the identity theorem (87) holds there; in particular J is holomorphic in $\operatorname{Re} s > a$. Now J has a pole at the real point β, because Φ and E are holomorphic there. Hence $a \geq \beta$, and a is finite. If $a > \beta$, then J would be

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holomorphic at the real point $a \in (\beta, 1]$ and would continue I to a neighbourhood of a, contrary to Lemma 14. Thus $a = \beta$.

Since $A \geq 0$, for every $\delta > 0$

$$|J(\beta + \delta + i\gamma)| = |I(\rho + \delta)| \leq I(\beta + \delta) = J(\beta + \delta).$$

Multiply by δ^r and let $\delta \downarrow 0$. On the left, the first term of J is bounded because $\gamma \neq 0$, the term E is bounded, and $\delta^r \Phi(\rho + \delta) \to A_\rho$; the limit is $|A_\rho|$. On the right, Φ and E are bounded near β and the limit is $c\,(r-1)!$. Therefore $c \geq |A_\rho|/(r-1)! = C_\rho$, a contradiction.

For the second inequality apply the same argument to the function

$$cX^\beta (\log X)^{r-1} + B(X),$$

for which

$$J(s) = \frac{c\,(r-1)!}{(s-\beta)^r} + \Phi(s) - E(s).$$

□

In particular $\limsup_{X\to\infty} |B(X)| X^{-\beta} (\log X)^{1-r} \geq C_\rho$. The theorem gives lower bounds only. At a zero of large height the constant C_ρ carries the factor $|\Gamma(\rho)| \asymp |\gamma|^{\beta - 1/2} \mathrm{e}^{-\pi|\gamma|/2}$ and is extremely small, but the exponent β is unaffected.

Proposition 23 (The real pole and an eventual sign)**.** *With c_0 as in (82),*

$$\liminf_{X\to\infty} \frac{B(X)}{\sqrt{X}} \leq -c_0,$$

unconditionally. If moreover $B(X) \leq 0$ for all sufficiently large X, then the Riemann hypothesis holds, every nontrivial zero ρ is simple, and

$$\left| \frac{\Gamma(\rho)\zeta(2\rho)}{\zeta'(\rho)} \right| \leq c_0.$$

Proof. Suppose the limit inferior exceeds $-c_0$. Then there are $c \in (0, c_0)$ and $X_1 > 1$ with $B(X) + c\sqrt{X} \geq 0$ for $X \geq X_1$. As in the proof of Theorem 36, the integral $I(s) = \int_{X_1}^\infty (B(X) + c\sqrt{X}) X^{-s}\, \mathrm{d}X/X$ equals, for $\operatorname{Re} s > 1$,

$$J(s) = \Phi(s) + \frac{c}{s - 1/2} + E(s)$$

with E entire. The function J has a simple pole at $s = 1/2$ with residue $c - c_0 < 0$ and is holomorphic at every real point of $(1/2, \infty)$. As before, the pole gives $a \geq 1/2$ for the abscissa of I, and Lemma 14 excludes $a > 1/2$. So $I(s) = J(s)$ for real $s > 1/2$. But $I(s) \geq 0$ there, while $J(s) \to -\infty$ as $s \downarrow 1/2$.

Assume that $B(X) \leq 0$ for $X \geq X_1$ and apply the same reasoning to $I(s) = \int_{X_1}^\infty (-B(X)) X^{-s}\, \mathrm{d}X/X$, which equals $J(s) = -\Phi(s) + E(s)$ for $\operatorname{Re} s > 1$. The pole of J at $1/2$ and its regularity on $(1/2, \infty)$ give abscissa $1/2$. Hence Φ is holomorphic in $\operatorname{Re} s > 1/2$, so ζ has no zero there, and the functional equation gives the Riemann hypothesis. Let $\rho = 1/2 + i\gamma$ be a zero of order r. Since $\zeta(2\rho) = \zeta(1 + 2i\gamma) \neq 0$, the function Φ has a pole of order exactly r at ρ. Positivity gives

$$|J(\rho + \delta)| \leq J(1/2 + \delta) = \frac{c_0}{\delta} + O(1) \qquad (\delta \downarrow 0).$$

The left side is of exact order δ^{-r}, so $r = 1$, and comparison of the coefficients of δ^{-1} gives the stated inequality. □

An eventual sign would thus give more than the square-root bound with an X^ε loss: it would give simplicity of all zeros and a uniform bound for the residues. Neither the sign nor the upper

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bound is proved here. The first assertion bounds B from one side only and does not control positive excursions.[18]

12.5. A Lower Bound for the Local Energy

For a locally square-integrable F on $(0,\infty)$ define the local energy over one octave on each side,

$$\mathcal{H}_F(X) = \int_{X/2}^{2X} |F(Y)|^2 \frac{\mathrm{d}Y}{Y}.$$

For the model $F_0(X) = -c_0\sqrt{X}$ one has $\mathcal{H}_{F_0}(X) = \frac{3}{2}c_0^2 X$ exactly.

Proposition 24 (Local-energy lower bound)**.** *Unconditionally,*

$$\limsup_{X\to\infty} \frac{\mathcal{H}_B(X)}{X} \geq \frac{3}{2}c_0^2 = \frac{3\pi}{8\,\zeta(1/2)^2} = 0.5524\ldots$$

The proof is in Appendix F: a bound $\mathcal{H}_B(X) \leq CX$ gives a bound for the Mellin transform of $|B|^2$ near its abscissa, and the Cauchy–Schwarz inequality compares it with the residue (82).

The constant is the one dictated by the real residue, as the model F_0 shows. The proposition constrains any proposed upper bound for $\mathcal{H}_B$ from below; it gives no upper bound. Table 15 collects the statements of this section.

Table 15. What is known about $B(X) = \sum_n \lambda(n)\mathrm{e}^{-n/X}$. Here $\rho = \beta + i\gamma$ is a zero of order r with $\beta > 1/2$.

Hypothesis	Conclusion	Reference
none	$\lvert B(X)\rvert \leq X\exp(-c\log X/\ell(\log X))$	Theorem 35
none	$\liminf B(X)/\sqrt{X} \leq -c_0$	Proposition 23
none	$\limsup \mathcal{H}_B(X)/X \geq \frac{3}{2}c_0^2$	Proposition 24
$\zeta \neq 0$ in $\mathrm{Re}\, s > \theta$	$B(X) \ll X^{\theta+\varepsilon}$, and conversely	Proposition 22
a zero ρ exists	$B(X) = \Omega_\pm\big(X^\beta(\log X)^{r-1}\big)$	Theorem 36
$B(X) \leq 0$ eventually	RH, simple zeros, bounded residues	Proposition 23

13. Reciprocal Dilates in Weighted Norms

The criterion of Nyman and Beurling expresses the Riemann hypothesis as an approximation problem in $L^2(0,1)$ for dilates of the fractional part of $1/x$, and Báez-Duarte showed that integer dilates suffice [19]; see also [20,24,30]. This section places the problem in a one-parameter family of weighted norms. In that family the set of exponents at which approximation succeeds is exactly $[\Theta, 1)$, the approximation succeeds unconditionally at the exponent 1 with an explicit rate, and at the critical exponent every fixed taper of the natural coefficients fails.

13.1. Balanced Dilates and Their Mellin Transforms

For $n \geq 1$ define

$$g_n(x) = \left\{\frac{1}{nx}\right\} - \frac{1}{n}\left\{\frac{1}{x}\right\} \qquad (x > 0).$$

Then $|g_n| \leq 1$, $g_1 = 0$, and $g_n(x) = 0$ for $x > 1$, since both fractional parts are then equal to their arguments. Writing $y = 1/x$,

$$g_n(1/y) = \frac{\lfloor y\rfloor}{n} - \left\lfloor \frac{y}{n}\right\rfloor \qquad (y > 0). \tag{88}$$

[18] The sign of $B(X)$ is not constant: $B(1) > 0.195$ and $B(3) < -0.079$, by the elementary evaluation in Appendix K.

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For $\sigma \geq 1/2$ and a measurable u on $(0,1)$ set

$$\|u\|_\sigma^2 = \int_0^1 |u(x)|^2 x^{2\sigma-1}\,\mathrm{d}x, \qquad d_{M,\sigma} = \inf_{c_1,\dots,c_M\in\mathbb{C}} \Big\|1 - \sum_{n\leq M} c_n g_n\Big\|_\sigma.$$

At $\sigma = 1/2$ this is the norm of $L^2(0,1)$. The norms decrease as σ increases, and $d_{M,\sigma}$ is nonincreasing in M.

Lemma 15 (Exact Mellin transform)**.** *For* $\operatorname{Re} s > 0$,

$$\int_0^1 g_n(x) x^{s-1}\,\mathrm{d}x = -\big(n^{-s} - n^{-1}\big)\frac{\zeta(s)}{s},$$

and both sides are holomorphic in $\operatorname{Re} s > 0$*; at* $s = 1$ *the right side is* $(\log n)/n$.

Proof. The left side is holomorphic in $\operatorname{Re} s > 0$ because g_n is bounded and supported in $(0,1]$. The right side is holomorphic there because the factor $n^{-s} - n^{-1}$ vanishes at the pole of ζ. It suffices to prove the identity for $0 < \operatorname{Re} s < 1$. The substitution $y = 1/(nx)$ gives

$$\int_0^\infty \Big\{\frac{1}{nx}\Big\} x^{s-1}\,\mathrm{d}x = n^{-s}\int_0^\infty \{y\} y^{-s-1}\,\mathrm{d}y.$$

On $0 < y < 1$ the integral is $1/(1-s)$. On $y > 1$ it is $1/(s-1) - \zeta(s)/s$, by the formula $\zeta(s) = s/(s-1) - s\int_1^\infty \{y\} y^{-s-1}\,\mathrm{d}y$, valid for $\operatorname{Re} s > 0$ [57, §2.1]. The sum is $-\zeta(s)/s$. Subtracting n^{-1} times the case $n = 1$ gives the lemma. □

Consequently, if $c_n = -\mu(n)w_n$ and $D(s) = \sum_{n\leq M} \mu(n) w_n n^{-s}$, the residual $u = 1 - \sum_{n\leq M} c_n g_n$ has the Mellin transform

$$\widehat{u}(s) = \int_0^1 u(x) x^{s-1}\,\mathrm{d}x = \frac{1 - \zeta(s)\big(D(s) - D(1)\big)}{s} \qquad (\operatorname{Re} s > 0). \tag{89}$$

If $|w_n| \leq 1$ we shall use the trivial bound

$$|D(1)| \leq \sum_{n\leq M} \frac{1}{n} \leq 1 + \log M. \tag{90}$$

Lemma 16 (Plancherel and evaluation)**.** *Let* $\sigma \geq 1/2$ *and let* u *be bounded and measurable on* $(0,1)$*. Then*

$$\|u\|_\sigma^2 = \frac{1}{2\pi}\int_{-\infty}^{\infty} |\widehat{u}(\sigma + it)|^2\,\mathrm{d}t,$$

and for every s_0 *with* $\operatorname{Re} s_0 > \sigma$,

$$|\widehat{u}(s_0)| \leq \frac{\|u\|_\sigma}{\sqrt{2(\operatorname{Re} s_0 - \sigma)}}.$$

Proof. Put $v(y) = \mathrm{e}^{-\sigma y} u(\mathrm{e}^{-y})$ for $y > 0$ and $v(y) = 0$ for $y \leq 0$. Then $v \in L^1 \cap L^2(\mathbb{R})$, $\int |v|^2 = \|u\|_\sigma^2$, and $\int v(y)\mathrm{e}^{-ity}\,\mathrm{d}y = \widehat{u}(\sigma + it)$; the first assertion is Plancherel's theorem. The second is the Cauchy–Schwarz inequality applied to $u(x)x^{\sigma-1/2}\cdot x^{s_0-\sigma-1/2}$, since $\int_0^1 x^{2(\operatorname{Re} s_0-\sigma)-1}\,\mathrm{d}x = 1/(2(\operatorname{Re} s_0 - \sigma))$. □

Upper bounds for the residuals need the size of ζ and $1/\zeta$ to the right of a zero-free line. We state the result for Dirichlet L-functions, uniformly in the character, because Section 14 uses it in that form. For $q = 1$ the bound for the reciprocal is Lemma A2.

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Lemma 17 (Subpower growth in a zero-free half-plane). *Let $1/2 \le \theta < 1$, $0 < \delta \le 1/4$ and $\kappa > 0$. Let χ be a primitive Dirichlet character of conductor $q \ge 1$ and suppose that $L(s,\chi)$ has no zero in $\operatorname{Re} s > \theta$. Then, for $\theta + \delta \le \sigma \le 2$ and all real t, with the restriction $|t| \ge 4$ when $q = 1$,*

$$\big(q(2+|t|)\big)^{-\kappa} \ll_{\theta,\delta,\kappa} |L(\sigma+it,\chi)| \ll_{\theta,\delta,\kappa} \big(q(2+|t|)\big)^{\kappa}.$$

The implied constants do not depend on χ or q. When $q = 1$, the lower bound for $|\zeta(\sigma+it)|$ holds for all real t, the reciprocal being extended by 0 at $s = 1$.

The proof, by the Borel–Carathéodory inequality and the three-circles theorem as in [57, §14.2], is in Appendix D.

13.2. Convergence in Shifted Norms

The Riesz-weighted Möbius coefficients give the residuals

$$u_M = 1 + \sum_{n\le M} \mu(n)\Big(1-\frac{n}{M}\Big)g_n, \qquad D_M(s) = \sum_{n\le M} \mu(n)\Big(1-\frac{n}{M}\Big)n^{-s}. \tag{91}$$

Theorem 37 (Convergence in shifted norms). *Suppose $1/2 \le \theta < \sigma < 1$ and $\zeta(s) \ne 0$ for $\operatorname{Re} s > \theta$. Then for every $0 < \delta < \sigma - \theta$,*

$$d_{M,\sigma} \le \|u_M\|_\sigma \ll_{\sigma,\theta,\delta} M^{-\delta}.$$

Conversely, let $\sigma \ge 1/2$ and let ρ be a zero of ζ with $\operatorname{Re}\rho > \sigma$. Then, for every M and every choice of coefficients,

$$\Big\|1 - \sum_{n\le M} c_n g_n\Big\|_\sigma^2 \ge \frac{2(\operatorname{Re}\rho - \sigma)}{|\rho|^2}, \qquad \text{so} \qquad d_{M,\sigma}^2 \ge \frac{2(\operatorname{Re}\rho-\sigma)}{|\rho|^2}. \tag{92}$$

Proof. For $y > 0$ one has $\frac{1}{2\pi i}\int_{\operatorname{Re} w=2} y^w \frac{\mathrm{d}w}{w(w+1)} = (1-1/y)_+$: for $y > 1$ close the contour to the left, with residues 1 and $-1/y$; for $y < 1$ close it to the right; the case $y = 1$ follows by continuity. Hence, for $\operatorname{Re} s \ge 1/2$,

$$D_M(s) = \frac{1}{2\pi i}\int_{\operatorname{Re} w=2} \frac{M^w}{w(w+1)}\,\frac{\mathrm{d}w}{\zeta(s+w)}. \tag{93}$$

Fix $s = \sigma + it$ and $\kappa \in (0, 1/4)$, and move the line to $\operatorname{Re} w = -\delta$. On and to the right of the new line $\operatorname{Re}(s+w) \ge \sigma - \delta > \theta$, so Lemma A2, applied with the margin $\min(\sigma-\delta-\theta, 1/5)$, gives $1/\zeta(s+w) \ll (1+|t|+|\operatorname{Im} w|)^\kappa$ there; for $\operatorname{Re}(s+w) \ge 2$ the reciprocal is bounded. The horizontal integrals at height $\pm V$ are $O(M^2 V^{\kappa-2})$ for fixed t and tend to zero. The only pole crossed is $w = 0$, with residue $1/\zeta(s)$; the pole at $w = -1$ is not reached because $\delta < 1$. Therefore

$$\Big|D_M(s) - \frac{1}{\zeta(s)}\Big| \ll M^{-\delta}\int_{-\infty}^{\infty} \frac{(1+|t|+|v|)^\kappa}{|-\delta+iv|\,|1-\delta+iv|}\,\mathrm{d}v \ll_{\sigma,\theta,\delta,\kappa} M^{-\delta}(1+|t|)^\kappa.$$

The same computation at $s = 1$, where the integrand is holomorphic at $w = 0$ because $1/\zeta(1+w)$ vanishes there, gives $D_M(1) \ll M^{-\delta}$.

By (89),

$$\widehat{u}_M(s) = \frac{-\zeta(s)\big(D_M(s) - 1/\zeta(s)\big) + \zeta(s) D_M(1)}{s}.$$

On $\operatorname{Re} s = \sigma$ we have $\zeta(s) \ll (1+|t|)^\kappa$: for $|t| \ge 4$ this is Lemma 17, and for $|t| \le 4$ the function is bounded because $\sigma < 1$. Thus $\widehat{u}_M(\sigma+it) \ll M^{-\delta}(1+|t|)^{2\kappa-1}$, which is square integrable since $\kappa < 1/4$. Lemma 16 gives the upper bound.

For the converse, let $u = 1 - \sum c_n g_n$. By Lemma 15, $\widehat{u}(\rho) = 1/\rho$, because the transform of each g_n vanishes at a zero of ζ. The evaluation bound of Lemma 16 gives $|\rho|^{-1} \le \|u\|_\sigma (2(\operatorname{Re}\rho - \sigma))^{-1/2}$. □

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A point on the line $\operatorname{Re} s = \sigma$ itself is not a bounded evaluation in the weighted space, so (92) says nothing about zeros on that line. Theorem 37 leaves open the endpoint $\sigma = \theta$, where zeros may lie on the line of integration. The next theorem settles it.

13.3. The Endpoint

Theorem 38 (Endpoint equivalence)**.** *Let* $1/2 \le \sigma < 1$. *Then*

$$\lim_{M\to\infty} d_{M,\sigma} = 0 \quad\Longleftrightarrow\quad \zeta(s) \neq 0 \text{ for } \operatorname{Re} s > \sigma.$$

Hence $\{\sigma \in [1/2, 1) : d_{M,\sigma} \to 0\} = [\Theta, 1)$, *and* $d_{M,\sigma} \ll_{\sigma,\delta} M^{-\delta}$ *for every* $\delta < \sigma - \Theta$.

At $\sigma = 1/2$ this is the theorem of Báez-Duarte [19], which sharpened the criterion of Nyman and Beurling. Beurling [24] treated the closure problem in $L^p(0,1)$ and proved that it is solvable exactly when ζ has no zero in $\operatorname{Re} s > 1/p$; the weighted norms $\|\cdot\|_\sigma$ are the Hilbert-space counterpart of his scale. The argument below is Báez-Duarte's, carried over to the weighted norms.[19]

Proof. If $d_{M,\sigma} \to 0$, then (92) shows that no zero has real part greater than σ.

Assume $\zeta(s) \neq 0$ for $\operatorname{Re} s > \sigma$. For $0 < \varepsilon < 1/2$ consider

$$u_{M,\varepsilon} = 1 + \sum_{n\le M} \mu(n) n^{-\varepsilon}\Big(1 - \frac{n}{M}\Big) g_n.$$

By Lemma 15, its transform is

$$U_{M,\varepsilon}(s) = \frac{1 - \zeta(s)\big(D_M(s+\varepsilon) - D_M(1+\varepsilon)\big)}{s}.$$

In (93) with $s + \varepsilon$ in place of s, move the line to $\operatorname{Re} w = -\varepsilon/2$. Then $\operatorname{Re}(s + \varepsilon + w) \ge \sigma + \varepsilon/2$, and Lemma A2 applies in the zero-free half-plane $\operatorname{Re} s > \sigma$ with the margin $\varepsilon/2 < 1/4$. As in the proof of Theorem 37, for $\operatorname{Re} s = \sigma$ and any $\kappa > 0$,

$$D_M(s+\varepsilon) = \frac{1}{\zeta(s+\varepsilon)} + O_{\varepsilon,\kappa}\big(M^{-\varepsilon/2}(1+|t|)^{\kappa}\big),$$
$$D_M(1+\varepsilon) = \frac{1}{\zeta(1+\varepsilon)} + O_{\varepsilon}\big(M^{-\varepsilon/2}\big).$$

Put

$$U_\varepsilon(s) = \frac{1}{s}\Big(1 - \frac{\zeta(s)}{\zeta(s+\varepsilon)}\Big) + \frac{\zeta(s)}{s\,\zeta(1+\varepsilon)}.$$

Then $U_{M,\varepsilon}(s) - U_\varepsilon(s) \ll_{\varepsilon,\kappa} M^{-\varepsilon/2}|\zeta(s)|(1+|t|)^{\kappa}/|s|$ on $\operatorname{Re} s = \sigma$. The convexity bound $\zeta(\sigma + it) \ll_\kappa (1+|t|)^{1/4+\kappa}$ for $\sigma \ge 1/2$, $|t| \ge 1$ [57, Ch. V] makes the square of this error at most a constant times $M^{-\varepsilon}(1+|t|)^{-3/2+4\kappa}$. This is integrable provided $\kappa < 1/8$, and we fix such a κ. By Lemma 16,

$$\limsup_{M\to\infty} d_{M,\sigma} \le \limsup_{M\to\infty} \|u_{M,\varepsilon}\|_\sigma \le \Big(\frac{1}{2\pi}\int_{-\infty}^{\infty} |U_\varepsilon(\sigma+it)|^2\,\mathrm{d}t\Big)^{1/2}. \tag{94}$$

It remains to show that the right side tends to zero with ε.

The second term of U_ε has L^2 norm $O(1/\zeta(1+\varepsilon)) = O(\varepsilon)$, because $\zeta(s)/s$ is square integrable on the line by the convexity bound and $\zeta(1+\varepsilon) > 1/\varepsilon$.

[19] Burnol [30], sharpening [20], proved that the distance in $L^2(0,1)$ from 1 to the span of the functions $\{\vartheta/x\} - \vartheta\{1/x\}$ with $1/M \le \vartheta \le 1$ is at least $(C + o(1))/\sqrt{\log M}$, where $C > 0$ is given by a sum over the zeros on the critical line. The functions g_n are the cases $\vartheta = 1/n$, so the same lower bound holds for $d_{M,1/2}$. No power rate is therefore possible at $\sigma = 1/2$, in contrast with the interior of $[\Theta, 1)$.

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For the first term, let $\xi(s) = \frac{1}{2}s(s-1)\pi^{-s/2}\Gamma(s/2)\zeta(s)$. Hadamard's factorisation [57, §2.12] is $\xi(s) = \mathrm{e}^{b_0+b_1 s}\prod_\rho(1-s/\rho)\mathrm{e}^{s/\rho}$, the product over the nontrivial zeros converging absolutely. Since $\sum_\rho |\rho|^{-2} < \infty$, the series $\sum_\rho \operatorname{Re}(1/\rho)$ converges absolutely, and

$$\left|\frac{\xi(s)}{\xi(s+\varepsilon)}\right| = \exp\Bigl(-\varepsilon\Bigl(\operatorname{Re} b_1 + \sum_\rho \operatorname{Re}\frac{1}{\rho}\Bigr)\Bigr)\prod_\rho \frac{|s-\rho|}{|s+\varepsilon-\rho|}.$$

The exponential factor is bounded by an absolute constant C_0 for $0 < \varepsilon \le 1/2$.[20] Every zero $\rho = \beta + i\gamma$ has $\beta \le \sigma = \operatorname{Re} s$, so

$$|s+\varepsilon-\rho|^2 = (\sigma-\beta+\varepsilon)^2 + (t-\gamma)^2 \ge (\sigma-\beta)^2 + (t-\gamma)^2 = |s-\rho|^2.$$

Each factor of the product thus has modulus at most 1, and $|\xi(s)| \le C_0|\xi(s+\varepsilon)|$. Consequently

$$\begin{aligned}\left|\frac{\zeta(s)}{\zeta(s+\varepsilon)}\right| &\le C_0\left|\frac{(s+\varepsilon)(s+\varepsilon-1)}{s(s-1)}\right|\left|\frac{\Gamma((s+\varepsilon)/2)}{\Gamma(s/2)}\right| \\ &\le C_\sigma(1+|t|)^{\varepsilon/2} \le C_\sigma(1+|t|)^{1/4}.\end{aligned}$$

Here the rational factor is at most $2\,(1+1/(2(1-\sigma)))$ because $|s| \ge 1/2$ and $|s-1| \ge 1-\sigma$, and the Gamma quotient is $O((1+|t|)^{\varepsilon/2})$ uniformly for $0 < \varepsilon \le 1/2$ by Stirling's formula [57, §4.12]. So the first term of U_ε is dominated by $(1 + C_\sigma(1+|t|)^{1/4})/|s|$, which is square integrable and independent of ε. As $\varepsilon \downarrow 0$ it tends to zero at every point of the line where $\zeta(s) \ne 0$, hence almost everywhere. Dominated convergence shows that its L^2 norm tends to zero. Since the left side of (94) does not depend on ε, it is zero.

The description of the set of exponents follows from the definition of Θ, and the rate is Theorem 37 with $\theta = \Theta$. □

Remark 11. *The theorem identifies the statement "$d_{M,\sigma} \to 0$" for one $\sigma < 1$, with or without a rate, with the statement that ζ has no zero in $\operatorname{Re} s > \sigma$. No zero-free half-plane $\operatorname{Re} s > \theta$ with $\theta < 1$ has been established, so no exponent $\sigma < 1$ is at present known to belong to the convergence set. The exponent $\sigma = 1$ is different.*

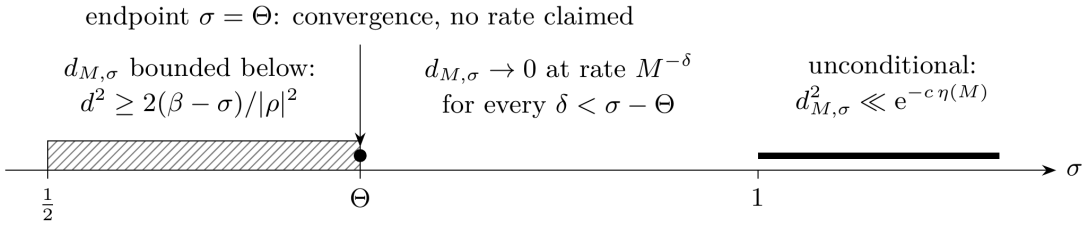

Figure 11. The convergence set of the reciprocal-dilate problem in the weighted norms $\|\cdot\|_\sigma$. For $\frac{1}{2} \le \sigma < \Theta$ a zero $\rho = \beta + i\gamma$ with $\beta > \sigma$ keeps the distance $d_{M,\sigma}$ bounded below by (92); for $\Theta < \sigma < 1$ the distance tends to 0 at a power rate (Theorem 37); at $\sigma = \Theta$ there is convergence and no rate is claimed (Theorem 38); for $\sigma \ge 1$ the rate is unconditional (Theorem 42).

13.4. The Endpoint Rate

At $\sigma = 1/2$ Theorem 38 gives no rate. The rate is known for the distance of Báez-Duarte, and it transfers to $d_{M,1/2}$. Write $\varphi_n(x) = \{1/(nx)\}$ and $\chi = \mathbf{1}_{(0,1]}$ on $(0,\infty)$, let $\|\cdot\|$ be the norm of $L^2(0,\infty)$, let d_M^{BD} be the distance from χ to the span of $\varphi_1, \dots, \varphi_M$, and let γ_0 be Euler's constant. In this notation $g_n = \varphi_n - \varphi_1/n$.

[20] It equals 1, since $b_1 = -\sum_\rho \operatorname{Re}(1/\rho)$; the bound is all that is used.

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Lemma 18 (Comparison with the Báez-Duarte distance). *Let* $\kappa = \|\varphi_1\|$. *Then* $\kappa^2 = \log 2\pi - \gamma_0 = 1.26066\ldots$, *and*

$$d_M^{BD} \le d_{M,1/2} \le (1+\kappa)\, d_M^{BD}.$$

Proof. For $c \in \mathbb{C}^M$ put $r_c = \chi - \sum_{n\le M} c_n\varphi_n$ and $A(c) = \sum_{n\le M} c_n/n$. For $x > 1$ we have $0 < 1/(nx) < 1$, so $r_c(x) = -A(c)/x$. Since $\sum c_n g_n = \sum c_n\varphi_n - A(c)\varphi_1$, the function $1 - \sum c_n g_n$ on $(0,1]$, extended by 0, equals $r_c + A(c)\varphi_1$ on $(0,\infty)$. Hence $\|1 - \sum c_n g_n\|_{1/2} \le \|r_c\| + \kappa|A(c)|$, and $|A(c)|^2 = \int_1^\infty |r_c|^2 \le \|r_c\|^2$. Taking c optimal for d_M^{BD} gives the upper bound. Conversely, given $a_2,\ldots,a_M$, put $c_n = a_n$ for $n \ge 2$ and $c_1 = -\sum_{n\ge2} a_n/n$; then $A(c) = 0$ and $\|1 - \sum a_n g_n\|_{1/2} = \|r_c\| \ge d_M^{BD}$. Finally $\kappa^2 = 1 + \int_1^\infty \{y\}^2 y^{-2}\,\mathrm{d}y = \log 2\pi - \gamma_0$. □

For $0 < \operatorname{Re} s < 1$ one has $\int_0^\infty \varphi_n(x) x^{s-1}\,\mathrm{d}x = -n^{-s}\zeta(s)/s$ (proof of Lemma 15), so by Mellin–Plancherel on $\operatorname{Re} s = \frac12$,

$$\|r_c\|^2 = \frac{1}{2\pi}\int_{-\infty}^{\infty} \bigl|1 - \zeta(s)D_c(s)\bigr|^2 \frac{\mathrm{d}t}{|s|^2}, \qquad D_c(s) = -\sum_{n\le M} c_n n^{-s}, \quad s = \tfrac12 + it. \tag{95}$$

The upper bound at the endpoint comes from the mollifier $V_M(s) = \sum_{n\le M} \mu(n)\bigl(1 - \frac{\log n}{\log M}\bigr) n^{-s}$ of Bettin, Conrey and Farmer. They obtain the mean square of $1 - \zeta V_M$ under the Riemann hypothesis and $\sum_{|\gamma|\le T} |\zeta'(\rho)|^{-2} \ll T^{3/2-\delta}$ [23, Theorem 1]. The only place where they need the exponent $3/2$ is a pointwise bound for the sum over pairs of zeros; integrating that sum instead gives the following weaker condition. For $1 < p \le 2$ and $\delta > 0$ consider

$$\sum_{0<|\gamma|\le T} |\zeta'(\rho)|^{-p} \ll T^{p-\delta}. \tag{96}$$

It implies that every zero is simple. For $p = 2$ it follows from the hypothesis of Bettin, Conrey and Farmer, and by Hölder's inequality that hypothesis also gives (96) for $4/3 < p \le 2$. It also follows, with $p = 2$, from $J_{-1}(T) \ll T$, and by Corollary 15(i) it implies the hypothesis of Theorem 14(b).

Lemma 19 (A sum over zeros). *Assume the Riemann hypothesis and* (96) *for some* $1 < p \le 2$ *and* $\delta > 0$. *Write* $a_\rho = |\zeta'(\rho)|^{-1}$, *fix* $0 < \varepsilon \le \frac14$, *and put*

$$S(t) = \sum_\rho \frac{a_\rho}{\varepsilon^2 + (t-\gamma)^2}.$$

There is $\eta_1 > 0$, *depending on* p *and* δ, *such that* $a_\rho \ll (|\gamma|+2)^{1-\eta_1}$,

$$\sum_\rho a_\rho (|\gamma|+2)^{-2+\eta_1} < \infty, \qquad \int_{-\infty}^{\infty} S(t)^2 (|t|+2)^{-2+\eta_1} \log^A(|t|+2)\,\mathrm{d}t < \infty \quad (A \ge 0).$$

Proof. Replacing δ by $\min(\delta, p(p-1)/2)$ we may assume $\delta < p(p-1)$. Put $c = p - \delta$ and $\eta_1 = \delta/p$, so that $c/p = 1 - \eta_1$. A single term of (96) is at most the whole sum, so $a_\rho^p \ll (|\gamma|+2)^c$, which is the first assertion. By Hölder's inequality and $N(2T) \ll T\log T$,

$$\sum_{T<|\gamma|\le 2T} a_\rho \le \Bigl(\sum_{|\gamma|\le 2T} a_\rho^p\Bigr)^{1/p} N(2T)^{1-1/p} \ll T^{2-(1+\delta)/p}\log T,$$

and $2 - (1+\delta)/p - 2 + \eta_1 = -1/p < 0$; summing over dyadic ranges gives the second assertion.

For the third put $K_\rho(t) = (\varepsilon^2 + (t-\gamma)^2)^{-1}$. Since there are $O(\log(|t|+2))$ zeros with $|\gamma - t| \le 1$, $\sum_\rho K_\rho(t) \ll \varepsilon^{-2}\log(|t|+2)$. Let $G(t) = \sum_\rho a_\rho^p K_\rho(t)$. The zeros with $|\gamma| \le 2|t|+2$ contribute at most $(2|t|+4)^c \sum_\rho K_\rho(t)$; the others satisfy $|t-\gamma| \ge |\gamma|/2$, so they contribute at most $4\sum_\rho a_\rho^p \gamma^{-2}$, which is

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finite by partial summation since $c<2$. Hence $G(t)\ll(|t|+2)^{c}\log(|t|+2)$. Hölder's inequality with exponents p and $p/(p-1)$ gives $S\le G^{1/p}\big(\sum_\rho K_\rho\big)^{1-1/p}$, so, as $2/p-1\ge 0$,

$$S(t)^2\le G(t)\,G(t)^{2/p-1}\Big(\sum_\rho K_\rho(t)\Big)^{2-2/p}\ll G(t)\,(|t|+2)^{c(2/p-1)}\log^2(|t|+2).$$

Put $\beta=2-\eta_1-c(2/p-1)$. Since $c(2/p-1)<p(2/p-1)=2-p$ and $\delta<p(p-1)$, we have $\beta>p-\delta/p>1$; also $\beta<2$. On $|t-\gamma|\le(|\gamma|+2)/2$ we have $(|\gamma|+2)/2\le|t|+2\le 2(|\gamma|+2)$ and $\int K_\rho=\pi/\varepsilon$; on the complement $K_\rho(t)\le 4(|\gamma|+2)^{-2}$ and $\int(|t|+2)^{-\beta}\log^A(|t|+2)\,\mathrm{d}t<\infty$. Hence

$$\int K_\rho(t)(|t|+2)^{-\beta}\log^{A+2}(|t|+2)\,\mathrm{d}t\ll(|\gamma|+2)^{-\beta}\log^{A+2}(|\gamma|+2),$$

and the integral in question is $\ll\sum_\rho a_\rho^p(|\gamma|+2)^{-\beta}\log^{A+2}(|\gamma|+2)$. By partial summation against (96) this converges as soon as $c<\beta$, that is $2c/p<2-\eta_1$, that is $2-2\delta/p<2-\delta/p$, which holds. □

Proposition 25 (Mean square of the mollified zeta function)**.** *Assume the Riemann hypothesis and* (96) *for some* $1<p\le 2$ *and* $\delta>0$*. Then, with* $s=\frac12+it$*,*

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\big|1-\zeta(s)V_M(s)\big|^2\frac{\mathrm{d}t}{|s|^2}=\frac{2+\gamma_0-\log 4\pi}{\log M}+O\big((\log M)^{-2}\big).$$

Proof. Write $L=\log M$, $E(s)=1-\zeta(s)V_M(s)$ and $Z=\zeta'/\zeta$, fix $\varepsilon\in(0,\frac14]$, let η_1 be as in Lemma 19, and fix $\eta>0$ with $\varepsilon+2\eta<\eta_1/2$ (replacing ε by a smaller number if necessary). Implied constants depend on $p,\delta,\varepsilon,\eta$. Under the Riemann hypothesis, for $s=\frac12\pm\varepsilon+it$, we use $\zeta(\frac12+\varepsilon+it)\ll(|t|+2)^{\eta}$, $\zeta(\frac12-\varepsilon+it)\ll(|t|+2)^{\varepsilon+\eta}$ and $Z(\frac12\pm\varepsilon+it)\ll\log(|t|+2)$ [57] [§§14.2, 14.5]; the last bound on the left line follows from the one on the right line and $Z(s)=\chi'(s)/\chi(s)-Z(1-s)$, where χ is the factor of the functional equation.

Since V_M has real coefficients, $\overline{E(s)}=E(1-s)$ on $\operatorname{Re}s=\frac12$, and $|s|^2=s(1-s)$ there. So the mean square equals $\frac{1}{2\pi i}\int_{(1/2)}E(s)E(1-s)\frac{\mathrm{d}s}{s(1-s)}$. The integrand is holomorphic in $\frac12-\varepsilon\le\operatorname{Re}s\le\frac12$ and, by the convexity bound, $O_M(|t|^{-3/2+\eta})$ there, so the line of integration may be moved to $\operatorname{Re}s=\frac12-\varepsilon$.

By [23] [Lemma 2], for $0<\operatorname{Re}s<1$,

$$V_M(s)=\frac{1}{\zeta(s)}\Big(1-\frac{Z(s)}{L}\Big)+\frac{B(s)}{L}+\frac{F_s}{L},\qquad B(s)=\sum_\rho\frac{M^{\rho-s}}{\zeta'(\rho)(\rho-s)^2},$$

where $F_s\ll M^{-\operatorname{Re}s-2}(|t|+1)^{-2}$; the zeros are simple, and the series B converges absolutely by Lemma 19. Hence

$$E(s)=L^{-1}\big(Z(s)-\zeta(s)B(s)-\zeta(s)F_s\big).$$

On $\operatorname{Re}s=\frac12-\varepsilon$ we have $|M^{\rho-s}|=M^{\varepsilon}$ and $|\rho-s|^{-2}=K_\rho(t)$, so $|B(s)|\le M^{\varepsilon}S(t)$; since the zeros are symmetric about the real axis, $|B(1-s)|\le M^{-\varepsilon}S(-t)=M^{-\varepsilon}S(t)$. Multiply out $E(s)E(1-s)$ and integrate each of the nine products against $\mathrm{d}s/(s(1-s))$, where $|s(1-s)|\asymp(|t|+2)^2$.

- $Z(s)Z(1-s)$ contributes $O\big(\int\log^2(|t|+2)(|t|+2)^{-2}\,\mathrm{d}t\big)=O(1)$.
- In $\zeta(s)B(s)\zeta(1-s)B(1-s)$ the factors $M^{\pm\varepsilon}$ cancel, and the contribution is $\ll\int S(t)^2(|t|+2)^{-2+\varepsilon+2\eta}\,\mathrm{d}t=O(1)$ by Lemma 19.
- $Z(s)\zeta(1-s)B(1-s)$ contributes $\ll M^{-\varepsilon}\int S(t)(|t|+2)^{-2+\eta}\log(|t|+2)\,\mathrm{d}t=O(M^{-\varepsilon})$, by the Cauchy–Schwarz inequality and Lemma 19, since $2\eta-\eta_1<1$.
- Every product containing F_s or F_{1-s} is $O(M^{-2})$.

The remaining product is $-Z(1-s)\zeta(s)B(s)$, and it remains to show that

$$I_0=-\frac{1}{2\pi i}\int_{(1/2-\varepsilon)}Z(1-s)\zeta(s)B(s)\frac{\mathrm{d}s}{s(1-s)}=L\sum_\rho\frac{1}{|\rho|^2}+O(M^{-\varepsilon}),$$

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since then the mean square is $L^{-2}(I_0+O(1))$ and $\sum_\rho |\rho|^{-2} = \sum_\rho 2\,\mathrm{Re}(1/\rho) = 2+\gamma_0-\log 4\pi$ [32] [Ch. 12].

On $\mathrm{Re}\, s = \frac12+\varepsilon$ we have $|B(s)| \le M^{-\varepsilon}S(t)$, and the integral over that line is $O(M^{-\varepsilon})$ as in the third item. Between the two lines the integrand is meromorphic with poles only at the zeros. For each n choose $T_n \in [n, n+1]$ with $|T_n - \gamma| \gg 1/\log n$ for every γ. On the segments $\mathrm{Re}\, s \in [\frac12-\varepsilon, \frac12+\varepsilon]$, $\mathrm{Im}\, s = \pm T_n$, we have $Z(1-s) \ll \log^2 T_n$ [32, Ch. 15] and $\zeta(s) \ll T_n^{\varepsilon+\eta}$; in $B(s)$ the $O(\log T_n)$ zeros with $|\gamma \mp T_n| \le 1$ contribute $\ll M^\varepsilon T_n^{1-\eta_1}\log^3 T_n$, the zeros with $|\gamma| \le 2T_n+2$ contribute $\ll M^\varepsilon T_n^{2-1/p}\log T_n$ by the proof of Lemma 19, and the rest contribute $O(M^\varepsilon)$. So the integrand is $\ll M^\varepsilon T_n^{\varepsilon+\eta-1/p}\log^3 T_n$, which tends to 0 because $\varepsilon+\eta < \frac12 \le 1/p$. By the residue theorem,

$$I_0 = \lim_{n\to\infty} \sum_{|\gamma|<T_n} \operatorname*{Res}_{s=\rho}\Big(Z(1-s)\zeta(s)B(s)\frac{1}{s(1-s)}\Big) + O(M^{-\varepsilon}).$$

Fix ρ and write $s = \rho+u$. Since $1-\rho = \bar\rho$, $Z(1-s) = -u^{-1} + \bar c_\rho + O(u)$ with $c_\rho = \zeta''(\rho)/(2\zeta'(\rho))$. The term of B belonging to ρ gives $\zeta(s)M^{\rho-s}\zeta'(\rho)^{-1}(\rho-s)^{-2} = u^{-1} + c_\rho - L + O(u)$, and the other terms of B are holomorphic at ρ, so after multiplication by $\zeta(s)$ they cancel the pole of $Z(1-s)$. With $h(s) = 1/(s(1-s))$, $h(\rho) = 1/|\rho|^2$, the residue is

$$\frac{L - c_\rho + \bar c_\rho}{|\rho|^2} - h'(\rho).$$

For the conjugate zero $\bar\rho$ we have $c_{\bar\rho} = \bar c_\rho$ and $h'(\bar\rho) = \overline{h'(\rho)} = -h'(\rho)$, since $h'(\rho) = (2\rho-1)/(\rho\bar\rho)^2$ is purely imaginary. So the residues at ρ and $\bar\rho$ add up to $2L/|\rho|^2$, and the sum over $|\gamma| < T_n$ is $L\sum_{|\gamma|<T_n}|\rho|^{-2}$. Letting $n\to\infty$ gives the formula for I_0. □

Theorem 39 (Endpoint constant). *Assume the Riemann hypothesis and* (96) *for some* $1 < p \le 2$ *and* $\delta > 0$. *Then, with* $u_M^{\log} = 1 + \sum_{n\le M}\mu(n)(1-\frac{\log n}{\log M})g_n$,

$$\lim_{M\to\infty}(\log M)\,d_{M,1/2}^2 = \lim_{M\to\infty}(\log M)\,\|u_M^{\log}\|_{1/2}^2 = 2+\gamma_0-\log 4\pi = 0.046191\ldots.$$

Proof. Write $C_0 = 2+\gamma_0-\log 4\pi$. Take $c_n = -\mu(n)(1-\log n/\log M)$. Then $D_c = V_M$, and Proposition 25 together with (95) gives $\|r_c\|^2 \sim C_0/\log M$. Next $A(c) = -\sum_{n\le M}\mu(n)/n + (\log M)^{-1}\sum_{n\le M}\mu(n)\log n/n$. Under the Riemann hypothesis $\sum_{n\le y}\mu(n) \ll y^{1/2+\eta}$ [57, §14.25], so by partial summation $\sum\mu(n)n^{-s}$ and $\sum\mu(n)(\log n)n^{-s}$ converge at $s=1$ to 0 and to $\lim_{s\to1}\zeta'(s)/\zeta(s)^2 = -1$, with tails $\ll M^{-1/2+2\eta}$. Hence $A(c) = -1/\log M + O(M^{-1/2+2\eta})$. By the proof of Lemma 18, $\|u_M^{\log}\|_{1/2} \le \|r_c\| + \kappa|A(c)| = (C_0/\log M)^{1/2}(1+o(1))$, so $\limsup(\log M)d_{M,1/2}^2 \le \limsup(\log M)\|u_M^{\log}\|_{1/2}^2 \le C_0$.

For the lower bound, the functions φ_n, $n\le M$, are among Burnol's functions $x\mapsto\{\vartheta/x\}$ with $1/M \le \vartheta \le 1$. By Lemma 18 and [30, Theorem 1.3], $\liminf(\log M)d_{M,1/2}^2 \ge \sum_\rho m_\rho^2/|\rho|^2$, summed over distinct zeros with multiplicities m_ρ. The hypothesis forces $\zeta'(\rho)\neq 0$, so $m_\rho = 1$. Under the Riemann hypothesis $1/|\rho|^2 = 1/\rho + 1/\bar\rho$ summed symmetrically, and $\sum_\rho \mathrm{Re}(1/\rho) = 1 + \frac12\gamma_0 - \frac12\log 4\pi$ [32, Ch. 12]. So the sum is C_0. □

Theorem 40 (Endpoint rate under the Riemann hypothesis). *Assume the Riemann hypothesis, fix* $0 < \delta \le \frac12$, *and put* $\varepsilon_M = 50(\log\log M)^{5/2+\delta}(\log M)^{-1/2}$. *For the functions* $u_{M,\varepsilon}$ *of the proof of Theorem 38,*

$$d_{M,1/2}^2 \le \|u_{M,\varepsilon_M}\|_{1/2}^2 \ll_\delta (\log\log M)^{5/2+\delta}(\log M)^{-1/2} \qquad (M \ge M_0(\delta)).$$

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Proof. Write $\varepsilon = \varepsilon_M$, $s = \frac{1}{2} + it$, $M_x(s) = \sum_{n\le x} \mu(n) n^{-s}$ and $D_M(s) = \sum_{n\le M} \mu(n)(1 - n/M)n^{-s}$. Under the Riemann hypothesis $\zeta(s+\varepsilon) \neq 0$, and by (89) the transform of $u_{M,\varepsilon}$ on the line is

$$\frac{1 - \zeta(s)/\zeta(s+\varepsilon)}{s} - \frac{\zeta(s)\big(D_M(s+\varepsilon) - 1/\zeta(s+\varepsilon)\big)}{s} + \frac{\zeta(s) D_M(1+\varepsilon)}{s}.$$

By Lemma 16, $\|u_{M,\varepsilon}\|_{1/2} \le J_\varepsilon^{1/2} + I^{1/2} + \kappa |D_M(1+\varepsilon)|$, where $J_\varepsilon = \frac{1}{2\pi}\int |1 - \zeta(s)/\zeta(s+\varepsilon)|^2 \frac{\mathrm{d}t}{|s|^2}$ and $I = \frac{1}{2\pi}\int |\zeta(s)|^2 |D_M(s+\varepsilon) - 1/\zeta(s+\varepsilon)|^2 \frac{\mathrm{d}t}{|s|^2}$. By [21, Proposition 2], $J_\varepsilon \ll \varepsilon$.

Since $\sum_{n\le M} a_n(1 - n/M) = M^{-1}\int_0^M \sum_{n\le x} a_n \,\mathrm{d}x$, the triangle inequality in L^2 gives $I^{1/2} \le M^{-1}\int_0^M I_x^{1/2}\,\mathrm{d}x$, where I_x is I with M_x in place of D_M. For $\sqrt{M} \le x \le M$, $N = \lfloor x \rfloor$ satisfies $\log N \ge \frac{1}{4}\log M$ and $\log\log N \le \log\log M$, so $\varepsilon \ge 25(\log\log N)^{5/2+\delta}(\log N)^{-1/2}$, and [21, Proposition 3] gives $I_x \ll_\delta N^{-\varepsilon/2} \le M^{-\varepsilon/8}$. For $x < \sqrt{M}$, $|M_x(s+\varepsilon)| \le 2\sqrt{x}$, $\frac{1}{2\pi}\int |\zeta(s)|^2 \frac{\mathrm{d}t}{|s|^2} = \kappa^2 \le 2$ and $|\zeta(s)/\zeta(s+\varepsilon)| \le 1 + |1 - \zeta(s)/\zeta(s+\varepsilon)|$, so $I_x^{1/2} \le 2\sqrt{2}\sqrt{x} + 1 + J_\varepsilon^{1/2}$. Hence $I^{1/2} \ll_\delta M^{-1/4} + M^{-\varepsilon/16}$.

Finally $\zeta(1+\varepsilon) > 1/\varepsilon$, so $0 < 1/\zeta(1+\varepsilon) < \varepsilon$, and for $x \ge 1$, $\sum_{n>x} \mu(n) n^{-1-\varepsilon} \ll_\eta x^{-1/2+\eta}$ uniformly in ε, by partial summation. Averaging over x as above gives $|D_M(1+\varepsilon)| \le 2\varepsilon + O(M^{-1/2+\eta})$. Since $M^{-\varepsilon_M/16} = \exp\big(-\frac{25}{8}(\log\log M)^{5/2+\delta}(\log M)^{1/2}\big) \ll \varepsilon_M$, all terms are $\ll_\delta \varepsilon_M^{1/2}$. □

Remark 12. *By Lemma 18, the bound of Theorem 40 also follows, with the factor $(1+\kappa)^2$, from the theorem of Balazard and de Roton [21] for d_M^{BD}; Theorem 40 shows that the construction of Theorem 38 attains it. Whether $d^2_{M,1/2} \ll 1/\log M$ follows from the Riemann hypothesis alone, or with simple zeros, is not known.*

Remark 13. *For $1 < p \le 2$, (96) implies $J_{-1}(T) \ll T^{2-2\delta/p}$, since $|\zeta'(\rho)|^{-p} \ll |\gamma|^{p-\delta}$ for each ρ and so $\sum |\zeta'(\rho)|^{-2} \le \max|\zeta'(\rho)|^{p-2} \sum |\zeta'(\rho)|^{-p}$. Conversely $J_{-1}(T) \ll T^{2-\delta}$ is (96) with $p = 2$. Hence Theorem 39 holds under the Riemann hypothesis and $J_{-1}(T) \ll T^{2-\delta}$ for some $\delta > 0$, a condition of the same type as in Corollary 15.*

Theorem 41 (Multiple zeros and polynomial tapers)**.** *Assume the Riemann hypothesis. Let P be a polynomial with $P(1) = 1$ that vanishes to order exactly $k \ge 0$ at 0, and put $V_M^P(s) = \sum_{n\le M} \mu(n) P(\log(M/n)/\log M) n^{-s}$ and $u_M^P = 1 + \sum_{n\le M} \mu(n) P(\log(M/n)/\log M) g_n$. If ζ has a zero of multiplicity $m \ge k+1$, then*

$$\limsup_{M\to\infty} (\log M)^{k+1-m} \|u_M^P\|_{1/2} > 0 \quad \textit{and} \quad \limsup_{M\to\infty} (\log M)^{k+1-m} \|1 - \zeta V_M^P\| > 0,$$

where $\|F\|^2 = \frac{1}{2\pi}\int_{-\infty}^{\infty} |F(\frac{1}{2}+it)|^2 |\frac{1}{2}+it|^{-2}\,\mathrm{d}t$. In particular, if $\|u_M^{\log}\|_{1/2} \to 0$ then every zero of ζ is simple.

Proof. Write $L = \log M$, $s = \frac{1}{2} + it$, $E = 1 - \zeta V_M^P$, $P(x) = \sum_{j=k}^d p_j x^j$ with $p_k \neq 0$, and let $\rho_0 = \frac{1}{2} + i\gamma_0$ have multiplicity $m \ge k+1$. Fix a real $\theta \in C_c^\infty(\mathbb{R})$ and put $H(u) = \frac{1}{2\pi}\int \zeta(\frac{1}{2}+it)\theta(t) \mathrm{e}^{iut}\,\mathrm{d}t$, a Schwartz function. Pairing E with $M^{-it}\theta(t)|s|^2$ and applying the Cauchy–Schwarz inequality gives

$$\Big|\sum_j p_j L^{-j} U_j(M)\Big| \le C_\theta \|E\| + O_A(L^{-A}),$$

where

$$U_j(X) = \sum_{n\le X} \frac{\mu(n)}{\sqrt{n}} h_j\Big(\log\frac{X}{n}\Big), \quad h_j(u) = u^j H(u),$$

with $C_\theta^2 = \frac{1}{2\pi}\int \theta^2 |s|^2\,\mathrm{d}t$. Since $\mathcal{M}(x) \ll x^{1/2+\eta}$ [57, §14.25], partial summation gives $U_j(\mathrm{e}^y) \ll \mathrm{e}^{\eta y}$. Hence, for $\operatorname{Re} z > 0$,

$$\int_0^\infty U_j(\mathrm{e}^y)\mathrm{e}^{-yz}\,\mathrm{d}y = \frac{\tilde h_j(z)}{\zeta(\frac{1}{2}+z)}, \qquad \tilde h_j(z) = \int_0^\infty u^j H(u)\mathrm{e}^{-uz}\,\mathrm{d}u,$$

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first for $\operatorname{Re} z > \frac{1}{2}$ and then by analytic continuation, both sides being holomorphic in $\operatorname{Re} z > 0$. The functions $\tilde{h}_j$ and all their derivatives are continuous on $\operatorname{Re} z \geq 0$. Since $\zeta(\frac{1}{2}+it)$ vanishes to order $m \geq k+1$ at $t=\gamma_0$, dominated convergence gives $\tilde{h}_k(i\gamma_0) = \frac{k!}{2\pi}\int \zeta(\frac{1}{2}+it)\theta(t)(i(\gamma_0-t))^{-k-1}\,\mathrm{d}t$, and we choose θ so that this is not 0. Let $W(y) = \sum_j p_j y^{d-j} U_j(\mathrm{e}^y)$. By Leibniz' rule and $1/\zeta(\rho_0+w) = c_{-m}w^{-m}+\dots$ with $c_{-m}\neq 0$, its Laplace transform at $z = i\gamma_0+\sigma$ is $p_k\tilde{h}_k(i\gamma_0)c_{-m}m(m+1)\cdots(m+d-k-1)\sigma^{-(m+d-k)} + O(\sigma^{-(m+d-k)+1})$ as $\sigma\to 0+$. If $L^{k+1-m}\|E\|\to 0$, then $W(y) = o(y^{d+m-k-1})$, so the Laplace transform is $o(\sigma^{-(d+m-k)})$, a contradiction. Finally, by the proof of Lemma 18, $\|u_M^P\|_{1/2} \geq \|E\| - \kappa|A(c)|$, and $A(c)\to 0$ by partial summation from $\sum_{n\leq x}\mu(n)/n \ll x^{-1/2+\eta}$, so the same conclusion holds for u_M^P. □

Remark 14. *In the polynomial model of the outlook, let $w \in C^2[0,1]$, $w(0)=1$, and $W_N = \sum_{n\leq N} w(n/N)a_n z^n$. The coefficients of $1-P_qW_N$ are $-(w_n-w_{n-1})(1+q^n) - qa_{n-2}(w_n-2w_{n-1}+w_{n-2})$ with $w_n = w(n/N)$ for $0\leq n\leq N$ and $w_n = 0$ otherwise. If $w(1)=w'(1)=0$, this gives $N\|1-P_qW_N\|^2 \leq (2\|w'\|_\infty + \|w''\|_\infty)^2 + \|w''\|_\infty^2/4$ uniformly in $|q|=1$; for $w(t)=(1-t)^2$ and $q=1$, $N\|1-P_1W_N\|^2\to 4$, the factor m^2 of Burnol's lower bound [30] for a zero of multiplicity $m=2$. Thus a taper vanishing to second order at $n=M$ is unaffected by close pairs in the model, while by Theorem 41 no fixed polynomial taper can cope with zeros of unbounded multiplicity, which the Riemann hypothesis does not exclude.*

13.5. The Exponent One

Write $\mathcal{M}(x) = \sum_{n\leq x}\mu(n)$ for the summatory function of the Möbius function; the letter M is reserved for the length of the approximant. The classical estimate of Walfisz [59] is

$$\mathcal{M}(x) \ll x\exp\bigl(-c_0'\,\eta(x)\bigr), \qquad \eta(x) = (\log x)^{3/5}(\log\log x)^{-1/5} \qquad (x\geq 3), \tag{97}$$

with an absolute constant $c_0' > 0$.

Theorem 42 (Unconditional rate at $\sigma=1$)**.** *Let $u_M^\sharp = 1+\sum_{n\leq M}\mu(n)g_n$. There is an absolute constant $c>0$ such that, for $M\geq 3$ and every $\sigma\geq 1$,*

$$d_{M,\sigma}^2 \leq \|u_M^\sharp\|_\sigma^2 \leq \|u_M^\sharp\|_1^2 \ll \exp\bigl(-c\,\eta(M)\bigr).$$

Proof. The first two inequalities are immediate. Put $R_M(s) = \sum_{n>M}\mu(n)n^{-s}$ for $\operatorname{Re} s\geq 1$. By (97) and partial summation, this series and the series of its termwise derivative converge uniformly on compact subsets of the line $\operatorname{Re} s = 1$, and

$$|R_M(1+it)| + |R_M'(1+it)| \ll (1+|t|)\,\varepsilon_1(M), \qquad \varepsilon_1(M) \ll \exp\bigl(-c_1'\eta(M)\bigr), \tag{98}$$

for some absolute $c_1' > 0$. Moreover $\sum_{n\geq 1}\mu(n)n^{-1-it} = 1/\zeta(1+it)$ for all real t, with the value 0 at $t=0$. By (89) the transform of $u_M^\sharp$ on the line $\operatorname{Re} s = 1$ is therefore

$$U_M(s) = \frac{\zeta(s)\bigl(R_M(s)-R_M(1)\bigr)}{s}.$$

Three ranges of t are treated separately. For $|t|\leq 1$, the mean value theorem and (98) give $|R_M(s)-R_M(1)| \ll |t|\varepsilon_1$, and $t\zeta(1+it)$ is bounded, so $U_M \ll \varepsilon_1$. For $1\leq|t|\leq T$, the bound $\zeta(1+it)\ll \log(2+|t|)$ [57, Theorem 3.5] and (98) give $U_M \ll \varepsilon_1\log(2+|t|)$. For $|t|>T$, the trivial bound $|\sum_{n\leq M}\mu(n)n^{-1-it}| \leq 1+\log M$ gives $U_M \ll (\log M)\log(2+|t|)/|t|$. By Lemma 16,

$$\|u_M^\sharp\|_1^2 \ll \varepsilon_1^2\,T\log^2(2+T) + \frac{(\log M)^2\log^2(2+T)}{T}.$$

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The choice $T = 1/\varepsilon_1$ gives $\|u_M^\sharp\|_1^2 \ll \varepsilon_1 (\log M)^2 \log^2(2/\varepsilon_1)$, and the logarithmic factors are absorbed by taking $c < c_1'$. The details are in Appendix G. □

The rate is that of the prime number theorem with the Vinogradov–Korobov error term. A power rate $d_{M,1} \ll M^{-\delta}$ is not excluded by any lower bound in this paper, but our proof of any such rate would pass through a zero-free half-plane.

13.6. Fixed Tapers at the Critical Norm

By Theorem 38, under the Riemann hypothesis the residuals u_M of (91) tend to zero in $\|\cdot\|_\sigma$ for every $\sigma > 1/2$. They do not tend to zero in the critical norm, and neither do the residuals of any other fixed taper, whether or not the Riemann hypothesis holds.

Theorem 43 (Every fixed taper of bounded variation)**.** *Let $w : [0,1] \to \mathbb{C}$ have bounded variation and satisfy $w(0+) = 1$. Put*

$$U_M = 1 + \sum_{n \le M} \mu(n)\, w(n/M)\, g_n.$$

For every nontrivial zero ρ of ζ with $\beta = \operatorname{Re}\rho \ge 1/2$,

$$\limsup_{M\to\infty} M^{1/2-\beta} \|U_M\|_{1/2} > 0,$$

and the limit superior is infinite if ρ is a multiple zero. In particular $\|U_M\|_{1/2}$ does not tend to zero. This applies to the Riesz tapers $w(t) = (1-t)^k$ of every real order $k > 0$ and to the sharp cut-off $w = 1$.

The proof is in Appendix H. Its steps are these. The identity $\sum_{n\le y} \mu(n) \lfloor y/n \rfloor = 1$ turns $U_M(1/y)$ into a sum of $\mu(n)$ against a dilated kernel. Integration against a test function W of vanishing first moment removes the term that carries $D(1)$ and leaves a sum $H(X) = \sum_n \mu(n) k_W(n/X)$ with

$$|H(X)| \le \sqrt{X}\, \|vW\|_{L^2} \|U_X\|_{1/2} + O(\log X),$$

whose Mellin transform is $\widehat{k}_W(s)/\zeta(s)$. The test function can be chosen with $\widehat{k}_W(\rho) \ne 0$; this is where the bounded variation of w and the normalisation $w(0+) = 1$ are used. The pole at ρ then prevents $H(X) = o(X^\beta)$.

For the Riesz family the exact rate of growth can be read off.

Corollary 20 (Exact growth of the linear residual)**.** *For the residuals u_M of (91),*

$$\limsup_{M\to\infty} \|u_M\|_{1/2} > 0 \qquad \text{and} \qquad \limsup_{M\to\infty} \frac{\log(1 + \|u_M\|_{1/2})}{\log M} = \Theta - \frac{1}{2}.$$

Thus the Riemann hypothesis is equivalent to $\|u_M\|_{1/2} \ll_\varepsilon M^\varepsilon$ for every $\varepsilon > 0$, although these norms do not tend to zero. For every taper w as in Theorem 43 the second limit superior is at least $\Theta - 1/2$.

Proof. The first assertion is Theorem 43 at a zero on the critical line. The lower bound in the second follows from the same theorem applied to zeros with real part approaching Θ.

For the upper bound take $s = 1/2 + it$ in (93) and move the line to $\operatorname{Re} w = \Theta - 1/2 + \delta$ with $0 < \delta < 1/4$. This line lies to the right of $w = 0$, so no pole is crossed. On it $\operatorname{Re}(s+w) = \Theta + \delta$. If $\Theta < 1$, Lemma A2 gives $1/\zeta(s+w) \ll (1 + |t| + |\operatorname{Im} w|)^\kappa$ for any $\kappa > 0$; if $\Theta = 1$, the lemma is not available, and we use instead $|1/\zeta(s+w)| \le \zeta(1+\delta)$. In both cases

$$|D_M(1/2 + it)| \ll_{\delta,\kappa} M^{\Theta - 1/2 + \delta} (1 + |t|)^\kappa.$$

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With (90) and the convexity bound $\zeta(1/2+it) \ll (1+|t|)^{1/4+\kappa}$, the transform (89) is square integrable on the critical line with norm $O_\delta(M^{\Theta-1/2+\delta})$, provided $\kappa < 1/8$. Lemma 16 gives $\|u_M\|_{1/2} \ll_\delta M^{\Theta-1/2+\delta}$. Let $\delta \downarrow 0$. □

13.7. The Harmonic Cost of the Coefficients

Theorem 43 concerns a profile held fixed as M grows. The following necessary condition applies to every finite choice of coefficients.

Theorem 44 (Harmonic coefficient cost). *Let $U = 1 + \sum_{n=2}^{M} a_n g_n$, put $d = \|U\|_{1/2}$ and $a_n = 0$ for $n > M$. For every integer $K \geq 2$,*

$$\Big(\sum_{n\geq 2} \frac{|a_n|^2}{n}\Big)^{1/2} \geq \Big(\sum_{2\leq n\leq K} \frac{\mu(n)^2}{n}\Big)^{1/2} - 2d\sqrt{K} \sum_{m\leq K} \frac{1}{m}.$$

Consequently, along any family of approximants with $d \to 0$,

$$\sum_{n\geq 2} \frac{|a_n|^2}{n} \geq \Big(\frac{12}{\pi^2} - o(1)\Big) \log\frac{1}{d},$$

and for $0 < d \leq \mathrm{e}^{-4}$,

$$\sum_{n\geq 2} \frac{|a_n|^2}{n} \geq \frac{12}{\pi^2} \log\frac{1}{d} - \frac{24}{\pi^2} \log\log\frac{1}{d} - 12.4. \tag{99}$$

The proof, in Appendix H, expresses $a_n - \mu(n)$ by Möbius inversion through the increments of the step function $y \mapsto U(1/y)$, whose weighted mean square is controlled by d.

A bounded harmonic norm $\sum |a_n|^2/n$ therefore excludes convergence at the critical exponent, whatever the shape of the coefficients. The Möbius coefficients themselves have $\sum_{n\leq M} \mu(n)^2/n \sim (6/\pi^2)\log M$. The theorem is a constraint on constructions and gives no lower bound for $d_{M,1/2}$ by itself.

13.8. The Logarithmic Taper

A taper that changes shape with M lies outside Theorem 43. The simplest is logarithmic:

$$u_M^{\log} = 1 + \sum_{n\leq M} \mu(n)\Big(1 - \frac{\log n}{\log M}\Big) g_n, \qquad D_M^{\log}(s) = \sum_{n\leq M} \mu(n)\Big(1 - \frac{\log n}{\log M}\Big) n^{-s}.$$

Let $\psi(y) = \sum_{n\leq y} \Lambda(n)$ and $E(y) = \lfloor y \rfloor - \psi(y)$. Expanding the floors in (88) and using $\sum_{n|m} \mu(n) = \mathbf{1}_{m=1}$ and $\sum_{n|m} \mu(n)\log n = -\Lambda(m)$ gives the exact formula

$$(\log M)\, u_M^{\log}(1/y) = (\log M) D_M^{\log}(1)\, \lfloor y \rfloor - \psi(y) \qquad (1 \leq y < M). \tag{100}$$

Proposition 26 (Logarithmic taper in shifted norms). *Suppose $1/2 \leq \theta < \sigma < 1$ and $\zeta(s) \neq 0$ for $\operatorname{Re} s > \theta$. Then for every $0 < \delta < \sigma - \theta$,*

$$\big\|(\log M)\, u_M^{\log} - E(1/\,\cdot\,)\big\|_\sigma \ll_{\sigma,\theta,\delta} M^{-\delta},$$

and the Mellin transform of $x \mapsto E(1/x)$ on $\operatorname{Re} s = \sigma$ is $(\zeta(s) + \zeta'(s)/\zeta(s))/s$. In particular $\|u_M^{\log}\|_\sigma \sim \|E(1/\,\cdot\,)\|_\sigma / \log M$.

The proof is in Appendix H; it repeats that of Theorem 37 with the kernel $M^w/(w^2 \log M)$, whose double pole at $w = 0$ produces the derivative of $1/\zeta$.

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The logarithmic taper thus converges in the shifted norms only at the rate $1/\log M$, and its scaled limit has a pole of its transform at every zero on the line $\operatorname{Re} s = \sigma$. At the critical exponent the relevant quantity is the interior mean square of E. For integer $M \geq 2$ put

$$\widetilde{u}_M = u_M^{\log} + M\Big(\frac{1}{\log M} - D_M^{\log}(1)\Big) g_M, \qquad I(Y) = \int_1^Y |E(y)|^2 \frac{\mathrm{d}y}{y^2}.$$

Since $g_M(1/y) = \lfloor y \rfloor / M$ for $1 \leq y < M$, formula (100) gives $(\log M)\widetilde{u}_M(1/y) = E(y)$ on that range, and so

$$\|\widetilde{u}_M\|^2_{L^2(1/M,1)} = \frac{I(M)}{(\log M)^2}. \tag{101}$$

Proposition 27 (Growth forced by an off-line zero)**.** *Let ρ be a zero of ζ of order m with $\beta = \operatorname{Re}\rho > 1/2$. Then*

$$\limsup_{Y\to\infty} \frac{I(Y)}{Y^{2\beta-1}} \geq \frac{m^2}{(2\beta-1)|\rho|^2}.$$

Consequently the Riemann hypothesis holds if and only if $\|\widetilde{u}_M\|_{L^2(1/M,1)}$ is bounded, and in that case its square is $O(1/\log M)$.

The proof is in Appendix H.

Since I is nondecreasing, a zero with $\beta > 1/2$ gives

$$\limsup_{M\to\infty} M^{1-2\beta} (\log M)^2 \, \|\widetilde{u}_M\|^2_{L^2(1/M,1)} > 0.$$

Table 16 compares the families.

Table 16. Explicit coefficient families $c_n = -\mu(n) w_n$, $n \leq M$. The second column assumes $\zeta \neq 0$ in $\operatorname{Re} s > \theta$ and $\theta < \sigma < 1$, $\delta < \sigma - \theta$, except in the fifth row, which is unconditional. The last column gives the number of the theorem, proposition or corollary.

Taper w_n	Norm $\|\cdot\|_\sigma$, $\sigma > \theta$	Critical norm	Reference
$1 - n/M$	$\ll M^{-\delta}$	exponent $\Theta - \frac{1}{2}$; no decay	37, 20
$w(n/M)$, w fixed	—	$\limsup M^{1/2-\beta}\|U_M\| > 0$	43
$n^{-\varepsilon}(1 - n/M)$	$\to 0$ also at $\sigma = \theta$ ($\varepsilon \to 0$)	—	38
$1 - \log n/\log M$	$\asymp 1/\log M$	interior norm $O(1)$ iff RH	26, 27
1, with $\sigma \geq 1$	$\ll \exp(-c\,\eta(M))$	no decay	42, 43
arbitrary a_n	—	$\sum \frac{\lvert a_n\rvert^2}{n} \geq (\frac{12}{\pi^2} - o(1)) \log \frac{1}{d}$	44

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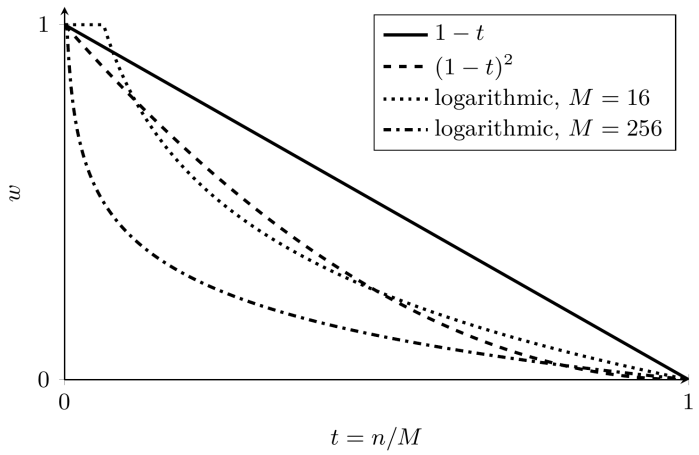

Figure 12. Fixed taper profiles $1 - t$ and $(1 - t)^2$, and the logarithmic profile $\min(1, -\log t/\log M)$ for $M = 16$ and $M = 256$. The logarithmic profile depends on M and tends to 0 at each fixed $t > 0$ as $M \to \infty$. It therefore lies outside Theorem 43 and is treated separately in Proposition 26.

13.9. The Ordinary-Dilate Model

The balanced reciprocal dilates should not be confused with the ordinary dilates $\{kx\}$. For those the approximation problem has an exact solution that does not involve the zeros.

Proposition 28 (Ordinary dilates)**.** *Let*

$$d_M^{\mathrm{ord}} = \inf_{c_1,\dots,c_M} \Big\| 1 - \sum_{k\le M} c_k \{kx\} \Big\|_{L^2(0,1)},$$

let φ be Euler's function and $J_2(d) = d^2 \prod_{p\mid d}(1 - p^{-2})$. Then

$$\big(d_M^{\mathrm{ord}}\big)^2 = \frac{1}{1 + 3S_M}, \qquad S_M = \sum_{d\le M} \frac{\varphi(d)^2}{J_2(d)} \sim M \prod_p \Big(1 - \frac{2}{p(p+1)}\Big).$$

The proof is a diagonalisation of the Gram matrix $\int_0^1 \{jx\}\{kx\}\,\mathrm{d}x = \frac{1}{4} + \gcd(j,k)^2/(12jk)$ and is given in Appendix H.

So $d_M^{\mathrm{ord}} \asymp M^{-1/2}$ whatever the zeros of ζ do. The difference is visible on the Mellin side:

$$\int_0^1 \{kx\} x^{s-1}\,\mathrm{d}x = \frac{k}{s+1} - \frac{k-1}{s} + \frac{k^{-s}}{s} \sum_{r=1}^{k-1} r^s \qquad (\mathrm{Re}\, s > 0),$$

which is not a multiple of $k^{-s}\zeta(s)/s$; at $k = 1$ it is $1/(s + 1)$. The transforms of the ordinary dilates have no common zero at the zeros of ζ, and the obstruction (92) has no analogue for them.

14. Dirichlet Characters

The criterion of Theorem 6 extends to twists by Dirichlet characters and then characterises the generalised Riemann hypothesis. Averaging over characters does not make the problem easier: the character of conductor one is always present.

14.1. The Signed Energy of a Twist

For a Dirichlet character χ, including the trivial character of conductor 1, put

$$L_\chi(x) = \sum_{n\le x} \lambda(n)\chi(n), \qquad Q_\chi(N) = \frac{1}{N} \sum_{t\in\mathbb{Z}} \Big| \sum_{\substack{t<n\le t+N \\ 1\le n\le N}} \lambda(n)\chi(n) \Big|^2.$$

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The window count in the proof of Proposition 1 does not use that the sequence is real, and gives

$$Q_\chi(N) = \frac{1}{N}\Big(|L_\chi(N)|^2 + \sum_{j=1}^{N-1}\big(|L_\chi(j)|^2 + |L_\chi(N) - L_\chi(j)|^2\big)\Big).$$

Since $|z|^2 + |w-z|^2 \geq |w|^2/2$,

$$\tfrac{1}{2}|L_\chi(N)|^2 \leq Q_\chi(N) \leq 5 \max_{j\leq N} |L_\chi(j)|^2. \tag{102}$$

Theorem 45 (Signed energy for Dirichlet characters)**.** *Let $1/2 \leq \theta \leq 1$. The following are equivalent.*

1. *No Dirichlet L-function has a zero in $\operatorname{Re} s > \theta$.*
2. *For every character χ and every $\varepsilon > 0$, $L_\chi(N) \ll_{\chi,\varepsilon} N^{\theta+\varepsilon}$.*
3. *For every character χ and every $\varepsilon > 0$, $Q_\chi(N) \ll_{\chi,\varepsilon} N^{2\theta+\varepsilon}$.*

The characters in the second and third conditions may be restricted to primitive ones. At $\theta = 1/2$ each condition is equivalent to the generalised Riemann hypothesis for all Dirichlet L-functions.

Proof. The second and third conditions are equivalent by (102).

Assume the second condition for a character χ. For $\operatorname{Re} s > 1$ the Euler factor $(1+\chi(p)p^{-s})^{-1} = (1-\chi(p)p^{-s})/(1-\chi(p)^2p^{-2s})$ gives

$$F_\chi(s) = \sum_{n\geq 1} \lambda(n)\chi(n)n^{-s} = \frac{L(2s,\chi^2)}{L(s,\chi)},$$

and by partial summation $F_\chi(s) = s\int_1^\infty L_\chi(x)x^{-s-1}\,\mathrm{d}x$ is holomorphic in $\operatorname{Re} s > \theta$. The identity $L(s,\chi)F_\chi(s) = L(2s,\chi^2)$ persists there. At a zero ρ of $L(s,\chi)$ with $\operatorname{Re}\rho > \theta \geq 1/2$ the left side would vanish, while $L(2\rho,\chi^2) \neq 0$ by the Euler product, since $\operatorname{Re}(2\rho) > 1$. Hence $L(s,\chi)$ has no zero in $\operatorname{Re} s > \theta$. A character and the primitive character inducing it have the same zeros in $\operatorname{Re} s > 0$, so primitive characters suffice.

Assume the first condition and fix χ modulo q, induced by the primitive character χ^*. If $\theta = 1$ the second condition is trivial. Let $\theta < 1$. Since $L(s,\chi) = L(s,\chi^*)\prod_{p\mid q}(1-\chi^*(p)p^{-s})$ and each factor of the finite product has modulus between $1-2^{-1/2}$ and 2 on $\operatorname{Re} s \geq 1/2$, Lemma 17 gives, for $0 < \delta \leq 1/4$ and $\kappa > 0$,

$$\frac{1}{L(\sigma+it,\chi)} \ll_{\chi,\delta,\kappa} (2+|t|)^\kappa \qquad (\theta+\delta \leq \sigma \leq 2),$$

where for χ^* trivial the reciprocal vanishes at $s = 1$. The numerator $L(2s,\chi^2)$ is bounded by $\zeta(2\theta+2\delta)$ in this strip. So F_χ is holomorphic for $\operatorname{Re} s \geq \theta+\delta$ and is $O((2+|t|)^\kappa)$ there. The proof of Proposition A1 uses only this bound and $|\lambda(n)\chi(n)| \leq 1$; it gives $L_\chi(x) \ll_{\chi,\varepsilon} x^{\theta+\varepsilon}$.

For the last assertion, the functional equation of a primitive L-function maps its zeros in the critical strip to their reflections in the line $\operatorname{Re} s = 1/2$, so the absence of zeros in $\operatorname{Re} s > 1/2$ puts all nontrivial zeros on that line. □

The constants are allowed to depend on χ. For the generalised Riemann hypothesis the bounds must nevertheless hold for every character.

14.2. Large Conductors

For $X > 0$ define the smoothed twists and their weighted mean square over primitive characters,

$$B_\chi(X) = \sum_{n\geq 1} \lambda(n)\chi(n)\mathrm{e}^{-n/X}, \qquad A_*(X) = \sum_{q\geq 1} \frac{1}{q^2\varphi(q)} \sideset{}{^*}\sum_{\chi \bmod q} |B_\chi(X)|^2.$$

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The star restricts the sum to primitive characters, and $q = 1$ contributes $|B(X)|^2$. By (80), $|B_\chi(X)| \le X$, so $A_*(X) \le \zeta(2)X^2$.

Theorem 46 (Large conductors). *For $X > 0$ and $Q \ge 1$,*

$$\sum_{q \ge Q} \frac{1}{q^2\varphi(q)} \sideset{}{^*}\sum_{\chi \bmod q} |B_\chi(X)|^2 \ll \frac{X^2}{Q^3} + \frac{X}{Q},$$

with an absolute implied constant. In particular the conductors $q \ge X^{1/3}$ contribute $O(X)$ for $X \ge 1$. The same bound holds with λ replaced by any sequence of modulus at most 1.

Proof. Let (a_n) have $|a_n| \le 1$ and put $T_\chi(y) = \sum_{n \le y} a_n \chi(n)$. On a block $R \le q < 2R$ we have $1/(q^2\varphi(q)) \le R^{-3} q/\varphi(q)$, and the multiplicative large sieve inequality [48], [41, Theorem 7.13] gives

$$\sum_{R \le q < 2R} \frac{1}{q^2\varphi(q)} \sideset{}{^*}\sum_{\chi \bmod q} |T_\chi(y)|^2 \le R^{-3}(y + 4R^2)\, y.$$

Partial summation gives $\sum_n a_n \chi(n) \mathrm{e}^{-n/X} = X^{-1} \int_0^\infty T_\chi(y) \mathrm{e}^{-y/X}\, \mathrm{d}y$. By Minkowski's integral inequality in the weighted ℓ^2 norm over the block, the square root of the block's contribution is at most

$$\frac{1}{XR^{3/2}} \int_0^\infty (y + 2R\sqrt{y}) \mathrm{e}^{-y/X}\, \mathrm{d}y = \frac{X}{R^{3/2}} + \sqrt{\frac{\pi X}{R}}.$$

Squaring and summing over $R = 2^j Q$, $j \ge 0$, gives the result. □

For general completely multiplicative signs the term X^2/Q^3 cannot be removed. Take an odd prime $r \in [Q, 3Q]$ and let f be the completely multiplicative function with $f(p) = \chi_r(p)$ for $p \ne r$ and $f(r) = 1$, where χ_r is the Legendre symbol. Then $f\chi_r$ is the indicator of the integers prime to r, and $\sum_{(n,r)=1} \mathrm{e}^{-n/X} \ge X/3$ for $X \ge 2$. This one character contributes at least $X^2/(9r^3) \gg X^2/Q^3$.

14.3. The Exact Exponent Across Conductors

Theorem 47 (Exact exponent across conductors). *Let Θ_D be the supremum of the real parts of the zeros of all primitive Dirichlet L-functions, ζ included. Then*

$$\limsup_{X \to \infty} \frac{\log(1 + A_*(X))}{\log X} = 2\Theta_D.$$

If $\Theta_D < 1$, then for every $\sigma \in (\Theta_D, 1)$ and $\kappa > 0$,

$$|B_\chi(X)| \le C(\sigma, \kappa)\, q^\kappa X^\sigma \qquad (X \ge 1) \tag{103}$$

for every primitive character χ of every conductor q, with $C(\sigma, \kappa)$ independent of χ and q.

The proof is in Appendix F. The lower bound uses one character at a time. The upper bound moves the contour in the Mellin inversion formula for B_χ to $\operatorname{Re} s = \sigma$ and uses Lemma 17; the uniformity of that lemma in χ and q is essential, because the resulting bound (103) is summed over all conductors.

By the functional equations, the generalised Riemann hypothesis for Dirichlet L-functions is the statement $\Theta_D = 1/2$. Theorems 46 and 47 reduce it to the bound

$$\sum_{q < X^{1/3}} \frac{1}{q^2\varphi(q)} \sideset{}{^*}\sum_{\chi \bmod q} |B_\chi(X)|^2 \ll_\varepsilon X^{1+\varepsilon} \qquad (\varepsilon > 0),$$

which is open. The term $q = 1$ is $|B(X)|^2$, so the positive average contains the original problem.

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14.4. Orthogonality and Square Divisors

Let $A(X)$ be defined as $A_*(X)$ but with the sum over all characters modulo q. Orthogonality of characters gives

$$A(X) = \sum_{n,m\ge 1} \lambda(n)\lambda(m)\mathrm{e}^{-(n+m)/X} \sum_{\substack{q \mid n-m \\ (q,n)=1}} \frac{1}{q^2},$$

where every q divides 0; the inner sum is at most $\zeta(2)$, which justifies the rearrangement. Since $\sum_{(q,n)=1} q^{-2} = \zeta(2)\sum_{d\mid n}\mu(d)d^{-2}$, the diagonal $n = m$ equals

$$\zeta(2)\sum_{d\ge 1}\frac{\mu(d)}{d^2\,(\mathrm{e}^{2d/X}-1)} = \frac{\zeta(2)}{2\zeta(3)}X + O(1),$$

by $(\mathrm{e}^y-1)^{-1} = y^{-1} + O(1)$ for $y > 0$. The diagonal is thus of exact order X, and all further information lies in the signed off-diagonal terms, which include $|B(X)|^2 - \sum_n \mathrm{e}^{-2n/X}$ from $q = 1$.

The identities $\lambda(n) = \sum_{d^2\mid n}\mu(n/d^2)$ and $\mu(n) = \sum_{d^2\mid n}\mu(d)\lambda(n/d^2)$ relate B_χ to $M_\chi(X) = \sum_n \mu(n)\chi(n)\mathrm{e}^{-n/X}$:

$$B_\chi(X) = \sum_{d\ge 1}\chi(d)^2 M_\chi(X/d^2), \qquad M_\chi(X) = \sum_{d\ge 1}\mu(d)\chi(d)^2 B_\chi(X/d^2).$$

Multiplication by $\chi(d)^2$ is a contraction in the weighted ℓ^2 norm over characters, and the terms are bounded by X/d^2. By Minkowski's inequality and $\sum_d d^{-1-2\varepsilon} < \infty$, a bound $O_\varepsilon(X^{1/2+\varepsilon})$ for the weighted norm of either family, valid for all $X > 0$, implies the same bound for the other. The Liouville and Möbius forms of the averaged problem are thus equivalent, and in both the cancellation at small conductors sits in the term $d = 1$.

14.5. Why Increasing the Modulus Retains Conductor One

Theorem 46 is indexed by the primitive conductor, not by the modulus. For the principal character $\chi_{0,p}$ modulo a prime p, removing the multiples of p gives

$$B_{\chi_{0,p}}(X) = B(X) + B(X/p), \tag{104}$$

since $\lambda(pm) = -\lambda(m)$. If $X \ge 2$ and $p \ge 2X\log X$, then by (80)

$$|B_{\chi_{0,p}}(X) - B(X)| \le \frac{1}{\mathrm{e}^{p/X}-1} \le \frac{1}{X^2-1} \le \frac{2}{X^2}.$$

A twist of arbitrarily large modulus can thus reproduce $B(X)$ almost exactly. It has conductor one and is not part of the large-conductor tail. On the Mellin side (104) multiplies $\Phi(s)$ by $1 + p^{-s}$, which has no zero in $\mathrm{Re}\, s > 0$ and removes no pole.

Nor can the tail be used to reconstruct the conductor-one term by an inequality that is uniform in the coefficients. For a sequence a with $|a_n| \le 1$ let $\mathrm{Tail}_a(X, Q)$ denote the left side of Theorem 46 with a in place of λ; then $\mathrm{Tail}_a(X, X^{1/3}) \ll X$. Suppose that

$$\Big|\sum_{n\ge 1} a_n \mathrm{e}^{-n/X}\Big|^2 \le K(X)\,\mathrm{Tail}_a(X, X^{1/3})$$

held for all such a. The constant sequence $a_n = 1$ has untwisted sum $(\mathrm{e}^{1/X}-1)^{-1} \sim X$, so $K(X) \gg X$, and the inequality would return only the trivial bound $O(X^2)$. Any reconstruction that does better must use arithmetic cancellation special to λ.

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15. What Is Unconditional

The equivalences of the preceding sections have the form "a bound with exponent θ holds if and only if ζ has no zero in $\operatorname{Re} s > \theta$". This section records what can be proved with no hypothesis, and then lists the conditional statements in one table. The unconditional results are transfers of known theorems; we state them with their sources.

15.1. The Full Window

Theorem 48 (Unconditional bounds for L and Q_λ)**.** *There is an absolute constant $c > 0$ such that, for $x \geq 3$ and $N \geq 3$,*

$$L(x) \ll x \exp\bigl(-c\,\eta(x)\bigr), \qquad Q_\lambda(N) \ll N^2 \exp\bigl(-2c\,\eta(N)\bigr),$$

where $\eta(x) = (\log x)^{3/5}(\log\log x)^{-1/5}$. Explicitly, for all $x \geq 1$ and all integers $N \geq 1$,

$$|L(x)| \leq \frac{x}{2641} + 8.3\sqrt{x}, \qquad Q_\lambda(N) \leq 5\Bigl(\frac{N}{2641} + 8.3\sqrt{N}\Bigr)^2.$$

Proof. Since $\lambda = \mu * \mathbf{1}_\square$, where $\mathbf{1}_\square$ is the indicator of the squares,

$$L(x) = \sum_{d \leq \sqrt{x}} \mathcal{M}\Bigl(\frac{x}{d^2}\Bigr). \tag{105}$$

The rate. For $3 \leq x < 25$ the bound is trivial. Let $x \geq 25$. The function η is increasing on $[5, \infty)$. For $d \leq x^{1/4}$ we have $x/d^2 \geq \sqrt{x} \geq 5$ and $\eta(x/d^2) \geq \eta(\sqrt{x}) \geq 2^{-3/5}\eta(x)$, so by (97) these terms contribute at most a constant times $\zeta(2)\,x\exp(-2^{-3/5}c_0'\eta(x))$. For $d > x^{1/4}$ the trivial bound $|\mathcal{M}(y)| \leq y$ gives a contribution of at most $x\sum_{d>x^{1/4}} d^{-2} \leq 2x^{3/4}$. This proves the bound for L with $c = 2^{-3/5}c_0'$. The bound for Q_λ follows from $Q_\lambda(N) \leq 5\max_{j\leq N}|L(j)|^2$, which is a consequence of (6), because $y\exp(-c\eta(y))$ is increasing for large y and tends to infinity.

Explicit constants. Cohen, Dress and El Marraki [31] proved

$$|\mathcal{M}(y)| \leq \frac{y}{4345} \qquad (y \geq y_0 = 2\,160\,535). \tag{106}$$

For $1 \leq y \leq 3\cdot 10^6$ one has $|\mathcal{M}(y)| \leq \sqrt{y}$; this is a finite verification, summarised in Table A8 of Appendix J. In (105) the terms with $x/d^2 \geq y_0$ contribute at most $\zeta(2)x/4345 < x/2641$, since $\zeta(2)/4345 = 1/2641.44\ldots$ The remaining terms have $\sqrt{x/y_0} < d \leq \sqrt{x}$ and $|\mathcal{M}(x/d^2)| \leq \sqrt{x}/d$, and the sum of $1/d$ over this range is at most $1 + \frac{1}{2}\log y_0 = 8.2929\ldots < 8.3$. The bound for Q_λ follows as before, the majorant being increasing. Appendix J gives the details. □

The rate for $\mathcal{M}$ is classical, and the transfer to L and Q_λ is routine; only the explicit constants for Q_λ seem not to have been recorded. Inserting the bound $|\mathcal{M}(y)| \leq 0.013\,y/\log y$ for $y \geq 1.1\cdot 10^6$ of Ramaré [55] in (105) gives in the same way an explicit saving of one logarithm. The explicit inequality is far from the truth in the range where L can be computed: for $x \leq 3\cdot 10^6$ the ratio $|L(x)|/(x/2641 + 8.3\sqrt{x})$ does not exceed 0.159.

15.2. Short Windows

For windows of length $H < N$ the theorem of Matomäki and Radziwiłł on multiplicative functions in short intervals gives a saving for almost all windows, and hence for the energy. The following statement is their theorem in the notation of this paper; the saving is theirs, and our contribution is the bookkeeping of the boundary windows.

Theorem 49 (Short windows, after Matomäki and Radziwiłł)**.** *For integers $16 \leq H \leq N$,*

$$\mathcal{Q}_\lambda(N,H) \ll NH\biggl(\Bigl(\frac{\log\log H}{\log H}\Bigr)^2 + (\log N)^{-1/100}\biggr).$$

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The trivial bound is $\mathcal{Q}_\lambda(N,H) \le 2NH$.

The proof is in Appendix I. Their Theorem 1 controls, outside an exceptional set, all windows that lie inside $[\max(H,\sqrt{N}),N]$, and the windows near the two ends of $[1,N]$ are estimated by Theorem 48.

The saving does not persist up to $H=N$: there the first term is of size $(\log\log N/\log N)^2$, and Theorem 48 is stronger.

15.3. The Barrier

Proposition 29 (Any power saving is a zero-free half-plane). *Let* $\delta>0$ *and let* $H(N)$ *be integers with* $1\le H(N)\le N$. *If*

$$\mathcal{Q}_\lambda\big(N,H(N)\big) \ll N^{1-\delta}H(N) \qquad \text{for all } N,$$

then $\zeta(s)\neq 0$ *for* $\operatorname{Re} s > 1-\delta/2$.

Proof. Each $n\le N$ lies in H windows and at most $N+H-1$ windows meet $[1,N]$, so $HL(N)=\sum_t S(t)$ and the Cauchy–Schwarz inequality give $|L(N)|^2 \le (N+H-1)\mathcal{Q}_\lambda(N,H)/H$; this is the lower bound in Proposition 1. With $H=H(N)$ it yields $|L(N)|^2 \le 2N\,\mathcal{Q}_\lambda(N,H)/H \ll N^{2-\delta}$, and Theorem 6 with $\theta=1-\delta/2$ applies. □

Theorems 48 and 49 save less than any power of N, and by Proposition 29 this is the limit of what can be proved without a zero-free half-plane. The same holds for the other objects of this paper; Table 17 lists the unconditional statements.

Table 17. Unconditional bounds. Here $\eta(x)=(\log x)^{3/5}(\log\log x)^{-1/5}$ and $\ell(t)=(\log t)^{2/3}(\log\log t)^{1/3}$.

Quantity	Bound	Input	Theorem
$L(x)$	$\le x/2641+8.3\sqrt{x}$	[31]	48
$L(x)$	$\ll x\exp(-c\,\eta(x))$	[59]	48
$Q_\lambda(N)$	$\le 5(N/2641+8.3\sqrt{N})^2$	[31]	48
$Q_\lambda(N)$	$\ll N^2\exp(-2c\,\eta(N))$	[59]	48
$\mathcal{Q}_\lambda(N,H)$	$\ll NH\big((\frac{\log\log H}{\log H})^2+(\log N)^{-1/100}\big)$	[46]	49
$B(X)$	$\le X\exp(-c\log X/\ell(\log X))$	[35], Lemma 13	35
$d^2_{M,\sigma}$, $\sigma\ge 1$	$\ll \exp(-c\,\eta(M))$	[59]	42
$A_*(X)$, $q\ge X^{1/3}$	$\ll X$	[48]	46

15.4. Below the Vinogradov–Korobov Scale

Proposition 29 shows that a power saving in the energy is a zero-free half-plane. The same holds at the finer scale of Theorem 48: a saving in the energy beyond the shape $\exp(-c\,\eta(N))$, even at one sequence of N or on dyadic averages, is a zero-free region beyond the shape of Lemma 13. The tool is the kernel of Proposition 3 with the factor $(s-\rho_0)^{-\nu}$ replaced by $(s-\rho_0)^{-1}$; the pole then is simple and every constant can be made uniform in ρ_0. Write $V(N)$ as in (10), η and ℓ as in Table 17, and

$$\delta(T)=\min\{1-\beta:\ \zeta(\beta+i\gamma)=0,\ 0<\gamma\le T\} \qquad (T\ge 15).$$

Lemma 13 gives $\delta(T)\ge c_1/\ell(T)$.

Lemma 20 (A uniform lower bound from one zero). *Let* $\rho_0=\beta_0+i\gamma_0$ *be a zero of* ζ *with* $\beta_0\ge\frac{7}{8}$, *and put* $\delta_0=1-\beta_0$. *For every integer* $N\ge 3\cdot 10^{13}|\gamma_0|^{8/15}$,

$$V(N)\ge 4.8\cdot 10^{-8}\,|\gamma_0|^{-11/2}\,N^{3-7\delta_0/2}.$$

Proof. Every zero has $|\gamma_0|>14$. Put $W(s)=s(s-1)\zeta(s)\mathrm{e}^{(s-\rho_0)^2}/(s-\rho_0)$. It is entire, since $\zeta(\rho_0)=0$, and in every vertical strip it decays like $\mathrm{e}^{-(t-\gamma_0)^2}$ times a power of $|t|$. Hence $w(u)=$

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$\frac{1}{2\pi i}\int_{(\kappa)} W(s)u^{-s}\,\mathrm{d}s$ converges absolutely and does not depend on κ. With $M(s)=\zeta(2s)/(s\zeta(s))=\int_1^\infty L(x)x^{-s-1}\,\mathrm{d}x$ for $\operatorname{Re}s>1$, Fubini's theorem gives, for $Y>0$,

$$I(Y):=\int_1^\infty L(x)\,w(x/Y)\,\frac{\mathrm{d}x}{x}=\frac{1}{2\pi i}\int_{(2)} G(s)Y^s\,\mathrm{d}s,$$

where

$$G(s)=W(s)M(s)=\frac{(s-1)\zeta(2s)\mathrm{e}^{(s-\rho_0)^2}}{s-\rho_0}.$$

In $\operatorname{Re}s\ge\frac34$ the only pole of G is the simple pole at ρ_0, with residue $(\rho_0-1)\zeta(2\rho_0)$. Moving the line to $\operatorname{Re}s=\frac34$, the horizontal integrals vanish by the Gaussian factor, and $I(Y)=(\rho_0-1)\zeta(2\rho_0)Y^{\rho_0}+R(Y)$.

Main term. $|\rho_0-1|\ge|\gamma_0|$, and $|\zeta(2\rho_0)|\ge\zeta(4\beta_0)/\zeta(2\beta_0)>1/\zeta(7/4)$, with $\zeta(7/4)=1.9623\ldots$

Remainder. On $\operatorname{Re}s=\frac34$ we have $|s-1|\le\frac14+|t|$, $|\zeta(2s)|\le\zeta(3/2)$, $|\mathrm{e}^{(s-\rho_0)^2}|\le\mathrm{e}^{1/16}\mathrm{e}^{-(t-\gamma_0)^2}$ and $|s-\rho_0|\ge\frac18$. With $\int(\frac14+|t|)\mathrm{e}^{-(t-\gamma_0)^2}\,\mathrm{d}t\le(\frac14+|\gamma_0|)\sqrt{\pi}+1\le1.876|\gamma_0|$ this gives $|R(Y)|\le 6.65\,|\gamma_0|Y^{3/4}$. Since $\beta_0-\frac34\ge\frac18$, it follows that

$$|I(Y)|\ge\frac{|\gamma_0|\,Y^{\beta_0}}{2\zeta(7/4)}\qquad(Y\ge Y_0=2.2\cdot10^{11}).\tag{107}$$

Two norms of w. By Mellin–Plancherel on $\operatorname{Re}s=-\frac12$, $C_w:=\int_0^\infty|w(u)|^2u^{-2}\,\mathrm{d}u=\frac{1}{2\pi}\int|W(-\frac12+it)|^2\,\mathrm{d}t$. The functional equation and $|\Gamma(\frac32+it)|^2=\pi(\frac14+t^2)/\cosh\pi t$ give $|\zeta(-\frac12+it)|\le\zeta(3/2)(\frac14+t^2)^{1/2}/(\sqrt{2}\pi)$. With $|s-\rho_0|\ge\frac{11}{8}$, $|\mathrm{e}^{(s-\rho_0)^2}|\le\mathrm{e}^{9/4}\mathrm{e}^{-(t-\gamma_0)^2}$ and $\frac32+|t|\le(\frac32+|\gamma_0|)(1+|t-\gamma_0|)$ one obtains $C_w\le50.5(\frac32+|\gamma_0|)^6\le93|\gamma_0|^6$. On $\operatorname{Re}s=3$ the same estimates give $|w(u)|\le C_3u^{-3}$ with $C_3\le\frac{1}{2\pi}\zeta(3)\mathrm{e}^{(17/8)^2}\frac12(\frac32\sqrt{\pi}+2)(3+|\gamma_0|)^2\le60.1|\gamma_0|^2$.

Splitting. By the Cauchy–Schwarz inequality and $\int_1^{N+1}L(x)^2\,\mathrm{d}x=V(N)$,

$$\Big|\int_1^{N+1}L(x)w(x/Y)\frac{\mathrm{d}x}{x}\Big|\le\Big(\frac{C_wV(N)}{Y}\Big)^{1/2},\qquad\Big|\int_{N+1}^\infty L(x)w(x/Y)\frac{\mathrm{d}x}{x}\Big|\le\frac{C_3Y^3}{2N^2},$$

the second by $|L(x)|\le x$. Put $K=(236\,|\gamma_0|N^{\delta_0})^{1/2}\ge1$ and $Y=N/K$. Then $Y^{3-\beta_0}\le N^{3-\beta_0}K^{-2}=N^2/(236|\gamma_0|)$, so the tail is at most $|\gamma_0|Y^{\beta_0}/(4\zeta(7/4))$. The condition on N and $\delta_0\le\frac18$ give $Y\ge N^{15/16}(236|\gamma_0|)^{-1/2}\ge Y_0$, and (107) yields

$$V(N)\ge\frac{|\gamma_0|^2Y^{1+2\beta_0}}{16\,\zeta(7/4)^2C_w}\ge\frac{|\gamma_0|^2N^{1+2\beta_0}K^{-3}}{16\,\zeta(7/4)^2\cdot93|\gamma_0|^6}=\frac{|\gamma_0|^{-11/2}N^{3-7\delta_0/2}}{16\,\zeta(7/4)^2\cdot93\cdot236^{3/2}},$$

where $Y^{1+2\beta_0}\ge N^{1+2\beta_0}K^{-3}$ because $K\ge1$ and $1+2\beta_0\le3$. The constant is $4.81\ldots\cdot10^{-8}$. □

Theorem 50 (Energy below the Vinogradov–Korobov scale)**.** (a) *Let $N\ge1$ and $F\ge1$ with $V(N)\le N^3/F$. Every zero $\beta+i\gamma$ with $|\gamma|\le(N/(3\cdot10^{13}))^{15/8}$ satisfies*

$$1-\beta\ge\min\Big\{\frac18,\ \frac{2}{7\log N}\Big(\log F-\frac{11}{2}\log|\gamma|-17\Big)\Big\}.$$

(b) *Suppose that $Q_\lambda(N)\le N^2\exp(-\psi(N)\eta(N))$ for all N in an infinite set $\mathcal{N}$, or that $\frac1X\sum_{X<N\le2X}Q_\lambda(N)\le X^2\exp(-\psi(X)\eta(X))$ for all X in an infinite set $\mathcal{N}$, where $\psi(N)\to\infty$ as $N\to\infty$ in $\mathcal{N}$. Then $\limsup_{T\to\infty}\ell(T)\,\delta(T)=\infty$.*

(c) *Let $0<a<1$. If $V(N)\le N^3\exp(-(\log N)^a)$ for infinitely many N, then $\delta(T)\ge\frac{1}{14}(11\log T)^{1-1/a}$ for infinitely many T.*

Proof. (a) Let $\zeta(\beta+i\gamma)=0$ with $|\gamma|$ in the stated range, so that $N\ge3\cdot10^{13}|\gamma|^{8/15}$. If $\beta<\frac78$ there is nothing to prove. Otherwise Lemma 20 and the hypothesis give $4.8\cdot10^{-8}|\gamma|^{-11/2}N^{3-7(1-\beta)/2}\le N^3/F$, and $\log(1/(4.8\cdot10^{-8}))<17$.

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(b) Under the first hypothesis, (11) gives $V(N) \le (2N+1)Q_\lambda(N) \le 5N^3 \exp(-\psi(N)\eta(N))$ for $N \in \mathcal{N}$. Under the second, for $X \in \mathcal{N}$ some $N' \in (X, 2X]$ has $Q_\lambda(N') \le X^2 \exp(-\psi(X)\eta(X))$, and V is nondecreasing, so $V(X) \le V(N') \le (2N'+1)Q_\lambda(N') \le 5X^3 \exp(-\psi(X)\eta(X))$. In both cases $V(N) \le 5N^3 \exp(-\psi(N)\eta(N))$ for $N \in \mathcal{N}$. Put $\lambda_N = \log N$, $g = \min(\psi(N)\eta(N) - \log 5, \lambda_N^{2/3})$, $\psi' = g/\eta(N)$ and $\log T = g/11$. As $N \to \infty$ in $\mathcal{N}$ we have $\psi' \to \infty$, $T \to \infty$, $T \le (N/(3 \cdot 10^{13}))^{15/8}$ and $g \ge 68$. Part (a) with $\log F = g$ and $\frac{11}{2}\log|\gamma| \le \frac{g}{2}$ gives $\delta(T) \ge \min(\frac{1}{8}, (g-34)/(7\lambda_N)) \ge g/(14\lambda_N) = 11\log T/(14\lambda_N)$. Now $(\log T)^{5/3} = (\psi'/11)^{5/3}\lambda_N(\log\lambda_N)^{-1/3}$ and $\log\log T \ge \log\eta(N) - \log 11 \ge \frac{1}{2}\log\lambda_N$ for large N. Hence

$$\ell(T)\delta(T) \ge \frac{11}{14} \cdot \frac{(\log T)^{5/3}(\log\log T)^{1/3}}{\lambda_N} \ge \frac{11}{14 \cdot 2^{1/3} \cdot 11^{5/3}} \psi'^{5/3} \longrightarrow \infty.$$

(c) Take $\log F = g = \lambda_N^a$ and $\log T = g/11$ in (a). For large N the range condition holds and $g \ge 68$, so $\delta(T) \ge g/(14\lambda_N) = (11\log T)^{1-1/a}/14$. □

Corollary 21 (The scale of the energy)**.** *At least one of the following holds.*

(i) *There is C with $N^2 \exp(-C\eta(N)) \le Q_\lambda(N) \ll N^2 \exp(-2c\,\eta(N))$ for all $N \ge 3$.*

(ii) $\limsup_{T\to\infty} \ell(T)\delta(T) = \infty$*; that is, the zero-free region of Lemma 13 can be widened by an unbounded factor at infinitely many heights.*

Proof. The upper bound in (i) is Theorem 48. If the lower bound fails for every C, then $\psi(N) = \log(N^2/Q_\lambda(N))/\eta(N)$ is unbounded, and Theorem 50(b), applied to an infinite set on which $\psi \to \infty$, gives (ii). □

No unconditional proof of alternative (ii) is known; the best unconditional statement is $\delta(T) \ge c_1/\ell(T)$ of Lemma 13. So no unconditional upper bound for $Q_\lambda(N)$ or $V(N)$, along one sequence of N, on dyadic averages, or on a set of positive density, can improve the shape $\exp(-c\,\eta(N))$ of Theorem 48, even by a factor $\psi(N) \to \infty$ in the exponent, without widening the Vinogradov–Korobov region at infinitely many heights. In the other direction, Theorem 50(a) applied to the bound of Theorem 48 returns a region of the shape $\delta(T) \gg 1/\ell(T)$, so the two transfers lose only constants in the exponent. Short-window information does not enter: $V(N)$ involves only the long sums $L(x)$, $x \le N$, and the saving of Theorem 49 concerns windows of length $o(N)$.

15.5. Conditional Inputs and What They Give

Table 18 assembles the consequences of a zero-free half-plane. In every row the conclusion, required for all $\varepsilon > 0$ or for the stated range of exponents, is equivalent to the hypothesis. No value $\theta < 1$ is known for which the hypothesis holds, and $\theta = 1/2$ is the Riemann hypothesis.

Table 18. Conditional inputs and what they give. Hypothesis for the first six rows: $\zeta(s) \ne 0$ for $\operatorname{Re} s > \theta$, where $1/2 \le \theta < 1$. Hypothesis for the last two rows: no Dirichlet L-function has a zero in $\operatorname{Re} s > \theta$.

Object	Conclusion	Reference
Liouville sum	$L(x) \ll x^{\theta+\varepsilon}$	Theorem 6
Signed energy	$Q_\lambda(N) \ll N^{2\theta+\varepsilon}$	Theorem 6
Smoothed sum	$B(X) \ll X^{\theta+\varepsilon}$	Proposition 22
Reciprocal dilates	$d_{M,\sigma} \to 0$ for $\theta \le \sigma < 1$	Theorem 38
Reciprocal dilates	$d_{M,\sigma} \ll M^{-\delta}$ for $\delta < \sigma - \theta$	Theorem 37
Riesz residual	$\|u_M\|_{1/2} \ll M^{\theta-1/2+\varepsilon}$	Corollary 20
Twisted sums	$L_\chi(N) \ll_\chi N^{\theta+\varepsilon}$, $Q_\chi(N) \ll_\chi N^{2\theta+\varepsilon}$	Theorem 45
Conductor average	$A_*(X) \ll X^{2\theta+\varepsilon}$	Theorem 47

Two kinds of published information about the zeros bear on these tables, and neither moves an exponent. The first is numerical: by Platt and Trudgian [54], all zeros with $0 < \operatorname{Im} s \le 3 \cdot 10^{12}$ lie on

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the critical line. A zero $\rho = \beta + i\gamma$ with $\beta > 1/2$ therefore has $|\gamma| > 3 \cdot 10^{12}$. The lower bounds it forces are correspondingly small in their constants, with $|\Gamma(\rho)|$ of order $e^{-\pi|\gamma|/2}$ in Theorem 36 and $|\rho|^{-2} < 10^{-24}$ in (92), but the exponents β and 2β are unchanged, and a verification to a finite height cannot bound Θ.

The second is the zero-density theory. Guth and Maynard [37] proved that the number $N(\sigma, T)$ of zeros with real part at least σ and height at most T is at most $T^{30(1-\sigma)/13+o(1)}$. This bounds the number of zeros to the right of a line. It does not exclude any one of them, and a single zero ρ with $\beta > \theta$ already forces $B(X) = \Omega_{\pm}(X^{\beta})$ by Theorem 36, $d^2_{M,\sigma} \geq 2(\beta - \sigma)/|\rho|^2$ for $\sigma < \beta$ by (92), and $Q_\lambda(N) > N^{2\sigma}$ for all large N and each $\sigma \in [1/2, \beta)$ by Theorem 8. Removing a sparse set of scales or averaging over them does not help. The exponents in Table 18 respond only to a zero-free half-plane.

16. The Energy Principle for Families

The engine of this paper is a single principle: *the growth exponent of one sign-definite quantity decides a zero-free half-plane.* For the Liouville function it is Theorems 1 and 2. This section states the principle for a finite family of Dirichlet characters, proves it in both directions, and shows that the announced proof [4] is an instance of it: its final step is the windowed form of the family principle (Section 17), and everything before that step proves the family growth bound on windows.

16.1. The Family Energy

Let $\Lambda(n)$ be the von Mangoldt function and, for $y \geq 1$,

$$S_\chi(y) = \sum_{n \leq y} \Lambda(n)\ \mathrm{Re}\,\chi(n)\Big(\frac{y}{n} - 1\Big),$$

the Cesàro-smoothed Chebyshev function of χ.[21] Put

$$G_1(y) = S_1(y) - y\log y + (1+\gamma)y, \qquad G_\chi(y) = S_\chi(y) + y\ \mathrm{Re}\,\frac{L'}{L}(1,\chi) \quad (\chi \neq 1),$$

where γ is Euler's constant. The subtracted terms are the contributions of $s = 1$, so that $G_\chi(y) = o(y)$ for each χ by the prime number theorem in progressions.

Definition 3. *A* family *is a finite multiset $\mathcal{F}$ of non-trivial primitive Dirichlet characters that is closed under $\chi \mapsto \bar\chi$ (with multiplicities), together with an integer $R \geq 1$. Its* family energy *is*

$$E_{\mathcal{F}}(y) = R\,G_1(y) + \sum_{\chi \in \mathcal{F}} G_\chi(y).$$

The Dirichlet coefficients behind $E_{\mathcal{F}}$ are $\Lambda(n)\big(R + \sum_{\chi\in\mathcal{F}} \mathrm{Re}\,\chi(n)\big)$. When $\mathcal{F}$ together with R copies of the trivial character is a union of groups of characters, this is $\Lambda(n)$ times a non-negative integer, by orthogonality, as for the Liouville energy. We do not need this positivity of the coefficients: the principle below needs only a one-sided bound for $E_{\mathcal{F}}$.

16.2. The Principle

Theorem 51 (Family energy principle)**.** *Let $(\mathcal{F}, R)$ be a family and $0 < \theta < 1$.*

1. *If there are C, y_0 with $E_{\mathcal{F}}(y) \leq Cy^\theta$ for all $y \geq y_0$, then $\zeta(s) \neq 0$ and $L(s,\chi) \neq 0$ for every $\chi \in \mathcal{F}$ and every s with $\mathrm{Re}\,s > \theta$.*
2. *Conversely, if $\zeta(s)$ and every $L(s,\chi)$, $\chi \in \mathcal{F}$, have no zeros in $\mathrm{Re}\,s > \theta$, with $\theta \geq 1/2$, then $|E_{\mathcal{F}}(y)| \ll_{\mathcal{F}} y^\theta$ for $y \geq 2$.*

[21] The weight $(y/n - 1)_+$ is the Cesàro mean of order one. Its Mellin transform decays like $|s|^{-2}$, so every sum over zeros below converges absolutely.

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In particular, the family has no zeros in $\operatorname{Re} s > 1/2$ *if and only if* $E_{\mathcal{F}}(y) \le y^{1/2+\varepsilon}$ *for every* $\varepsilon > 0$ *and all large* y.

Part (i) is a one-sided statement, like Theorem 8: an upper bound alone suffices. Its proof is Landau's argument for integrals with non-negative integrands [6]; its statement and its Lean proof are in [4]. Part (ii) is the converse, which completes the analogy with Theorem 1.

Proof. *The Mellin transforms.* For $\operatorname{Re} s > 1$ and $n \ge 1$, $\int_n^\infty (y/n - 1) y^{-s-1}\,\mathrm{d}y = n^{-s}/(s(s-1))$. Summing over n, and using $\sum \Lambda(n)\chi(n)n^{-s} = -L'/L(s,\chi)$ and $\int_1^\infty y \log y\, y^{-s-1}\,\mathrm{d}y = (s-1)^{-2}$, $\int_1^\infty y\, y^{-s-1}\,\mathrm{d}y = (s-1)^{-1}$, we get for $\operatorname{Re} s > 1$

$$\int_1^\infty E_{\mathcal{F}}(y)\, y^{-s-1}\,\mathrm{d}y = -\frac{1}{s(s-1)}\,\frac{P'}{P}(s) + \frac{a}{(s-1)^2} + \frac{b}{s-1} =: M(s),$$

where $P(s) = \big((s-1)\zeta(s)\big)^R \prod_{\chi\in\mathcal{F}} L(s,\chi)$ and a, b are constants.[22] The function P is entire. Expanding $-P'/P = -R/(s-1) - R\zeta'/\zeta + \dots$ at $s = 1$ shows that the subtracted main terms in G_1 and G_χ cancel the double and simple poles of M at $s = 1$, so M is holomorphic at $s = 1$. Hence M continues meromorphically to $\mathbb{C}$, and in $\operatorname{Re} s > 0$ its only poles are at the zeros of P. At a zero β of P of multiplicity $m \ge 1$,

$$M(s) = -\frac{m}{\beta(\beta-1)}\,\frac{1}{s-\beta} + O(1) \qquad (s \to \beta). \tag{108}$$

(i) Put $h(y) = Cy^\theta - E_{\mathcal{F}}(y)$, which is non-negative for $y \ge y_0$. On $[1, y_0]$ the functions G_1 and G_χ are bounded, so

$$I(s) = \int_{y_0}^\infty h(y)\, y^{-s-1}\,\mathrm{d}y = \frac{Cy_0^{\theta-s}}{s-\theta} - M(s) + B(s)$$

for $\operatorname{Re} s > 1$, where $B(s) = \int_1^{y_0} E_{\mathcal{F}}(y) y^{-s-1}\,\mathrm{d}y$ is entire. Let σ_c be the abscissa of convergence of I. Because $h \ge 0$, Landau's theorem [6] says that I is holomorphic in $\operatorname{Re} s > \sigma_c$ and cannot be continued holomorphically to a neighbourhood of the real point σ_c.[23] Suppose $\sigma_c > \theta$. Then $\sigma_c \le 1$, since M is holomorphic in $\operatorname{Re} s > 1$, and σ_c must be a singularity of $-M$, that is, a real zero $\beta = \sigma_c \in (\theta, 1)$ of P. For real $s \downarrow \beta$ the integral $I(s)$ is non-negative, but by (108), since $\beta(\beta-1) < 0$,

$$I(s) = -\frac{m}{|\beta(\beta-1)|}\,\frac{1}{s-\beta} + O(1) \to -\infty.$$

This contradiction shows $\sigma_c \le \theta$. Then I is holomorphic in $\operatorname{Re} s > \theta$, hence so is M, and by (108) P has no zero there. Since $(s-1)\zeta(s)$ and each $L(s,\chi)$ are entire factors of P, none of them vanishes in $\operatorname{Re} s > \theta$.

(ii) The Cesàro-smoothed explicit formula [32, Ch. 17–19] reads, for $y \ge 2$ and $\chi \ne 1$ primitive,

$$S_\chi(y) + y \operatorname{Re}\frac{L'}{L}(1,\chi) = -\operatorname{Re}\sum_\rho \frac{y^\rho}{\rho(\rho-1)} + O_\chi(\log y),$$

the sum running over the non-trivial zeros ρ of $L(s,\chi)$, and similarly for G_1 with the zeros of ζ. It follows from Mellin inversion of M along $\operatorname{Re} s = 2$ and a shift of the contour to $\operatorname{Re} s = -1/2$: the residues at the zeros of P give the sum over ρ by (108), the residues at $s = 0$ and at the trivial zeros give the error term, and the shifted integral is $O(y^{-1/2})$ because $M(s) \ll |s|^{-2}\log|s|$ there. The sum over ρ converges absolutely, since $\sum_\rho |\rho|^{-2} < \infty$. If every ρ has $\operatorname{Re}\rho \le \theta$, each term is at most $y^\theta/|\rho(\rho-1)|$,

[22] Here we use that $\mathcal{F}$ is closed under conjugation, so that $\sum_{\chi\in\mathcal{F}} \operatorname{Re}\frac{L'}{L}(s,\chi) = \sum_{\chi\in\mathcal{F}}\frac{L'}{L}(s,\chi)$ for real s, and by analytic continuation the real parts may be dropped throughout. The factor $(s-1)^R$ removes the pole of ζ^R, and the constants a, b absorb the resulting terms $R/(s-1)$ and the main terms of G_1 and G_χ.

[23] The proof: if I were holomorphic in a disc about σ_c, its Taylor series about a real point $\sigma > \sigma_c$ would have radius greater than $\sigma - \sigma_c$; its coefficients are $\int h(y)(-\log y)^k y^{-\sigma-1}\,\mathrm{d}y/k!$, all of one sign, so the series may be summed under the integral at a real point to the left of σ_c, where the integral would then converge.

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so $|G_\chi(y)| \le y^\theta \sum_\rho |\rho(\rho-1)|^{-1} + O(\log y) \ll y^\theta$, and summing over the family gives the bound for $E_{\mathcal{F}}$. □

Remark 15. *For the Liouville function the sign-definite quantity is $Q(N) \ge 0$, a quadratic form, and the exponent is 2Θ; for a family it is the one-sided error $Cy^\theta - E_{\mathcal{F}}(y) \ge 0$, a linear form, and the exponent is $\Theta_{\mathcal{F}}$, the largest real part of a zero in the family. The proofs share the same three steps: a Mellin transform with a sign-definite integrand, a real singularity forced by Landau's theorem, and the residue (108) of the wrong sign.*

16.3. The Single Input

By Theorem 51, Hypothesis H is equivalent to a statement about energies alone.

Definition 4. Hypothesis W *is the statement: every primitive character ψ belongs to some family $(\mathcal{F}, R)$ with $E_{\mathcal{F}}(y) \le Cy^{1/2+\varepsilon}$ for every $\varepsilon > 0$, a constant $C = C(\mathcal{F}, \varepsilon)$, and all large y.*

Corollary 22. *Hypotheses H and W are equivalent. Under either, the Riemann hypothesis, the generalized Riemann hypothesis for Dirichlet L-functions, and the critical bound $Q(N) \ll N^{1+\varepsilon}$ hold.*

Proof. W implies H by Theorem 51(i), applied for each ε with $\theta = 1/2 + \varepsilon$, together with the passage from primitive to imprimitive characters, which changes $L(s, \chi)$ only by finitely many Euler factors $1 - \chi^*(p)p^{-s}$, none of which vanishes in $\operatorname{Re} s > 0$. H implies W by Theorem 51(ii), with the family $\{\psi, \bar{\psi}\}$ (or $\mathcal{F} = \varnothing$ when $\psi = 1$) and $R = 1$. The rest is Theorem 56 and Corollary 24. □

The announced proof [4] does not prove Hypothesis W, which asks for the bound at every large y: it proves the family bound on a sequence of windows, which suffices by the windowed principle of Section 17. It takes for $\mathcal{F}$ the characters of the cyclotomic fields described in Appendix A, with $R = 3o + 1$ copies of the trivial character, where o is the size of the family, and it bounds $E_{\mathcal{F}}$ along that tower through the Galois cohomology of the fields, which is the window bound VWin. In the language of this paper, then, the announced proof is a bound for a family energy, and the whole of its 175,187 files goes into the inequality $E_{\mathcal{F}}(y) \le Cy^{1/2+\varepsilon}$ on those windows. The deduction from that inequality to the Riemann hypothesis is Theorem 52 and Section 20, and it is checked in Lean.

17. The Windowed Energy Principle

Theorem 51 asks for the energy bound at *every* large y. The announced proof [4] never proves that much. For each ε it proves the bound only on a sequence of windows $[Y, Y^{1+\delta}]$, one window at each level of a tower of cyclotomic fields, and nothing at all between the windows. This section shows that windows suffice. The proof is a version of Landau's theorem that reads off the zeros from high Taylor coefficients of the Mellin transform of the energy, in the spirit of Turán's power-sum method.[24]

17.1. Statement

In this section a *weighted family* is a finite list of Dirichlet characters $\chi_1, \dots, \chi_m$, each either the trivial character modulo 1 or non-trivial, with weights $a_1, \dots, a_m > 0$. Its energy is

$$E(y) = \sum_{i=1}^{m} a_i \, G_{\chi_i}(y),$$

with G_χ as in Section 16 (and $G_\chi = G_1$ for the trivial character). The families of Definition 3 are the case where the non-trivial χ_i are closed under conjugation and the trivial character carries weight R.

[24] Landau's theorem [6] says that a Dirichlet integral with a non-negative integrand has a singularity at the real point of its abscissa of convergence. Here the integrand is not non-negative and is only controlled on windows, so we work instead with the Taylor coefficients at a point far to the right, and compare them with the contribution of the nearest pole. The statement and the proof below follow the corresponding theorem of [4]; the exposition in terms of the energy is ours.

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Theorem 52 (Windowed energy principle). *Let E be the energy of a weighted family, and let $0 < \theta < 1$, $\delta > 0$ and C be real. Suppose that for every Y_0 there is a $Y \geq Y_0$ with*

$$E(y) \leq C\, y^{\theta} \qquad \text{for all } y \in [Y, Y^{1+\delta}]. \tag{109}$$

Then $L(s, \chi_i) \neq 0$ for every i and every s with $\operatorname{Re} s > \theta$.

Only an upper bound is assumed, and only on the windows. Between two windows E may be as large or as negative as it likes. The width δ is fixed, so on a logarithmic scale each window $[\log Y, (1+\delta)\log Y]$ has length proportional to its distance from the origin.

Corollary 23. *Suppose that for every $\varepsilon > 0$ there are $\delta, C > 0$ such that the bound $G_1(y) \leq C y^{1/2+\varepsilon}$ holds on windows $[Y, Y^{1+\delta}]$ for arbitrarily large Y. Then the Riemann hypothesis holds. The same with G_1 replaced by the energy of a weighted family containing a primitive character χ gives the generalized Riemann hypothesis for $L(s, \chi)$.*

Proof. Theorem 52 with $\theta = 1/2 + \varepsilon$ gives a zero-free half-plane $\operatorname{Re} s > 1/2 + \varepsilon$ for every ε, hence for $\operatorname{Re} s > 1/2$. Theorem 56 turns this into the statements on the critical line. □

17.2. The Mellin Transform of the Energy

Put $u = \log y$ and $g(u) = E(\mathrm{e}^u)$, and let

$$\Phi(s) = \int_0^{\infty} g(u)\, \mathrm{e}^{-su}\, \mathrm{d}u = \int_1^{\infty} E(y)\, y^{-s-1}\, \mathrm{d}y.$$

Since $|S_\chi(y)| \leq y \sum_{n \leq y} \Lambda(n)/n \ll y \log y$, we have $|g(u)| \leq C_0 \mathrm{e}^{2u}$ for $u \geq 0$, and the integral converges absolutely for $\operatorname{Re} s > 2$.[25]

Lemma 21. *Φ extends to a meromorphic function on $\operatorname{Re} s > 0$. Its poles are exactly the points ρ with $0 < \operatorname{Re} \rho < 1$ at which some $L(s, \chi_i)$ or $L(s, \bar\chi_i)$ vanishes, and all of them are simple. The residue at ρ is*

$$c(\rho) = -\frac{1}{2\rho(\rho - 1)} \sum_{i=1}^{m} a_i \big(m_i(\rho) + \bar m_i(\rho) \big),$$

where $m_i(\rho)$ and $\bar m_i(\rho)$ are the orders of vanishing of $L(s, \chi_i)$ and $L(s, \bar\chi_i)$ at ρ. In particular $c(\rho) \neq 0$, and $c(\beta) > 0$ at a real pole β.

Proof. For $n \geq 1$ and $\operatorname{Re} s > 1$,

$$\int_1^{\infty} \Big(\frac{y}{n} - 1 \Big)_+ y^{-s-1}\, \mathrm{d}y = \frac{n^{-s}}{s(s-1)},$$

$$\int_1^{\infty} y \cdot y^{-s-1}\, \mathrm{d}y = \frac{1}{s-1}, \qquad \int_1^{\infty} y \log y \cdot y^{-s-1}\, \mathrm{d}y = \frac{1}{(s-1)^2}.$$

Summing the first over n with the coefficients $\Lambda(n)\chi(n)$, and using $\operatorname{Re} \chi(n) = \tfrac{1}{2}(\chi(n) + \bar\chi(n))$, the transform of G_χ for $\chi \neq 1$ is

$$\Phi_\chi(s) = \frac{1}{2} \Big(-\frac{L'}{L}(s, \chi) - \frac{L'}{L}(s, \bar\chi) \Big) \frac{1}{s(s-1)} + \operatorname{Re} \frac{L'}{L}(1, \chi)\, \frac{1}{s-1}.$$

[25] The exponent 2 is wasteful; any exponent above 1 would do. It is chosen only so that the constants below are simple.

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At $s=1$ the two terms cancel to first order: the first is $-\operatorname{Re}\frac{L'}{L}(1,\chi)/(s-1)+O(1)$, because $\frac{L'}{L}(1,\bar\chi)$ is the complex conjugate of $\frac{L'}{L}(1,\chi)$. So Φ_χ is regular at $s=1$. For the trivial character, $-\frac{\zeta'}{\zeta}(s)=\frac{1}{s-1}-\gamma+O(s-1)$ and $\frac{1}{s(s-1)}=\frac{1}{s-1}-1+O(s-1)$, so

$$-\frac{\zeta'}{\zeta}(s)\frac{1}{s(s-1)}=\frac{1}{(s-1)^2}-\frac{1+\gamma}{s-1}+O(1),$$

and the terms $-y\log y+(1+\gamma)y$ in G_1 remove exactly this principal part. In both cases the transform continues meromorphically to $\mathbb{C}$ (by the continuation of L'/L), and on $\operatorname{Re}s>0$ its only poles are at the zeros of $L(s,\chi)$ and $L(s,\bar\chi)$, since $L(s,\chi)\neq0$ on $\operatorname{Re}s\geq1$ and $s(s-1)$ does not vanish in $0<\operatorname{Re}s$ except at $s=1$, which we have just seen is regular. Near a zero ρ of order m, $-\frac{L'}{L}(s,\chi)=-\frac{m}{s-\rho}+O(1)$, which gives the residue. All terms of the sum have the same sign and at least one is non-zero, so $c(\rho)\neq0$. For real $\beta\in(0,1)$ we have $\beta(\beta-1)<0$, so $c(\beta)>0$. Finally $\Phi=\sum_i a_i\Phi_{\chi_i}$. □

The positivity $c(\beta)>0$ is the only point where a sign enters. It is the analogue, for families, of the positivity of the Liouville energy in Theorem 2: a real zero to the right of θ forces the energy to be large and *positive* on average, so that an upper bound alone rules it out.

17.3. Taylor Coefficients

For $\operatorname{Re}s>2$ we may differentiate under the integral sign, and the k-th Taylor coefficient of Φ at s is

$$c_k(s):=\frac{(-1)^k\Phi^{(k)}(s)}{k!}=\int_0^\infty g(u)\,\frac{u^k}{k!}\,\mathrm{e}^{-su}\,\mathrm{d}u. \tag{110}$$

Two elementary facts about the kernel $u^k\mathrm{e}^{-au}/k!$ will be used.

Lemma 22. *1. For $a>0$ and $k\geq0$, $\sup_{u>0}\dfrac{u^k\mathrm{e}^{-au}}{k!}\leq a^{-k}$.*

2. For every $\delta>0$ there are $\sigma>2$ and C_1 such that for every $U\geq1$ there is an integer $k\geq U$ with

$$\int_{(0,\infty)\setminus[U,(1+\delta)U]}\frac{u^k}{k!}\,\mathrm{e}^{-(\sigma-2)u}\,\mathrm{d}u\leq\frac{C_1}{\sigma^k}.$$

Proof. (i) The maximum is at $u=k/a$ and equals $a^{-k}k^k\mathrm{e}^{-k}/k!$, and $k^k\mathrm{e}^{-k}\leq k!$ because $k^k/k!$ is one term of the series for e^k.

(ii) Put $\lambda=\sigma-2$. The function $\lambda^{k+1}u^k\mathrm{e}^{-\lambda u}/k!$ is the density of a sum of $k+1$ independent exponential variables of rate λ, with mean $(k+1)/\lambda$. Take $k=\lceil(1+\delta/2)\lambda U\rceil$, so that $k\geq U$ once $\lambda\geq1$ and the mean lies in the middle of $[U,(1+\delta)U]$. Chernoff's bound for this sum gives, for a constant $\kappa=\kappa(\delta)>0$,

$$\int_{(0,\infty)\setminus[U,(1+\delta)U]}\frac{u^k}{k!}\mathrm{e}^{-\lambda u}\,\mathrm{d}u\leq\frac{2\mathrm{e}^{-\kappa k}}{\lambda^{k+1}}.$$

This is at most $C_1\sigma^{-k}$ as soon as $(\sigma/(\sigma-2))\mathrm{e}^{-\kappa}\leq1$, that is, for σ large in terms of δ.[26] □

Lemma 23 (The window inequality)**.** *Assume (109), and let σ be as in Lemma 22(ii). There is a constant C_2 such that for every $\eta>0$ and every U_0 there are $U\geq U_0$ and an integer $k\geq U$ with*

$$|c_k(\sigma+i\gamma)|+\operatorname{Re}c_k(\sigma)\leq C_2\,\frac{\mathrm{e}^{\eta U}}{(\sigma-\theta)^k}\qquad\text{for every }\gamma\in\operatorname{Re}. \tag{111}$$

[26] The Lean proof makes the choice explicit: with $l=2+6/\delta$ it takes $\sigma=l(1+\delta)(2+l)$ and bounds the two tails by direct comparison with geometric series, without probability.

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Proof. Take $Y \geq \mathrm{e}^{U_0}$ from (109) and put $U = \log Y$, so that $g(u) \leq C\mathrm{e}^{\theta u}$ on $[U, (1+\delta)U]$. Write $g^+ = \max(g, 0)$. By (110),

$$|c_k(\sigma + i\gamma)| + \operatorname{Re} c_k(\sigma) \leq \int_0^\infty \big(|g(u)| + g(u)\big) \frac{u^k}{k!} \mathrm{e}^{-\sigma u}\, \mathrm{d}u = 2\int_0^\infty g^+(u) \frac{u^k}{k!} \mathrm{e}^{-\sigma u}\, \mathrm{d}u,$$

because $|\mathrm{e}^{-(\sigma+i\gamma)u}| = \mathrm{e}^{-\sigma u}$. This is the step that uses only an upper bound for g: the combination $|g| + g$ sees only the positive part. On the window, $g^+ \leq C^+\mathrm{e}^{\theta u}$ with $C^+ = \max(C, 0)$, so by Lemma 22(i) with $a = \sigma - \theta$ the window contributes at most

$$2C^+ \int_U^{(1+\delta)U} \frac{u^k}{k!} \mathrm{e}^{-(\sigma-\theta)u}\, \mathrm{d}u \leq 2C^+ \delta U\, (\sigma - \theta)^{-k} \leq 2C^+ \delta\, \mathrm{e}^{\eta U} (\sigma - \theta)^{-k}$$

once U is large in terms of η. Off the window, $g^+ \leq C_0 \mathrm{e}^{2u}$, and Lemma 22(ii) bounds the contribution by $2C_0C_1\sigma^{-k} \leq 2C_0C_1(\sigma - \theta)^{-k}$. Take $C_2 = 2C^+\delta + 2C_0C_1$. □

17.4. The Nearest Pole

The Taylor coefficients of Φ at a point z grow like $|z - \rho|^{-k}$, where ρ is the pole nearest to z. The window inequality says that the coefficients grow no faster than $(\sigma - \theta)^{-k}$, up to $\mathrm{e}^{\eta U}$. So no pole may be closer to the line $\operatorname{Re} s = \sigma$ than $\sigma - \theta$, which is the claim. Two facts make this precise.

Lemma 24. *Let Φ be holomorphic on a disc $B(z, r)$ except for a simple pole at $\rho \in B(z, r)$ with residue c, and let $|z - \rho| < r' < r$. Then $c_k(z) = c\,(z - \rho)^{-k-1} + O(r'^{-k})$ as $k \to \infty$. If Φ is holomorphic on all of $B(z, r)$, then $c_k(z) = O(r'^{-k})$.*

Proof. $\Phi(s) - c/(s - \rho)$ is holomorphic on $B(z, r)$, so by Cauchy's estimate on the circle of radius r' its k-th Taylor coefficient at z is $O(r'^{-k})$. The k-th Taylor coefficient of $c/(s - \rho)$ at z, with the sign $(-1)^k$ of (110), is $c(z - \rho)^{-k-1}$. □

Lemma 25. *Let $0 < d < e \leq \varepsilon_1$, $c > 0$ and M, C be real. It is impossible that for every $\eta > 0$ and every U_0 there are $U \geq U_0$ and $k \geq U$ with*

$$\frac{c}{d^k} \leq \frac{M}{e^k} + \frac{C\,\mathrm{e}^{\eta U}}{\varepsilon_1^k}.$$

Proof. Choose $\eta > 0$ with $d\,\mathrm{e}^{\eta} < e$. Since $k \geq U$, the right side is at most $(|M| + |C|)\mathrm{e}^{\eta k} e^{-k}$, and $(|M| + |C|)(d\mathrm{e}^{\eta}/e)^k < c$ for k large. □

17.5. Proof of Theorem **??**

On $\operatorname{Re} s \geq 1$ no $L(s, \chi_i)$ vanishes, so we must show that Φ has no pole ρ_0 with $\theta < \operatorname{Re}\rho_0 < 1$. Suppose it has one. Fix σ as in Lemma 22(ii); every pole has real part below $1 < \sigma$. The poles in the closed disc $\overline{B}(\sigma, \sigma - \theta)$ are finite in number, since they are zeros of finitely many L-functions in a compact subset of $\operatorname{Re} s > 0$. One of two things happens.

Case 1: the pole of Φ nearest to σ is real, at $\beta > \theta$, and it is the only pole in a disc $B(\sigma, r)$ with $\sigma - \beta < r \leq \sigma - \theta$. By Lemma 21, $c(\beta) > 0$. By Lemma 24, $\operatorname{Re} c_k(\sigma) \geq c(\beta)(\sigma - \beta)^{-k-1} - Me^{-k}$ with $e = (\sigma - \beta + r)/2$. The term $|c_k(\sigma + i\gamma)|$ in (111) is non-negative, so

$$\frac{c(\beta)/(\sigma - \beta)}{(\sigma - \beta)^k} \leq \frac{M}{e^k} + \frac{C_2\,\mathrm{e}^{\eta U}}{(\sigma - \theta)^k},$$

for every η and arbitrarily large U, with $k \geq U$. Since $\sigma - \beta < e \leq \sigma - \theta$, this contradicts Lemma 25.

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Case 2: otherwise. Then there are $D \le \sigma - \theta$ such that the disc $B(\sigma, D)$ contains no pole, and a pole ρ_1 with $\sigma - \operatorname{Re}\rho_1 < D$.[27] Move along the vertical line $\operatorname{Re} s = \sigma$ to a point $z = \sigma + i\gamma$ with a *unique* nearest pole ρ at distance $d = |z - \rho| < D$: start at $z_0 = \sigma + i \operatorname{Im}\rho_1$, whose distance to ρ_1 is $\sigma - \operatorname{Re}\rho_1 < D$; the poles at minimal distance from z_0 lie on a circle centred on the line and to its left, so no two of them have the same imaginary part; moving z_0 a little up or down, towards the one with the largest (or smallest) imaginary part, makes that one strictly nearest, because $|z_0 + it - \rho|^2 = |z_0 - \rho|^2 - 2t \operatorname{Im}(\rho - z_0) + t^2$. Choose r with $d < r \le D$ such that ρ is the only pole in $B(z, r)$, and put $e = (d + r)/2$. By Lemma 24 at z and at σ,

$$|c_k(z)| \ge \frac{|c(\rho)|}{d^{k+1}} - \frac{M}{e^k}, \qquad \operatorname{Re} c_k(\sigma) \ge -\frac{M'}{e^k},$$

the second because Φ is holomorphic on $B(\sigma, D)$ and $e < D$. Adding, and using (111) with γ fixed,

$$\frac{|c(\rho)|/d}{d^k} \le \frac{M + M'}{e^k} + \frac{C_2\, \mathrm{e}^{\eta U}}{(\sigma - \theta)^k},$$

and since $d < e < D \le \sigma - \theta$ this again contradicts Lemma 25. □

17.6. Where the Windows Come from

In [4] the windows are the scales of a tower. For a primitive character ψ of order o, the family is the set of powers of ψ with the trivial character, with weights 1 and $(3o + 1)^{-1}$, the cyclic family of ψ. At each large x in a window, the proof chooses a finite abelian group A, the Galois group of an auxiliary abelian extension (the *host*), and splits the characters of A into a kept set, which contains the powers of ψ, and a discarded set. The family energy at x is then bounded in two parts:

$$E(x) \le x^{1/2 + (e_D + e_L + 3)\varepsilon} + \Sigma_{\mathrm{disc}}(x), \qquad \Sigma_{\mathrm{disc}}(x) \le 4o\, x^{1/2 + \varepsilon},$$

with exponents e_D, e_L depending only on the host constants. The second bound uses only the host condition (E) of Theorem 58, which bounds the error of the explicit formula for each discarded character. The first bound is a character-sum estimate; its inputs are a gap bound for the kept characters, a thin-set condition at the scale of the window and a cost bound. Those inputs come from the window bound of Appendix A, which is derived from the Galois cohomology of the host. Combined with Theorem 52 for $\theta = 1/2 + (e_D + e_L + 7)\varepsilon$ and every ε, it gives Hypothesis H. Sections 21, 22 and 23 write this conversion out.

17.7. The Lean Check

The Lean formalization restates the theorem in the notation of this section and derives Corollary 23 for ζ:

Lean name	Statement
`windowed_energy_principle`	Theorem 52
`riemannHypothesis_of_windowBound`	Corollary 23 for $\zeta \Rightarrow$ `RiemannHypothesis`

It contains the 30 files of the proof of this theorem in [4], adapted only in their import lines, and `#print axioms` reports only `propext`, `Classical.choice` and `Quot.sound` for both theorems.

[27] If there is no pole in $\overline{B}(\sigma, \sigma - \theta)$, take $D = \sigma - \theta$ and $\rho_1 = \rho_0$. Otherwise let D be the distance from σ to the nearest pole. If a nearest pole ρ_1 is not real, then $\sigma - \operatorname{Re}\rho_1 < |\sigma - \rho_1| = D$. If the only nearest pole is real, at β, then $\beta \ge \theta$; if $\beta > \theta$ we are in Case 1, and if $\beta = \theta$ then $D = \sigma - \theta$ and $\rho_1 = \rho_0$ serve, since ρ_0 is not real and lies outside $B(\sigma, D)$.

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18. The Critical Bound and the Half-Plane

The results above fix the shape of the remaining problem. The Riemann hypothesis is equivalent to

$$Q(N) \ll_\varepsilon N^{1+\varepsilon} \qquad \text{for every } \varepsilon > 0, \tag{112}$$

and by Theorem 8 it is equally equivalent to (112) along any one infinite sequence of N. The bound (112) is an upper bound for a positive definite Toeplitz form at the single vector λ (Proposition 2), and the three ways of attacking it that the paper examines each meet a precise obstruction.

Correlations. Fejér's identity writes $Q(N)$ through the shifted correlations of λ. Bounding these one shift at a time, uniformly in the endpoint, needs the collar hypothesis of Section 7, which implies the generalized Riemann hypothesis for every Dirichlet character, is false at its endpoint for λ (Section 9), and in its sharp form without the factor N^ε fails along every infinite set of dyadic scales for every sign sequence (Theorem 25). Any successful use of correlations must therefore keep the cancellation between different shifts.

Properties stable under perturbation. By Theorem 3 the exponent of the energy becomes at least 2β when λ is altered on a suitable set of x^β primes, for every $\beta \in (1/2, 1)$, while every property in Table 6 is kept. A proof of (112) must use information about λ that such a change destroys, that is, information that distinguishes λ from the functions λ_S of that theorem, with S of size x^β and $\beta > 1/2$.

Approximation. The reciprocal-dilate problem of Section 13 has the same convergence set as the energy (Theorem 38). At $\sigma = 1/2$ no fixed taper of bounded variation converges at all (Theorem 43), convergence forces the harmonic norm of the coefficients to grow like $(12/\pi^2)\log(1/d)$ (Theorem 44), and the interior norm of the logarithmic taper stays bounded if and only if the Riemann hypothesis holds (Proposition 27). The approximation side offers an equivalent target, not an easier one.

Three further questions concern the critical scale.

(1) *Boundedness of $Q(N)/N$.* By Theorem 10 the set $B_{1/2}$ is infinite, so under the Riemann hypothesis B_Θ is infinite; whether its complement is infinite is not known. By Proposition 7, $Q(N)/N$ is bounded exactly when $L(x) \ll \sqrt{x}$. It is unbounded if the positive ordinates are linearly independent over $\mathbb{Q}$ (Corollary 11), and already if they satisfy no integer relation with coefficients in $\{-2, \dots, 2\}$ (Corollary 14); unconditionally it exceeds 4.6242 on a set of positive lower logarithmic density (Corollary 17) and stays above 0.2175 for large N (Theorem 17). Under the Riemann hypothesis and $J_{-1}(T) \ll T^\theta$ for some $\theta < 2$ its logarithmic mean is the finite number S of Theorem 14, so averaging cannot decide the question; conversely a finite logarithmic mean forces $J_{-1}(T) = o(T^2(\log\log T)^2)$ (Corollary 15). For $N \le 3 \cdot 10^7$ the largest value is $Q(N)/N = 1.1534$ at $N = 12\,897\,104$ (Table A12).

(2) *The endpoint rate.* Under the Riemann hypothesis $d^2_{M,1/2} \ll (\log\log M)^{5/2+\delta}(\log M)^{-1/2}$ (Theorem 40), and if in addition $\sum_{|\gamma|\le T} |\zeta'(\rho)|^{-p} \ll T^{p-\delta}$ for some $1 < p \le 2$ and $\delta > 0$, which amounts to $J_{-1}(T) \ll T^{2-\delta}$ (Remark 13), then $(\log M)\, d^2_{M,1/2} \to 2 + \gamma_0 - \log 4\pi$ (Theorem 39), so the lower bound of Burnol [30] has the right order. An upper bound of order $(\log M)^{-1/2}$ for $d_{M,1/2}$ from the Riemann hypothesis alone, or with simple zeros, is open. Simplicity alone cannot suffice for the mollifier V_M: in the polynomial model of [23], with $P(z) = (1-z)(1-qz)$ and $q = \mathrm{e}^{-i\eta}$, $1/P = \sum_n a_n z^n$, the analogue $W_N = \sum_{n\le N}(1 - n/N)a_n z^n$ satisfies, in the normalized L^2 norm of the unit circle, the exact identity $N^2\|1 - PW_N\|^2 = |1+q|^2 + \sum_{j=2}^{N}|1+q^j|^2 + |1-q^N|^2/|1-q|^2$, so the error is of order 1 rather than $1/N$ once two zeros are closer than $1/N$. Simplicity is necessary for V_M (Theorem 41) but, by this identity, not sufficient. A taper vanishing to second order at $n = M$ removes the close-pair obstruction in the model (Remark 14), but no fixed polynomial taper can cope with zeros of unbounded multiplicity, which the Riemann hypothesis does not exclude. A proof from the Riemann hypothesis alone must therefore control clusters of zeros at scale $1/\log M$.

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(3) *Explicit constants.* Theorem 48 gives $|L(x)| \le x/2641 + 8.3\sqrt{x}$ from the bound of Cohen, Dress and El Marraki for the Mertens function. Any improvement of the constant 4345 in that bound, or of the range in which $|M(y)| \le \sqrt{y}$ has been checked, improves the constants for L and Q at once.

Answers to these questions would sharpen the picture at the critical scale, but the bound (112) is a statement about the rightmost zero of ζ, and Theorems 1–3 show that the energy reaches it exactly through a zero-free half-plane. Part II therefore proves that half-plane from a single cited window theorem, and derives from it the critical bound, the twisted family bounds, and the Riemann hypothesis and its generalization.

Part II

The proofs of RH and GRH

This part proves the Riemann hypothesis and the generalized Riemann hypothesis for Dirichlet L-functions. Section 19 states the one cited input, the window theorem, and the published theorems on which it rests. Section 20 proves in full that the zero-free half-plane gives the Riemann hypothesis and the generalized Riemann hypothesis. Sections 21, 22 and 23 describe how the window theorem is obtained, and Section 24 assembles the proof.

19. The Window Theorem

19.1. Galois Representations

Let E be a CM field and π a regular algebraic cuspidal automorphic representation of $\mathrm{GL}_a(\mathbb{A}_E)$ that is polarized. For every prime ℓ and every isomorphism $\iota : \overline{\mathbb{Q}}_\ell \cong \mathbb{C}$ there is a continuous semisimple representation $r_{\ell,\iota}(\pi) : \mathrm{Gal}(\overline{E}/E) \to \mathrm{GL}_a(\overline{\mathbb{Q}}_\ell)$ whose Frobenius eigenvalues at unramified places are the Satake parameters of π, and which satisfies local–global compatibility at every finite place. This is due to Harris and Taylor [64] and Shin [67] in the cases coming from compact Shimura varieties, to Chenevier and Harris [63] in general, and the local–global compatibility at all places is due to Caraiani [61,62]; see also [65,66] for the non-self-dual case. We use these representations only through the theorem of the next subsection.

19.2. The Window Theorem

For a primitive Dirichlet character ψ of order o, the cyclic family $\mathcal{F}_\psi$ consists of the powers ψ^j, $0 \le j < o$, with the trivial character taken modulo 1, the trivial character carrying weight 1 and the others weight $(3o+1)^{-1}$. Its energy $E_\psi(y)$ is the family energy of Section 16.

Theorem 53 (Window theorem [4]). [28] *For every primitive Dirichlet character ψ and every sufficiently small $\varepsilon > 0$ there are $\delta > 0$ and $C > 0$ such that, for every Y_0, there is $Y \ge Y_0$ with*

$$E_\psi(y) \le C\, y^{1/2+\varepsilon} \qquad (Y \le y \le Y^{1+\delta}).$$

In [4], Theorem 53 is obtained from the following arithmetic statement, which is where the Galois representations enter. For a number field E, a prime ℓ and an integer e, let $\mathrm{Sha}^1_e(E,\ell)$ be the kernel of the restriction map

$$H^1\big(G_{E,S_\ell},\, \mathbb{Q}_\ell/\mathbb{Z}_\ell(-e)\big) \longrightarrow \prod_{v \in S_\ell} H^1\big(G_{E_v},\, \mathbb{Q}_\ell/\mathbb{Z}_\ell(-e)\big),$$

[28] This is the one theorem we take from [4], in the way any paper uses a cited result. Some Lean proofs of our deductions are adapted from the same project, as noted where they occur. The energy framework, the equivalences, the reading of the family and windowed principles as energy principles, and the deductions of this paper are ours.

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where S_ℓ is the set of places of E above ℓ, G_{E,S_ℓ} is the Galois group of the maximal extension of E unramified outside S_ℓ and the infinite places, and $(-e)$ is the twist by the $(-e)$th power of the ℓ-adic cyclotomic character; write len for its length as a $\mathbb{Z}_\ell$-module.

Theorem 54 (Window theorem, cohomological form [4]). *Let ψ be a primitive Dirichlet character, let $0 < \varepsilon < 1/52$, and let $e(x)$ be integers with $x^{2+8\varepsilon} \le 2^{e(x)} \le 2x^{2+8\varepsilon}$. There are $J \ge 1$ and X_0 with the following property. Let $X \ge X_0$, put $Y_1 = X^{1/J}$ and $u = Y_1^{\varepsilon/5}$, and let q, d be primes with*

$$q \equiv 3 \pmod 8, \quad d \equiv 7 \pmod{16}, \quad \tfrac{u}{2} < \log q \le u, \quad \tfrac{u(\log Y_1)^2}{2} < \log d \le u(\log Y_1)^2,$$

$q \mid d+1$, and $p \mid d+1$ for every odd prime $p \le u$. Then there is Y with $Y_1 \le Y$ and $Y^2 \le X$ such that, for every $x \in [Y, Y^2]$ and every number field $E \subset \mathbb{Q}(\zeta_N)$, with $2^{e(x)+2}$, the conductor of ψ, q and d dividing N, whose Galois group is cut out by

$$a \equiv \pm 1 \pmod{2^{e(x)+2}}, \qquad \psi(a) = 1, \qquad \left(\frac{a}{q}\right) = \left(\frac{a}{d}\right) = 1,$$

the groups $\mathrm{Sha}^1_e(E,\ell)$ are finite for $1 \le e < t$, $t = \lfloor x^{401} \rfloor$, and for every finite set $\mathcal{S}$ of primes

$$\sum_{\ell \in \mathcal{S}} \log \ell \sum_{1 \le e < t} \operatorname{len} \mathrm{Sha}^1_e(E,\ell) \;\le\; \exp\left(\tfrac{1}{2}x^{\varepsilon}\right).$$

The fields E are CM fields, and the bound is proved by a modularity argument: the Galois representations of the first subsection, attached to theta lifts between definite unitary groups over E, control the Selmer groups of the twists $\mathbb{Q}_\ell/\mathbb{Z}_\ell(-e)$, and the Greenberg–Wiles Euler characteristic formula [68] [Theorem 2.19] converts that control into the length bound. Theorem 58 then converts the length bound into the energy bound of Theorem 53. Sections 22 and 23 give a full account of the proof of Theorem 54 in [4].

The theorem is proved in [4] from the Galois representations above, in three steps, which Sections 21–24 describe: an explicit bound for the lengths of Tate–Shafarevich groups of Galois modules over CM fields E in a cyclotomic tower attached to ψ, with auxiliary primes in narrow ranges, obtained by a modularity argument and the Greenberg–Wiles Euler characteristic formula; the supply of hosts that this bound gives; and the host decomposition, which turns it into the energy bound. The proof is that of [4], and is not ours.

19.3. Status of the Cited Theorem

The proof in [4] is a Lean 4 project over Mathlib of 175,187 files. Its final statement is Mathlib's own statement of the Riemann hypothesis, it contains no unproved steps and no new axioms, and it has been reported to compile with only the standard axioms [16,17]. It has not yet been refereed, and its Galois-cohomological step has no published human-readable account. The deduction from Theorem 53 to the Riemann hypothesis, on the other hand, is short and is given in full in this paper (Theorem 66), with a human-readable Lean check.

19.4. Consequences for the Energy

The results of Part I turn the zero-free half-plane into exact statements about Q.

Theorem 55 (Consequences of the half-plane). *Suppose that no Dirichlet L-function vanishes in $\operatorname{Re} s > 1/2$, as Theorem 66 proves. Then:*

1. *$\Theta = 1/2$, and $\log Q(N)/\log N \to 1$;*
2. *for every $\varepsilon > 0$, $Q(N) \le N^{1+\varepsilon}$ for all sufficiently large N, and the critical bound (112) holds;*
3. *for every Dirichlet character χ and every $\varepsilon > 0$, $Q_\chi(N) \ll_{\chi,\varepsilon} N^{1+\varepsilon}$;*

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4. *for every set S of primes with $\#\{p \le x : p \in S\} \le x^{1/2+o(1)}$, the energy Q_{λ_S} has growth exponent 1, while for each $\beta \in (1/2, 1)$ some S of size x^β has exponent at least 2β;*
5. *the Nyman–Beurling distance satisfies $d^2_{M,1/2} \ll (\log M)^{-1/2+o(1)}$.*

Proof. The case $\chi = 1$ gives $\zeta(s) \neq 0$ in $\operatorname{Re} s > 1/2$, so $\Theta = 1/2$, and (i) is Corollary 5. Part (ii) is the second half of Theorem 8 with $\sigma = 1/2 + \varepsilon/2$, and (112) follows. Part (iii) is Theorem 45 at $\theta = 1/2$. Part (iv) is Theorem 3 with $\Theta = 1/2$. Part (v) is Theorem 40. □

19.5. How the Window Theorem Meets the Energy

The argument of [4] works through positivity. Its accompanying remark describes a definite sign obtained from the Guinand–Weil explicit formula with a variance test function, and a multiplicity argument of Stepanov type obtained from iterated theta lifts. The energy is itself a variance: by the Fejér identity of Section 2.1, $Q(N)$ is the mean square of the window sums of λ, a positive definite Toeplitz form at the single vector λ (Proposition 2). A positivity argument for the explicit formula is therefore the dual of an upper bound for this form, and the zero-one law (Theorem 8) explains why such an argument may work along a sparse family of test functions, as the formal argument does along a tower of cyclotomic fields: one infinite sequence suffices.

The argument respects Theorem 3. The modified functions λ_S share every statistical property of λ in Table 6 but have no functional equation and no automorphic origin, and for $\beta > 1/2$ some of them have energy exponent above 1. Any proof must therefore use structure that λ_S lacks. The window theorem does, through theta series and the automorphy of $L(s, \chi)$. Theorem 3 is thus a necessary condition that it meets, and a test that any further proof through the energy must also pass.

The theorem does not settle the questions that are sharper than the Riemann hypothesis. Whether $Q(N)/N$ is bounded is, by Proposition 7, the question whether $L(x) \ll \sqrt{x}$, and that would force every zero to be simple and the positive ordinates to satisfy integer linear relations (Theorem 4); it is expected to be false. The logarithmic mean of $Q(N)/N^2$ in Theorem 4 still needs the hypothesis on $\sum |\zeta'(\rho)|^{-2}$.

20. The Complete Reduction

This section proves in full the steps that join the zero-free half-plane to the classical statements. Each step is also checked in Lean 4 over Mathlib; Table 19 lists the theorems.

Throughout, χ is a Dirichlet character modulo $N \ge 1$, and $L(s, \chi)$ is its L-function, continued to $\mathbb{C}$ (to $\mathbb{C} \setminus \{1\}$ if χ is trivial).[29]

Definition 5 (The input)**.** Hypothesis H *is the statement: for every $N \ge 1$, every Dirichlet character χ modulo N and every s with $\operatorname{Re} s > 1/2$, excluding $s = 1$ when χ is trivial, $L(s, \chi) \neq 0$.*

Hypothesis H is proved in Theorem 66.

20.1. From the Half-Plane to the Critical Line

Theorem 56. *Assume Hypothesis H. Then:*

1. *every zero of ζ other than $-2, -4, -6, \ldots$ has real part $1/2$;*
2. *for every primitive Dirichlet character χ, every zero of $L(s, \chi)$ with $0 < \operatorname{Re} s < 1$ has real part $1/2$.*

Proof. (i) Let $\zeta(s) = 0$, with s not a negative even integer and $s \neq 1$. If $\operatorname{Re} s > 1/2$, Hypothesis H with $N = 1$ gives $\zeta(s) \neq 0$, which is impossible. Suppose $\operatorname{Re} s < 1/2$. Then $s \neq 0$, because $\zeta(0) = -1/2$.

[29] For a trivial character modulo N the function $L(s, \chi) = \zeta(s) \prod_{p|N} (1 - p^{-s})$ has a simple pole at $s = 1$ and, besides the zeros of ζ, zeros on the line $\operatorname{Re} s = 0$ from the Euler factors. These lie outside the half-plane $\operatorname{Re} s > 1/2$ and play no part below.

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Put $w = 1 - s$, so $\operatorname{Re} w > 1/2$ and $w \neq 1$, and w is not a non-positive integer. The functional equation in Riemann's asymmetric form [57, §2.1],

$$\zeta(1-w) = 2(2\pi)^{-w}\,\Gamma(w)\cos\Big(\frac{\pi w}{2}\Big)\zeta(w),$$

gives $0 = \zeta(s) = 2(2\pi)^{-w}\Gamma(w)\cos(\pi w/2)\zeta(w)$. The first two factors are non-zero, and $\Gamma(w) \neq 0$ because Γ has no zeros.[30] The cosine vanishes only if w is an odd integer, that is, if $s = -2k$ for an integer k. Since $\operatorname{Re} s < 1/2$ and $s \neq 0$, this would make s a negative even integer, which is excluded. Hence $\zeta(w) = 0$ with $\operatorname{Re} w > 1/2$, against Hypothesis H. So $\operatorname{Re} s = 1/2$.

(ii) Let χ be primitive modulo N, let $\kappa \in \{0, 1\}$ with $\chi(-1) = (-1)^\kappa$, and let

$$\Lambda(s,\chi) = \Gamma_{\mathbb{R}}(s+\kappa)\,L(s,\chi), \qquad \Gamma_{\mathbb{R}}(s) = \pi^{-s/2}\Gamma(s/2),$$

be the completed L-function, which is entire for $N > 1$. It satisfies [32, Ch. 9]

$$\Lambda(1-s,\bar\chi) = \varepsilon(\chi)\,N^{s-1/2}\,\Lambda(s,\chi), \qquad |\varepsilon(\chi)| = 1,$$

and $\bar\chi$ is primitive of the same conductor. For $\operatorname{Re} s > 0$ the factor $\Gamma_{\mathbb{R}}(s+\kappa)$ is finite and non-zero, so in the strip $0 < \operatorname{Re} s < 1$ the zeros of $L(s,\chi)$ and of $\Lambda(s,\chi)$ coincide, and likewise at $1-s$ for $\bar\chi$.[31] Let $L(s,\chi) = 0$ with $0 < \operatorname{Re} s < 1$. If $\operatorname{Re} s > 1/2$, Hypothesis H is contradicted directly. If $\operatorname{Re} s < 1/2$, then $\Lambda(s,\chi) = 0$, hence $\Lambda(1-s,\bar\chi) = 0$, hence $L(1-s,\bar\chi) = 0$ with $\operatorname{Re}(1-s) > 1/2$ and $1-s \neq 1$, against Hypothesis H. So $\operatorname{Re} s = 1/2$. □

Corollary 24. *Under Hypothesis H every conclusion of Theorem 55 holds; in particular $Q(N) \leq N^{1+\varepsilon}$ for every $\varepsilon > 0$ and all large N.*

Proof. Theorem 55 assumes exactly Hypothesis H. □

20.2. From a Holomorphic Logarithmic Derivative to the Half-Plane

A weaker-looking form of Hypothesis H asks only that the logarithmic derivative $L'/L(s,\chi)$ be holomorphic on $\operatorname{Re} s > 1/2$ apart from the pole at $s = 1$. We show that this already implies Hypothesis H. The point is a general fact about holomorphic functions.

Lemma 26. *Let $V \subset \mathbb{C}$ be open and connected, let P and q be holomorphic on V with $P' = qP$ on V, and suppose $P(w) \neq 0$ for some $w \in V$. Then P has no zero in V.*

Proof. Suppose $P(s_0) = 0$ for some $s_0 \in V$. Since V is connected and P does not vanish identically on V, the identity theorem says that P does not vanish identically near s_0, so P has a zero of some finite order $m+1 \geq 1$ at s_0: $P(s) = (s-s_0)^{m+1}h(s)$ with h holomorphic and $h(s_0) \neq 0$. Then

$$P'(s) = (s-s_0)^m\big((m+1)h(s) + (s-s_0)h'(s)\big),$$

and the bracket equals $(m+1)h(s_0) \neq 0$ at s_0, so P' has order exactly m at s_0. On the other hand $P' = qP$, and q is holomorphic at s_0, so P' has order at least $m+1$ there. This is a contradiction. □

Theorem 57. *Suppose that for every Dirichlet character χ there is a function q_χ, holomorphic on $U = \{s : \operatorname{Re} s > 1/2,\ s \neq 1\}$, with $L'(s,\chi) = q_\chi(s)L(s,\chi)$ on U. Then Hypothesis H holds, and with it the Riemann hypothesis and the generalized Riemann hypothesis for primitive Dirichlet L-functions.*

[30] $1/\Gamma$ is entire, and $\Gamma(w)$ is finite here because $\operatorname{Re} w > 0$.

[31] The same holds for $N = 1$, where $\Lambda(s,1) = \Gamma_{\mathbb{R}}(s)\zeta(s)$ has poles only at 0 and 1, outside the open strip.

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Proof. Let $\operatorname{Re} s_0 > 1/2$, with $s_0 \neq 1$ if χ is trivial. If $\operatorname{Re} s_0 \geq 1$, then $L(s_0, \chi) \neq 0$ by the theorem of Hadamard and de la Vallée Poussin and its extension to Dirichlet characters [32, Ch. 4]. Suppose $\operatorname{Re} s_0 < 1$, and let $t = 1$ if $\operatorname{Im} s_0 \geq 0$ and $t = -1$ otherwise. The set

$$V_t = \{s : \operatorname{Re} s > 1/2,\ \operatorname{Re} s - t \operatorname{Im} s < 1\}$$

is the intersection of two open half-planes, so it is open and convex, hence connected. It contains s_0, because $\operatorname{Re} s_0 - t \operatorname{Im} s_0 \leq \operatorname{Re} s_0 < 1$. It does not contain $s = 1$, so $V_t \subset U$ and $L(s, \chi)$ is holomorphic on V_t. It contains $w = 2 + 2ti$, because $2 - 2t^2 = 0 < 1$, and $L(w, \chi) \neq 0$ because $\operatorname{Re} w > 1$. Lemma 26, with $P = L(\cdot, \chi)$ and $q = q_\chi$, gives $L(s_0, \chi) \neq 0$. The last assertion is Theorem 56. □

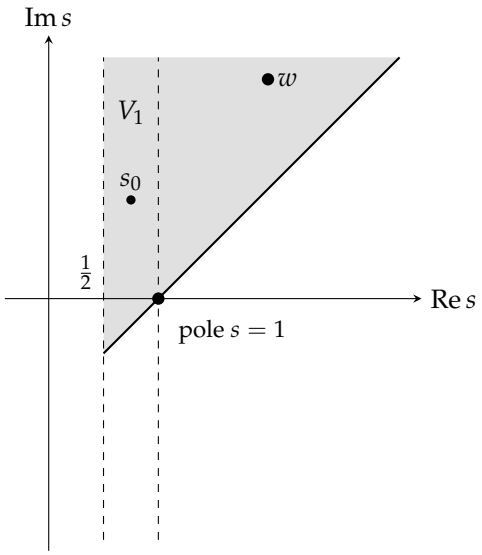

Figure 13. The region V_1 used in Theorem 57 (shaded): the part of the half-plane $\operatorname{Re} s > 1/2$ above the line $\operatorname{Re} s - \operatorname{Im} s = 1$ through the pole. It is convex, it contains the point s_0 in the critical strip with $\operatorname{Im} s_0 \geq 0$, and it reaches the point $w = 2 + 2i$ where $L(w, \chi) \neq 0$ is classical.

20.3. The Lean Check

Table 19 lists the theorems that formalise this section. Each is stated with Mathlib's own ζ, $L(s, \chi)$, primitivity and `RiemannHypothesis`, and for each the command `#print axioms` reports only `propext`, `Classical.choice` and `Quot.sound`. The Lean proofs of Theorem 56 and of the order argument of Lemma 26 are adapted from [4]; the region V_t is ours.

Table 19. The Lean theorems of Section 20, built with Lean 4.33.1 over Mathlib.

Lean name	Statement	In the text
`rh_of_right`	$\zeta \neq 0$ on $\operatorname{Re} s > 1/2 \Rightarrow$ `RiemannHypothesis`	Theorem 56(i)
`rh_of_H`	Hypothesis H $\Rightarrow$ `RiemannHypothesis`	Theorem 56(i)
`critLine`	Hypothesis H $\Rightarrow$ zeros of primitive $L(s, \chi)$ in $0 < \operatorname{Re} s < 1$ lie on $\operatorname{Re} s = \frac{1}{2}$	Theorem 56(ii)
`zero_free_of_logDeriv`	Lemma 26	Lemma 26
`hypothesisH_of_logDeriv`	holomorphic L'/L on $U \Rightarrow$ Hypothesis H	Theorem 57
`riemannHypothesis_of_logDeriv`, `grh_of_logDeriv`	holomorphic L'/L on $U \Rightarrow$ RH and GRH	Theorem 57
`windowed_energy_principle`	window bounds $\Rightarrow$ the weighted family is zero-free above θ	Theorem 52
`riemannHypothesis_of_windowBound`, `grh_of_windowBound`	window bound for every primitive cyclic family $\Rightarrow$ RH and GRH	Theorem 66

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20.4. What Remains

With this section the logical structure is complete: the paper proves

$$\text{Hypothesis H} \implies \text{RH and GRH for Dirichlet } L\text{-functions} \implies Q(N) \ll_\varepsilon N^{1+\varepsilon},$$

and the second arrow is in fact an equivalence for the Riemann hypothesis (Theorems 1 and 2). Hypothesis H itself follows from the window theorem by the windowed energy principle; this is Theorem 66.

21. The Host Decomposition

Section 17 reduced Hypothesis H to a bound for the energy of the cyclic family $\langle\psi\rangle$ on windows. This section proves the theorem that produces that bound from two inputs, an analytic one and an arithmetic one, and states both inputs precisely. The theorem and its proof are elementary; the inputs are where the window theorem of Section 19 does its work. The decomposition is the one used in [4], and we follow it, writing it in the language of the energy.

21.1. Hosts and Their Characters

Fix a primitive character ψ of order o and let $\mathcal{F}_\psi$ be its cyclic family: the powers ψ^j, $0 \le j < o$, with the trivial character taken modulo 1, the trivial character with weight 1 and the others with weight $(3o+1)^{-1}$. Its energy is

$$E_\psi(x) = G_1(x) + \frac{1}{3o+1} \sum_{0<j<o} \operatorname{Re}\, G_{\psi^j}(x), \tag{113}$$

with G_χ the ramp energy of Section 16.

A *host* at scale x consists of the following data.

(i) Two primes $q \equiv 3 \pmod 8$ and $d \equiv 7 \pmod{16}$ with $q \mid d+1$, an integer $e \ge 2$ and an integer Q with $2^{e+2} \mid Q$, $(Q, qd) = 1$, and $\left(\frac{-d}{l}\right) = 1$ for every odd prime $l \mid Q$. The *host field* E is the subfield of $\mathbb{Q}(\zeta_{Qqd})$ fixed by the classes $a \in (\mathbb{Z}/Qqd)^\times$ with

$$a \equiv \pm 1 \pmod{2^{e+2}}, \qquad \psi(a) = 1, \qquad \left(\frac{a}{q}\right) = \left(\frac{a}{d}\right) = 1.$$

It is a Galois CM field. Through its characters, $\mathrm{Gal}(E/\mathbb{Q})$ is identified with the group $E_0 \times A$ of (ii), so $[E:\mathbb{Q}] = 4\,\mathrm{ord}$ and its maximal real subfield F has degree $2\,\mathrm{ord}$.

(ii) A finite abelian group A of order ord, which carries the 2-power and ψ parts of $\mathrm{Gal}(E/\mathbb{Q})$, the Klein group $E_0 = (\mathbb{Z}/2)^2$ that records the two quadratic characters $(\cdot/q)$ and $(\cdot/d)$, and the cyclic subgroup $A_0 \le A$ generated by a distinguished element a_0, with index $[A:A_0] = o$. Put $\Gamma = E_0 \times (A/A_0)$, so $\#\Gamma = 4o$. The characters of $E_0 \times A$ that are trivial on $\{0\} \times A_0$ are the characters of Γ; they form a set Ψ_{A_0}, and by duality of finite abelian groups $\#\Psi_{A_0} = \#\Gamma = 4o$.

(iii) A partition of Ψ_{A_0} into a *kept* set K, which contains the trivial character and the characters attached to the powers ψ^j, and a *discarded* set $\Psi_{A_0} \setminus K$, compatible with the family in the sense of (115) below.

(iv) A finite set $T \subset A$ of *test points* that is equidistributed over the cosets of A_0 in the following quantitative sense: with $\lambda(\gamma) = 4/(3o+1)$ for $\gamma \in A_0$ and $\lambda(\gamma) = 3/(3o+1)$ otherwise,

$$\left|\#\{t \in T :\ t - \gamma \in A_0\} - \lambda(\gamma)\,\#T\right| \le x^{-1+\varepsilon}\,\#T \qquad (\gamma \in A). \tag{114}$$

The congruence conditions on q and d make $-d$ a square modulo every odd prime dividing Q and make q split in $\mathbb{Q}(\sqrt{-d})$; they are the conditions under which the auxiliary primes of Section 19 exist. The test points T are the places at which the energy of the host is sampled.

For a character $\theta \in \Psi_{A_0}$, viewed as a Dirichlet character through the Artin map, write θ^* for the primitive character inducing it and $G_\theta = G_{\theta^*}$; write $\theta(t)$ for $\theta(0,t)$ when $t \in A$. Put $w(\theta) = 1$ if θ is

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trivial on $\{0\} \times A$ and $w(\theta) = c = (3o+1)^{-1}$ otherwise. The compatibility required of a host in (iii) is that the kept characters reproduce the family with these weights:

$$E_\psi(x) = \sum_{\theta \in K} w(\theta)\ \mathrm{Re}\ G_\theta(x). \tag{115}$$

This identity is one of the host axioms of [4], and it is the only place where the cyclic family enters; the rest of the section concerns the host.

21.2. The Two Inputs

Definition 6 (Explicit-formula control)**.** *A host at scale x satisfies condition (E) with constant C if*

(E1) every discarded character $\theta \in \Psi_{A_0} \setminus K$ satisfies $|G_\theta(x)| \le x^{1/2+\varepsilon}$;

(E2) every discarded character satisfies $\left|\frac{L'}{L}(1,\theta^*)\right| \le C \log x$, and every kept non-trivial character satisfies $\left|\frac{L'}{L}(1,\theta^*)\right| \le K_0$ for a constant K_0 depending only on ψ;

(E3) the test points satisfy (114).

Condition (E1) is a square-root bound for the energies of the discarded characters at the single scale x. It is not a hypothesis on their zeros: by the explicit formula, $G_\theta(x)$ is $-\sum_\rho x^\rho/(\rho(\rho-1))$ over the zeros of $L(s,\theta^*)$ plus bounded terms, and (E1) is implied by, but weaker than, the absence of zeros with real part above $1/2+\varepsilon$. In [4] it is derived from the host conditions of Definition 8 below. Condition (E2) is the standard bound for the logarithmic derivative at $s=1$; for the kept characters it is a consequence of the zero-free region near $\mathrm{Re}\, s = 1$ with constants depending on ψ.

Definition 7 (Core bounds)**.** *A host at scale x satisfies condition (K) with exponents e_D, e_L if there are: a finite set $T_2 \subset A_0$ of* thick *points; positive integers m and N; for each $t \in T$ a sample point a_t with $|a_t/b - x| \le 1$ for a common scale $b > 0$; for each $u \in A_0$ and $j < m$ a window weight $w_{u,j}$ on A, with $b \le w_{u,j} \le b+\eta+1$ at the thick points and $0 < w_{u,j} \le 2b'$ elsewhere, where $b'X' \le xb$ and $a_t/w_{u,j} \le N$; and constants $\eta, X', \vartheta \ge 0$; such that*

(K1) (gap bound) *for every $u \in A_0$,*

$$\sum_{t\in T}\sum_{j<m}\Big(S_{u,j}(a_t;t) - \mathrm{Re}\ M_{u,j}(a_t)\Big) \ \le\ x^{(e_D+e_L)\varepsilon}\,\mathcal{E}(x),$$

where $S_{u,j}(a;t)$ is the weighted sum of $\Lambda(n)(a/n-1)$ over the integers $n \le a$ whose Frobenius class lies in $\bar{t} = (0, t+A_0) \in \Gamma$, weighted by $w_{u,j}$, $M_{u,j}(a)$ is its main term, and $\mathcal{E}(x) = \#T\,(x+\eta+1)\,(1 + m^2/k) + m\,(f\,m\,x^\varepsilon + N_c)$ is the error bracket of the windows, with $k \ge xm$ and f, N_c the window counts;

(K2) (thin scales) *the sums over n outside the thick windows are at most X' in total and the scales below ϑ contribute nothing;*

(K3) (cost) *the window weights satisfy the cost bound of Section 17 at scale x with parameters b, η, ϑ;*

(K4) (rest) *$m \le x^{1/2+\varepsilon/2}$, and the remaining sizes $\#A, \#T, \#T_2, m, b, k, a$ satisfy the polynomial bounds of the formal statement, which we use in the two forms $\mathcal{E}(x) \le x^{1+3\varepsilon}$ and $\#T\,b\,m \ge 4o\,x^{1/2}$.*

Condition (K) is a statement about the distribution of primes in the Frobenius classes of E, sampled at the test points and smoothed by the window weights: the gap bound (K1) says that, averaged over the test points and the windows, the prime sums in each class are within $x^{(e_D+e_L)\varepsilon}\mathcal{E}(x)$ of their main terms. It is the arithmetic input, and it is what the length bound of Theorem 54 supplies through the connectedness statement of Definition 9.

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21.3. The Decomposition Theorem

Theorem 58 (Host decomposition)**.** *Let $e_D, e_L, C \geq 0$ and $\varepsilon > 0$. There is x_1 such that for every $x \geq x_1$ and every host at scale x satisfying (E) with constant C and (K) with exponents e_D, e_L,*

$$E_\psi(x) \leq x^{1/2+(e_D+e_L+7)\varepsilon}.$$

The proof is in two steps. The first bounds the discarded part, the second the kept part in terms of the full host energy and the discarded part.

Lemma 27 (The discarded part)**.** *Under (E),*

$$\Sigma_{\mathrm{disc}}(x) := \sum_{\theta \in \Psi_{A_0} \setminus K} |G_\theta(x)| \leq 4o\, x^{1/2+\varepsilon}.$$

Proof. Each term is at most $x^{1/2+\varepsilon}$ by (E1), and there are at most $\#\Psi_{A_0} = 4o$ terms. The count holds because a character of $E_0 \times A$ trivial on $\{0\} \times A_0$ is the same as a character of the quotient $E_0 \times (A/A_0)$, and a finite abelian group has exactly as many characters as elements, so $\#\Psi_{A_0} = \#E_0 \cdot \#(A/A_0) = 4 \cdot o$. □

Lemma 28 (The kept part)**.** *Let $c = (3o+1)^{-1}$. There is x_1, depending on $e_D, e_L, C, \varepsilon, o$ and K_0, such that for $x \geq x_1$ and every host at scale x satisfying (E) and (K),*

$$\sum_{\theta \in K} w(\theta)\ \mathrm{Re}\ G_\theta(x)\ \leq\ x^{1/2+(e_D+e_L+3)\varepsilon} + \Sigma_{\mathrm{disc}}(x).$$

Proof. Write $\Psi = \Psi_{A_0}$. For $\theta \in \Psi$ and a test point $t \in T$ the value $\theta(t)$ depends only on the class $\bar{t} \in \Gamma$, and orthogonality on the finite group Γ, whose character group is Ψ, gives, for every function g on Γ,

$$\sum_{\theta \in \Psi} \overline{\theta(\bar{t})}\, \widehat{g}(\theta) = \#\Gamma\, g(\bar{t}), \qquad \widehat{g}(\theta) = \sum_{s \in \Gamma} \theta(s) g(s). \tag{116}$$

Apply this with g the prime sum in the class: for $a > 0$ and a window weight w, let

$$S_w(a;s) = \sum_{\substack{n \leq a \\ \mathrm{Frob}(n) \in s}} \Lambda(n)\, w(\mathrm{Frob}(n)) \Big(\frac{a}{n} - 1\Big),$$

$$\widehat{S}_w(a;\theta) = \sum_{n \leq a} \Lambda(n)\theta(n) w(\mathrm{Frob}(n)) \Big(\frac{a}{n} - 1\Big).$$

When $w \equiv 1$, $\widehat{S}_1(a;\theta)$ is the ramp sum whose difference from its main term is $G_\theta(a)$; the main term is $a \log a - (1+\gamma_0)a$ for θ trivial and $-a\,\frac{L'}{L}(1,\theta^*)$ otherwise (Section 16). Thus

$$\mathrm{Re}\ G_\theta(a) = \mathrm{Re}\ \widehat{S}_1(a;\theta) - \mathrm{Re}\ M(a;\theta),$$

and (116) applied to $S_1(a;\cdot) - M(a;\cdot)$, where $M(a;s)$ is the corresponding sum of main terms, gives for each $t \in T$

$$\sum_{\theta \in \Psi} \overline{\theta(t)}\ \mathrm{Re}\ G_\theta(a) = \#\Gamma \big(S_1(a;\bar{t}) - \mathrm{Re}\ M(a;\bar{t})\big). \tag{117}$$

The right side is the gap at the test point t: the difference between the prime sum in the class $\bar{t}$ and its main term.

Now average (117) over $t \in T$ and over the windows $j < m$ at $a = a_t$ with weights $w_{u,j}$, for each $u \in A_0$. The left side becomes

$$\sum_{\theta \in \Psi} \Big(\sum_{t \in T} \overline{\theta(t)}\Big) \sum_{j<m} \mathrm{Re}\ G_\theta^{(u,j)}(a_t),$$

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where $G_\theta^{(u,j)}$ is the ramp energy with the weight $w_{u,j}$ inserted. By (K2) and (K3), replacing $w_{u,j}$ by the constant b on the thick windows and discarding the thin ones changes each term by at most the thin and cost contributions, which are at most $x^{1+2\varepsilon}$ in total by (K4) once $x \geq x_1$. By (E3), the character sums over the test points are

$$\sum_{t\in T} \overline{\theta(t)} = \begin{cases} \#T & \theta \text{ trivial on } A, \\ c\,\#T + O\big(Mx^{-1+\varepsilon}\#T\big) & \text{otherwise,} \end{cases}$$

with $M = \#\Psi = 4o$, since the shares $\lambda(\gamma)$ of (114) are exactly the Fourier coefficients of the weight $w(\theta)$: a character trivial on A has weight 1 and the others have weight $c = (3o+1)^{-1}$. The left side is therefore

$$\#T\,b\,m\Big(\sum_{\theta\in K} w(\theta)\,\mathrm{Re}\;G_\theta(x) + \sum_{\theta\in\Psi\setminus K} w(\theta)\,\mathrm{Re}\;G_\theta(x)\Big) + O\big(Mx^{-1+\varepsilon}\#T\,b\,m\sum_{\theta\in\Psi}|G_\theta(x)|\big) + O(x^{1+2\varepsilon}),$$

after replacing a_t by x, which costs $|a_t/b - x| \leq 1$ per term. The right side is $\#\Gamma$ times the gap sum of (K1), hence at most $\#\Gamma\, x^{(e_D+e_L)\varepsilon}\mathcal{E}(x) \leq 4o\, x^{1+(e_D+e_L+3)\varepsilon}$ by (K4).

Divide by $\#T\,b\,m$. The kept sum is $E_\psi(x)$ by (115). The discarded sum is bounded in absolute value by $\Sigma_{\mathrm{disc}}(x)$, since $|w(\theta)| \leq 1$. The error from the character sums is at most $Mx^{-1+\varepsilon}\big(\Sigma_{\mathrm{disc}}(x) + \sum_{\theta\in K}|G_\theta(x)|\big)$, and $|G_\theta(x)| \leq x\log x$ trivially, so this is at most $4oM\,x^\varepsilon\log x$. The normalising factor $\#T\,b\,m$ is at least $x^{1/2}$ by (K4), and the remaining $O(x^{1+2\varepsilon})$ term divided by it is at most $x^{1/2+2\varepsilon}$. Collecting terms,

$$E_\psi(x) \leq 4o\,x^{1/2+(e_D+e_L+3)\varepsilon}\,\frac{x^{1/2}}{\#T\,b\,m} + \Sigma_{\mathrm{disc}}(x) + x^{1/2+2\varepsilon} + 4oM\,x^\varepsilon\log x,$$

and (K4) gives $\#T\,b\,m \geq 4o\,x^{1/2}$, so the first term is at most $x^{1/2+(e_D+e_L+3)\varepsilon}$. The last two terms are absorbed by enlarging x_1, since $(e_D+e_L+3)\varepsilon > 2\varepsilon$. □

Proof of Theorem 58. By Lemma 28 and Lemma 27,

$$E_\psi(x) \leq x^{1/2+(e_D+e_L+3)\varepsilon} + 4o\,x^{1/2+\varepsilon}.$$

For $x \geq x_1$ with $x_1^\varepsilon \geq 1+4o$, the right side is at most $(1+4o)x^{1/2+(e_D+e_L+3)\varepsilon} \leq x^{1/2+(e_D+e_L+4)\varepsilon}$, which is below $x^{1/2+(e_D+e_L+7)\varepsilon}$. □

Read with the energy in mind, Theorem 58 says that the energy of the family is controlled by the energy of the *other* characters of a larger abelian field, measured at the same scale, together with the distribution of primes in the Frobenius classes of that field. The bound transfers along the host. It does not create the square-root size: (E1) already asks for square-root size for the discarded characters, and (K1) asks for a square-root-type gap in the Frobenius classes. The theorem is exact about where the arithmetic lies.

21.4. The Two Deep Inputs

Theorem 52 and Theorem 58 together reduce Hypothesis H to the following two statements, which are the final inputs of [4].

Definition 8 (Host supply)**.** Host supply *is the statement: for every ψ and every small ε there are scales x forming windows of a fixed width, arbitrarily far out, and at each such scale a host satisfying (E), together with the* length bound*: the total length $\sum_\ell \log\ell \sum_{1\leq e<t} \mathrm{len}\,\mathrm{Sha}^1_e(E,\ell)$ of Theorem 54 for the host field E is at most* $\exp(\frac{1}{2}x^\varepsilon)$.

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Definition 9 (Connectedness)**.** Connectedness *is the statement: every class of families whose hosts satisfy the length bound contains a family that satisfies (K) on all of its windows.*

Host supply is where the Galois representations of Section 19 enter, through the length bound of Theorem 54; connectedness is where the length bound is converted into the gap bound (K1). Both are proved in [4] and are described in Sections 22–24; neither is in the classical literature in this form. Together they are the whole content of the window theorem.

Corollary 25. *Host supply and connectedness imply Hypothesis H, and hence the Riemann hypothesis and the generalized Riemann hypothesis for Dirichlet L-functions.*

Proof. By Theorem 58 the cyclic family of each ψ satisfies $E_\psi(x) \le x^{1/2+(e_D+e_L+7)\varepsilon}$ on windows of fixed width at arbitrarily large heights, for every small ε. Theorem 52 with $\theta = 1/2 + (e_D + e_L + 7)\varepsilon$ gives $L(s, \psi^j) \neq 0$ for $\operatorname{Re} s > 1/2 + (e_D + e_L + 7)\varepsilon$, the exponents e_D, e_L being fixed. Letting $\varepsilon \to 0$ gives $L(s, \psi) \neq 0$ in $\operatorname{Re} s > 1/2$ for every primitive ψ, hence Hypothesis H, since an imprimitive L-function differs from the primitive one only by Euler factors without zeros in $\operatorname{Re} s > 0$. Theorem 56 gives the rest. □

The theorem of this section is checked in Lean on its own, in the project [10]: the host decomposition and its two lemmas compile over Mathlib with only the standard axioms, with (E) and (K) as explicit hypotheses.

22. From Tate–Shafarevich Lengths to K-groups and Bernoulli Numbers

The quantity controlled by the cohomological window theorem is

$$\sum_{\ell} \log \ell \sum_{1 \le e < t} \operatorname{len}_{\mathbb{Z}_\ell} \operatorname{Sha}^1_e(E, \ell), \tag{118}$$

where E is a host field, $\operatorname{Sha}^1_e(E, \ell)$ is the kernel of the localisation map on $H^1(G_{E,S_\ell}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(-e))$, and t is a large parameter. This section rewrites (118) in terms of objects of classical arithmetic. The rewriting has three steps, each of which is either proved in full or reduced to a precisely cited theorem. First, Poitou–Tate duality identifies each $\operatorname{Sha}^1_e(E, \ell)$ with the dual of a subgroup of the second étale cohomology group $H^2(\mathcal{O}_E[1/\ell], \mathbb{Z}_\ell(1+e))$, and the difference between the two is computed exactly in terms of local roots of unity (Proposition 31). Second, the norm residue theorem of Rost and Voevodsky, in the form of the Quillen–Lichtenbaum conjecture for rings of integers, identifies these cohomology groups with the ℓ-primary parts of the even K-groups $K_{2e}(\mathcal{O}_E)$ (Theorem 59), so that (118) becomes $\sum_{e<t} \log \#K_{2e}(\mathcal{O}_E)$ minus an explicit, non-negative defect (Corollary 26). Third, the decomposition of these finite Galois modules under the abelian group $\operatorname{Gal}(E/\mathbb{Q})$ separates each $K_{2e}(\mathcal{O}_E)$ into two halves. One half, the *Bernoulli half*, has order given by generalised Bernoulli numbers through the Iwasawa main conjecture of Mazur and Wiles (Theorem 60); the other, the *regulator half*, has order given by the index of Soulé's cyclotomic elements (Theorem 61). We then prove, from the functional equation of Dirichlet L-functions and an elementary integrality statement for Bernoulli numbers, that the Bernoulli half of $\sum_{e<t} \log \#K_{2e}(\mathcal{O}_E)$ is at most $3\,[E : \mathbb{Q}]\, t^2 \log(tN)$ (Proposition 33). The regulator half is not bounded in this section; Remark 16 states exactly what is left.

Throughout, log is the natural logarithm, len denotes the length of a finite $\mathbb{Z}_\ell$-module (so that $\#M = \ell^{\operatorname{len} M}$), v_ℓ is the ℓ-adic valuation normalised by $v_\ell(\ell) = 1$ and extended to $\overline{\mathbb{Q}}_\ell$, and $M^\vee = \operatorname{Hom}(M, \mathbb{Q}/\mathbb{Z})$ is the Pontryagin dual of a locally compact abelian group M.

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22.1. Standing Notation

Let ψ be a primitive Dirichlet character, $m \geq 0$ an integer, and q, d primes with $q \equiv 3 \pmod 8$ and $d \equiv 7 \pmod{16}$. Let N be a positive integer divisible by 2^{m+2}, by the conductor of ψ, by q and by d, and let $H \leq (\mathbb{Z}/N)^\times$ be the subgroup of classes a with

$$a \equiv \pm 1 \pmod{2^{m+2}}, \qquad \psi(a) = 1, \qquad \left(\frac{a}{q}\right) = \left(\frac{a}{d}\right) = 1.$$

The host field E is the fixed field of H in $\mathbb{Q}(\zeta_N)$, and $G = \mathrm{Gal}(E/\mathbb{Q}) \cong (\mathbb{Z}/N)^\times / H$. The map $a \mapsto (a \bmod 2^{m+2}, \psi(a), (a/q), (a/d))$ embeds G into $\big((\mathbb{Z}/2^{m+2})^\times/\{\pm 1\}\big) \times \mu_{\mathrm{ord}(\psi)} \times \{\pm 1\}^2$, so

$$D := [E : \mathbb{Q}] \text{ divides } 2^{m+2}\,\mathrm{ord}(\psi), \qquad D \mid \varphi(N), \tag{119}$$

and every odd prime dividing D divides $\mathrm{ord}(\psi)$. Since $(-1/q) = -1$ for $q \equiv 3 \pmod 4$, the class of -1 is not in H, so complex conjugation $c \in G$ is a non-trivial element of order 2; it is the same element for every complex embedding of E because G is abelian. Thus E is a CM field: it is totally imaginary, with $r_1(E) = 0$ real and $r_2(E) = D/2$ complex places, and its maximal real subfield is the fixed field of c. Every character $\chi\colon G \to \overline{\mathbb{Q}}^\times$ is a primitive Dirichlet character of conductor $f_\chi \mid N$, and $\chi(c) = \chi(-1)$. Exactly half of the D characters of G are even ($\chi(-1) = 1$) and half are odd, because $\chi \mapsto \chi(c)$ is a surjection of $\widehat{G}$ onto $\{\pm 1\}$.

For a prime ℓ let S_ℓ be the set of places of E above ℓ, $S = S_\ell \cup S_\infty$, and $G_S = G_{E,S}$ the Galois group of the maximal extension of E unramified outside S. Since $E/\mathbb{Q}$ is Galois, the completions E_v, $v \in S_\ell$, are pairwise isomorphic; we write $g_\ell = \#S_\ell \leq D$. Let $\chi_{\mathrm{cyc}}\colon G_\mathbb{Q} \to \mathbb{Z}_\ell^\times$ be the ℓ-adic cyclotomic character, and for $j \in \mathbb{Z}$ put

$$A_j = \mathbb{Q}_\ell/\mathbb{Z}_\ell(j), \qquad T_j = \mathbb{Z}_\ell(j), \qquad M_{k,j} = \mathbb{Z}/\ell^k(j) = A_j[\ell^k] = T_j/\ell^k.$$

The Cartier dual $M' = \mathrm{Hom}(M, \mu_{\ell^k})$ of $M_{k,j}$ is $M_{k,1-j}$; in particular the finite layers $A_{-e}[\ell^k]$ of A_{-e} and T_{1+e}/ℓ^k of T_{1+e} are Cartier dual to each other. For a finite G_S-module M and $i \in \{1, 2\}$ put

$$\mathrm{Sha}^i(G_S, M) = \ker\Big(H^i(G_S, M) \to \bigoplus_{v \in S} H^i(E_v, M)\Big),$$

and the same for discrete torsion or compact $\mathbb{Z}_\ell$-modules, with $H^i(G_S, T_j) := \varprojlim_k H^i(G_S, M_{k,j})$ for the compact modules. Because all infinite places of E are complex and $H^i(\mathbb{C}, M) = 0$ for $i \geq 1$, the sum over S may be replaced by the sum over S_ℓ for $i \geq 1$; so $\mathrm{Sha}^1_e(E, \ell) = \mathrm{Sha}^1(G_S, A_{-e})$ in this notation, and no modification of the cohomology at infinite places is needed anywhere below, not even for $\ell = 2$.

For ℓ-primary torsion modules M and $S \supseteq S_\ell \cup S_\infty$, the Galois cohomology of G_S coincides with the étale cohomology of the scheme $\mathrm{Spec}\,\mathcal{O}_E[1/\ell]$,

$$H^i(G_S, M) \cong H^i_{\mathrm{et}}(\mathcal{O}_E[1/\ell], M), \tag{120}$$

by [104, Ch. II, Proposition 2.9]; passing to inverse limits over k gives the same for T_j, where $H^i_{\mathrm{et}}(\mathcal{O}_E[1/\ell], \mathbb{Z}_\ell(j)) = \varprojlim_k H^i_{\mathrm{et}}(\mathcal{O}_E[1/\ell], \mu_{\ell^k}^{\otimes j})$ is the definition used in the K-theoretic literature [112, §5.4]. We use the two notations interchangeably.

22.2. Poitou–Tate Duality and the Exact Formula

We begin with two elementary facts about limits.

Lemma 29. (a) *Let $(A_k \to B_k \to C_k)_{k \geq 1}$ be an inverse system of exact sequences of finite abelian groups. Then $\varprojlim A_k \to \varprojlim B_k \to \varprojlim C_k$ is exact. In particular an inverse limit of kernels is the kernel of the inverse limit map.* (b) *Let $(A_k \to B_k)_{k \geq 1}$ be a direct system of homomorphisms of abelian groups. Then $\varinjlim \ker(A_k \to B_k) = \ker(\varinjlim A_k \to \varinjlim B_k)$, and $(\varinjlim A_k)^\vee = \varprojlim A_k^\vee$ when the A_k are finite.*

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Proof. (a) Let $b = (b_k) \in \varprojlim B_k$ map to 0 in $\varprojlim C_k$. For each k the set $X_k = \{a \in A_k : a \mapsto b_k\}$ is non-empty by exactness at B_k, finite, and the transition maps of the system (A_k) carry X_{k+1} into X_k. An inverse limit of non-empty finite sets is non-empty (choose $a_k \in X_k$ for each k; a diagonal argument, or compactness of $\prod X_k$, produces a compatible family), so b comes from $\varprojlim A_k$. That the image of $\varprojlim A_k$ lies in the kernel is clear. The statement about kernels is the case $A_k = \ker(B_k \to C_k)$. (b) Filtered colimits are exact in the category of abelian groups, and Pontryagin duality interchanges direct and inverse limits of finite groups. □

Proposition 30 (Poitou–Tate). *Let ℓ be a prime and $e \geq 1$. There is a canonical isomorphism*

$$\mathrm{Sha}^1(G_S, \mathbb{Q}_\ell/\mathbb{Z}_\ell(-e)) \;\cong\; \mathrm{Sha}^2(G_S, \mathbb{Z}_\ell(1+e))^\vee,$$

and an exact sequence

$$\begin{aligned} 0 \to \mathrm{Sha}^2(G_S, \mathbb{Z}_\ell(1+e)) \to H^2(\mathcal{O}_E[1/\ell], \mathbb{Z}_\ell(1+e)) \\ \to \bigoplus_{v \in S_\ell} H^0(E_v, \mathbb{Q}_\ell/\mathbb{Z}_\ell(-e))^\vee \to H^0(E, \mathbb{Q}_\ell/\mathbb{Z}_\ell(-e))^\vee \to 0. \end{aligned} \tag{121}$$

All groups in (121) *are finite. Consequently* $\mathrm{Sha}^1_e(E, \ell)$ *is finite and*

$$\operatorname{len} \mathrm{Sha}^1_e(E, \ell) = \operatorname{len} H^2(\mathcal{O}_E[1/\ell], \mathbb{Z}_\ell(1+e)) - \delta_e(\ell), \tag{122}$$

where

$$\delta_e(\ell) := \sum_{v \in S_\ell} \operatorname{len} H^0(E_v, \mathbb{Q}_\ell/\mathbb{Z}_\ell(-e)) - \operatorname{len} H^0(E, \mathbb{Q}_\ell/\mathbb{Z}_\ell(-e)) \;\geq 0. \tag{123}$$

Proof. Fix $k \geq 1$ and put $M = M_{k,1+e} = \mathbb{Z}/\ell^k(1+e)$, so that $M' = \mathbb{Z}/\ell^k(-e) = A_{-e}[\ell^k]$. The Poitou–Tate theorem [82, Theorem 8.6.7] gives a perfect pairing $\mathrm{Sha}^1(G_S, M') \times \mathrm{Sha}^2(G_S, M) \to \mathbb{Q}/\mathbb{Z}$, natural in M, and the nine-term exact sequence [82, Theorem 8.6.10]

$$\begin{aligned} 0 \to H^0(G_S, M) \to P^0_S(M) \to H^2(G_S, M')^\vee \to H^1(G_S, M) \to P^1_S(M) \\ \to H^1(G_S, M')^\vee \to H^2(G_S, M) \to P^2_S(M) \to H^0(G_S, M')^\vee \to 0, \end{aligned}$$

where $P^i_S(M) = \prod'_{v \in S} H^i(E_v, M)$ is the restricted product of local cohomology groups, with Tate-modified cohomology at the infinite places. Since S is finite and all infinite places of E are complex, the Tate cohomology of $\mathrm{Gal}(\mathbb{C}/\mathbb{C}) = 1$ vanishes in every degree and $P^i_S(M) = \bigoplus_{v \in S_\ell} H^i(E_v, M)$. By local Tate duality [82, Theorem 7.2.6], $H^2(E_v, M) \cong H^0(E_v, M')^\vee$, naturally in M, and the map $P^2_S(M) \to H^0(G_S, M')^\vee$ is the sum of the duals of the restriction maps $H^0(G_S, M') \to H^0(E_v, M')$. All groups in the sequence are finite: the global ones by [82, Ch. VIII, §3], since $S \supseteq S_\ell \cup S_\infty$ and M is a finite ℓ-primary module, the local ones by [82, Ch. VII, §1].

Now let k vary. The modules $M_{k,1+e}$ form an inverse system under reduction, the modules $A_{-e}[\ell^k]$ a direct system under inclusion, and the two systems are Cartier dual to one another. By Lemma 29(a) the inverse limit over k of the last four terms of the nine-term sequence is exact; its terms are, by definition, $H^2(G_S, T_{1+e})$ and the inverse limits of the finite groups $H^0(E_v, A_{-e}[\ell^k])^\vee$ and $H^0(G_S, A_{-e}[\ell^k])^\vee$, which by Lemma 29(b) are $H^0(E_v, A_{-e})^\vee$ and $H^0(E, A_{-e})^\vee$, because H^0 commutes with direct limits of discrete modules. The inverse limit of $\mathrm{Sha}^2(G_S, M_{k,1+e})$ is the kernel of $H^2(G_S, T_{1+e}) \to \bigoplus_v H^2(E_v, T_{1+e})$ by Lemma 29(a), that is, $\mathrm{Sha}^2(G_S, T_{1+e})$. This gives (121), with $H^2(G_S, T_{1+e})$ rewritten by (120).

On the other side, cohomology of the profinite group G_S with discrete coefficients commutes with direct limits [82, Proposition 1.5.1], as does the local cohomology, and the direct sum over the finite

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set S_ℓ commutes with direct limits. Lemma 29(b) then gives $\mathrm{Sha}^1(G_S, A_{-e}) = \varinjlim_k \mathrm{Sha}^1(G_S, A_{-e}[\ell^k])$. Dualising and using the naturality of the Poitou–Tate pairing in M,

$$\mathrm{Sha}^1(G_S, A_{-e})^\vee = \varprojlim_k \mathrm{Sha}^1(G_S, A_{-e}[\ell^k])^\vee = \varprojlim_k \mathrm{Sha}^2(G_S, M_{k,1+e}) = \mathrm{Sha}^2(G_S, T_{1+e}),$$

which is the asserted duality.

Finiteness of $H^2(\mathcal{O}_E[1/\ell], \mathbb{Z}_\ell(1+e))$ for $1+e \geq 2$ is Soulé's theorem [88]; see [112, Proposition 42]. The groups $H^0(E_v, A_{-e})$ and $H^0(E, A_{-e})$ are finite because they are computed in Lemma 31 below (they are cyclic of order a finite power of ℓ). Hence every term of (121) is finite, Sha^2 is finite, and so is its dual $\mathrm{Sha}^1_e(E, \ell)$. Taking lengths along the exact sequence (121), the length of Sha^2 is the length of H^2 minus the length of the image of H^2 in $\bigoplus_v H^0(E_v, A_{-e})^\vee$, and that image is the kernel of the surjection onto $H^0(E, A_{-e})^\vee$, of length $\sum_v \operatorname{len} H^0(E_v, A_{-e}) - \operatorname{len} H^0(E, A_{-e})$. Since Pontryagin duality preserves lengths of finite groups, (122) follows. The non-negativity of $\delta_e(\ell)$ is contained in the exactness (the map $\bigoplus_v H^0(E_v, A_{-e})^\vee \to H^0(E, A_{-e})^\vee$ is surjective); directly, it holds because $H^0(E, A_{-e}) \to H^0(E_v, A_{-e})$ is injective for any single $v \in S_\ell$, both groups being subgroups of $\mathbb{Q}_\ell/\mathbb{Z}_\ell$. □

We now compute the local terms. For a field K of characteristic 0 with absolute Galois group G_K write $U_K = \chi_{\mathrm{cyc}}(G_K) \subseteq \mathbb{Z}_\ell^\times$ for the image of the cyclotomic character, a closed subgroup.

Lemma 30. *Let ℓ be a prime, $u \in \mathbb{Z}_\ell$ with $v_\ell(u-1) = s$, where $s \geq 1$, and $s \geq 2$ if $\ell = 2$. Then $v_\ell(u^j - 1) = s + v_\ell(j)$ for every integer $j \geq 1$.*

Proof. Write $u = 1 + \ell^s y$ with $y \in \mathbb{Z}_\ell^\times$. Then $u^\ell - 1 = \ell^{s+1} y + \sum_{2 \leq i \leq \ell} \binom{\ell}{i} \ell^{is} y^i$. For $2 \leq i < \ell$ the binomial coefficient is divisible by ℓ, so the ith term has valuation at least $1 + is \geq s+2$; the term $i = \ell$ has valuation ℓs, and $\ell s \geq s+2$ because $(\ell-1)s \geq 2$ under our hypotheses. Hence $v_\ell(u^\ell - 1) = s+1$, and by induction $v_\ell(u^{\ell^r} - 1) = s + r$ for all $r \geq 0$ (the hypotheses are preserved since $s + r \geq s$). If $j = \ell^r j'$ with $\ell \nmid j'$, put $w = u^{\ell^r}$; then $u^j - 1 = (w-1)(1 + w + \cdots + w^{j'-1})$ and the second factor is congruent to j' modulo ℓ, hence a unit. So $v_\ell(u^j - 1) = v_\ell(w-1) = s + r$. □

Lemma 31 (Local roots of unity)**.** *Let ℓ be a prime, $K/\mathbb{Q}_\ell$ a finite extension and $j \neq 0$ an integer. Then $H^0(K, \mathbb{Q}_\ell/\mathbb{Z}_\ell(j))$ is a cyclic group of order ℓ^a with*

$$a = a_\ell(K, j) := \min_{u \in U_K} v_\ell(u^j - 1) < \infty,$$

and $a_\ell(K, j) = a_\ell(K, -j)$. Moreover:

(i) *If $K \subseteq \mathbb{Q}_\ell(\zeta_N)$ and $\ell \nmid N$, then $U_K = \mathbb{Z}_\ell^\times$; for ℓ odd one then has $a_\ell(K, j) = 1 + v_\ell(j)$ if $(\ell-1) \mid j$ and $a_\ell(K, j) = 0$ otherwise.*

(ii) *If $K \subseteq \mathbb{Q}_\ell(\zeta_N)$ and $\ell^r \parallel N$ with $r \geq 1$ ($r \geq 2$ if $\ell = 2$), then $U_K \supseteq 1 + \ell^r \mathbb{Z}_\ell$ and $a_\ell(K, j) \leq r + v_\ell(j)$.*

The same statements hold for the global field E in place of K, with $U_E = \chi_{\mathrm{cyc}}(G_E)$, and $a_\ell(E, j) \leq a_\ell(E_v, j)$ for every $v \in S_\ell$.

Proof. An element $x \in \mathbb{Q}_\ell/\mathbb{Z}_\ell$ is fixed by $\sigma \in G_K$ acting through $\chi_{\mathrm{cyc}}(\sigma)^j$ if and only if $(\chi_{\mathrm{cyc}}(\sigma)^j - 1)x = 0$. Hence $H^0(K, \mathbb{Q}_\ell/\mathbb{Z}_\ell(j)) = \{x : (u^j - 1)x = 0 \ \forall u \in U_K\} = \ell^{-a}\mathbb{Z}_\ell/\mathbb{Z}_\ell$ with $a = \min_u v_\ell(u^j - 1)$, provided this minimum is finite. The group U_K is the Galois group of $K(\mu_{\ell^\infty})/K$, which is an infinite extension since $[K : \mathbb{Q}_\ell] < \infty$ and $[\mathbb{Q}_\ell(\mu_{\ell^k}) : \mathbb{Q}_\ell] \to \infty$; so U_K is an infinite closed subgroup of $\mathbb{Z}_\ell^\times$ and contains some u with $u^j \neq 1$, whence $a < \infty$. Since $v_\ell(u^{-j} - 1) = v_\ell(u^j - 1)$ for a unit u, we get $a_\ell(K, j) = a_\ell(K, -j)$.

(i) Suppose $\ell \nmid N$. Then $\mathbb{Q}_\ell(\zeta_N)/\mathbb{Q}_\ell$ is unramified, while $\mathbb{Q}_\ell(\mu_{\ell^\infty})/\mathbb{Q}_\ell$ is totally ramified; hence $K \cap \mathbb{Q}_\ell(\mu_{\ell^\infty}) = \mathbb{Q}_\ell$, and restriction $\mathrm{Gal}(K(\mu_{\ell^\infty})/K) \to \mathrm{Gal}(\mathbb{Q}_\ell(\mu_{\ell^\infty})/\mathbb{Q}_\ell)$ is an isomorphism, that is,

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$U_K = \mathbb{Z}_\ell^\times$. Let now ℓ be odd. Choose $u_0 \in \mathbb{Z}_\ell^\times$ whose image generates $\mathbb{F}_\ell^\times$. If $(\ell-1) \nmid j$ then $u_0^j \not\equiv 1 \pmod{\ell}$, so $a = 0$. If $(\ell-1) \mid j$ then for every $u \in \mathbb{Z}_\ell^\times$ we have $u^{\ell-1} \equiv 1 \pmod{\ell}$ by Fermat, and Lemma 30 applied to $u^{\ell-1}$ (if $u^{\ell-1} \neq 1$) and the exponent $j/(\ell-1)$, whose ℓ-adic valuation is $v_\ell(j)$ because $\ell \nmid \ell - 1$, gives $v_\ell(u^j - 1) = v_\ell(u^{\ell-1} - 1) + v_\ell(j) \geq 1 + v_\ell(j)$, with equality if $v_\ell(u^{\ell-1} - 1) = 1$. For $u = 1 + \ell$, Lemma 30 gives $v_\ell((1+\ell)^{\ell-1} - 1) = 1 + v_\ell(\ell - 1) = 1$. Hence $a = 1 + v_\ell(j)$.

(ii) Write $N = N'\ell^r$ with $\ell \nmid N'$. Then $\mathbb{Q}_\ell(\zeta_N) = \mathbb{Q}_\ell(\zeta_{N'})\mathbb{Q}_\ell(\mu_{\ell^r})$ with $\mathbb{Q}_\ell(\zeta_{N'})$ unramified. Let $L = \mathbb{Q}_\ell(\zeta_N) \cap \mathbb{Q}_\ell(\mu_{\ell^\infty})$. Then $L \supseteq \mathbb{Q}_\ell(\mu_{\ell^r})$, and $L/\mathbb{Q}_\ell(\mu_{\ell^r})$ is at the same time totally ramified (as a subextension of $\mathbb{Q}_\ell(\mu_{\ell^\infty})/\mathbb{Q}_\ell(\mu_{\ell^r})$) and unramified (as a subextension of $\mathbb{Q}_\ell(\zeta_N)/\mathbb{Q}_\ell(\mu_{\ell^r})$, which is generated by $\zeta_{N'}$); so $L = \mathbb{Q}_\ell(\mu_{\ell^r})$. Since $K \subseteq \mathbb{Q}_\ell(\zeta_N)$, the group U_K contains $U_{\mathbb{Q}_\ell(\zeta_N)} = \chi_{\mathrm{cyc}}(\mathrm{Gal}(\overline{\mathbb{Q}}_\ell/\mathbb{Q}_\ell(\zeta_N))) = \mathrm{Gal}(\mathbb{Q}_\ell(\mu_{\ell^\infty})/L) = 1 + \ell^r\mathbb{Z}_\ell$. Therefore $a_\ell(K, j) \leq \min_{u \in 1+\ell^r\mathbb{Z}_\ell} v_\ell(u^j - 1) = r + v_\ell(j)$ by Lemma 30, the minimum being attained at $u = 1 + \ell^r$.

For the global field, the same computation applies with U_E, which is infinite because $E(\mu_{\ell^\infty})/E$ is infinite. Since $G_{E_v} \subseteq G_E$ (as a decomposition group), $U_{E_v} \subseteq U_E$ and so $a_\ell(E, j) \leq a_\ell(E_v, j)$. □

In the terminology of [112, Definition 12, Lemma 13], $\ell^{a_\ell(E,j)}$ is the ℓ-part $w_j^{(\ell)}(E)$ of the integer $w_j(E)$, the largest integer w such that $\mathrm{Gal}(E(\mu_w)/E)$ has exponent dividing j. We shall only need the upper bounds. To state them uniformly we recall an elementary form of Chebyshev's bound.

Lemma 32. *For every real $x \geq 1$, $\prod_{p \leq x} p < 4^x$, the product over primes.*

Proof. It suffices to prove $\prod_{p \leq n} p < 4^n$ for integers $n \geq 1$, since $\prod_{p \leq x} p = \prod_{p \leq \lfloor x \rfloor} p$. For $n = 1, 2$ the product is 1 and 2. Let $n \geq 3$ and assume the claim for all smaller integers. If n is even, n is not prime and $\prod_{p \leq n} p = \prod_{p \leq n-1} p < 4^{n-1}$. If $n = 2m + 1$ is odd, every prime p with $m + 2 \leq p \leq 2m + 1$ divides $\binom{2m+1}{m} = (2m+1)!/(m!(m+1)!)$, because p divides the numerator and not the denominator. Moreover $2\binom{2m+1}{m} = \binom{2m+1}{m} + \binom{2m+1}{m+1} \leq \sum_i \binom{2m+1}{i} = 2^{2m+1}$, so $\binom{2m+1}{m} \leq 4^m$. Hence $\prod_{p \leq n} p \leq \binom{2m+1}{m} \prod_{p \leq m+1} p < 4^m \cdot 4^{m+1} = 4^n$. □

Proposition 31 (Exact formula and the local defect)**.** *Let $e \geq 1$. For every prime ℓ,*

$$\operatorname{len} \mathrm{Sha}^1_e(E, \ell) = \operatorname{len} H^2(\mathcal{O}_E[1/\ell], \mathbb{Z}_\ell(1+e)) - \delta_e(\ell), \qquad 0 \leq \delta_e(\ell) \leq g_\ell\, \alpha_\ell(e),$$

where $g_\ell = \#S_\ell \leq D$ and

$$\alpha_\ell(e) = \begin{cases} 1 + v_\ell(e) & \text{if } \ell \nmid N \text{ and } (\ell-1) \mid e, \\ 0 & \text{if } \ell \nmid N \text{ and } (\ell-1) \nmid e, \\ v_\ell(N) + v_\ell(e) & \text{if } \ell \mid N. \end{cases}$$

The total defect $\Delta_e := \sum_\ell \log \ell \cdot \delta_e(\ell)$ satisfies

$$0 \leq \Delta_e \leq D\, \beta(e), \qquad \beta(e) := (e+1)\log 4 + 2\log e + \log N. \tag{124}$$

In particular

$$\sum_\ell \log \ell \cdot \operatorname{len} \mathrm{Sha}^1_e(E, \ell) \leq \sum_\ell \log \ell \cdot \operatorname{len} H^2(\mathcal{O}_E[1/\ell], \mathbb{Z}_\ell(1+e)) \\ = \log \#\Big(\bigoplus_\ell H^2(\mathcal{O}_E[1/\ell], \mathbb{Z}_\ell(1+e)) \Big).$$

Proof. The formula for the length and $\delta_e(\ell) \geq 0$ are (122). By Lemma 31, $\operatorname{len} H^0(E_v, \mathbb{Q}_\ell/\mathbb{Z}_\ell(-e)) = a_\ell(E_v, e)$, and $E_v \subseteq \mathbb{Q}_\ell(\zeta_N)$ because $E \subseteq \mathbb{Q}(\zeta_N)$. If $\ell \nmid N$ then ℓ is odd (as $2^{m+2} \mid N$) and part (i) of the lemma gives $a_\ell(E_v, e) = \alpha_\ell(e)$. If $\ell \mid N$ then $r = v_\ell(N) \geq 1$, and $r \geq m + 2 \geq 2$ when $\ell = 2$, so part

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(ii) gives $a_\ell(E_v,e)\le\alpha_\ell(e)$. Dropping the non-negative term $\operatorname{len}H^0(E,\mathbb{Q}_\ell/\mathbb{Z}_\ell(-e))$ from $\delta_e(\ell)$ yields $\delta_e(\ell)\le g_\ell\alpha_\ell(e)$.

For the total defect, $g_\ell\le D$ for all ℓ, so $\Delta_e\le D\sum_\ell\alpha_\ell(e)\log\ell$. The primes $\ell\nmid N$ with $\alpha_\ell(e)\neq0$ satisfy $\ell-1\le e$, so by Lemma 32

$$\sum_{\ell\nmid N}\alpha_\ell(e)\log\ell\le\sum_{\ell\le e+1}\log\ell+\sum_\ell v_\ell(e)\log\ell<(e+1)\log4+\log e,$$

while $\sum_{\ell\mid N}\alpha_\ell(e)\log\ell\le\sum_{\ell\mid N}v_\ell(N)\log\ell+\sum_\ell v_\ell(e)\log\ell=\log N+\log e$. Adding gives (124). The last display follows from $\delta_e(\ell)\ge0$; the sum over ℓ is finite because $H^2(\mathcal{O}_E[1/\ell],\mathbb{Z}_\ell(1+e))=0$ for all but finitely many ℓ, as follows from Theorem 59 below together with the finiteness of $K_{2e}(\mathcal{O}_E)$. □

The proposition is slightly sharper than the inequality in its last display: the defect Δ_e is an explicit, non-negative quantity, linear in e and in $\log N$, which is subtracted rather than added. It can be summed over $e<t$:

$$\sum_{1\le e<t}\Delta_e\le D\Big(\frac{t(t+1)}{2}\log4+2(t-1)\log t+(t-1)\log N\Big)\le2D\,t^2\log(tN)\tag{125}$$

for $t\ge2$, using $t(t+1)/2\le\frac34t^2$ and $\log4\le\log N$, which hold since $t\ge2$ and $N\ge2^{m+2}qd\ge84$.

22.3. The Quillen–Lichtenbaum Isomorphism

The second step replaces the étale cohomology groups by K-groups. For a number field F with ring of integers $\mathcal{O}_F$, the groups $K_n(\mathcal{O}_F)$ are finitely generated (Quillen [106]), of rank computed by Borel [95]: for $n\ge2$ the rank of $K_n(\mathcal{O}_F)$ is 0 if n is even, r_1+r_2 if $n\equiv1\pmod4$, and r_2 if $n\equiv3\pmod4$. In particular $K_{2e}(\mathcal{O}_F)$ is finite for $e\ge1$, and $K_{2e}(\mathcal{O}_F)\otimes\mathbb{Z}_\ell$ is its ℓ-primary part. The localisation sequence of Quillen, together with his computation $K_{2i}(\mathbb{F}_q)=0$, $K_{2i-1}(\mathbb{F}_q)\cong\mathbb{Z}/(q^i-1)$ for finite fields [105], shows that the map $K_{2e}(\mathcal{O}_F)\otimes\mathbb{Z}_\ell\to K_{2e}(\mathcal{O}_F[1/\ell])\otimes\mathbb{Z}_\ell$ is an isomorphism for $e\ge1$: its kernel is the image of $\bigoplus_{v\mid\ell}K_{2e}(k_v)\otimes\mathbb{Z}_\ell=0$, and its cokernel embeds in $\bigoplus_{v\mid\ell}K_{2e-1}(k_v)\otimes\mathbb{Z}_\ell=\bigoplus_{v\mid\ell}\mathbb{Z}/(q_v^e-1)\otimes\mathbb{Z}_\ell$, which vanishes because q_v^e-1 is prime to ℓ when q_v is a power of ℓ (see [112, §§5.4–5.7]). So nothing is lost by passing to $\mathcal{O}_F[1/\ell]$, where ℓ is invertible and the étale sheaves $\mu_{\ell^k}^{\otimes j}$ are locally constant.

Dwyer and Friedlander [97] constructed natural maps from $K_n(\mathcal{O}_F[1/\ell])\otimes\mathbb{Z}_\ell$ to étale K-theory, and hence étale Chern characters

$$\begin{aligned}K_{2i-2}(\mathcal{O}_F[1/\ell])\otimes\mathbb{Z}_\ell&\longrightarrow H^2_{\text{ét}}(\mathcal{O}_F[1/\ell],\mathbb{Z}_\ell(i)),\\K_{2i-1}(\mathcal{O}_F[1/\ell])\otimes\mathbb{Z}_\ell&\longrightarrow H^1_{\text{ét}}(\mathcal{O}_F[1/\ell],\mathbb{Z}_\ell(i)),\end{aligned}\tag{126}$$

for $i\ge2$, which Soulé [88] had shown to be surjective (this is also the source of the finiteness of H^2 used above). The Quillen–Lichtenbaum conjecture for rings of integers asserts that these maps are isomorphisms. It is a consequence of the Bloch–Kato conjecture on the norm residue homomorphism, now the norm residue isomorphism theorem of Rost and Voevodsky: Voevodsky proved it for $\ell=2$ in [109] and for all ℓ in [110], relying on Rost's work on norm varieties, and a gap in the literature was filled by Weibel [113]; the complete proof is in the book of Haesemeyer and Weibel [100]. The deduction of the Quillen–Lichtenbaum conjecture from the norm residue theorem, via the motivic spectral sequence, is also explained in [100] and in [112]. For $\ell=2$ the identification of the 2-adic K-theory of rings of integers is due to Rognes and Weibel [107], who deduced it from Voevodsky's theorem at the prime 2. We shall use the following precise form.

Theorem 59 (Quillen–Lichtenbaum for rings of integers)**.** *Let F be a number field, ℓ a prime and $i\ge2$.*

(a) *If ℓ is odd, the maps (126) are isomorphisms. In particular $K_{2i-2}(\mathcal{O}_F)\otimes\mathbb{Z}_\ell\cong H^2_{\text{ét}}(\mathcal{O}_F[1/\ell],\mathbb{Z}_\ell(i))$* [112, Theorem 70].

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(b) *If $\ell = 2$ and F is totally imaginary, then*

$$K_{2i-2}(\mathcal{O}_F) \otimes \mathbb{Z}_2 \cong H^2_{\text{et}}(\mathcal{O}_F[1/2], \mathbb{Z}_2(i))$$

[107, Theorem 0.4(a)], [112, Theorem 73].

The isomorphisms are natural in F; in particular, when $F/\mathbb{Q}$ is Galois they are equivariant for $\mathrm{Gal}(F/\mathbb{Q})$ acting on both sides by functoriality.

The host field E is a CM field, hence totally imaginary, so both parts apply to it, and we obtain the following reformulation of (118).

Corollary 26. *For every $e \geq 1$,*

$$\sum_{\ell} \log \ell \cdot \operatorname{len} \mathrm{Sha}^1_e(E, \ell) = \log \#K_{2e}(\mathcal{O}_E) - \Delta_e, \qquad 0 \leq \Delta_e \leq D\,\beta(e), \tag{127}$$

with Δ_e and $\beta(e)$ as in Proposition 31. Consequently, for $t \geq 2$,

$$\log \prod_{1 \leq e < t} \#K_{2e}(\mathcal{O}_E) \;-\; 2Dt^2 \log(tN) \leq \sum_{\ell} \log \ell \sum_{1 \leq e < t} \operatorname{len} \mathrm{Sha}^1_e(E, \ell) \;\leq\; \log \prod_{1 \leq e < t} \#K_{2e}(\mathcal{O}_E).$$

Proof. The finite group $K_{2e}(\mathcal{O}_E)$ is the direct sum of its ℓ-primary parts $K_{2e}(\mathcal{O}_E) \otimes \mathbb{Z}_\ell$, so $\log \#K_{2e}(\mathcal{O}_E) = \sum_\ell \log \ell \cdot \operatorname{len}(K_{2e}(\mathcal{O}_E) \otimes \mathbb{Z}_\ell)$. By Theorem 59 with $i = e + 1 \geq 2$, applicable at every prime because E is totally imaginary, the ℓ-primary part has the same length as $H^2(\mathcal{O}_E[1/\ell], \mathbb{Z}_\ell(1+e))$. Now sum (122) over ℓ and use (124) and (125). □

Thus the problem of bounding (118) is, up to the explicit defect, the problem of bounding the orders of the even K-groups of $\mathcal{O}_E$ in the range $2 \leq 2e < 2t$. The remainder of the section bounds the part of $\log \#K_{2e}(\mathcal{O}_E)$ that is governed by Bernoulli numbers.

22.4. Character Decomposition and the Two Halves

Fix an odd prime ℓ and an integer $n \geq 2$ (we shall take $n = 1 + e$), and put

$$M = M_\ell(n) := H^2(\mathcal{O}_E[1/\ell], \mathbb{Z}_\ell(n)), \qquad M^{\pm} := \{x \in M : \; cx = \pm x\}.$$

Here $G = \mathrm{Gal}(E/\mathbb{Q})$ acts on M by functoriality, through its action on $\operatorname{Spec} \mathcal{O}_E[1/\ell]$, and $c \in G$ is complex conjugation. Because ℓ is odd, $\frac{1 \pm c}{2} \in \mathbb{Z}_\ell[G]$ are complementary idempotents and $M = M^+ \oplus M^-$. By Theorem 59, $M \cong K_{2n-2}(\mathcal{O}_E) \otimes \mathbb{Z}_\ell$ as $\mathbb{Z}_\ell[G]$-modules, so the same decomposition holds for the ℓ-primary part of $K_{2n-2}(\mathcal{O}_E)$.

Definition 10. *Let ℓ be odd and $n \geq 2$. The* Bernoulli half *of $M_\ell(n)$ is the eigenspace $M_\ell(n)^{(-1)^n}$ on which c acts by $(-1)^n$; the* regulator half *is the eigenspace $M_\ell(n)^{(-1)^{n-1}}$. Writing $n = 1 + e$, the Bernoulli half of $K_{2e}(\mathcal{O}_E) \otimes \mathbb{Z}_\ell$ is the eigenspace of c with eigenvalue $(-1)^{e+1}$, and the regulator half that with eigenvalue $(-1)^e$. Let $\mathcal{L}$ denote the set of odd primes not dividing $D = [E : \mathbb{Q}]$. For $e \geq 1$ put*

$$\mathfrak{B}_e := \sum_{\ell \in \mathcal{L}} \log \ell \cdot \operatorname{len}\big(K_{2e}(\mathcal{O}_E) \otimes \mathbb{Z}_\ell\big)^{(-1)^{e+1}},$$

$$\mathfrak{R}_e := \sum_{\ell \in \mathcal{L}} \log \ell \cdot \operatorname{len}\big(K_{2e}(\mathcal{O}_E) \otimes \mathbb{Z}_\ell\big)^{(-1)^{e}},$$

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and let $\mathfrak{T}_e$ *be the sum of* $\log \ell \cdot \operatorname{len}(K_{2e}(\mathcal{O}_E) \otimes \mathbb{Z}_\ell)$ *over the primes* $\ell \notin \mathcal{L}$*, that is, over* $\ell = 2$ *and the odd primes dividing* $\operatorname{ord}(\psi)$*. Then, by Theorem 59,*

$$\log \#K_{2e}(\mathcal{O}_E) = \mathfrak{B}_e + \mathfrak{R}_e + \mathfrak{T}_e. \tag{128}$$

The names are justified by Theorems 60 and 61 below. For the trivial character and $E = \mathbb{Q}$ the Bernoulli half is $H^2(\mathbb{Z}[1/\ell], \mathbb{Z}_\ell(n))$ for even n, whose order is the ℓ-part of the numerator of $\zeta(1-n) = -B_n/n$ [112, Example 44], and the regulator half is $H^2(\mathbb{Z}[1/\ell], \mathbb{Z}_\ell(n))$ for odd n, which is conjectured to vanish (this is equivalent to Vandiver's conjecture for all ℓ [112, §5.1], [102]).

To refine the parity decomposition into characters we need the idempotents of $\mathbb{Z}_\ell[G]$, which exist integrally exactly when $\ell \nmid D$. So let $\ell \in \mathcal{L}$. Let $\mathcal{O} = \mathbb{Z}_\ell[\mu_D]$ be the ring of integers of $\mathbb{Q}_\ell(\mu_D)$, an unramified extension of $\mathbb{Q}_\ell$ of some degree d since $\ell \nmid D$; its maximal ideal is $\ell\mathcal{O}$ and $\operatorname{len}_{\mathcal{O}}(\mathcal{O}/\ell^k\mathcal{O}) = k$. Every character $\chi \in \widehat{G}$ takes values in $\mu_D \subset \mathcal{O}^\times$, once an embedding $\iota\colon \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_\ell$ is fixed; we fix ι once and for all and suppress it from the notation when no confusion can arise. The elements

$$e_\chi = \frac{1}{D} \sum_{\sigma \in G} \chi(\sigma)^{-1} \sigma \in \mathcal{O}[G]$$

are orthogonal idempotents with $\sum_\chi e_\chi = 1$ and $\sigma e_\chi = \chi(\sigma) e_\chi$. For a finite $\mathbb{Z}_\ell[G]$-module A write $A_{\mathcal{O}} = A \otimes_{\mathbb{Z}_\ell} \mathcal{O}$; then $A_{\mathcal{O}} = \bigoplus_\chi e_\chi A_{\mathcal{O}}$, and $e_\chi A_{\mathcal{O}}$ is the χ-isotypic part $\{x \in A_{\mathcal{O}} : \sigma x = \chi(\sigma) x \ \forall \sigma\}$.

Lemma 33. *Let* $\ell \in \mathcal{L}$ *and let* A *be a finite* $\mathbb{Z}_\ell[G]$*-module. Then* $\operatorname{len}_{\mathcal{O}}(A_{\mathcal{O}}) = \operatorname{len}_{\mathbb{Z}_\ell}(A)$*, and for* $\varepsilon \in \{\pm 1\}$

$$\operatorname{len}_{\mathbb{Z}_\ell}(A^\varepsilon) = \sum_{\chi(-1)=\varepsilon} \operatorname{len}_{\mathcal{O}}(e_\chi A_{\mathcal{O}}).$$

Proof. Both sides of the first identity are additive in A, and for $A = \mathbb{Z}/\ell^k$ we have $A_{\mathcal{O}} = \mathcal{O}/\ell^k\mathcal{O}$, of $\mathcal{O}$-length k. For the second, $e_\chi c = \chi(c) e_\chi$ shows that e_χ kills $(A^\varepsilon)_{\mathcal{O}}$ unless $\chi(c) = \chi(-1) = \varepsilon$, so $(A^\varepsilon)_{\mathcal{O}} = \bigoplus_{\chi(-1)=\varepsilon} e_\chi A_{\mathcal{O}}$, and we apply the first identity to A^ε. □

We also record the structure of the torsion in H^1, which enters the statements below.

Lemma 34. *For every prime* ℓ *and* $n \geq 1$*,* $H^1(\mathcal{O}_E[1/\ell], \mathbb{Z}_\ell(n))_{\mathrm{tors}} \cong H^0(E, \mathbb{Q}_\ell/\mathbb{Z}_\ell(n))$ *as* $\mathbb{Z}_\ell[G]$*-modules; this is a cyclic group of order* $w_n^{(\ell)}(E) = \ell^{a_\ell(E,n)}$*, on which* G *acts through a character* θ_n *with* $\theta_n(c) = (-1)^n$*. Moreover* $\sum_\ell \log \ell \cdot \operatorname{len} H^0(E, \mathbb{Q}_\ell/\mathbb{Z}_\ell(n)) \leq \beta(n)$*, with* β *as in* (124)*.*

Proof. Since $n \neq 0$, the module $\mathbb{Z}_\ell(n)$ has no non-zero G_S-invariants. The exact sequence $0 \to \mathbb{Z}_\ell(n) \xrightarrow{\ell^k} \mathbb{Z}_\ell(n) \to \mathbb{Z}/\ell^k(n) \to 0$ of compact G_S-modules therefore gives an exact sequence

$$0 \to H^0(E, \mathbb{Z}/\ell^k(n)) \to H^1(G_S, \mathbb{Z}_\ell(n)) \xrightarrow{\ell^k} H^1(G_S, \mathbb{Z}_\ell(n))$$

(the long exact sequence for compact coefficients, [82, Ch. II, §7]; it can also be obtained from the finite-level sequences $0 \to \mathbb{Z}/\ell^k(n) \to \mathbb{Z}/\ell^{k+j}(n) \to \mathbb{Z}/\ell^j(n) \to 0$ by passing to the limit over j with Lemma 29). Hence the ℓ^k-torsion of $H^1(G_S, \mathbb{Z}_\ell(n))$ is isomorphic to $H^0(E, \mathbb{Z}/\ell^k(n))$, compatibly in k and G-equivariantly, and taking the union over k gives the first claim. The description of $H^0(E, \mathbb{Q}_\ell/\mathbb{Z}_\ell(n))$ as a cyclic group of order $\ell^{a_\ell(E,n)}$ is Lemma 31. The group G acts on it through the restriction of χ_{cyc}^n to the group $\mathrm{Gal}(E(\mu_{\ell^\infty})/\mathbb{Q})$ followed by reduction, and $\chi_{\mathrm{cyc}}(c) = -1$, so $\theta_n(c) = (-1)^n$. Finally $a_\ell(E,n) \leq a_\ell(E_v, n) \leq \alpha_\ell(n)$ by Lemma 31 and the proof of Proposition 31, and $\sum_\ell \alpha_\ell(n) \log \ell \leq \beta(n)$ was shown there. □

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We can now state the two theorems that give the names to the two halves. For a primitive Dirichlet character χ of conductor f and $n \geq 1$, the generalised Bernoulli number is

$$B_{n,\chi} = f^{n-1} \sum_{a=1}^{f} \chi(a) B_n\Big(\frac{a}{f}\Big), \qquad B_n(x) = \sum_{k=0}^{n} \binom{n}{k} B_k x^{n-k}, \tag{129}$$

where B_k are the ordinary Bernoulli numbers ($B_0 = 1$, $B_1 = -\frac{1}{2}$, $B_2 = \frac{1}{6}$, $B_k = 0$ for odd $k \geq 3$) and $B_n(x)$ is the Bernoulli polynomial; this is [111, Proposition 4.1], and for $\chi = 1$ (so $f = 1$) and $n \geq 2$ it gives $B_{n,1} = B_n(1) = B_n$. The basic fact, due to Hurwitz and Hecke in this form, is

$$L(1-n,\chi) = -\frac{B_{n,\chi}}{n} \qquad (n \geq 1) \tag{130}$$

[111, Theorem 4.2]. In particular $L(1-n,\chi)$ lies in the field $\mathbb{Q}(\chi)$ generated by the values of χ, and for every automorphism σ of $\overline{\mathbb{Q}}$ one has $L(1-n,\sigma\circ\chi) = \sigma(L(1-n,\chi))$, as is clear from (129). For $\chi \in \widehat{G}$ we write $\bar{\chi} = \chi^{-1}$ for the complex conjugate character.

Theorem 60 (The Bernoulli half). *Let $\ell \in \mathcal{L}$, $n \geq 2$ and $\chi \in \widehat{G}$ with $\chi(-1) = (-1)^n$. Then $L(1-n,\bar{\chi}) \neq 0$ and*

$$\operatorname{len}_{\mathcal{O}} e_\chi H^2(\mathcal{O}_E[1/\ell], \mathbb{Z}_\ell(n))_{\mathcal{O}} \;=\; \operatorname{len}_{\mathcal{O}} e_\chi H^0(E, \mathbb{Q}_\ell/\mathbb{Z}_\ell(n))_{\mathcal{O}} \;+\; v_\ell\big(\iota L(1-n,\bar{\chi})\big). \tag{131}$$

Here $H^0(E,\mathbb{Q}_\ell/\mathbb{Z}_\ell(n)) = H^1(\mathcal{O}_E[1/\ell],\mathbb{Z}_\ell(n))_{\mathrm{tors}}$ by Lemma 34, and $v_\ell(\iota L(1-n,\bar{\chi}))$ is an integer, possibly negative, which is non-negative unless $\chi = \theta_n$.

This is the χ-component of the Iwasawa main conjecture for abelian fields, proved by Mazur and Wiles [81] (for the cyclotomic $\mathbb{Z}_\ell$-extension of abelian fields; Wiles [92] treats totally real fields), descended to the finite level. The descent from the Iwasawa-theoretic statement about characteristic ideals to the statement (131) about the finite groups $H^2(\mathcal{O}_E[1/\ell],\mathbb{Z}_\ell(n))$, with the torsion of H^1 accounting for the denominators of the Bernoulli numbers, was carried out by Kolster, Nguyen Quang Do and Fleckinger [79] under the hypothesis $\ell \nmid [E:\mathbb{Q}]$ that we have imposed through $\ell \in \mathcal{L}$; in this form it appears as the "cohomological Lichtenbaum conjecture" for abelian fields in Kolster's survey [101]. The same statement, including the primes $\ell \mid [E:\mathbb{Q}]$ (for which the idempotents e_χ are not available and the formulation has to be changed), is the Bloch–Kato Tamagawa number conjecture [71] for the Dirichlet motive $\mathbb{Q}(\chi)(n)$, proved up to powers of 2 by Huber and Kings [75, Theorem 1.3.1] and, in the equivariant form over $\mathbb{Z}_\ell[G]$, by Burns and Greither [96]; see also Benois and Nguyen Quang Do [94]. The Euler factor $(1-\bar{\chi}(\ell)\ell^{n-1})$ that relates $L(1-n,\bar{\chi})$ to the value of the ℓ-adic L-function is an ℓ-adic unit for $n \geq 2$ (it is congruent to 1 modulo ℓ), so it does not appear in (131); see §22.6 for this and for the imprimitive Euler factors. Whether the right-hand side of (131) involves χ or $\bar{\chi}$ depends on the normalisation of the Galois action on M (left action through automorphisms of $\operatorname{Spec}\mathcal{O}_E[1/\ell]$ versus the contragredient action); the normalisation above is the one under which the trivial character and $E = \mathbb{Q}$ reproduce [112, Example 44], and under which the Herbrand–Ribet pattern $e_{\omega^{1-n}}A \neq 0 \Leftrightarrow \ell \mid B_n$ for the ℓ-class group A of $\mathbb{Q}(\mu_\ell)$ is recovered. The bound proved below is insensitive to this choice, because $\chi \mapsto \bar{\chi}$ permutes the characters of a given parity. The non-vanishing $L(1-n,\bar{\chi}) \neq 0$ is proved independently in Lemma 35 below; the last sentence of the theorem follows from Lemma 34, since the left-hand side of (131) is non-negative and the torsion term vanishes unless $\chi = \theta_n$.

Theorem 61 (The regulator half). *Let $\ell \in \mathcal{L}$, $n \geq 2$ and $\chi \in \widehat{G}$ with $\chi(-1) = (-1)^{n-1}$. Then*

$$e_\chi H^0(E,\mathbb{Q}_\ell/\mathbb{Z}_\ell(n))_{\mathcal{O}} = 0, \qquad e_\chi H^1(\mathcal{O}_E[1/\ell],\mathbb{Z}_\ell(n))_{\mathcal{O}} \cong \mathcal{O},$$

and

$$\#\, e_\chi H^2(\mathcal{O}_E[1/\ell],\mathbb{Z}_\ell(n))_{\mathcal{O}} = \big[\, e_\chi H^1(\mathcal{O}_E[1/\ell],\mathbb{Z}_\ell(n))_{\mathcal{O}} \;:\; \mathcal{O}\cdot e_\chi c_n(E)\,\big], \tag{132}$$

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where $c_n(E) \in H^1(\mathcal{O}_E[1/\ell], \mathbb{Z}_\ell(n))$ is Soulé's cyclotomic element, the image under the twisted Kummer map of the norm-compatible system of cyclotomic units $(\zeta_{N\ell^k} - 1)_k$ of the tower $E(\mu_{\ell^\infty})$ [88,108].

The vanishing of the torsion term is Lemma 34: the character θ_n has parity $(-1)^n$, not $(-1)^{n-1}$. That $e_\chi H^1_{\mathcal{O}}$ is free of rank one follows from Borel's rank computation and Theorem 59 (or Soulé's theorem that $H^1(\mathcal{O}_E[1/\ell], \mathbb{Q}_\ell(n))$ has dimension $r_2 = D/2$ for $n \geq 2$ and totally imaginary E), together with the fact that G acts on $H^1 \otimes \mathbb{Q}_\ell$ through the characters of parity $(-1)^{n-1}$, each with multiplicity one; this is the statement that the regulator half has "rank one per character". The index formula (132) is the other half of the Iwasawa main conjecture (the formulation through cyclotomic units rather than through ℓ-adic L-functions), again descended to the finite level; it is proved in [79] for $\ell \nmid [E : \mathbb{Q}]$ and in [75,96] in general. We stress that Theorem 61 is stated here only to identify the regulator half; it is not used in the bound, and the index on the right of (132) is not estimated in this section.

Combining the two theorems with Theorem 59 and Lemma 34 gives, for $\ell \in \mathcal{L}$, the two formulae

$$\operatorname{len}\big(K_{2n-2}(\mathcal{O}_E) \otimes \mathbb{Z}_\ell\big)^{(-1)^n} = \operatorname{len} H^0(E, \mathbb{Q}_\ell/\mathbb{Z}_\ell(n)) + \sum_{\chi(-1)=(-1)^n} v_\ell\big(\iota L(1-n,\chi)\big), \tag{133}$$

$$\operatorname{len}\big(K_{2n-2}(\mathcal{O}_E) \otimes \mathbb{Z}_\ell\big)^{(-1)^{n-1}} = \sum_{\chi(-1)=(-1)^{n-1}} \operatorname{len}_{\mathcal{O}}\Big(\frac{e_\chi H^1_{\mathcal{O}}}{\mathcal{O}\, e_\chi c_n(E)}\Big), \tag{134}$$

where $H^1 = H^1(\mathcal{O}_E[1/\ell], \mathbb{Z}_\ell(n))$ in (134). In (133) we have summed (131) over the characters of parity $(-1)^n$ with Lemma 33, used that the χ-component of $H^0(E, \mathbb{Q}_\ell/\mathbb{Z}_\ell(n))_{\mathcal{O}}$ vanishes for all but the single character θ_n of parity $(-1)^n$, and replaced $\bar\chi$ by χ in the sum. The formula (133) is what will be estimated.

22.5. The Bernoulli Bound

We now estimate the right-hand side of (133), summed over $\ell \in \mathcal{L}$ with weights $\log \ell$. Two inputs are needed: an archimedean bound for $|L(1-n,\chi)|$, which comes from the functional equation, and a control of the denominators of $L(1-n,\chi)$, which comes from the von Staudt–Clausen theorem. Both are proved in full.

Lemma 35 (Functional equation at negative integers)**.** *Let χ be a primitive Dirichlet character of conductor $f \geq 1$ (with $\chi = 1$, $f = 1$ allowed, in which case $L(s,\chi) = \zeta(s)$), let $\delta \in \{0,1\}$ be defined by $\chi(-1) = (-1)^\delta$, and let $n \geq 2$ be an integer with $n \equiv \delta \pmod 2$, that is, $\chi(-1) = (-1)^n$. Then*

$$|L(1-n,\chi)| = \frac{2\,(n-1)!\,f^{n-1/2}}{(2\pi)^n}\,|L(n,\bar\chi)|, \tag{135}$$

and $0 < |L(n,\bar\chi)| \leq \zeta(n) \leq \zeta(2) = \pi^2/6$. In particular $L(1-n,\chi) \neq 0$, and

$$|B_{n,\chi}| = n\,|L(1-n,\chi)| = \frac{2\,n!\,f^{n-1/2}}{(2\pi)^n}\,|L(n,\bar\chi)| \leq \frac{2\,n!\,f^{n-1/2}}{(2\pi)^n}\,\zeta(n).$$

Proof. Put $\xi(s,\chi) = (f/\pi)^{(s+\delta)/2}\Gamma\big(\frac{s+\delta}{2}\big)L(s,\chi)$. For $f > 1$ the functional equation of the primitive character χ reads

$$\xi(1-s,\bar\chi) = \frac{i^\delta f^{1/2}}{\tau(\chi)}\,\xi(s,\chi), \qquad \tau(\chi) = \sum_{a=1}^{f} \chi(a) e^{2\pi i a/f}, \qquad |\tau(\chi)| = f^{1/2},$$

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Ref. [32] [Ch. 9]; the constant has absolute value 1, so $|\xi(1-s,\bar\chi)| = |\xi(s,\chi)|$ for all s at which both sides are finite. For $f = 1$ the same identity is Riemann's $\xi(1-s) = \xi(s)$ with $\delta = 0$. Apply it at $s = n$, exchanging the roles of χ and $\bar\chi$ (which have the same conductor and parity):

$$(f/\pi)^{(1-n+\delta)/2}\,\big|\Gamma\big(\tfrac{1-n+\delta}{2}\big)\big|\,|L(1-n,\chi)| = (f/\pi)^{(n+\delta)/2}\,\Gamma\big(\tfrac{n+\delta}{2}\big)\,|L(n,\bar\chi)|.$$

Since $n \equiv \delta \pmod 2$, the number $z = (1-n+\delta)/2$ is a half-odd integer, $z \le -\frac12$, so $\Gamma(z)$ is finite and non-zero, and the reflection formula $\Gamma(z)\Gamma(1-z) = \pi/\sin(\pi z)$ with $|\sin \pi z| = 1$ gives $|\Gamma(z)|^{-1} = \pi^{-1}\Gamma(1-z) = \pi^{-1}\Gamma\big(\frac{1+n-\delta}{2}\big)$. Hence

$$|L(1-n,\chi)| = \Big(\frac{f}{\pi}\Big)^{n-1/2}\frac{\Gamma\big(\frac{n+\delta}{2}\big)\Gamma\big(\frac{n+1-\delta}{2}\big)}{\pi}\,|L(n,\bar\chi)|.$$

As $\delta \in \{0,1\}$, the two Gamma arguments are $n/2$ and $(n+1)/2$ in some order, and Legendre's duplication formula gives $\Gamma(n/2)\Gamma((n+1)/2) = 2^{1-n}\pi^{1/2}\Gamma(n) = 2^{1-n}\pi^{1/2}(n-1)!$. Substituting,

$$\begin{aligned}|L(1-n,\chi)| &= f^{\,n-1/2}\pi^{-(n-1/2)}\cdot 2^{1-n}\pi^{1/2}(n-1)!\cdot\pi^{-1}\,|L(n,\bar\chi)| \\ &= \frac{2\,(n-1)!\,f^{\,n-1/2}}{(2\pi)^n}\,|L(n,\bar\chi)|,\end{aligned}$$

which is (135). (As a check, for $\chi = 1$, $n = 2$ it gives $|\zeta(-1)| = 2\cdot 1\cdot(2\pi)^{-2}\zeta(2) = 1/12$.) For $\operatorname{Re} s > 1$ the Euler product $L(s,\bar\chi) = \prod_p (1-\bar\chi(p)p^{-s})^{-1}$ converges absolutely and has no zero factor, so $L(n,\bar\chi) \neq 0$, and $|L(n,\bar\chi)| \le \sum_{k\ge1} k^{-n} = \zeta(n) \le \zeta(2)$. The statement for $B_{n,\chi}$ follows from (130). □

Lemma 36 (Archimedean size)**.** *Under the hypotheses of Lemma 35,*

$$\log|L(1-n,\chi)| \le n\log n + n\log f - n(1+\log 2\pi) + 1 + \log\frac{\pi^2}{3} \;\le\; n\log(nf).$$

Proof. Since log is increasing, $\log k \le \int_k^{k+1}\log x\,\mathrm{d}x$ for $k \ge 1$, so $\log(n-1)! = \sum_{k=2}^{n-1}\log k \le \int_1^n \log x\,\mathrm{d}x = n\log n - n + 1$. Taking logarithms in (135) and using $|L(n,\bar\chi)| \le \pi^2/6$ and $f^{n-1/2} \le f^n$,

$$\log|L(1-n,\chi)| \le \log 2 + (n\log n - n + 1) + n\log f - n\log(2\pi) + \log\frac{\pi^2}{6},$$

which is the first inequality since $\log 2 + \log(\pi^2/6) = \log(\pi^2/3)$. For the second, $1 + \log 2\pi > 2.83$ and $1 + \log(\pi^2/3) < 2.20$, so $-n(1+\log 2\pi) + 1 + \log(\pi^2/3) < 0$ for every $n \ge 1$. □

Lemma 37 (Denominators)**.** *Let χ be a primitive Dirichlet character of conductor $f \ge 1$, $n \ge 1$, and $P_n = \prod_{p\le n+1} p$. Then $f\,P_n\,B_{n,\chi}$ is an algebraic integer in $\mathbb{Q}(\chi)$; consequently $n\,f\,P_n\,L(1-n,\chi)$ is an algebraic integer, and $P_n < 4^{n+1}$.*

Proof. By the von Staudt–Clausen theorem [111, Ch. 5], the denominator of B_k for even $k \ge 2$ is $\prod_{(p-1)\mid k} p$, and $B_1 = -\frac12$; all other B_k with $k \ge 1$ odd vanish. Every prime p dividing the denominator of some B_k with $k \le n$ satisfies $p - 1 \le k \le n$ (for $k = 1$, $p = 2$), so $p \le n+1$ and $p \mid P_n$; as these denominators are squarefree, $P_n B_k \in \mathbb{Z}$ for all $0 \le k \le n$. From (129),

$$f\,P_n\,B_{n,\chi} = \sum_{a=1}^{f}\chi(a)\sum_{k=0}^{n}\binom{n}{k}\,(P_nB_k)\,a^{n-k}f^k,$$

in which every term is an algebraic integer ($\binom{n}{k}$, P_nB_k, a^{n-k}, $f^k \in \mathbb{Z}$ and $\chi(a)$ is a root of unity or 0). The statement for $L(1-n,\chi)$ follows from (130), and $P_n < 4^{n+1}$ is Lemma 32. □

We can now bound the Bernoulli half for a fixed twist.

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Proposition 32. *Let $n \geq 2$ and $\varepsilon = (-1)^n$. Then*

$$\sum_{\ell \in \mathcal{L}} \log \ell \cdot \operatorname{len}\big(K_{2n-2}(\mathcal{O}_E) \otimes \mathbb{Z}_\ell\big)^{\varepsilon} \leq \frac{D}{2}\Big(n \log(nN) + \log n + \log N + (n+1)\log 4\Big) + \beta(n), \quad (136)$$

where $\beta(n) = (n+1)\log 4 + 2\log n + \log N$. Equivalently, with $n = 1 + e$,

$$\mathfrak{B}_e \leq \frac{D}{2}\Big((e+1)\log\big((e+1)N\big) + \log(e+1) + \log N + (e+2)\log 4\Big) + \beta(e+1).$$

Proof. Multiply (133) by $\log \ell$ and sum over $\ell \in \mathcal{L}$. The contribution of the torsion terms is $\sum_{\ell \in \mathcal{L}} \log \ell \cdot \operatorname{len} H^0(E, \mathbb{Q}_\ell/\mathbb{Z}_\ell(n)) \leq \beta(n)$ by Lemma 34. It remains to bound

$$\Sigma := \sum_{\ell \in \mathcal{L}} \log \ell \sum_{\chi(-1)=\varepsilon} v_\ell\big(\iota L(1-n, \chi)\big).$$

Group the characters of parity ε into orbits under $\operatorname{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ acting on values; the parity is preserved, since $\chi(-1) = \pm 1$ is rational. The orbit of χ consists of the characters $\sigma \circ \chi$, $\sigma \in \operatorname{Gal}(\mathbb{Q}(\chi)/\mathbb{Q})$, which are pairwise distinct because the values of χ generate $\mathbb{Q}(\chi)$; so the orbit has $[\mathbb{Q}(\chi) : \mathbb{Q}]$ elements. Put $\lambda_\chi = L(1-n, \chi) \in \mathbb{Q}(\chi)^\times$ (non-zero by Lemma 35). Then $L(1-n, \sigma \circ \chi) = \sigma(\lambda_\chi)$, and since $v_\ell \circ \iota$ is a valuation on $\overline{\mathbb{Q}}$ extending v_ℓ on $\mathbb{Q}$,

$$\sum_\sigma v_\ell\big(\iota\sigma(\lambda_\chi)\big) = v_\ell\Big(\iota \prod_\sigma \sigma(\lambda_\chi)\Big) = v_\ell(r_\chi), \qquad r_\chi := N_{\mathbb{Q}(\chi)/\mathbb{Q}}(\lambda_\chi) \in \mathbb{Q}^\times.$$

For a non-zero rational number $r = a/b$ in lowest terms,

$$\sum_{\ell \in \mathcal{L}} \log \ell \cdot v_\ell(r) \leq \sum_{\ell :\, v_\ell(r) > 0} \log \ell \cdot v_\ell(r) = \log |a| = \log |r| + \log b,$$

because the terms with $v_\ell(r) < 0$ are negative and we have only enlarged the set of primes. By Lemma 37, $n f_\chi P_n \lambda_\chi$ is an algebraic integer, hence so is its norm $(n f_\chi P_n)^{[\mathbb{Q}(\chi):\mathbb{Q}]} r_\chi$, which is therefore a rational integer; so the denominator of r_χ is at most $(n f_\chi P_n)^{[\mathbb{Q}(\chi):\mathbb{Q}]}$. Further, $|r_\chi| = \prod_\sigma |\sigma(\lambda_\chi)| = \prod_{\chi'} |L(1-n, \chi')|$, the product over the orbit of χ, all of whose members have conductor f_χ and satisfy $|L(1-n, \chi')| \leq (n f_\chi)^n$ by Lemma 36. Hence, for one orbit,

$$\sum_{\ell \in \mathcal{L}} \log \ell \cdot v_\ell(r_\chi) \leq [\mathbb{Q}(\chi) : \mathbb{Q}]\Big(n \log(n f_\chi) + \log n + \log f_\chi + \log P_n\Big).$$

Summing over the orbits, and using $f_\chi \leq N$ (as $f_\chi \mid N$), $P_n < 4^{n+1}$, and the fact that there are exactly $D/2$ characters of parity ε (§22.1), we get

$$\Sigma \leq \frac{D}{2}\Big(n \log(nN) + \log n + \log N + (n+1)\log 4\Big),$$

which together with the torsion contribution proves (136). □

Summing over the range of twists gives the main result of this section.

Proposition 33 (Bernoulli bound)**.** *Let E be a host field of degree $D = [E : \mathbb{Q}]$, contained in $\mathbb{Q}(\zeta_N)$, and let $t \geq 2$. The Bernoulli half of the sum $\sum_{1 \leq e < t} \log \# K_{2e}(\mathcal{O}_E)$, in the sense of Definition 10, satisfies*

$$\sum_{1 \leq e < t} \mathfrak{B}_e \leq 3\, D\, t^2 \log(tN). \quad (137)$$

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In particular, writing $\mathfrak{R}_e$ and $\mathfrak{T}_e$ for the regulator half and the contribution of the primes outside $\mathcal{L}$ as in (128),

$$\sum_{\ell} \log \ell \sum_{1 \le e < t} \operatorname{len} \operatorname{Sha}^1_e(E,\ell) = \sum_{1 \le e < t} \big(\mathfrak{B}_e + \mathfrak{R}_e + \mathfrak{T}_e - \Delta_e \big) \le 3\, D\, t^2 \log(tN) + \sum_{1 \le e < t} \big(\mathfrak{R}_e + \mathfrak{T}_e \big). \quad (138)$$

Proof. Put $n = 1 + e$, which runs over $2 \le n \le t$. By Proposition 32, $\sum_{e<t} \mathfrak{B}_e$ is at most

$$\frac{D}{2} \sum_{n=2}^{t} \Big(n \log(nN) + \log n + \log N + (n+1)\log 4 \Big) + \sum_{n=2}^{t} \Big((n+1)\log 4 + 2\log n + \log N \Big).$$

Every logarithm occurring here is at most $\log(tN)$: indeed $n \le t$, $\log 4 \le \log N$ since $N \ge 84$, and $\log(nN) \le \log(tN)$. Using $\sum_{n=2}^{t} n = \frac{t(t+1)}{2} - 1$ and $\sum_{n=2}^{t}(n+1) = \frac{(t+1)(t+2)}{2} - 3$, the whole expression is at most $\log(tN)$ times

$$\frac{D}{2}\Big(\frac{t(t+1)}{2} - 1 + 2(t-1) + \frac{(t+1)(t+2)}{2} - 3 \Big) + \Big(\frac{(t+1)(t+2)}{2} - 3 + 3(t-1) \Big) = \frac{D}{2}\big(t^2 + 4t - 5\big) + \frac{1}{2}\big(t^2 + 9t - 10\big).$$

For $t \ge 2$ one has $4t - 5 \le 2t^2$ and $9t - 10 \le 4t^2$, so this is at most $\frac{3D}{2}t^2 + \frac{5}{2}t^2 \le \big(\frac{3}{2} + \frac{5}{4}\big) D t^2 < 3Dt^2$, using $D \ge 2$. This proves (137). The identity in (138) is (127) combined with (128), and the inequality uses $\Delta_e \ge 0$ and (137). □

The bound (137) is polynomial in t and logarithmic in N, with the degree D as a linear factor. For the host fields of §22.1, $D \le 2^{m+2}\operatorname{ord}(\psi)$ and $D \le \varphi(N) < N$ by (119), so the right-hand side is at most $3 \cdot 2^{m+2}\operatorname{ord}(\psi)\, t^2 \log(tN)$; in the application the parameters satisfy $2^{m+2} \le 8x^{2+8\varepsilon}$ and $t \le x^{401}$ with $\log N$ polynomial in x, so the Bernoulli half is of size at most a fixed power of x, far below the target $\exp(\frac{1}{2}x^{\varepsilon})$. The bound is also of the right shape: for $E = \mathbb{Q}$ and even n the two sides of (131) are exactly the numerator and denominator of $\zeta(1-n) = -B_n/n$, and $\log|B_n| \sim n \log n$ by (135), so the term $n \log n$ in Proposition 32 cannot be improved beyond constants.

22.6. The Prime 2 and the Primes Dividing N

Three features of the host fields deserve comment: the prime $\ell = 2$, which always divides $[E : \mathbb{Q}]$ and for which the parity decomposition is not available; the primes dividing N, at which the characters of G are ramified; and the shape of the global formula for $\#K_{2e}(\mathcal{O}_E)$ that one obtains by putting all characters together.

The prime 2

The identification $K_{2e}(\mathcal{O}_E) \otimes \mathbb{Z}_2 \cong H^2(\mathcal{O}_E[1/2], \mathbb{Z}_2(1+e))$ of Theorem 59(b) holds because E is totally imaginary [107, Theorem 0.4(a)]; for fields with a real place the 2-adic K-groups follow an eight-fold periodic pattern involving the signature defect [107, Theorem 0.6], and the 2-part of $\operatorname{Sha}^1_e(E,\ell)$ would also require the Tate-modified cohomology at real places in Proposition 30. Neither complication arises for CM fields, and so Corollary 26 includes $\ell = 2$ without change. What is lost at $\ell = 2$ is the decomposition into halves: c acts on $M_2(n) = H^2(\mathcal{O}_E[1/2], \mathbb{Z}_2(n))$, but $\frac{1 \pm c}{2} \notin \mathbb{Z}_2[G]$, and in general $M_2(n)^+ \oplus M_2(n)^-$ is a proper subgroup of $M_2(n)$ with 2-torsion quotient. The arithmetic input at 2 is nevertheless complete. The 2-adic Iwasawa main conjecture for abelian fields was proved by Greither [99], and from it Kolster (appendix to [107]) and Rognes–Weibel derived the 2-adic Lichtenbaum formula for totally real abelian fields: for F totally real abelian with r_1 real places and $R = \mathcal{O}_F[1/2]$, and even $i > 0$, the rational numbers $\zeta_F(1-i)$ and $2^{r_1}\#K_{2i-2}(R)\{2\}/\#K_{2i-1}(R)\{2\}$

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have the same 2-adic valuation [107, Theorem 0.2]; this is the precise form of the "power of 2" in the Birch–Tate and Lichtenbaum formulas for such fields. Flach [98] completed the proof of the equivariant Tamagawa number conjecture for Tate motives over absolutely abelian fields at the prime 2, by establishing a sharpened form of the 2-adic cyclotomic main conjecture; together with Burns–Greither [96] for odd primes, this gives the equivariant conjecture for $h^0(\operatorname{Spec} E)(n)$, E abelian, at every prime. For our CM field E this determines the $\mathbb{Z}_2[G]$-module $M_2(n)$ up to the information encoded in a determinant over $\mathbb{Z}_2[G]$, hence in particular its order, in terms of the leading term of ζ_E at $1-n$ and of a 2-adic regulator. Extracting from this an explicit bound for $\operatorname{len} M_2(n)$ would again require a bound for the regulator half, which is not attempted here; the term $\mathfrak{T}_e$ in (128) therefore stays on the right-hand side of (138).

Odd primes dividing $[E:\mathbb{Q}]$

By (119), an odd prime dividing D divides $\operatorname{ord}(\psi)$; these primes form a finite set depending only on ψ, and for them the idempotents e_χ are not integral. The equivariant theorems of [75,96] cover these primes as well, but in the language of Fitting ideals or determinants over $\mathbb{Z}_\ell[G]$ rather than of isotypic components; that bookkeeping is not needed here, and these primes are kept in $\mathfrak{T}_e$ and excluded from $\mathcal{L}$. When $\operatorname{ord}(\psi)$ is a power of 2, as for quadratic ψ, there are no such primes and $\mathfrak{T}_e$ is purely 2-adic.

Primes dividing N and Euler factors

The formula (131) uses the *primitive* L-function of $\bar\chi$, and no restriction on the ramification of ℓ in E is needed: Mazur–Wiles and [75,79,96] treat all abelian fields $E \subseteq \mathbb{Q}(\zeta_N)$, whether or not $\ell \mid N$, and the primes $\ell \mid N$ with $\ell \nmid D$ belong to $\mathcal{L}$ and were included in Propositions 32 and 33. Euler factors enter in two ways. First, the Iwasawa-theoretic statement is phrased through the ℓ-adic L-function $L_\ell(s,\chi\omega^n)$, whose value at $1-n$ is $(1-\bar\chi(\ell)\ell^{n-1})L(1-n,\bar\chi)$ when χ has the parity $(-1)^n$ (here ω is the Teichmüller character, and $\bar\chi(\ell)=0$ if $\ell \mid f_\chi$) [111, Theorem 5.11]. For $n \geq 2$ the factor $1-\bar\chi(\ell)\ell^{n-1}$ is congruent to 1 modulo ℓ, hence an ℓ-adic unit, and it has been omitted from (131) without affecting the lengths. Second, the Dedekind zeta function of E factors *exactly* as

$$\zeta_E(s) = \prod_{\chi \in \widehat{G}} L(s,\chi) \tag{139}$$

with primitive L-functions, because $E/\mathbb{Q}$ is abelian; if instead one uses the characters of $(\mathbb{Z}/N)^\times$ lifted from G, or the zeta function of $\mathcal{O}_E[1/N]$, each $L(s,\chi)$ is replaced by $L(s,\chi)\prod_{p\mid N,\, p\nmid f_\chi}(1-\chi(p)p^{-s})$, and at $s=1-n$ the extra factor is $\prod_{p\mid N,\, p\nmid f_\chi}(1-\chi(p)p^{n-1})$. This is a non-zero algebraic integer, since $|\chi(p)p^{n-1}| = p^{n-1} \geq 2$, of absolute value at most $\prod_{p\mid N}(1+p^{n-1}) \leq \prod_{p\mid N} p^n \leq N^n$, because $1+p^{n-1} \leq 2p^{n-1} \leq p^n$. Consequently the imprimitive version of Proposition 32 remains true if $n\log(nN)$ is replaced by $n\log(nN)+n\log N$, which does not change the shape of (137); the primitive formulation is simply sharper, and it is the one we have used.

Lichtenbaum's formula

Multiplying (131) over the characters of parity $(-1)^n$ and (132) over those of parity $(-1)^{n-1}$, summing over all primes, and using (139) together with $\prod_\ell w_n^{(\ell)}(E) = w_n(E)$, one obtains the cohomological form of Lichtenbaum's conjecture [103] for abelian fields [75, Theorem 1.4.1]: for every number field F for which it is known, and in particular for all abelian F, and every $n \geq 2$,

$$\zeta_F^*(1-n) = \pm 2^{?}\, R_n(F) \prod_\ell \frac{\#H^2(\mathcal{O}_F[1/\ell], \mathbb{Z}_\ell(n))}{\#H^1(\mathcal{O}_F[1/\ell], \mathbb{Z}_\ell(n))_{\mathrm{tors}}}, \tag{140}$$

where $\zeta_F^*(1-n)$ is the leading coefficient of the Taylor expansion of ζ_F at $s=1-n$ (the order of vanishing being r_2 for even n and r_1+r_2 for odd n), $R_n(F)$ is the Beilinson regulator of $K_{2n-1}(\mathcal{O}_F)\otimes\mathbb{R}$,

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and $2^?$ denotes a power of 2. By Theorem 59 and Lemma 34, for totally imaginary F this is the statement, with $n = 1 + e$,

$$\#K_{2e}(\mathcal{O}_F) = \frac{w_{e+1}(F)\,|\zeta_F^*(-e)|}{R_{e+1}(F)} \qquad \text{up to a power of 2,} \tag{141}$$

which is the form stated in the surveys of Kolster [101] and Weibel [112]; for K_2 and totally real F it is the Birch–Tate formula $\zeta_F(-1) = (-1)^{r_1}\,\#K_2(\mathcal{O}_F)/w_2(F)$ of [112, Conj. 31]. The power of 2 in (141) is the one left undetermined by the normalisation of the regulator and by the odd-primary methods; it is pinned down by the 2-adic results of Rognes–Weibel for totally real fields [107, Theorem 0.1, 0.2] and, for abelian F in general, by the 2-adic equivariant Tamagawa number conjecture of Flach [98], once $R_n(F)$ is normalised as in the Bloch–Kato conjecture. In the formula (141) the Bernoulli half of $\#K_{2e}(\mathcal{O}_F)$ is the part of $|\zeta_F^*(-e)|$ coming from the $D/2$ factors $L(-e,\chi)$ with $\chi(-1) = (-1)^{e+1}$, which are finite non-zero numbers, and the regulator half is the part coming from the $D/2$ factors with $\chi(-1) = (-1)^e$, each of which vanishes to first order at $s = -e$ and contributes its derivative $L'(-e,\chi)$, from which the corresponding component of the regulator is divided out; this is the content of (133) and (134).

22.7. What Is Bounded and What Is Not

Remark 16. *The identity* (138) *is exact, and Proposition 33 bounds the term* $\sum_{e<t}\mathfrak{B}_e$ *by* $3Dt^2\log(tN)$*, while the defect* $\sum_{e<t}\Delta_e$ *is non-negative and so can be dropped from an upper bound. The terms that are* not *bounded in this section are the following. First and foremost, the regulator half* $\sum_{e<t}\mathfrak{R}_e$*: by* (134) *and Theorem 61, it is the sum over odd primes* $\ell \nmid D$*, over* $1 \le e < t$*, and over the* $D/2$ *characters* χ *with* $\chi(-1) = (-1)^e$*, of* $\log\ell$ *times the index of Soulé's cyclotomic element in the* χ*-component of* $H^1(\mathcal{O}_E[1/\ell],\mathbb{Z}_\ell(1+e))/\mathrm{tors}$*. Bounding it amounts to showing that Soulé's elements are close to generators, uniformly in* ℓ*,* e *and* χ*; no bound for this quantity follows from the results quoted here, and for* $E = \mathbb{Q}$ *its vanishing is equivalent to Vandiver's conjecture. Second, the contribution* $\sum_{e<t}\mathfrak{T}_e$ *of the prime* 2 *and of the odd primes dividing* $\mathrm{ord}(\psi)$*, for which the arithmetic input (Flach, Burns–Greither, Huber–Kings) is available in equivariant form but has not been converted here into an estimate for the lengths. Everything else in* (118)*, that is, the passage from Tate–Shafarevich groups to K-groups and the Bernoulli half of the K-groups, has been bounded explicitly, by* $3Dt^2\log(tN)$*, with no hypothesis beyond the cited theorems.*

23. The Regulator Half, the Assembly of the Length Bound, and the Passage to the Energy

This section completes the account of how the cohomological window theorem (Theorem 54) is proved and how it is then used. The argument is that of [4]. We state its steps in the ordinary language of the subject and place each of them in the published literature on which it rests; the steps attributed to [4] are proved there and are not reproved here. When we write that [4] proves a statement, the proof is the one given in that project; when we describe a mechanism, it is the published method of which the statement is an instance. A closing remark lists the steps that have been checked by hand.

The previous section dealt with the minus part of the Tate–Shafarevich groups, whose lengths are governed by generalized Bernoulli numbers. The present section deals with the plus part, whose lengths are governed by congruences between Eisenstein series and cusp forms, and then assembles the two halves, together with the prime 2 and the finitely many bad primes, into the length bound of Theorem 54. Its last part explains how that length bound enters the analytic argument of Sections 21 and 24: as an additive error term in a place-weighted sum on the geometric side of the host construction.

Notation

Throughout, χ is a primitive Dirichlet character of conductor q_χ and order o; this is the character called ψ in Theorem 54, and we rename it here because the letter ψ will be needed for the characters of

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the host field. A *host* in this section is the arithmetic datum behind Definition 8: an integer $e_0 \geq 2$ (the 2-power level is 2^{e_0+2}) and two auxiliary primes q, d with

$$q \equiv 3 \pmod 8, \qquad d \equiv 7 \pmod{16}, \qquad (qd, q_\chi) = 1.$$

Its level is

$$N = \operatorname{lcm}(2^{e_0+2}, q_\chi) \cdot q\, d,$$

and its field $E = E_{\chi,e_0,q,d}$ is the subfield of $\mathbb{Q}(\zeta_N)$ fixed by the subgroup $H \leq (\mathbb{Z}/N)^\times$ of classes a with

$$a \equiv \pm 1 \pmod{2^{e_0+2}}, \qquad \chi(a) = 1, \qquad \left(\frac{a}{q}\right) = \left(\frac{a}{d}\right) = 1.$$

Since $q \equiv d \equiv 3 \pmod 4$ the two Legendre symbols are odd characters, so E contains $\mathbb{Q}(\sqrt{-q})$ and $\mathbb{Q}(\sqrt{-d})$ and is a CM field; it is Galois over $\mathbb{Q}$ with abelian group $(\mathbb{Z}/N)^\times/H$. The characters of $\mathrm{Gal}(E/\mathbb{Q})$ are the Dirichlet characters ψ modulo N that are trivial on H; for an integer $e \geq 1$ we write X_e for the set of those with $\psi(-1) = (-1)^e$, with values taken in the ring of integers $\mathcal{O}$ of a finite extension of $\mathbb{Q}_\ell$ when a prime ℓ has been fixed.

For a prime ℓ, a CM field E and $e \geq 1$, the group $\mathrm{Sha}^1_e(E, \ell)$ was defined before Theorem 54; it is a $\mathbb{Z}_\ell$-module of finite length when finite. Complex conjugation acts on it, and when ℓ is odd it splits into the two eigenspaces

$$\mathrm{Sha}^1_e(E,\ell)^+ = \{x : \ cx = (-1)^e x\}, \qquad \mathrm{Sha}^1_e(E,\ell)^- = \{x : \ cx = (-1)^{e+1} x\},$$

whose lengths add up to the length of the whole.[32] The convention places on the "plus" side the eigenvalue that the cyclotomic character χ_ℓ^{-e} itself takes on complex conjugation, so that the plus part is the part of the Galois cohomology on which the full group $\mathrm{Gal}(E/\mathbb{Q})$ acts through *even* characters (relative to e), and the minus part through odd ones. The minus part was the subject of the previous section. When $\mathrm{Gal}(E/\mathbb{Q})$ is abelian, the plus part decomposes further under its characters, and the piece belonging to $\psi \in X_e$ is the ψ-isotypic part of the Tate–Shafarevich group of $\mathbb{Q}_\ell/\mathbb{Z}_\ell(-e) \otimes \psi^{-1}$ over $\mathbb{Q}$, unramified outside ℓN. Its length as an $\mathcal{O}$-module is written $V_e(\psi)$, and we normalise it to

$$\nu_e(\psi) = \frac{V_e(\psi)}{e(\mathcal{O}/\mathbb{Z}_\ell)},$$

where $e(\mathcal{O}/\mathbb{Z}_\ell)$ is the ramification index. The point of the normalisation is that $\nu_e(\psi)\log\ell$ is the logarithm of the cardinality of the module divided by the degree $[\mathcal{O} : \mathbb{Z}_\ell]$, so it is comparable with the logarithm of an ordinary integer, and it is unchanged when $\mathcal{O}$ is enlarged. A bound of the form $V_e(\psi) < \infty$ and $\nu_e(\psi)\log\ell \leq B$ will be written $V_e(\psi) \preceq B$.

By Poitou–Tate duality, the Tate–Shafarevich group of $\mathbb{Q}_\ell/\mathbb{Z}_\ell(-e)$ over E unramified outside ℓ is dual to the second cohomology group $H^2(\mathcal{O}_E[1/\ell], \mathbb{Z}_\ell(e+1))$, up to the local terms at the places above ℓ and at infinity [82, Ch. VIII], [89]. These second cohomology groups are finite for $e \geq 1$ by Soulé [88], and for the field $\mathbb{Q}$ and odd $e+1 \geq 3$ their order is the index of Soulé's cyclotomic elements in $H^1(\mathbb{Z}[1/\ell], \mathbb{Z}_\ell(e+1))$, a statement that is the Bloch–Kato conjecture for the Tate motive $\mathbb{Q}(e+1)$ and is proved from the main conjecture of Iwasawa theory [71,75,79]. The eigenspace lengths $V_e(\psi)$ are therefore the ψ-components of the index of cyclotomic elements, and the task of this half of the argument is to bound that index from above, uniformly in E, ℓ and e, by a quantity that is polynomial in N and e. The earlier parts of this exposition explained why such a bound, over the whole family of hosts and twists, is what the energy argument needs.

[32] The sign convention is the one of the project.

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23.1. The Regulator Half: Eisenstein Congruences

Ribet's method

The classical mechanism is Ribet's [83]. Let p be an odd prime and k an even integer with $2 \le k \le p-3$. Herbrand had shown that if the ω^{1-k}-eigenspace of the p-part of the class group of $\mathbb{Q}(\zeta_p)$ is nonzero then p divides the Bernoulli number B_k; Ribet proved the converse. His argument runs as follows. The Eisenstein series G_k has constant term $-B_k/2k$, so when $p \mid B_k$ it is congruent modulo p to a cusp form, and by a lemma of Deligne and Serre one may take that cusp form to be an eigenform f of weight k (or, after a weight-lowering step, of weight 2 and level p). The p-adic Galois representation ρ_f attached to f has semisimplified reduction $1 \oplus \omega^{k-1}$, and Ribet chooses a lattice for which the reduction is a non-split extension of ω^{k-1} by 1 (or of 1 by ω^{k-1}, after a twist). That extension class is a cohomology class in $H^1(\mathbb{Q}, \omega^{1-k})$ which is unramified at p, because f is ordinary and its restriction to a decomposition group is reducible with the correct ordering of characters, and unramified away from p because so is ρ_f. An unramified class in $H^1(\mathbb{Q}, \omega^{1-k})$ is, by class field theory, a nonzero element of the ω^{1-k}-eigenspace of the class group of $\mathbb{Q}(\zeta_p)$. Mazur's study of the Eisenstein ideal [80] put the congruence into the language of Hecke algebras, and Mazur and Wiles [81] pushed the method up the cyclotomic tower to prove the main conjecture of Iwasawa theory over $\mathbb{Q}$, which Wiles then extended to totally real fields [92].

The method has two directions, and it is the second that matters here. The first, Ribet's, produces a nonzero class in a Selmer group from a congruence: a cusp form congruent to an Eisenstein series yields a non-split lattice and hence a class. The second says that the Selmer group is no larger than the congruence module: the Eisenstein ideal of the Hecke algebra, or the congruence number measuring how far the cusp forms are from the Eisenstein series, bounds the order of the Selmer group from above. In the setting of GL_2 over $\mathbb{Q}$ this is the content of the main conjecture, and the two directions together give the equality of Mazur–Wiles. For the present argument an *upper* bound is what is wanted, since the index of cyclotomic elements must be shown to be small; and it is wanted in a form uniform enough to be summed over hosts, primes ℓ and twists e, so what is needed is not an equality of ideals but an explicit upper bound for the congruence number in terms of the level and the weight.

Unitary groups and GSp_4

The passage from GL_2 to larger groups is due to Skinner and Urban. In [84] they study p-adic deformations of automorphic representations of unitary groups, in the first instance $\mathrm{U}(3)$, and construct from an Eisenstein series on $\mathrm{U}(3)$ whose constant term vanishes a cuspidal representation congruent to it; the attached three-dimensional Galois representation then has a lattice whose reduction is a non-split extension involving a two-dimensional modular representation and a character, and this extension is a class in a Selmer group of a twist of the modular representation. The survey [85] explains the strategy, and the long paper [86] carries it out on $\mathrm{U}(2,2)$ to prove the main conjecture for GL_2 over $\mathbb{Q}$ at ordinary primes; the analogous argument for GSp_4, where the Eisenstein series is of Klingen or Siegel type and the congruence produces classes in the Selmer group of a modular form or of its twist, is in Urban [91], with the Saito–Kurokawa case treated by Brown [72]. In the residually reducible setting over $\mathbb{Q}$ the deformation-theoretic form of Ribet's method is Skinner–Wiles [87]. Bellaïche and Chenevier [69] used non-tempered forms on $\mathrm{U}(3)$ to construct Selmer classes for imaginary quadratic fields in the direction predicted by Bloch and Kato, and in [70] they developed the general theory of families of Galois representations and their reducibility loci, with the "generalized matrix algebra" formalism that makes the lattice argument work at a reducible point of an eigenvariety for a group of any rank.

In all of these the Galois side is the following. One has a reducible representation, the Eisenstein representation, which in the rank-three unitary case has the shape

$$\rho_{\mathrm{Eis}} = \Psi \oplus 1 \oplus \Psi^{-1}, \qquad \Psi = \chi_\ell^{e}\psi,$$

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a conjugate self-dual sum of three characters, and in the symplectic rank-four case the shape

$$\rho_{\mathrm{Eis}} = \theta \oplus \Psi \oplus \alpha \oplus 1, \qquad \theta = \alpha\Psi,$$

with α a character of Hodge–Tate weight one and Ψ of weight e, so that the four weights are $e+1, e, 1, 0$. A class in H^1 of the appropriate twist of a character by another is an extension of two of these constituents, and such an extension, if it is to be "automorphic", must be the reduction of a lattice in an irreducible automorphic Galois representation congruent to ρ_{Eis}. The Selmer class therefore lives in the tangent space of the Eisenstein point inside the space of all automorphic deformations, and the size of the Selmer group is bounded by the size of the congruence module there.

The automorphic inputs

Three further ingredients make the method yield an explicit upper bound rather than an existence statement, and all three appear among the hypotheses of the Lean statements described below.

The first is the existence of Galois representations attached to the automorphic representations that occur: the theorem of Harris–Taylor, Shin, Chenevier–Harris and Caraiani recalled in Section 19, together with local–global compatibility at every place. In the project this is a single cited statement (Appendix A, Input A); it is needed because the congruence module is computed on the automorphic side, as a product over automorphic representations of the differences between their Hecke eigenvalues and those of the Eisenstein series, and the Galois representations are what turn such a product into a statement about cohomology classes.

The second is a modularity input in the direction of Taylor–Wiles [90,93] and Kisin [78]: an *adjoint lifting* statement saying that every point of the relevant deformation ring, or at least every point of the components meeting the Eisenstein point, is automorphic. Without it the Selmer group could contain classes corresponding to deformations that are not seen by any automorphic form, and the congruence number would bound only the automorphic part. The Lean project contains a theorem of this kind, deduced from a package of published results on the cohomology of unitary Shimura varieties, and the statement that the Case II and Case III bounds below hold is proved from it together with a density statement for the image of the GSp_4 Galois representations. The adjoint here is the adjoint of the three- or four-dimensional representation, twisted by a power of the cyclotomic character, and the vanishing of its Bloch–Kato Selmer group is what makes the deformation ring smooth at an irreducible automorphic point; this is the standard way in which an $R = \mathbb{T}$ theorem controls a congruence module.

The third is Kato's theorem [77]: for a newform f of weight $k \geq 2$ and a critical integer r with $L(f, \varphi, r) \neq 0$, the Bloch–Kato Selmer group of $V_f(r) \otimes \varphi$ is finite, for a Dirichlet character φ. In the rank-three case there are two kinds of automorphic point near the Eisenstein point: irreducible ones, and "endoscopic" ones of the form $W \oplus 1$ where W is the Galois representation of a newform of weight $2e+1$ twisted by χ_ℓ^{-e}, restricted to the CM field. The endoscopic points are where the Bloch–Kato Selmer groups of twists of a rank-two modular representation enter, and the project assumes, in the precise form of Kato's theorem for twists by characters that kill the roots of unity of a bounded order, that these Selmer groups vanish. This is what rules out that the lattice argument produces a class that lives on the endoscopic branch rather than on the Eisenstein one.

Finally, a combinatorial input: J. A. Green's parametrisation of the irreducible characters of $\mathrm{GL}_d(\mathbb{F}_g)$ [74], in particular of its cuspidal characters by orbits of characters of $\mathbb{F}_{g^d}^\times$ under Frobenius, is used to control the local types of the theta lifts at the auxiliary primes, and so the local components of the Hecke algebras in which the congruences are measured. In the chain of Section 24 it is cited at step 4; it is mentioned here because the bounds below are stated for every set of characters $\psi \in X_e$ at once, and the uniformity over ψ comes from a count of local types.

The congruence number

The quantitative heart of the plus-part bound is a statement which we now state in the paper's notation, as a theorem of [4].

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Theorem 62 ([4], congruence number of rank three)**.** *Let K be an imaginary quadratic field. There are integers $c, A \geq 0$, depending on K, with the following property. Let $e \geq 1$, let $N \geq 1$, let $p \geq 7$ be a prime with $p \nmid N \operatorname{disc}(K)$, and let w be a place of K above p. Let Z be a finite set of roots of unity stable under $\mathrm{Aut}(\mathbb{C})$. Then there is a nonzero integer $\mathcal{N}$ with*

$$\log|\mathcal{N}| \leq c \cdot \#Z \cdot \big(N(e+1)\big)^{A} \log p$$

such that, for every prime $\ell \neq 2, p$, every $\iota : \overline{\mathbb{Q}}_\ell \cong \mathbb{C}$, every Frobenius element σ at w, and every family $(X_z)_{z \in Z}$ of finite sets of pairwise non-isomorphic admissible three-dimensional representations of $\mathrm{Gal}(\overline{K}/K)$ of level N and weight e, either irreducible and automorphic or endoscopic of the form $W \oplus 1$ with W attached to a newform of weight $2e+1$, one has

$$\operatorname{tr}\rho(\sigma^{-1}) \neq \iota^{-1}\big(\mathrm{N}w^{-e}z + 1 + \mathrm{N}w^{e}z^{-1}\big) \quad (\rho \in X_z),$$

and

$$\|\mathcal{N}\|_\ell \;\leq\; \prod_{z \in Z} \prod_{\rho \in X_z} \big\|\operatorname{tr}\rho(\sigma^{-1}) - \iota^{-1}\big(\mathrm{N}w^{-e}z + 1 + \mathrm{N}w^{e}z^{-1}\big)\big\|_\ell .$$

Here $\mathrm{N}w^{-e}z + 1 + \mathrm{N}w^{e}z^{-1}$ is the trace at Frobenius of the Eisenstein representation $\Psi \oplus 1 \oplus \Psi^{-1}$ whose character Ψ takes the value $z\,\mathrm{N}w^{-e}$ there, so the inequality says that the product of all the "congruence differences" between automorphic points and the Eisenstein point, at a single Frobenius, is bounded below ℓ-adically by a fixed integer of polynomial size. The integer $\mathcal{N}$ is independent of ℓ: it is the congruence number of the Eisenstein series with the cuspidal spectrum of the relevant level and weight, computed once over $\mathbb{C}$ by the automorphic theory, and the polynomial bound on its size comes from bounds for the dimension of the space of forms and for the size of their Hecke eigenvalues, which are polynomial in the level and the weight. The rank-four version is the same statement for GSp_4 over $\mathbb{Q}$, with the four-character Eisenstein trace and with a bound $c\,\varphi(k)(M(e+1))^{A}\log p$ where k is the order of the roots of unity involved.

The second half of the mechanism is the lattice argument, which turns a class in the ψ-eigenspace into a congruence at σ.

Theorem 63 ([4], one character)**.** *Let K be an imaginary quadratic field, ℓ an odd prime, $e \geq 1$, and ψ a character modulo N satisfying the admissibility conditions listed below. Let $\sigma \in \mathrm{Gal}(\overline{K}/K)$ and let $t = \Psi(\sigma) + 1 + \Psi(\sigma)^{-1}$ with $\Psi = \chi_\ell^{e}\psi$. If no admissible representation ρ as in Theorem 62 has $\operatorname{tr}\rho(\sigma) = t$, then $V_e(\psi)$ is finite and there is a finite set X of pairwise non-isomorphic admissible representations with*

$$v_e(\psi) \leq 2 \sum_{\rho \in X} v_\ell\big(\operatorname{tr}\rho(\sigma) - t\big),$$

where v_ℓ is the valuation normalised by $v_\ell(\ell) = 1$.

The admissibility conditions on (K, ℓ, e, ψ) are: $\ell \neq 2$; $e \geq 1$; N even; $\ell \nmid \mathrm{ord}(\psi)$; the ℓ-component of ψ has square one; $\psi(-1) = (-1)^e$; K is imaginary quadratic with $\ell \nmid \operatorname{disc}(K) h_K$ and $(\operatorname{disc}(K), N) = 1$; the fixed field of the kernel of $\chi_\ell \cdot \chi_P(\ell, \psi)$ is not contained in K (a genericity condition on the auxiliary character defined below); and $\ell - 1 \nmid 2e$. The factor 2 reflects the normalisation of lengths by the ramification index, and the sum over X is the ℓ-adic valuation of a product of congruence differences, which is exactly what Theorem 62 bounds. Composing the two, and choosing σ to be a Frobenius at a place above an auxiliary prime p coprime to $30N\operatorname{disc}(K)$, one obtains the bound in the form promised: for the whole orbit of characters with a given kernel H,

$$\sum_{\psi} v_e(\psi) \;\leq\; 2\, v_\ell(\mathcal{N}), \qquad \log|\mathcal{N}| \leq c\,\varphi([(\mathbb{Z}/N)^{\times} : H])\,\big(N(e+1)\big)^{A}\log p,$$

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with $\mathcal{N} \neq 0$ independent of ℓ. Summing over a finite set $\mathcal{L}$ of primes ℓ and using two auxiliary primes $p \neq p'$ so that each ℓ differs from at least one of them, the statement becomes

$$\sum_{\ell \in \mathcal{L}} \log \ell \sum_{\psi} v_e(\psi) \;\le\; 2c\,\varphi([(\mathbb{Z}/N)^\times : H])\,\big(N(e+1)\big)^A\,(\log p + \log p'),$$

since $\sum_\ell v_\ell(\mathcal{N}) \log \ell \le \log|\mathcal{N}|$ for a nonzero integer. This is the shape of every bound in the plus part: a nonzero integer of polynomially bounded size whose ℓ-adic valuations, summed against $\log \ell$, bound the eigenspace lengths.

The case split

The bound just described applies only under the admissibility conditions, and in particular only when $\ell - 1 \nmid 2e$ and ℓ is prime to the discriminant and class number of K. The remaining configurations of (ℓ, e, ψ) are handled by other arguments, and the division into cases is the organising principle of this half of the proof. We now give it in the paper's notation; the cases and the bounds are those of [4].

Fix the primitive character χ of conductor q_χ and order o. Let p_K be the least prime $p \ge 7$ with $p \equiv 3 \pmod 4$ and $p \nmid q_\chi$, let $K^* = \mathbb{Q}(\sqrt{-p_K})$ and h^* its class number, and let p_1 be the least prime with $p_1 \equiv 1$ modulo the odd part of o and $p_1 \nmid 2q_\chi p_K$. The *bad set* is

$$B_\chi = \{3\} \cup \{\text{odd primes dividing } o\, q_\chi\, p_K\, h^*\, p_1\}.$$

It depends on χ alone, not on the host. The field K^* is the imaginary quadratic field over which the unitary group of Theorem 62 is taken in Case I; the primes p_K, h^* are excluded because they violate the admissibility conditions; p_1 is an auxiliary prime at which a character of order ord ψ can be found, used in Case III.

For a character ψ modulo N and a prime ℓ, write $\psi = \theta_0\kappa$ with κ the ℓ-power-order part and θ_0 the part of order prime to ℓ; write ψ_2 for the component of ψ at the prime 2 and $c_2(\psi)$ for the 2-adic valuation of its conductor; and write $\chi_P(\ell, \psi)$ for θ_0 divided by its component at ℓ, which is the part of θ_0 of conductor prime to ℓ. Put $\theta_1 = \theta_0/(\theta_0)_2$, let f_0 be the order of ℓ modulo $\operatorname{lcm}(4, \operatorname{ord}\theta_1)$, and let $n_0 = v_2(\ell^{f_0} - 1)$. These auxiliary quantities are taken from [4]; they measure the 2-adic and ℓ-adic complexity of ψ and enter the exponents of the bounds. The cases, for ℓ odd, $1 \le e < t$ and $\psi \in X_e$, are the following. Case I is the case $\ell \notin B_\chi$ and $\ell - 1 \nmid 2e$; it is treated for all $\psi \in X_e$ at once. Case Z is the case $\ell \notin B_\chi$, $\ell - 1 \mid 2e$ and $\psi = 1$. Case II is the case $e \ge 2$, the order of $\chi_P(\ell, \psi)$ even, and either $\ell \in B_\chi$ or $\ell - 1 \mid 2e$. Case III is the case $\ell \notin B_\chi$, $\ell - 1 \mid 2e$, $\psi \neq 1$ and the order of ψ odd. Case IV is the case $\ell \in B_\chi$, $e \ge 2$ and the order of $\chi_P(\ell, \psi)$ odd. Case V is the case $\ell \in B_\chi$ and $e = 1$.

The cases cover every configuration that occurs for a host that is big for t, in the sense defined below, as the definitions show. If $\ell \in B_\chi$, then either $e = 1$, which is Case V, or $e \ge 2$, and then the order of $\chi_P(\ell, \psi)$ is even or odd, which are Cases II and IV. If $\ell \notin B_\chi$, then either $\ell - 1 \nmid 2e$, which is Case I, or $\ell - 1 \mid 2e$. In the latter case $\ell \ge 5$ because $3 \in B_\chi$, so $\ell - 1 \mid 2e$ forces $e \ge 2$; and ℓ does not divide N, because $\ell \nmid q_\chi$ (as $\ell \notin B_\chi$), $\ell \neq 2$, and $\ell \le 2e + 1 < 2t + 1 < q, d$ excludes $\ell = q$ and $\ell = d$. Hence ψ has no component at ℓ, and $\ell \nmid o$ gives that ψ has order prime to ℓ, so $\chi_P(\ell, \psi) = \psi$. Then $\psi = 1$ is Case Z, $\psi \neq 1$ of odd order is Case III, and ψ of even order is Case II with $\ell - 1 \mid 2e$. The bigness of the host is thus what makes the case split exhaustive, and it is used in exactly this way.

Theorem 64 ([4], the plus-part counts)**.** *Fix χ. There are a constant $C = C(\chi)$ and two primes r_1, r_2 depending on χ such that, for every host (e_0, q, d) of χ with $q, d > r_1, r_2$ and with $q, d > 2t + 1$ and $q, d > q_\chi, o, p_K, h^*, p_1$ ("the host is big for t"), the following hold for every odd prime ℓ, every $1 \le e < t$ and every $\psi \in X_e$. In Case I, for every finite set $\mathcal{L}$ of odd primes all in Case I for e,*

$$\sum_{\ell \in \mathcal{L}} \log \ell \sum_{\psi \in X_e} v_e(\psi) \le (Nt)^C.$$

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In Case Z, $V_e(1) = 0$. In Cases II and III, $V_e(\psi) \preceq (\ell Ne)^C$. In Case V, $V_e(\psi) \preceq (\ell Ne)^C$, the bound being obtained by two different arguments according as $R_1(\ell,\psi) = 16\,o\,(\ell q_\chi)^2 4^{n_0}$ is at most or exceeds $2^{c_2(\psi)}$. In Case IV, if B is a bound for the lengths over the layers of the cyclotomic $\mathbb{Z}_\ell$-tower over $\mathbb{Q}(\zeta_{\ell q_\chi})$ in the sense of Section 23.2, then

$$\sum_{1\le e<t} \nu_e(\psi) \le B + (Nt)^C.$$

The statements (I), (II), (III), (V) and (Z) are each a separate proposition of the project, and they are combined into the plus-part bound by elementary summation. Their provenance, as far as the statements show it, is as follows.

Case I is Theorem 62 composed with Theorem 63, over $K = K^*$, with H the subgroup cutting out E, and with the two auxiliary primes r_1, r_2 chosen once for all, prime to $30 q_\chi p_K$; the condition $\ell \notin B_\chi$ gives $\ell \nmid \operatorname{disc}(K^*) h^*$ and $\ell \nmid o$, the condition $\ell - 1 \nmid 2e$ is the last admissibility condition, and the genericity condition on $\chi_\ell \chi_P$ is where the primes q, d dividing N help, since the auxiliary quadratic characters make the kernel large. The exponent C absorbs the explicit polynomial $\operatorname{disc}(K^*)^5 N^{18} e^7$ of the project's intermediate statement, and the factor $\varphi([(\mathbb{Z}/N)^\times : H])$ is at most N.

Case Z is a vanishing statement for the trivial character: for $\ell \ge 5$ and even $e \ge 2$ with $\ell - 1 \mid 2e$, the $\psi = 1$ eigenspace vanishes identically. In this configuration the twist $\mathbb{Z}_\ell(e+1)$ is congruent modulo $\ell - 1$ either to $\mathbb{Z}_\ell(1)$ or to its quadratic twist, so the relevant eigenspace of the class group of $\mathbb{Q}(\zeta_\ell)$ is the trivial or the quadratic one, and both vanish for elementary reasons; the project proves the statement directly.

Cases II and III go through GSp_4 over $\mathbb{Q}$ rather than $\mathrm{U}(3)$ over K^*. The Eisenstein representation is $\theta \oplus \Psi \oplus \alpha \oplus 1$ with $\alpha = \chi_\ell\, \omega^{-1} \xi$, where ξ is an auxiliary odd character: in Case II a primitive odd character of conductor 2^n with $n = n_{\mathrm{II}}(\ell,\psi)$, a quantity of size $O(\log(\ell N 4^{n_0} 2^{c_2(\psi)} o \ell^3 q_\chi^2))$, and in Case III the product of such a character of conductor $2^{n_{\mathrm{III}}}$ with a character modulo p_1 of order exactly $\operatorname{ord}\psi$. The set of bad primes for the deformation problem is the set of primes dividing $2\ell N p_1$. The automorphy of the irreducible points of the deformation space, together with the finiteness of the Eisenstein component and the absence of "endoscopic" branches, is the content of a statement that the project derives from its adjoint lifting theorem and a density statement for the image of the symplectic Galois representations. The role of the 2-power conductor of ξ is to separate the four characters $\theta, \Psi, \alpha, 1$ from one another at the prime 2, which is why the 2-adic data $c_2(\psi)$ and n_0 enter.

Case V with $R_1(\ell,\psi) \le 2^{c_2(\psi)}$, that is when the 2-part of the conductor of ψ is large compared with the other data, goes through a rank-three unitary deformation ring over a CM field, with the automorphy of its points on the "alternative" branch supplied by the same kind of modularity theorem; the project's statement asserts that the universal deformation of the residual representation, restricted to the points of type I of a certain frame of local conditions, is automorphic on $\mathrm{U}(3)$. Case V with $R_1(\ell,\psi) > 2^{c_2(\psi)}$, when the 2-part is small, is not an Eisenstein congruence at all: it is proved from Poitou–Tate duality, local Tate duality and the finiteness of local cohomology, and from the Iwasawa-theoretic bound called "Lemma K" in the next subsection, because in this case the character ψ has small conductor and the whole eigenspace can be bounded through H^2 over a fixed small field. Case IV is likewise not an Eisenstein congruence: it is reduced to a bound over the layers of the cyclotomic tower, which is where $\ell \in B_\chi$ is harmless because there are finitely many such ℓ and the tower bound is allowed to depend on ℓ.

Why the window ranges make the auxiliary data behave

The conditions on q and d in Theorem 54 are of three kinds, and each is used in a specific place.

The congruence conditions $q \equiv 3 \pmod 8$ and $d \equiv 7 \pmod{16}$ make both Legendre symbols odd characters, so that E is a CM field containing $\mathbb{Q}(\sqrt{-q})$ and $\mathbb{Q}(\sqrt{-d})$, and they fix the behaviour of the prime 2 in these two quadratic fields: $-q \equiv 5 \pmod 8$ so 2 is inert in $\mathbb{Q}(\sqrt{-q})$, and $-d \equiv 1 \pmod 8$ so 2 splits in $\mathbb{Q}(\sqrt{-d})$. Together with the condition $a \equiv \pm 1 \pmod{2^{e_0+2}}$ defining the 2-power part of E, this pins down the 2-adic completion of E and of its characters, which is what the 2-adic

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auxiliary quantities $c_2(\psi)$, n_0 and $n_{\mathrm{II}}, n_{\mathrm{III}}$ depend on. The condition $q \mid d+1$ says that $d \equiv -1 \pmod{q}$, so $-d$ is a square modulo q and q splits in $\mathbb{Q}(\sqrt{-d})$; and the condition that every odd prime $p \le u$ divides $d+1$ says that every such p splits in $\mathbb{Q}(\sqrt{-d})$. This is the condition, recorded in Section 21, that every odd prime dividing the level $Q = \operatorname{lcm}(2^{e_0+2}, q_\chi)$ splits in $\mathbb{Q}(\sqrt{-d})$, and its role is on the automorphic side: the unitary groups are taken relative to $\mathbb{Q}(\sqrt{-d})$ and the relevant primes must be split for the local theory of the theta lifts and for Green's parametrisation to apply in the split form.

The size conditions $\frac{u}{2} < \log q \le u$ and $\frac{u}{2}(\log Y_1)^2 < \log d \le u(\log Y_1)^2$, with $u = Y_1^{\varepsilon/5}$, do two things. First, they make q and d exceed every quantity attached to χ alone, namely q_χ, o, p_K, h^*, p_1 and the two auxiliary primes r_1, r_2 of Case I, and they make $q, d > 2t+1$ where $t = \lfloor x^{401} \rfloor$ is the range of twists. The latter means that no prime ℓ with $\ell - 1 \mid 2e$ for some $e < t$ can equal q or d, and no prime of B_χ can equal q or d: the auxiliary primes never coincide with the primes at which the delicate Cases II–V occur, so they contribute to the level N and to the field E but never to the bad set. That is the content of the hypothesis "the host is big for t" in Theorem 64. Second, they make the level small enough for the polynomial bounds to be useful: $\log N \le (e_0+2)\log 2 + \log q_\chi + \log q + \log d$ is of size $O(u(\log Y_1)^2) = O(x^{\varepsilon/5}(\log x)^2)$ for x in the window $[Y, Y^2]$, and $\log t \le 401 \log x$, so every bound of the form $(Nt)^C$ or $(\ell N e)^C$ summed over the finitely many relevant ℓ is at most $\exp(O(x^{\varepsilon/5}(\log x)^2))$, which is far below the target $\exp(\frac{1}{2}x^\varepsilon)$. The windows are chosen so that q and d are large compared with χ and t but small compared with $\exp(x^\varepsilon)$, and there is room between these two requirements precisely because the plus-part bounds are polynomial in N and t.

23.2. Assembly of the Length Bound

The full Tate–Shafarevich length of a host field is assembled from three parts: the prime 2, the minus parts for odd ℓ, and the plus parts for odd ℓ. The assembly is a one-line lemma, which we state as such because it fixes the shape of the final bound.

Lemma 38 ([4]). *Let E be a CM field and $t \ge 1$. Suppose that*

$$\sum_{1 \le e < t} \operatorname{len} \mathrm{Sha}^1_e(E, 2) \le B_2,$$

$$\sum_{\ell \ne 2} \log \ell \sum_{1 \le e < t} \operatorname{len} \mathrm{Sha}^1_e(E, \ell)^- \le B_-,$$

$$\sum_{\ell \ne 2} \log \ell \sum_{1 \le e < t} \operatorname{len} \mathrm{Sha}^1_e(E, \ell)^+ \le B_+,$$

each sum over ℓ being over an arbitrary finite set and all the groups involved being finite. Then all $\mathrm{Sha}^1_e(E, \ell)$ for $1 \le e < t$ are finite and, for every finite set $\mathcal{S}$ of primes,

$$\sum_{\ell \in \mathcal{S}} \log \ell \sum_{1 \le e < t} \operatorname{len} \mathrm{Sha}^1_e(E, \ell) \le B_2 \log 2 + B_- + B_+.$$

The proof is the eigenspace decomposition of the preceding subsection for odd ℓ and the separation of the term $\ell = 2$.

The three inputs are supplied as follows.

The prime 2 and the bad layers

For $\ell = 2$ and for the finitely many odd primes $\ell \in B_\chi$, the bounds are not Eisenstein congruences but Iwasawa theory in cyclotomic $\mathbb{Z}_\ell$-towers. The basic input is the following finiteness and growth statement, which the project isolates as a hypothesis and calls Lemma K.

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Lemma 39 (Lemma K). *There is a constant C_K with the following property. Let ℓ be a prime, L a totally complex number field of degree n and absolute discriminant D containing μ_ℓ (or μ_4 when $\ell = 2$), and $e \geq 1$. Then $H^2(\mathcal{O}_L[1/\ell], \mathbb{Z}_\ell(e+1))$ is finite and*

$$\log \#H^2\big(\mathcal{O}_L[1/\ell], \mathbb{Z}_\ell(e+1)\big) \leq \exp\big((2en)^{C_K} \log(2D)\big).$$

Finiteness is Soulé's theorem [88]; the bound is of the shape one gets from Iwasawa's description of the growth of the ℓ-part of class groups in the cyclotomic $\mathbb{Z}_\ell$-extension [76], which is $\ell^{\mu\ell^n+\lambda n+\nu}$ at the nth layer, together with the vanishing of the μ-invariant for abelian fields proved by Ferrero and Washington [73], which removes the doubly exponential term, and with the relation between H^2 with twisted coefficients and the eigenspaces of class groups up the tower [82, Ch. XI]. The explicit form, with an absolute constant C_K, is taken from [4].

From Lemma K, Poitou–Tate duality, local Tate duality and local finiteness the project derives the bounds for $\ell = 2$ and for the layers of the cyclotomic towers over the fields $\mathbb{Q}(\zeta_{\ell q_\chi})$, $\ell \in B_\chi$, with Ferrero–Washington entering separately as the statement that the ℓ-rank of the class group is bounded along the tower. The resulting statement is formulated as a property of windows: for the abelian field M cut out by χ, for the finite set B' of odd primes with the fields $L_\ell = \mathbb{Q}(\zeta_{\ell q_\chi})$, for $0 < \varepsilon < 1/52$ and for admissible choices $e_0(x), t(x)$ with $2^{e_0(x)} \leq x^{A_1}$ and $t(x) \leq x^{C_t}$, there is X_0 such that for every $X \geq X_0$ and every host pair (q, d) with $\log(qd) \leq X^{\varepsilon/4J}$ there is Y with $X^{1/J} \leq Y$, $Y^2 \leq X$, and, for all $x \in [Y, Y^2]$, both the 2-adic bound

$$\sum_{1 \leq e < t(x)} \operatorname{len} \mathrm{Sha}^1_e(E, 2) \leq \exp\big(x^{4\varepsilon/5}\big)$$

for every field $E = E_{\chi, e_0(x), q, d}$, and the layer bounds

$$\sum_{1 \leq e < t(x)} \operatorname{len} \mathrm{Sha}^1_e(L_1, \ell) \leq \exp\big(x^{4\varepsilon/5}\big)$$

for every $\ell \in B'$ and every layer L_1 of the cyclotomic $\mathbb{Z}_\ell$-tower over L_ℓ of degree ℓ^j with $j = \lfloor \log_\ell R_\ell \rfloor + 3$, where R_ℓ is a rank bound depending on ℓ and L_ℓ. The exponent J is explicit, $J = 2^{R+2}(2C_K C_t/\varepsilon)^R$ with R the sum of the rank bounds, and it is the origin of the exponent J in Theorem 54. That the bound is available only on a window $[Y, Y^2]$ inside $[X^{1/J}, X]$, and not at every x, is a pigeonhole phenomenon: the layer bounds are uniform in the layer only up to a finite number of exceptional twists e, and the window is chosen so that those exceptional twists lie outside the range $[1, t(x))$. This is the mechanism by which the proof yields a window theorem rather than a bound at every height, and it is the reason that Section 17 was needed on the analytic side.

The minus part

The minus part is the subject of the previous section and we only recall the shape of its bound. For every CM field $E = E_{\chi, e_0, q, d}$ with $e_0 \geq 2$ and every $t \geq 2$,

$$\sum_{\ell \neq 2} \log \ell \sum_{1 \leq e < t} \operatorname{len} \mathrm{Sha}^1_e(E, \ell)^- \leq 3nt^2 \log(tN_E) + C \sum_{\substack{\ell \mid o \\ \ell \neq 2}} \ell\, n^2 t^2 \log(t\ell N_E)^2,$$

where $n = [E : \mathbb{Q}]$, N_E is the least N with $E \subset \mathbb{Q}(\zeta_N)$, and C is absolute; the first term comes from the primes $\ell \nmid n$, where the minus-part length is bounded by the valuation of a generalized Bernoulli number through the Stickelberger element, and the second from the primes dividing the order o, where the Stickelberger argument needs a correction. This bound holds for every x, with no window, and it is polynomial in N_E and t.

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The plus part

The plus part is Theorem 64, combined as follows. For a host that is big for t, the primes $\ell \notin B_\chi$ are handled by Cases I, Z, II and III, which together give

$$\sum_{\ell\in\mathcal{S}} \log \ell \sum_{1\le e<t} \operatorname{len} \mathrm{Sha}^1_e(E,\ell)^+ \le (Nt)^C$$

for every finite set $\mathcal{S}$ of odd primes outside B_χ; the primes $\ell \in B_\chi$ are handled by Cases II, IV and V, which give, for each such ℓ,

$$\log \ell \sum_{1\le e<t} \operatorname{len} \mathrm{Sha}^1_e(E,\ell)^+ \le (Nt)^C + C\, B_\ell,$$

where B_ℓ is the layer bound of the previous paragraph for the tower over $\mathbb{Q}(\zeta_{\ell q_\chi})$. Adding over the finitely many $\ell \in B_\chi$ gives the plus-part bound in the form $(Nt)^C + C \max_\ell B_\ell$, for a larger C depending on χ. On a window where the layer bounds are $B_\ell \le \exp(x^{4\varepsilon/5})$ this is

$$\sum_{\ell\in\mathcal{S}} \log \ell \sum_{1\le e<t(x)} \operatorname{len} \mathrm{Sha}^1_e(E,\ell)^+ \le \big(N_E t(x)\big)^C + C\exp\big(x^{4\varepsilon/5}\big),$$

for every finite set $\mathcal{S}$ of odd primes.

The resulting theorem

Putting the three parts into Lemma 38, with $t(x) = \lfloor x^{401} \rfloor$ and $2^{e_0(x)} \asymp x^{2+8\varepsilon}$, gives for every x in the window

$$\sum_{\ell\in\mathcal{S}} \log \ell \sum_{1\le e<t} \operatorname{len} \mathrm{Sha}^1_e(E,\ell) \le \exp(x^{4\varepsilon/5}) \log 2 + \big(Nt\big)^{O(1)} + C\exp(x^{4\varepsilon/5}) \le \exp\big(\tfrac{1}{2}x^{\varepsilon}\big)$$

for x large, since $\log(Nt) = O(x^{\varepsilon/5}(\log x)^2)$. This is Theorem 54, the cohomological window theorem, whose hypotheses on (q,d) are exactly those under which the three parts are available at once: the arithmetic conditions of Section 23.1 for the plus part, and the size condition $\log(qd) \le X^{\varepsilon/4J}$ for the 2-adic and layer windows, which the range $\log d \le u(\log Y_1)^2$ with $u = X^{\varepsilon/5J}$ ensures.[33] We record it in the form in which it is used in the next subsection.

Theorem 65 (Cohomological window theorem [4]). *Let χ be a primitive Dirichlet character, $0 < \varepsilon < 1/52$, and $e_0(x)$ integers with $x^{2+8\varepsilon} \le 2^{e_0(x)} \le 2x^{2+8\varepsilon}$. There are $J \ge 1$ and X_0 such that for every $X \ge X_0$ and every pair (q,d) satisfying the window conditions of Theorem 54 relative to $Y_1 = X^{1/J}$, there is $Y \in [Y_1, X^{1/2}]$ such that for all $x \in [Y, Y^2]$, the field $E = E_{\chi,e_0(x),q,d}$ satisfies: the groups $\mathrm{Sha}^1_e(E,\ell)$ are finite for all primes ℓ and all $1 \le e < \lfloor x^{401} \rfloor$, and for every finite set $\mathcal{S}$ of primes*

$$\sum_{\ell\in\mathcal{S}} \log \ell \sum_{1\le e<\lfloor x^{401}\rfloor} \operatorname{len} \mathrm{Sha}^1_e(E,\ell) \le \exp\big(\tfrac{1}{2}x^{\varepsilon}\big).$$

The statement is for every choice of the auxiliary local data (the places above ℓ and the identification of the cyclotomic character) with which the project's definition of the Tate–Shafarevich group is made; this is what the phrase "for every choice of local data" in Appendix A refers to, and it is why the bound is not satisfied trivially.

23.3. From the Length Bound to the Energy Bound

The last step is the one that connects the Galois cohomology with the analytic argument of Sections 21 and 24. Its structure is simple once the objects are named, and we describe it in three

[33] The hypotheses are Poitou–Tate duality, local finiteness and local Tate duality, Lemma K, the window statement for 2 and the bad layers, the minus-part bound, and the five counts of Theorem 64.

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stages: the datum on which the length bound is imposed, the sum in which the length enters, and the connectedness statement that results.

The family and its hosts

The analytic argument works with a *family* $\mathcal{F}$ indexed by scales x in a set W of windows unbounded above, and at each scale by a finite index set I_x; to each (x, i) is attached a host datum $D = D_{x,i}$ and a size datum $L = L_{x,i}$ at scale x with parameter ε. The host datum D carries in particular the CM field $E = E_D$ of the host, the rank data R with its twist range $t = t_R$, and a system of Galois-cohomological "rows" including the lengths $\operatorname{len} \operatorname{Sha}^1_e(E, \ell)$ for all ℓ and $1 \le e < t$, together with a totally complex Galois coefficient field $\mathbb{L}$ with its set of finite places Pl, each place w carrying a residue characteristic, a ramification index e_w over $\mathbb{Q}$, a complex conjugate $\bar{w}$ and a *local twist type* μ_w, which equals either the untwisted type μ_0 or the twisted type of w. The size datum L carries the scale x, the parameters $\mathcal{L} = x^\varepsilon$, $b = \exp(\mathcal{L})$, $a = xb$, a finite support set $\operatorname{supp} \subset \mathrm{Pl}$ of places, and various weights indexed by the *lifts* $j \in \{1, \dots, m\}$ of the host. The quantities m (the number of lifts), f (the number of test points) and n are the rank parameters of the host, and D_0 is the rank of the "zero space" at the untwisted type, a dimension that is the same for all lifts j on the families considered.

The length bound of Theorem 65 is imposed on the family in the following form, called the *Sha hypothesis* of the family: there is x_0 such that for all $x \ge x_0$ and all $i \in I_x$, the field E of the host $D_{x,i}$ satisfies the length bound with $t = t_R$ and with the right-hand side $\sqrt{b} = \exp(\frac{1}{2}x^\varepsilon)$. This is precisely the conclusion of Theorem 65 at the scales of the family, which is why the scales in that theorem are called windows and why the energy argument is run only on them.

A first elementary consequence of the Sha hypothesis is a *record* of the lengths of the host: writing

$$\mathrm{vv}_D(\ell) = \log \ell \sum_{1 \le e < t} \operatorname{len} \operatorname{Sha}^1_e(E, \ell)$$

for primes ℓ and $\mathrm{vv}_D(\ell) = 0$ otherwise, the bound for every finite $\mathcal{S}$ implies that the series $\sum_\ell \mathrm{vv}_D(\ell)$ converges and has sum at most $\sqrt{b}$, and that every $\operatorname{Sha}^1_e(E, \ell)$ with $1 \le e < t$ is finite. This infinite sum is the quantity that enters the geometric side.

The place-weighted sum

On the geometric side of the host construction the family carries, for each lift j and each place w of $\mathbb{L}$, a non-negative rational number $\eta_j(w)$, the *jth local defect* at w. The definitions tell us this much about it: it is non-negative; it vanishes at every place whose local twist type is the untwisted one, since those places carry no twist; on the families actually constructed in the project it is given as a sum of local contributions over a finite set of "locations" attached to (j, w); and in the generic and class records of the construction it is identically zero, the defect being nonzero only in the twisted records. Each place carries the weight

$$\lambda(w) = \frac{\log \mathrm{N}w}{[\mathbb{L} : \mathbb{Q}]},$$

the logarithm of the absolute norm normalised by the degree, and the ramification index e_w. The *place-weighted sum* at lift j is

$$\eta\text{-}\textstyle\sum_L(j) = \displaystyle\sum_{\substack{w \in \operatorname{supp} \\ \mu_w \ne \mu_0}} \lambda(w)\, e_w\, \eta_j(w),$$

a finite sum over the twisted places of the support. In the language of Section 21, this is the total local contribution, over the places at which the host's Galois module is twisted, to the error made when the lift j of the host is compared with its untwisted model; the normalisation by $\log \mathrm{N}w / [\mathbb{L} : \mathbb{Q}]$ is the one under which the sum is comparable with a logarithm of a discriminant or of a conductor, and the factor e_w accounts for ramification of $\mathbb{L}$ over $\mathbb{Q}$. The construction of $\eta_j(w)$ belongs to the host construction of [4]; only the properties listed above are used.

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The statement that links the Galois cohomology to this sum is the following, which is a hypothesis on the family in the project and is discharged by the host construction.

Proposition 34 ([4], the link)**.** *There are C and x_0 such that for all $x \geq x_0$, all $i \in I_x$ at which the* $\mathrm{Sha}^1_e(E, \ell)$, $1 \leq e < t$, *are finite, and all lifts j,*

$$\eta\text{-}\sum_L(j) \;\leq\; d_0(j)\Big(C\big(f\,m\,b\,\mathcal{L}^2 + (\log x)^2\,\mathcal{N}_c\big) + \sum_{\ell} \mathrm{vv}_D(\ell)\Big),$$

whenever the series on the right converges, where $d_0(j)$ is the rank of the zero space of the lift j at the untwisted type and $\mathcal{N}_c$ is a rank budget of the host, polynomial in the rank data and in x^ε.

The inequality says that the local defects at the twisted places are controlled by two things: a main term, polynomial in the rank data and in $b = \exp(x^\varepsilon)$, which is the cost of the construction itself, and an additive term equal to the total Tate–Shafarevich length of the host, weighted by $\log \ell$ and summed over all primes. This is the only place in the analytic argument where the Galois cohomology of the host appears, and it appears exactly as an additive error term: the length of a Tate–Shafarevich group measures the failure of a local-global principle for the twisted Galois module, and that failure is what prevents the local defects from being read off from the untwisted model. The convergence hypothesis is supplied by the record of the previous paragraph.

The division bound and connectedness

Combining Proposition 34 with the record gives the bound that the core of the host decomposition needs, which the project calls the *division bound*: there are C' and x_0 such that for all $x \geq x_0$, all $i \in I_x$ and all lifts j,

$$\eta\text{-}\sum_L(j) \;\leq\; D_0 \cdot C'\,x^\varepsilon\,(B_0 + \mathcal{N}_c), \qquad B_0 = f\,m\,b\,x^\varepsilon,$$

with B_0 the size budget of the host. The deduction is elementary. On the family, the scale of $L_{x,i}$ is x and its parameter is ε, so $\mathcal{L} = x^\varepsilon$ and $b = \exp(x^\varepsilon) \geq 1$; the Sha hypothesis and the record give $\sum_\ell \mathrm{vv}_D(\ell) \leq \sqrt{b} \leq b \leq B_0$; the rank equality $d_0(j) = D_0$ on all lifts is a separate hypothesis of the family ("small level rank equality"); and $(\log x)^2 \leq x^\varepsilon$ for x large. The three terms on the right of Proposition 34 are then each at most $x^\varepsilon(B_0 + \mathcal{N}_c)$ up to the constant $|C| + 1$, which is C'. The length bound enters at one point only: it says that the additive Galois-cohomological error is at most $\sqrt{b}$, and $\sqrt{b}$ is absorbed into the main term $B_0 = fmb\,x^\varepsilon$ because $B_0 \geq b$. A length bound of the form $\exp(x^{\varepsilon'})$ with any $\varepsilon' \leq \varepsilon$ would have served equally; the exponent $\frac{1}{2}$ in $\exp(\frac{1}{2}x^\varepsilon)$ is a convenience of the assembly in Section 23.2, not a requirement of this step.

The connectedness statement of Definition 9 is now the following. A *host requirement* is a property of the host at a scale, and for the character χ the requirement G_χ is the conjunction of two conditions: the explicit-formula condition (E) of Theorem 58 at the host, with the parameter 4, and the Sha bound $\mathrm{VVle}(E, t, \sqrt{b})$ of Definition 8. The class of families *satisfying* G_χ at ratio $\lambda > 1$ is the class of families whose scales form windows of ratio λ and whose hosts satisfy G_χ at every scale beyond the family threshold. For every family in this class the Sha hypothesis holds, by reading off the second component of G_χ. Connectedness is the statement that for every $\lambda > 1$ and every host requirement G for which hosts are supplied on windows of ratio λ (Definition 8), if every family satisfying G satisfies the Sha hypothesis, then the class of families satisfying G is *connected*: there are exponents e_D, e_L and $\varepsilon_1 > 0$ such that for every $0 < \varepsilon < \varepsilon_1$ the class contains a family at parameter ε whose chains are *tied* on all of its windows, with constants bounded in terms of e_D, e_L and the family. The tied chains are the hypothesis (K) of Theorem 58: the gap, thin-scale and cost conditions of the core, made explicit on each window. The division bound is one of the inputs to tying the chains, and the chain of implications is: Sha hypothesis, hence record, hence (with the link and the rank equality) division bound, hence (with the other inputs of the host construction, which are analytic and do not involve Galois cohomology) tied chains.

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The explicit-formula condition is a separate analytic condition. It asks that every discarded character of the host have $|G_\theta(x)| \le x^{1/2+\varepsilon}$ at the scale and that the logarithmic derivatives at 1 of the characters involved be bounded by $C \log x$; it is a condition at a single scale on finitely many characters of conductor at most N, and it is supplied by the host construction, not by the Galois cohomology. The two conditions feed the host decomposition theorem separately: (E) directly as its hypothesis, and the Sha bound through the division bound into (K).

Finally, the two ends of the argument are joined. On one side, host supply: from Theorem 65 for χ and from the existence of admissible rank data at every scale, the project proves that hosts satisfying G_χ are supplied on windows of ratio $3/2$. Here Theorem 65 is used for the second component of G_χ, and its window $[Y, Y^2]$, with $Y \ge X^{1/J}$ and $Y^2 \le X$, is what gives windows of a fixed ratio arbitrarily far out. On the other side, connectedness applied to G_χ gives, for each small ε, a family in the class whose hosts satisfy (E) and whose chains are tied; Theorem 58 then gives $E_\chi(x) \le x^{1/2+(e_D+e_L+7)\varepsilon}$ on all the windows of that family, Theorem 52 gives $L(s, \chi^j) \ne 0$ for $\operatorname{Re} s > 1/2 + (e_D + e_L + 7)\varepsilon$, and $\varepsilon \to 0$ gives Hypothesis H for χ. This is the composition recorded in Corollary 25.

Remark 17 (Scope of this section)**.** *The theorems, lemmas and propositions attributed above to [4], the cases of the case split and the bounds are proved in that project and are stated here in the notation of this paper; they are not reproved here. Four steps have been checked by hand: the exhaustiveness of the case split for a big host, from the definitions of the cases, of B_χ and of N; the deduction of the division bound from the link, the record, the rank equality and the Sha hypothesis; the assembly of Lemma 38; and the arithmetic of the window ranges in Section 23.1, namely that q, d lie above every quantity attached to χ and t, that 2 is inert in $\mathbb{Q}(\sqrt{-q})$ and split in $\mathbb{Q}(\sqrt{-d})$, that odd primes up to u split in $\mathbb{Q}(\sqrt{-d})$, and that $(Nt)^{O(1)} \le \exp(\frac{1}{2}x^\varepsilon)$ on the windows. The classical mechanisms named in this section (Ribet's method and its extensions by Mazur–Wiles, Wiles, Skinner–Wiles, Skinner–Urban, Urban and Bellaïche–Chenevier, the Taylor–Wiles–Kisin modularity lifting, Kato's theorem, Iwasawa's growth formula with Ferrero–Washington, Soulé's finiteness and Green's parametrisation) are those of the published literature cited; they explain what the statements of [4] are instances of, and the reasons given for the presence of particular hypotheses are our reading of that project.*

24. The Top of the Proof

This section follows the two inputs of Section 21 up to the final theorem of [4] and shows that both of them come from a single statement about Galois representations. The proof in [4] has four links.

24.1. The Chain

1. Hypothesis H is deduced from one statement on Galois representations, Input A of Appendix A: the existence of the Galois representations attached to polarized cuspidal automorphic representations over CM fields, with local–global compatibility at every place.
2. That statement is used in two places: through the three literature statements of Appendix A (local–global compatibility, the local Langlands correspondence, Galois representations for polarized pairs), and through the window bound of the next link.
3. Input A implies the window bound, that is, the Sha bound of Definition 8 on windows. This is a modularity argument. Through residual modularity at depth zero, base change for isobaric representations, and the abelian theory of the CM host, the Selmer groups of the host's Galois modules are controlled. The Greenberg–Wiles Euler characteristic formula then turns this control into the Sha bound.
4. The literature statements and the window bound imply Hypothesis H. The window bound gives host supply (Definition 8), and connectedness (Definition 9) comes from a shape theorem on windows. That theorem uses J. A. Green's parametrisation of the cuspidal characters of $\mathrm{GL}_d(\mathbb{F}_g)$ to control the local types of the theta lifts. From there the argument is Sections 21 and 17.

In the published literature, Input A is the theorem of Harris and Taylor [64], Shin [67] and Chenevier and Harris [63], with the local–global compatibility of Caraiani [61,62]. The form used in [4]

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includes local–global compatibility at every place, and [4] proves it itself rather than citing it. What is new in [4] is the passage from Input A to the window bound, links 3 and 4 above; together with Theorem 58 it gives Theorem 53.

24.2. The Main Theorem

Theorem 66 (The Riemann hypothesis and its generalization)**.** *Every nontrivial zero of $\zeta(s)$ has real part $1/2$. For every Dirichlet character χ, $L(s,\chi) \neq 0$ for $\operatorname{Re} s > 1/2$ (and $s \neq 1$ when χ is trivial), and every zero of a primitive Dirichlet L-function in the strip $0 < \operatorname{Re} s < 1$ lies on $\operatorname{Re} s = 1/2$. Moreover $Q(N) \ll_\varepsilon N^{1+\varepsilon}$ for every $\varepsilon > 0$.*

Proof. Let ψ be a primitive character and $\varepsilon > 0$ small. By Theorem 53 the energy of the cyclic family $\mathcal{F}_\psi$ satisfies $E_\psi(y) \leq Cy^{1/2+\varepsilon}$ on windows $[Y, Y^{1+\delta}]$ at arbitrarily large heights. Theorem 52 with $\theta = 1/2 + \varepsilon$ gives $L(s,\psi^j) \neq 0$ for $\operatorname{Re} s > 1/2 + \varepsilon$ and every j, in particular for ψ itself and, with ψ the character modulo 1, for ζ. Letting $\varepsilon \to 0$ gives the zero-free half-plane for every primitive character. An imprimitive character χ induced from χ^* has $L(s,\chi) = L(s,\chi^*)\prod_{p\mid N}(1-\chi^*(p)p^{-s})$, and the Euler factors do not vanish in $\operatorname{Re} s > 0$, so Hypothesis H holds. Theorem 56 gives the statements on the critical line, and Theorem 55(ii) gives the energy bound. □

The deduction in this proof, from the window bound to the Riemann hypothesis and to the critical line for primitive characters, is checked in the human-readable Lean repository of this paper [10], where it takes 32 files and about 2,700 lines, with only the standard axioms. The window theorem enters it as one explicitly stated hypothesis.

24.3. What This Paper Proves and What It Cites

Part I (including Section 17), Section 20 and the host decomposition of Section 21 are proved in full here, and Section 22 derives in full, from the published theorems it cites, the passage from Tate–Shafarevich groups to K-groups and Bernoulli numbers. The window theorem is cited from [4], where it is derived from the published theorems on Galois representations; it is the one statement of the paper that is not proved in the text.

Part III

Consequences

Every result of this part is proved in full from Theorem 66, or from the unconditional half-plane where stated, and each has its own Lean repository, cited in a footnote where the result appears.

25. Two Corollaries of RH and GRH

This section records two corollaries of the Riemann hypothesis and the generalized Riemann hypothesis in the form of Hypothesis H: the sharp Pólya–Vinogradov inequality (Corollary 27) and a power-saving averaged Chowla bound (Corollary 29). Without any form of the Riemann hypothesis, the zero-free half-plane announced by OpenAI gives the first of them as well (Theorem 67). The subsection on Chowla's conjecture also records two results that need no hypothesis on zeros: Chowla's conjecture for almost all shifts (Corollary 31) and both signs of $\lambda(n)\lambda(n+h)$ at every fixed shift (Corollary 32).

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25.1. The P ólya–Vinogradov Inequality

For a non-principal character χ modulo q, Pólya and Vinogradov proved in 1918 that $\max_x \left|\sum_{n\le x} \chi(n)\right| \ll \sqrt{q}\log q$.[34]

Corollary 27 (Pólya–Vinogradov under RH and GRH)**.** *Assume Hypothesis H (equivalently W), that is, RH and GRH for Dirichlet L-functions. There is an absolute constant c such that every primitive Dirichlet character χ modulo $q \ge 3$ satisfies*

$$M(\chi) := \max_{x\ge 1}\left|\sum_{n\le x}\chi(n)\right| \le c\,\sqrt{q}\,\log\log q.$$

This is best possible up to the constant: there are infinitely many quadratic characters with $M(\chi) \ge c'\sqrt{q}\log\log q$ for an absolute $c' > 0$.[35]

Proof. For every r and every character ψ modulo r, the character $\chi\psi$ is a Dirichlet character, so Hypothesis H gives $L(s,\chi\psi) \ne 0$ in $\operatorname{Re} s > 1/2$ (Lemma A7 shows that $\chi\psi$ is non-principal when $r < q$, so the pole at $s = 1$ does not occur). This is the hypothesis (A22) of the transfer theorem, Theorem A1, with $\theta = 1/2$ and every height, and that theorem gives $M(\chi) \le 2c_4\sqrt{q}\log\log q$. Appendix L proves it in full; the method is that of Montgomery and Vaughan [7], who proved the bound under the generalized Riemann hypothesis. The lower bound is Paley's [13], and it is unconditional. □

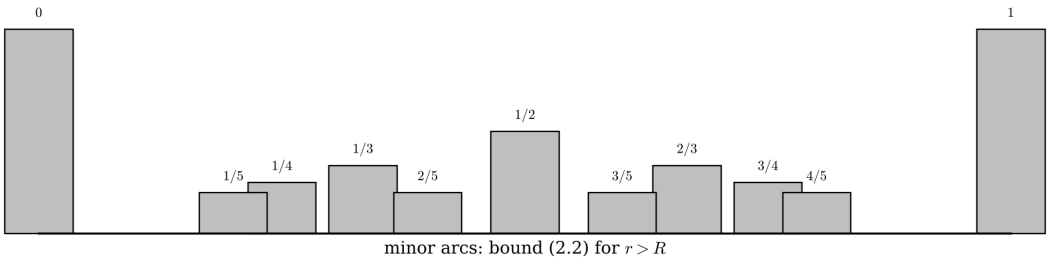

Figure 14. The dissection of $\mathbb{R}/\mathbb{Z}$ used in the proof of Corollary 27 (Appendix L), drawn with $R = 5$. Near a fraction a/r with $r \le R$ the character sum is a combination of prime sums of the twists $\chi\psi$, ψ mod r, which Hypothesis H makes small; away from such fractions it is small by the estimate of Montgomery and Vaughan for multiplicative coefficients.

The theorem settles, under Hypothesis H, the size of the largest character sums up to the value of the constant.[36]

[34] This classical inequality needs no hypothesis on zeros. In the form $\left|\sum_{M\le n<M+L}\chi(n)\right| \le \sqrt{q}\,(1+\log q)$ for every primitive χ modulo $q > 1$, it is verified in Lean 4, through the Gauss sum identity $|\tau(\chi)| = \sqrt{q}$ and the Fourier expansion of χ, in the repository github.com/anaxia-research/polya-vinogradov-lean, with only the standard axioms.

[35] The bound $M(\chi) \ll \sqrt{q}\log\log q$ from Hypothesis H is verified in Lean 4 in github.com/anaxia-research/character-sums-grh and in github.com/anaxia-research/Polya-rh, with only the standard axioms.

[36] Granville and Soundararajan [14] determined the optimal constant conditionally for characters of odd order and improved the unconditional Pólya–Vinogradov bound for them. The constant in Corollary 27 is not determined here.

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25.2. The Unconditional Pólya–Vinogradov Inequality

OpenAI has announced a proof [8,11] that no Dirichlet L-function, the Riemann zeta function included, has a zero with $\operatorname{Re} s > 7/8$.[37] With the transfer theorem of Appendix L this removes every hypothesis from the Pólya–Vinogradov bound, at the cost of the constant.

Theorem 67 (Without the Riemann hypothesis)**.** *There is an absolute constant c such that every primitive Dirichlet character χ modulo $q \geq 3$ satisfies*

$$\max_{x \geq 1} \Big| \sum_{n \leq x} \chi(n) \Big| \leq c \sqrt{q} \log\log q.$$

Proof. For every $r < q$ and every character ψ modulo r, the character $\chi\psi$ is non-principal (Lemma A7), so by the theorem of OpenAI $L(s, \chi\psi) \neq 0$ in $\operatorname{Re} s > 7/8$. This is the hypothesis (A22) of Theorem A1 with $\theta = 7/8$, at every height, and that theorem gives the bound with $c = 8c_4$. □

Corollary 28 (Without the Riemann hypothesis)**.** *For every $\varepsilon > 0$, $Q(N) \ll_\varepsilon N^{7/4+\varepsilon}$ and*

$$\sum_{h=1}^{N-1} \Big(1 - \frac{h}{N}\Big) C_h(N-h) \ll_\varepsilon N^{7/4+\varepsilon}.$$

Proof. The half-plane $\operatorname{Re} s > 7/8$ gives $\Theta \leq 7/8$, and Theorem 8 with $\sigma = 7/8 + \varepsilon/2$ gives $Q(N) \ll N^{7/4+\varepsilon}$. The second bound follows by the Fejér identity, as in the proof of Corollary 29 below. □

If the half-plane of [8] stands, these are the first power savings in the Liouville energy and in the averaged correlations of λ over all shifts; without it, only the Vinogradov–Korobov saving of Theorem 48 is available.

25.3. Chowla's Conjecture

Chowla's conjecture asserts that for each fixed $h \geq 1$ the correlation $C_h(N) = \sum_{n \leq N} \lambda(n)\lambda(n+h)$ is $o(N)$. The energy gives an averaged form with a power saving.

Corollary 29 (Averaged Chowla under RH and GRH)**.** *Assume Hypothesis H, that is, RH and GRH for Dirichlet L-functions. Then for every $\varepsilon > 0$,*

$$\sum_{h=1}^{N-1} \Big(1 - \frac{h}{N}\Big) C_h(N-h) \ll_\varepsilon N^{1+\varepsilon}.$$

Proof. By the Fejér identity (3) with $a = \lambda$ and $H = N$, $Q(N) = N + 2\sum_{h=1}^{N-1}(1 - h/N)C_h(N-h)$, and $Q(N) \ll N^{1+\varepsilon}$ by Corollary 22. □

The same averaging holds without the Fejér weights, at the cost of a weaker but still power-saving exponent.

[37] The first preprint [8] proves the half-plane $\operatorname{Re} s > 7/8$ with a proof written in Lean 4 and, by its account, checked by the Lean kernel over Mathlib. The second [11] gives a proof of ordinary length of the weaker half-plane $\operatorname{Re} s > 11/12$, through a mean square of twisted Möbius sums over the Eisenstein integers, Kubota's cubic theta function and the quadratic large sieve; it covers every finite-order Hecke L-function over $\mathbb{Q}(\sqrt{-3})$, and the Dirichlet case follows by base change, $L_K(s, \chi \circ \mathrm{N}) = L(s,\chi)L(s,\chi\chi_{-3})$. Neither has yet been refereed. Either half-plane suffices below, with a different constant. The results of this subsection are unconditional in the sense that they assume no form of the Riemann hypothesis; they rest on this announced theorem.

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Corollary 30 (Two-point correlations averaged over the shift)**.** *Assume Hypothesis H. Then for every $\varepsilon > 0$ there is C_ε such that for all $N, H \geq 1$*

$$\Big|\sum_{h=1}^{H} C_h(N)\Big| \leq C_\varepsilon N(N+H)^{1/2+\varepsilon}, \qquad \textit{in particular} \qquad \sum_{h=1}^{N} C_h(N) \ll_\varepsilon N^{3/2+\varepsilon}.$$

Proof. Exchanging the sums, $\sum_{h\leq H} C_h(N) = \sum_{n\leq N} \lambda(n) \sum_{m\leq H} \lambda(n+m)$, and $\sum_{m\leq H} \lambda(n+m) = L(n+H) - L(n)$. By Corollary 34(ii), $|L(y)| \leq Cy^{1/2+\varepsilon}$, so the inner sum is at most $2C(N+H)^{1/2+\varepsilon}$ for $n \leq N$, and $|\lambda(n)| = 1$ gives the bound.[38] □

For almost all shifts the full conjecture holds, with no hypothesis on zeros.

Corollary 31 (Chowla for almost all shifts)**.** *Assume the averaged Chowla theorem of Matomäki, Radziwiłł and Tao [47]. For every $k \geq 1$ and $\delta > 0$, for all large H and then all large X, at most δH^k tuples $(h_1, \dots, h_k) \in [1, H]^k$ have a sign pattern $\epsilon \in \{\pm 1\}^k$ whose frequency $\frac{1}{X}\#\{n \leq X : \lambda(n+h_i) = \epsilon_i \ (1 \leq i \leq k)\}$ differs from 2^{-k} by more than δ.*

Proof. The indicator of the pattern ϵ is $\prod_i \frac{1}{2}(1 + \epsilon_i \lambda(n+h_i)) = 2^{-k} \sum_S \prod_{i\in S} \epsilon_i \lambda(n+h_i)$, the sum over subsets $S \subseteq \{1, \dots, k\}$. The term $S = \varnothing$ gives 2^{-k}. For $S \neq \varnothing$ the theorem of [47] gives $\sum_{h\in[1,H]^S} \big|\sum_{n\leq X} \prod_{i\in S} \lambda(n+h_i)\big| \leq \eta H^{|S|} X$ for large H and then large X, with $\eta = \delta^2/2^k$. Summing over the 2^k subsets and applying Markov's inequality to the tuples where the deviation exceeds δ gives at most δH^k exceptional tuples.[39] □

The trivial bound for the left side is $N^2/2$, and the unconditional averaged results [47] save only a factor tending to infinity slowly. Corollary 29 saves a full power, but it bounds an average over h and says nothing about a single h. That is not a weakness of the method but a limitation of every argument through zeros, and Theorem 3 explains why. The sequences λ_S of that theorem with $\#S \cap [1, x] \leq x^{1/2+o(1)}$ have the same energy exponent as λ, so a zero-free half-plane cannot distinguish them from λ, while their correlations at a fixed h can be changed by the choice of S.[40] Chowla's conjecture for a single shift is therefore beyond the reach of Hypothesis H; the known cases [15,56] use the multiplicative structure of λ directly, not its zeros.

The simplest use of that structure already gives, with no hypothesis at all, both values of $\lambda(n)\lambda(n+h)$ at every fixed shift, in explicit windows.

Corollary 32 (Both signs at every fixed shift)**.** *Let $h, m \geq 1$. There is an integer n with $243hm \leq n < 256hm$ and $\lambda(n)\lambda(n+h) = -1$, and there is an integer n with $486hm \leq n < 512hm$ and $\lambda(n)\lambda(n+h) = 1$. Consequently, for every $h \geq 1$ each of the two values occurs for infinitely many n, and for $x \geq 512h$ each occurs for at least $\log_2(x/(512h))$ integers $n \leq x$.*

Proof. The function λ is completely multiplicative, with $\lambda(3^5) = (-1)^5 = -1$ and $\lambda(2^8) = (-1)^8 = 1$. Put $a = 243hm$ and $b = 256hm$. Then $\lambda(a) = -\lambda(hm)$ and $\lambda(b) = \lambda(hm)$, and $\lambda(hm) \neq 0$, so $\lambda(a) \neq \lambda(b)$. The points a, $a+h$, …, $a+13mh = b$ form a walk of $13m$ steps of length h (Figure 15). Its end values differ, so the value of λ changes at some step: there is j with $0 \leq j < 13m$ such that $n = a + jh$ satisfies $\lambda(n) \neq \lambda(n+h)$. Since both values lie in $\{\pm 1\}$, this means $\lambda(n)\lambda(n+h) = -1$, and $a \leq n < b$.

[38] Verified in Lean 4 in github.com/anaxia-research/averaged-chowla-grh, with only the standard axioms.

[39] Verified in Lean 4 in github.com/anaxia-research/chawla-rh, with the theorem of [47] as a named hypothesis and only the axioms the standard axioms.

[40] Changing $\lambda(p)$ for $p \in S$ changes $\lambda(n)$ on the multiples of S, and among $n \leq N$ these have density $\sum_{p\in S} 1/p$, which may be positive for a set S of size $x^{1/2+o(1)}$ chosen among small primes; the correlation $C_h(N)$ then moves by a positive proportion of N for suitable S. Thus no statement that depends only on the zeros, and is therefore shared by λ and these λ_S, can decide $C_h(N) = o(N)$ for a fixed h.

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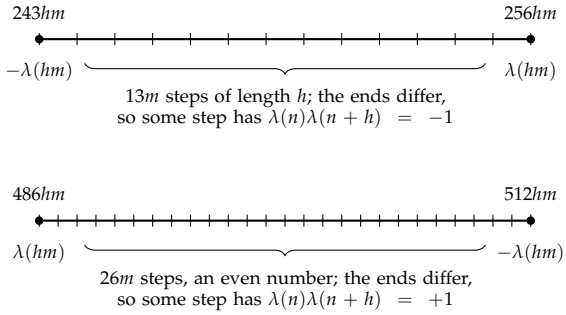

Figure 15. The two walks in the proof of Corollary 32, drawn for $m = 1$, with the value of λ below each end. The upper ends differ because $\lambda(3^5) = -1$ and $\lambda(2^8) = 1$. The lower walk is the upper one doubled, so its ends also differ; a walk that changed sign at every one of its $26m$ steps would end where it started.

For the second statement put $a' = 2a = 486hm$ and $b' = 2b = 512hm$. Since $\lambda(2) = -1$, we have $\lambda(a') = \lambda(hm)$ and $\lambda(b') = -\lambda(hm)$, so again $\lambda(a') \neq \lambda(b')$. The walk a', $a' + h$, …, $a' + 26mh = b'$ has $26m$ steps. Suppose that $\lambda(n)\lambda(n+h) = -1$ for each of the points $n = a' + jh$, $0 \le j < 26m$. Then λ changes sign at every step, and after an even number of steps $\lambda(b') = (-1)^{26m}\lambda(a') = \lambda(a')$, a contradiction. Hence some n with $a' \le n < b'$ has $\lambda(n)\lambda(n+h) = 1$.

For the count take $m = 2^i$ with $i \ge 0$. Since $256 < 2 \cdot 243$, the windows $[243h2^i, 256h2^i)$ are pairwise disjoint, and since $512 < 2 \cdot 486$ so are the windows $[486h2^i, 512h2^i)$. Each window of either kind contains an integer n of the corresponding sign, and both windows with index i lie in $[1, x]$ as soon as $512h2^i \le x$. The indices $0 \le i \le \log_2\big(x/(512h)\big)$ are at least $\log_2\big(x/(512h)\big)$ in number, which gives the count and, letting $x \to \infty$, the infinitude.[41] □

Corollary 32 is much weaker in its count than the theorem of Matomäki and Radziwiłł [46, Corollary 2], by which $\big|\sum_{n \le x} \lambda(n)\lambda(n+h)\big| \le (1 - \delta(h))x$ for large x, so that each value occurs for a positive proportion of $n \le x$. Its interest is that the windows are explicit, so that each interval $[243hm, 256hm)$ and each interval $[486hm, 512hm)$ contains a point of the required sign, and that the proof uses nothing but the two values $\lambda(3^5) = -1$ and $\lambda(2^8) = 1$. Neither result gives Chowla's conjecture $C_h(N) = o(N)$ at a fixed shift.

26. Further Corollaries

Hypothesis H has many classical consequences. This section derives the principal ones in full from a single analytic lemma, Lemma 40 below, which bounds $\log L(s, \chi)$ to the right of a zero-free half-plane. Every argument takes as input only a half-plane $\operatorname{Re} s > a$ free of zeros, so each corollary holds in two forms: with $a = 1/2$ under Hypothesis H, and with $a = 7/8$ under the half-plane announced by OpenAI [8], which assumes no form of the Riemann hypothesis. Table 20 lists the results and the Lean declarations that check them.[42]

Throughout, χ is a Dirichlet character modulo q, $s = \sigma + it$, and $\mathcal{L} = \log\big(2q(3 + |t|)^2\big)$. For non-principal χ the function $L(s, \chi)$ is entire. For $\operatorname{Re} s > 0$ partial summation gives $L(s, \chi) = s\int_1^\infty S(u)u^{-s-1}\,\mathrm{d}u$ with $S(u) = \sum_{n \le u} \chi(n)$ and $|S(u)| \le q$, so

$$|L(s, \chi)| \le \frac{q|s|}{\sigma} \le 2q|s| \qquad (\sigma \ge \tfrac{1}{2}). \tag{142}$$

[41] Verified in Lean 4 in full, with no hypothesis, for the two windows, the infinitude and the count, in the form "at least k integers $n \le 256h2^k$, respectively $n \le 512h2^k$", in github.com/anaxia-research/chawla-rh, with only the standard axioms.

[42] The Lean declarations are in the repository [10], which states Hypothesis H word for word as in [4]. The corollaries in the $a = 7/8$ form are proved from the half-plane hypothesis of the authors' Apache-licensed library [9], which uses general lemmas of OpenAI's library but not its half-plane, and Hypothesis H implies that hypothesis. So they hold under H with no use of the announced half-plane.

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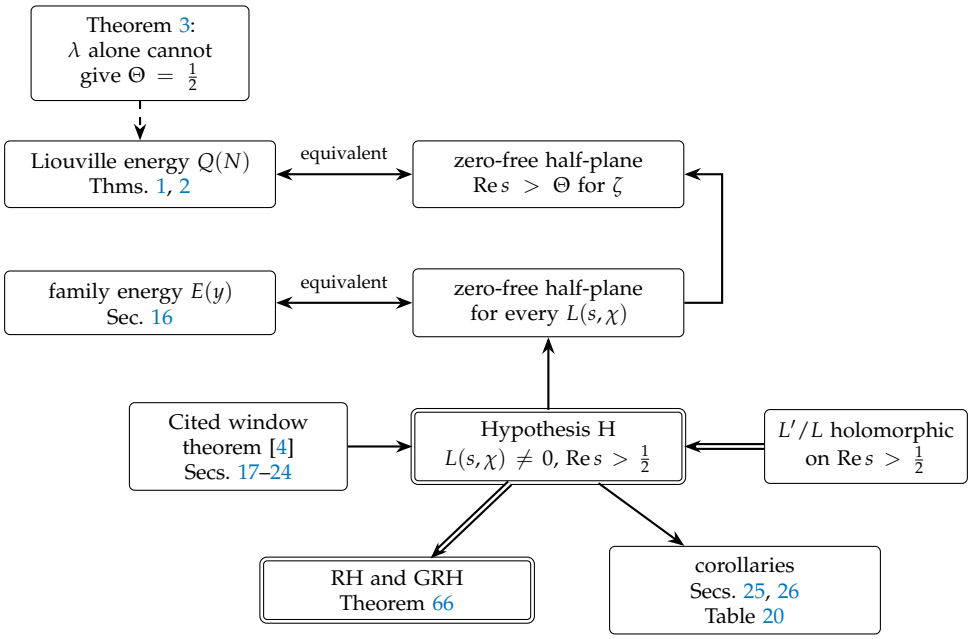

Figure 16. The paper as a whole. The energy of the Liouville function, and the family energy of the Dirichlet characters, are each equivalent to a zero-free half-plane (Theorems 1 and 2, and Section 16). Theorem 3 (dashed) shows that the energy of λ alone cannot reach $\Theta = \frac{1}{2}$, so the half-plane enters as Hypothesis H, supplied by the window theorem of [4] (Theorem 53); a holomorphic L'/L would also imply it. Double borders and double arrows mark the steps checked in Lean [10].

26.1. Logarithmic Control

Lemma 40 (Logarithmic control). *Let $\frac{1}{2} \le a < 1$ and $0 < e < \frac{1}{1000}$, and let χ be non-principal with $L(s,\chi) \neq 0$ for $\operatorname{Re} s > a$. There are constants $c_1(e)$ and, for every $\varepsilon > 0$, $c_2(e,\varepsilon)$, not depending on q, χ, a or t, such that for $a + 2e \le \sigma \le 2$*

$$\left|\log L(s,\chi)\right| \le \varepsilon\mathcal{L} + c_2(e,\varepsilon), \qquad \left|\frac{L'}{L}(s,\chi)\right| \le c_1(e)\,\mathcal{L},$$

where $\log L$ is the branch given on $\operatorname{Re} s > 1$ by $\sum_n \chi(n)\Lambda(n)n^{-s}/\log n$.

Proof. Fix t and put $c = 2 + it$ and $R = 2 - a - e/2$. The disc $|s - c| < R$ lies in $\operatorname{Re} s > a$, where L has no zeros, so $\log L$ extends holomorphically to it, and $|\log L(c)| \le \log\zeta(2) < 1$. On the disc, $|s| \le 4 + |t| \le (3 + |t|)^2$, and (142) gives $\operatorname{Re}\log L(s) = \log|L(s)| \le \mathcal{L}$.

Step 1 (Borel–Carathéodory). For $r = R - e/2 = 2 - a - e$, the Borel–Carathéodory inequality gives

$$\max_{|s-c|\le r} |\log L(s)| \le \frac{2r}{R-r} \max_{|s-c|\le R} \operatorname{Re}\log L(s) + \frac{R+r}{R-r}|\log L(c)| \le \frac{8}{e}(\mathcal{L}+1).$$

Step 2 (three circles). Let $r_1 = \frac{1}{2}$, $r_2 = 2 - \sigma$ and $r_3 = r$. If $\sigma \ge 3/2$, then $|\log L(s)| \le \log\zeta(3/2) < 2$ and there is nothing to prove. Otherwise $r_1 < r_2 \le r_3$, because $\sigma \ge a + 2e$. On $|s - c| \le r_1$ we have $\operatorname{Re} s \ge 3/2$, so $|\log L| \le 2$ there. Hadamard's three circles theorem gives $|\log L(\sigma + it)| \le 2^{1-\alpha}\big(8(\mathcal{L}+1)/e\big)^{\alpha}$ with

$$\alpha = \frac{\log(r_2/r_1)}{\log(r_3/r_1)} \le \frac{\log\big(2(2-a-2e)\big)}{\log\big(2(2-a-e)\big)} =: 1 - \delta,$$

and $\delta = \delta(e) > 0$ uniformly in $a \in [\frac{1}{2}, 1)$. Since $y^{1-\delta} \le \varepsilon' y + c(\varepsilon', \delta)$ for $y \ge 0$, the first bound follows.

Step 3 (Cauchy). For $\sigma \ge a + 2e$ the disc $|z - s| \le e/2$ lies in $|z - c| \le r$, so Cauchy's estimate and Step 1 give $|L'/L(s)| = |(\log L)'(s)| \le (2/e)(8/e)(\mathcal{L}+1) \le c_1(e)\mathcal{L}$. □

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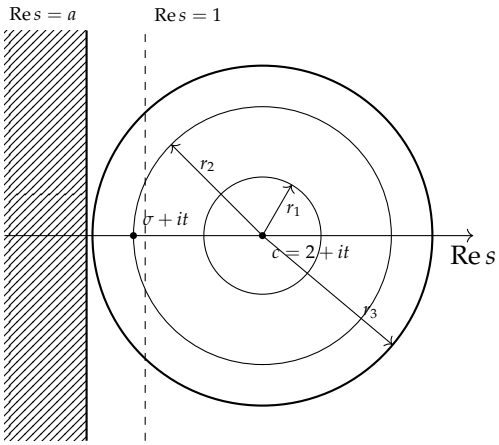

Figure 17. The three circles in the proof of Lemma 40, centred at $c = 2 + it$. On the inner circle ($r_1 = \frac{1}{2}$) $\log L$ is bounded by the Euler product. On the outer circle ($r_3 = 2 - a - e$), Borel–Carathéodory bounds $\log L$ by $O(\mathcal{L}/e)$, because the circle stays to the right of the zero-free line $\operatorname{Re} s = a$ (hatched region to its left). The middle circle ($r_2 = 2 - \sigma$) passes through $\sigma + it$, and Hadamard's theorem interpolates with an exponent $\alpha < 1$.

Corollary 33 (Lindelöf bounds). [43] *Under Hypothesis H, for every $w > 0$, every non-principal χ modulo q and every s with $\operatorname{Re} s \geq \frac{1}{2} + w$,*

$$|L(s,\chi)| + |L(s,\chi)|^{-1} \ll_w \big(q(3+|t|)\big)^w.$$

In particular the Lindelöf hypothesis holds: $\zeta(\frac{1}{2}+it) \ll_\varepsilon (3+|t|)^\varepsilon$, [44] *and $L(\frac{1}{2}+it,\chi) \ll_\varepsilon (q(3+|t|))^\varepsilon$ for primitive χ. Under the half-plane $\operatorname{Re} s > 7/8$ the first bound holds for $\operatorname{Re} s \geq 7/8 + w$.*

Proof. Lemma 40 with $a = 1/2$, $2e \leq w$ and $\varepsilon = w/2$ gives $|\log L| \leq \frac{w}{2}\mathcal{L} + c_2$ for $\frac{1}{2} + w \leq \sigma \leq 2$, and $e^{\frac{w}{2}\mathcal{L}} \leq 2^w (q(3+|t|))^w$. For $\sigma > 2$ both $|L|$ and $|L|^{-1}$ are at most $\zeta(2)$. For ζ the same argument applies to $|t| \geq 3$, where the disc of Lemma 40 does not reach the pole at $s = 1$.

On the critical line we use the functional equation. For primitive χ, $|L(\frac{1}{2} - w + it, \chi)| \asymp_w (q(3+|t|))^w |L(\frac{1}{2} + w - it, \bar{\chi})|$ by Stirling's formula, so $|L| \ll (q(3+|t|))^{2w}$ on both lines $\sigma = \frac{1}{2} \pm w$. The Phragmén–Lindelöf principle in the strip between them, applied to $L(s,\chi)(q(s+3))^{-2w}$ (and to $(s-1)\zeta(s)(s+3)^{-1-2w}$ for ζ), gives the same bound on $\sigma = \frac{1}{2}$. Since w is arbitrary, this is the Lindelöf hypothesis. □

26.2. Sums of Arithmetic Functions

Lemma 41 (Smoothing). *Let $F(s) = \sum_{n \geq 1} a_n n^{-s}$ converge absolutely for $\operatorname{Re} s > 1$, with $|a_n| \leq B \log(n+2)$. Let $\frac{1}{2} \leq \sigma_0 < 1$ and $0 < w < 1$. Suppose F continues holomorphically to $\operatorname{Re} s \geq \sigma_0$, except for at most a simple pole at $s = 1$ with residue ρ, and $|F(\sigma + it)| \leq K(3+|t|)^w$ for $\sigma_0 \leq \sigma \leq 2$. Then for $x \geq 2$*

$$\sum_{n \leq x} a_n = \rho x + O\big((K + B + |\rho|)\, x^{\sigma_0 + w} \log x\big),$$

with an absolute implied constant.

Proof. Put $R(y) = \sum_{n \leq y} a_n (y - n)$. For $y > 0$, Perron's formula with the absolutely convergent kernel $y^{s+1}/(s(s+1))$ gives $R(y) = \frac{1}{2\pi i} \int_{(2)} F(s) \frac{y^{s+1}}{s(s+1)}\, ds$. Move the line to $\operatorname{Re} s = \sigma_0$. The horizontal

[43] Verified in Lean 4 in full, for $\operatorname{Re} s \geq \frac{1}{2} + w$: in github.com/anaxia-research/liouville-energy-rh, with only the standard axioms. The form $\operatorname{Re} s \geq \frac{7}{8} + w$ is also in github.com/anaxia-research/character-sums-grh.

[44] Verified in Lean 4 in full at $\sigma = \frac{1}{2}$: in github.com/anaxia-research/lindelof-rh, with only the standard axioms. The Lean proof localises ζ at height t_0 by the factor $e^{(s-\frac{1}{2}-it_0)^2}$ and applies Hadamard's three-lines theorem on $\frac{1}{4} \leq \operatorname{Re} s \leq \frac{1}{2} + 8e$.

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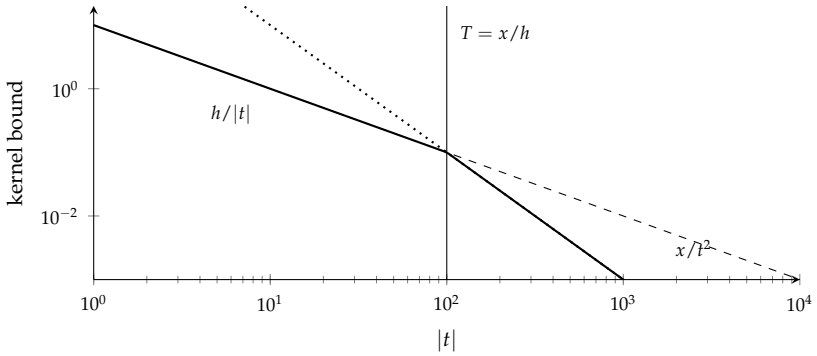

Figure 18. The kernel in the proof of Lemma 41, drawn with $x = 1000$ and $h = 10$. The difference $((x+h)^{s+1} - x^{s+1})/(s(s+1))$ is at most $(2x)^{\sigma_0}$ times $\min(h/|t|, 4x/t^2)$ (solid). The first bound (dashed) is used for $|t| \le T = x/h$ and the second (dotted) beyond. Both parts contribute $O(hT^w \log T)$, which gives the error $x^{\sigma_0+w} \log x$.

segments vanish in the limit by the growth bound, and the residue at $s = 1$ is $\rho y^2/2$, so $R(y) = \rho y^2/2 + I(y)$, where $I(y)$ is the integral over $\operatorname{Re} s = \sigma_0$.

For $1 \le h \le x$,

$$hS(x) = R(x+h) - R(x) - \sum_{x<n\le x+h} a_n(x+h-n), \qquad S(x) = \sum_{n\le x} a_n,$$

and the last sum is at most $(h+1)h\,B\log(2x+2)$. The main terms give $\rho((x+h)^2 - x^2)/2 = \rho hx + \rho h^2/2$. In $I(x+h) - I(x)$ the kernel is $((x+h)^{s+1} - x^{s+1})/(s(s+1))$, and $|(x+h)^{s+1} - x^{s+1}| = |s+1|\,\big|\int_x^{x+h} u^s\,\mathrm{d}u\big| \le |s+1|h(2x)^{\sigma_0}$, and also $\le 2(2x)^{\sigma_0+1}$. With $T = x/h$, the integral is therefore at most

$$K(2x)^{\sigma_0}\Big(\int_{|t|\le T} \frac{h(3+|t|)^w}{|\sigma_0+it|}\,\mathrm{d}t + \int_{|t|>T} \frac{4x(3+|t|)^w}{t^2}\,\mathrm{d}t\Big) \ll Kx^{\sigma_0} h\,T^w \log(3+T).$$

Dividing by h and taking $h = x^{\sigma_0}$, so that $T^w \le x^w$, gives $S(x) = \rho x + O(Kx^{\sigma_0+w}\log x + |\rho|x^{\sigma_0} + Bx^{\sigma_0}\log x)$. □

Corollary 34 (Primes, Möbius and Liouville sums). [45] *Under Hypothesis H, for every $\varepsilon > 0$:*

1. $\psi(x) = x + O_\varepsilon(x^{1/2+\varepsilon})$, *hence* $\pi(x) = \operatorname{li}(x) + O_\varepsilon(x^{1/2+\varepsilon})$;[46]
2. $M(x) = \sum_{n\le x}\mu(n) \ll_\varepsilon x^{1/2+\varepsilon}$ *and* $L(x) = \sum_{n\le x}\lambda(n) \ll_\varepsilon x^{1/2+\varepsilon}$;[47]
3. *for non-principal χ modulo q,* $\sum_{n\le x}\chi(n) \ll_\varepsilon q^\varepsilon x^{1/2+\varepsilon}$ *and* $\sum_{n\le x}\chi(n)\Lambda(n) \ll_\varepsilon x^{1/2+\varepsilon}\log(q+2)$;
4. *every interval $(x, x + x^{1/2+\varepsilon}]$ with $x \ge x_0(\varepsilon)$ contains a prime.*[48]

Under the half-plane $\operatorname{Re} s > 7/8$ the same statements hold with $\frac12$ replaced by $\frac78$.

Proof. Apply Lemma 41 with $\sigma_0 = \frac12 + \varepsilon/3$ and $w = \varepsilon/3$. The growth bounds come from Lemma 40 and Corollary 33: for (i), $F = -\zeta'/\zeta$, $\rho = 1$, with $|F| \ll \log(3+|t|)$ for $|t| \ge 3$ and F bounded on the compact segment $|t| \le 3$, which avoids $s = 1$; for (ii), $F = 1/\zeta$ and $F = \zeta(2s)/\zeta(s)$, with $\rho = 0$, since $|\zeta(2s)| \le \zeta(1+2\varepsilon/3)$; for (iii), $F = L(s,\chi)$ with $K \ll q^{\varepsilon/3}$, and $F = -L'/L(s,\chi)$

[45] Verified in Lean 4 for part (iii): the character sums in full in github.com/anaxia-research/liouville-energy-rh, and the twisted prime sums in the form $q^\varepsilon x^{1/2+\varepsilon}$ in github.com/anaxia-research/primes-in-progressions-grh, with only the standard axioms.

[46] Verified in Lean 4 for ψ and θ: in the repository github.com/anaxia-research/prime-counting-rh, with only the standard axioms. The Lean proof applies the smoothing lemma to $-\zeta'/\zeta - \zeta = \sum_n(\Lambda(n)-1)n^{-s}$, whose poles at $s = 1$ cancel; the passage to $\pi(x)$ by partial summation is in the text only.

[47] Verified in Lean 4 in full, with the exponent $\frac12 + \varepsilon$: for $M(x)$ and $L(x)$ in the repository github.com/anaxia-research/averaged-chowla-grh, with only the standard axioms. The same repository proves $\sum_{n\le N}\lambda(n+h) \ll_\varepsilon (N+h)^{1/2+\varepsilon}$ uniformly in h, hence the one-point case of Chowla's conjecture for every shift.

[48] Verified in Lean 4 in github.com/anaxia-research/prime-counting-rh, with only the standard axioms. This is the ε-form of Cramér's bound $p_{n+1} - p_n \ll p_n^{1/2}\log p_n$.

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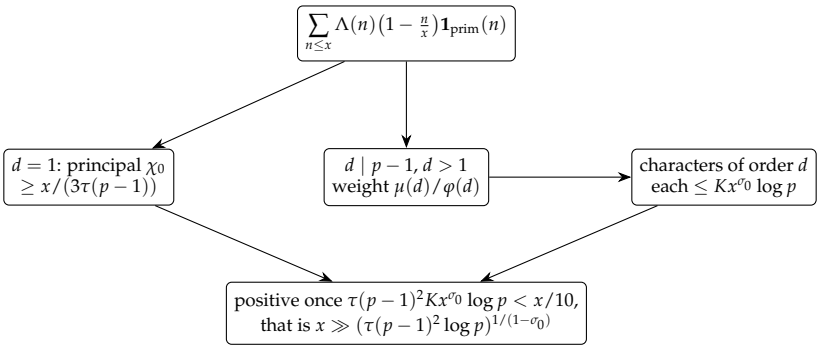

Figure 19. Vinogradov's detector in the proof of Corollary 37. The principal character gives the main term. The other characters, grouped by their order d, contribute at most $\tau(p-1)$ error terms, each controlled by the twisted prime sums of Corollary 34(iii).

with $K \ll \log(q+2)$. Partial summation turns (i) into the statement for $\pi(x)$. For (iv), (i) gives $\psi(x+h)-\psi(x) = h + O(x^{1/2+\varepsilon/2})$ with $h = x^{1/2+\varepsilon}$, and prime powers p^k, $k \ge 2$, contribute at most $O(x^{1/2}\log^2 x)$, so for large x the interval contains a prime. The case $a = 7/8$ is the same with $\sigma_0 = \frac{7}{8} + \varepsilon/3$. □

Part (ii) is the other direction of Theorem 6: the energy bound $Q(N) \ll N^{1+\varepsilon}$ follows from it by (7).

Corollary 35 (Primes in arithmetic progressions)**.** *Under Hypothesis H, for every $\varepsilon > 0$, uniformly in $q \ge 1$ and in a coprime to q:*

1. $\varphi(q)\,\psi(x;q,a) = x + O_\varepsilon(q^{1+\varepsilon}x^{1/2+\varepsilon})$ *for $x \ge 1$, where $\psi(x;q,a) = \sum_{n\le x,\ n\equiv a\ (q)}\Lambda(n)$;*
2. *the least prime $p \equiv a \pmod q$ satisfies $p \ll_\varepsilon q^{2+\varepsilon}$.*

Proof. (i) By orthogonality of characters, $\varphi(q)\psi(x;q,a) = \sum_\chi \overline{\chi}(a)\,\psi(x,\chi)$ with $\psi(x,\chi) = \sum_{n\le x}\chi(n)\Lambda(n)$, the sum over all characters modulo q.[49] For $\chi \ne \chi_0$, Lemma 41 with $F = -L'/L(s,\chi)$, $\rho = 0$ and the bound $|F| \ll \log(q(3+|t|)) \ll_\varepsilon q^{\varepsilon/3}(3+|t|)^{\varepsilon/3}$ of Lemma 40 gives $\psi(x,\chi) \ll_\varepsilon q^\varepsilon x^{1/2+\varepsilon}$; there are fewer than q such characters, so together they contribute $O(q^{1+\varepsilon}x^{1/2+\varepsilon})$. For the principal character, $\psi(x) - \psi(x,\chi_0) = \sum_{n\le x,\ (n,q)>1}\Lambda(n)$. A prime $p \mid q$ contributes $\log p$, so primes give at most $\log q$, and the proper powers $p^k \le x$, $k \ge 2$, give at most $\psi(x) - \theta(x) \le 2\sqrt{x}\log x$. With Corollary 34(i), $\psi(x,\chi_0) = x + O(x^{1/2+\varepsilon})$, and (i) follows.

(ii) Put $\eta = \varepsilon/20$ and let K be the constant of (i) for η. Suppose no prime $p \equiv a \pmod q$ lies in $[1,x]$. Then only proper prime powers enter $\psi(x;q,a)$, so $\varphi(q)\psi(x;q,a) \le q\big(\psi(x) - \theta(x)\big) \le 2q\sqrt{x}\log x \le (2/\eta)\,q^{1+\eta}x^{1/2+\eta}$. By (i), $x \le Mq^{1+\eta}x^{1/2+\eta}$ with $M = K + 2/\eta$, that is $x^r \le Mq^{1+\eta}$ with $r = \frac{1}{2} - \eta$. Take $x = Cq^{2+\varepsilon}$ with $C = (M+1)^{1/r}$. Then $x^r = (M+1)q^{(2+\varepsilon)r}$, and $(2+\varepsilon)r \ge 1+\eta$ for $0 < \varepsilon \le 1$, so $x^r \ge (M+1)q^{1+\eta} > Mq^{1+\eta}$, a contradiction. Hence a prime $p \equiv a \pmod q$ exists with $p \le Cq^{2+\varepsilon}$. □

Part (ii) is the conditional form of Linnik's theorem, whose unconditional exponent is 5 [5]. The sharper conditional bound $p \ll (\varphi(q)\log q)^2$ needs explicit formulas with zeros, which this paper does not use.

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26.3. Characters and Primitive Roots

Corollary 36 (Least non-residues). [50] *Under Hypothesis H, and also under the half-plane* $\operatorname{Re} s > 7/8$*, for every* $\varepsilon > 0$ *there is* C *such that for every prime* p *and every non-principal* χ *modulo* p *some* $n \le Cp^{\varepsilon}$ *has* $\chi(n) \neq 1$*. In particular the least quadratic non-residue modulo* p *is* $O_{\varepsilon}(p^{\varepsilon})$*, which is Vinogradov's conjecture.*

Proof. Suppose $\chi(n) = 1$ for every $n \le x$. Then $\sum_{n \le x} \chi(n) = \lfloor x \rfloor \ge x/2$. Corollary 34(iii), in the form with $\frac{7}{8}$, gives $x/2 \le C_w p^w x^{7/8+w}$, so $x^{1/8-w} \le 2C_w p^w$. With $w = \min(\varepsilon, 1)/32$ and $x = C' p^{\varepsilon}$, where $C' = (2C_w + 1)^{32/3}$, the left side is at least $x^{3/32} = (2C_w + 1) p^{3\varepsilon/32} > 2C_w p^w$, a contradiction. □

Corollary 37 (Least prime primitive root). [51] *Under Hypothesis H, for every* $\eta > 0$ *there is* C *such that every prime* p *has a prime primitive root* $\ell \le C\big(\tau(p-1)^2 \log(p+2)\big)^{2+\eta}$*. Under the half-plane* $\operatorname{Re} s > 7/8$ *the same holds with exponent* $8 + \eta$*. Consequently every prime has a prime primitive root* $\ell \ll_{\varepsilon} p^{\varepsilon}$*, and every prime* $p \ge p_0$ *has a prime primitive root* $\ell < p$ *(Erdős's Problem 985 for large* p*).*

Proof. By Vinogradov's formula, the indicator of the primitive roots modulo p is

$$\frac{\varphi(p-1)}{p-1} \sum_{d \mid p-1} \frac{\mu(d)}{\varphi(d)} \sum_{\operatorname{ord} \chi = d} \chi(n).$$

Weight the primes by $\Lambda(n)(1 - n/x)$. The term $d = 1$ is $\frac{\varphi(p-1)}{p-1} \sum_{n \le x} \Lambda(n)(1 - n/x) \ge \frac{x}{3\tau(p-1)}$ for $x \ge x_0$, because $\varphi(m)/m \ge 2^{-\omega(m)} \ge 1/\tau(m)$ and the Riesz mean of Λ is $x/2 + o(x)$ by the prime number theorem. Each of the other at most $\tau(p-1)$ characters contributes at most $B = Kx^{\sigma_0} \log(p+2)$, with $\sigma_0 = a + 8e$, by the Riesz form of Corollary 34(iii). If $\tau(p-1)^2 B < x/10$, the weighted count is positive, so some prime $\ell \le x$ is a primitive root. This holds for $x = C(\tau(p-1)^2 \log(p+2))^{1/(1-\sigma_0)}$, and $1/(1-\sigma_0) \le 2 + \eta$ for $a = \frac{1}{2}$ and $8e$ small, or $\le 8 + \eta$ for $a = \frac{7}{8}$. The bound $\tau(m) \ll m^{\nu}$ gives $\ell \ll p^{\varepsilon}$, and $\varepsilon = \frac{1}{2}$ gives $\ell < p$ once $p > C^2$. □

Corollary 38 (Character sums). [52] *Under Hypothesis H, for every* $\eta > 0$ *and* $\theta > \frac{1}{2}$ *there is* C *with* $\left|\sum_{n \le x} \chi(n)\right| \le Cx^{\theta}$ *for every non-principal* χ *modulo* q *and every* $x \ge q^{\eta}$*. This is Vinogradov's character sum conjecture. Under the half-plane* $\operatorname{Re} s > 7/8$ *the same holds for* $\theta > \frac{7}{8}$*.*

Proof. Corollary 34(iii) gives $\ll q^w x^{1/2+w}$. For $x \ge q^{\eta}$, $q^w \le x^{w/\eta}$, and $w = (\theta - \frac{1}{2})\eta/(1+\eta)$ makes the exponent θ. □

26.4. Values at $S = 1$

Corollary 39 (Logarithmic derivatives at $s = 1$). [53] *Under Hypothesis H, and also under the half-plane* $\operatorname{Re} s > 7/8$*, there is* C *such that* $|L'/L(1,\chi)| \le C \log\log(q+2)$ *for every non-principal* χ *modulo* q*.*

---

[49] Verified in Lean 4 in full in github.com/anaxia-research/primes-in-progressions-grh, with only the standard axioms. The Lean proof bounds each $\psi(x,\chi)$, $\chi \neq 1$, by $Kq^{\varepsilon}x^{1/2+\varepsilon}$, using $F = -L'/L(s,\chi)$ in the smoothing lemma, and then argues exactly as here.

[50] Verified in Lean 4 in github.com/anaxia-research/liouville-energy-rh (parent repository) and in its own repository github.com/anaxia-research/least-nonresidue-grh, with only the standard axioms.

[51] Verified in Lean 4 in github.com/anaxia-research/liouville-energy-rh (parent repository) and in its own repository github.com/anaxia-research/primitive-root-grh (Erdős Problem 985 also in github.com/anaxia-research/Erd-s-985), with only the standard axioms.

[52] Verified in Lean 4 in full, for every $\theta > \frac{1}{2}$: in github.com/anaxia-research/liouville-energy-rh, with only the standard axioms. The range $\theta > \frac{7}{8}$ is also in github.com/anaxia-research/character-sums-grh.

[53] Verified in Lean 4 in github.com/anaxia-research/liouville-energy-rh (parent repository) and in its own repository github.com/anaxia-research/euler-kronecker-grh, with only the standard axioms.

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Proof. For $\operatorname{Re} s > 1$, $-L'/L(s,\chi) = \sum_n \chi(n)\Lambda(n)n^{-s}$. Let $A(x) = \sum_{n\le x}\chi(n)\Lambda(n)(1-n/x)$ be the Riesz mean. By Corollary 34(iii) in its Riesz form, $|A(x)| \le Kx^{\theta_0}\log(q+2)$ with $\theta_0 = a + 8e < 1$. The Mellin transform of the Riesz kernel gives, for $\operatorname{Re} s > 1$,

$$-\frac{L'}{L}(s,\chi) = s(s+1)\int_1^\infty A(x)x^{-s-1}\,\mathrm{d}x,$$

and the bound on A continues this identity to $\operatorname{Re} s > \theta_0$, in particular to $s = 1$. Split the integral at $Y = (\log(q+2))^{2/(1-\theta_0)}$. On $[1,Y]$, $|A(x)| \le \sum_{n\le x}\Lambda(n) \le 2x$, so that part is at most $4\int_1^Y x^{-1}\,\mathrm{d}x = 4\log Y \ll \log\log(q+2)$. On $[Y,\infty)$ the bound for A gives at most $2K\log(q+2)\,Y^{\theta_0-1}/(1-\theta_0) \ll 1$. □

Corollary 40 (Euler–Kronecker constants of cyclotomic fields)**.** [54] *Under Hypothesis H, and also under the half-plane* $\operatorname{Re} s > 7/8$*, there is* C *such that for every prime* $q \ge 3$

$$-C\log q\log\log q \le \sum_{\chi\ne\chi_0}\frac{L'}{L}(1,\chi) \le C\log q,$$

the sum running over the non-principal characters modulo q*. Since* $\zeta_{\mathbb{Q}(\zeta_q)}(s) = \zeta(s)\prod_{\chi\ne\chi_0}L(s,\chi)$ *for prime* q*, the Euler–Kronecker constant* $\gamma_q = \gamma + \sum_{\chi\ne\chi_0}L'/L(1,\chi)$ *of* $\mathbb{Q}(\zeta_q)$ *satisfies the same bounds.*

Proof. Put $c(n) = \sum_{\chi\ne\chi_0}\chi(n) = (q-1)\mathbf{1}_{n\equiv 1\,(q)} - \mathbf{1}_{q\nmid n}$ and $B(x) = \sum_{n\le x}c(n)\Lambda(n)(1-n/x)$. Summing the identity of the previous proof over $\chi \ne \chi_0$ gives $S := -\sum_{\chi\ne\chi_0}L'/L(1,\chi) = 2\int_1^\infty B(x)x^{-2}\,\mathrm{d}x$. Let $Y = q^{1/(1-\theta_0)}$.

Tail. For $x \ge Y$ the bound for each character gives $|B(x)| \le (q-2)Kx^{\theta_0}\log(q+2)$, so $\left|2\int_Y^\infty B(x)x^{-2}\,\mathrm{d}x\right| \le \frac{2K}{1-\theta_0}(q-2)\log(q+2)\,Y^{\theta_0-1} \ll \log q$.

Head. For $x \le Y$ write $B = (q-1)B_1 - B_0$, where B_1 and B_0 are the Riesz means of Λ over $n \equiv 1 \pmod q$ and over $q \nmid n$. Both are non-negative, and $B_0(x) \le \psi(x) \le 2x$. So $2\int_1^Y B(x)x^{-2}\,\mathrm{d}x \ge -4\log Y \gg -\log q$, which is the upper bound $\sum_{\chi\ne\chi_0}L'/L(1,\chi) \le C\log q$. For the other bound, $B_1(x) = 0$ for $x \le q$. For $q < x \le eq$ at most two integers $n \le x$ satisfy $n \equiv 1 \pmod q$, so $(q-1)B_1(x) \le 2q\log(eq)$, and this range contributes $O(\log q)$. For $x > eq$ the Brun–Titchmarsh inequality, together with the trivial bound for the prime powers, gives $(q-1)B_1(x) \le \frac{2x\log x}{\log(x/q)} + O(\sqrt{x}\log^2 x)$. With $u = x/q$,

$$\int_{eq}^{Y}\frac{2\log x}{x\log(x/q)}\,\mathrm{d}x = \int_{\mathrm{e}}^{Y/q}\frac{2(\log q+\log u)}{u\log u}\,\mathrm{d}u \le 2\log q\,\log\log Y + 2\log Y \ll \log q\log\log q.$$

Hence $S \le C\log q\log\log q$, which is the lower bound for $\sum_{\chi\ne\chi_0}L'/L(1,\chi)$. □

[54] Verified in Lean 4 in github.com/anaxia-research/liouville-energy-rh (parent repository) and in its own repository github.com/anaxia-research/euler-kronecker-grh, with only the standard axioms.

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Table 20. The conjectures and classical statements proved in this paper from Hypothesis H, with the result that proves each one, the input it uses, the method, and its Lean declaration in [10] (rows 1–4 in the main project, the others in its corollaries project). "Lean" means the declaration builds with only the axioms `propext`, `Classical.choice` and `Quot.sound`; "Lean (form)" means the stated weaker form is checked in Lean and the full statement is proved in the text; "text" means the proof is given in full in the text only. "H" is Hypothesis H, the half-plane $\operatorname{Re} s > \frac{1}{2}$ of [4]; "L'/L holomorphic" is the hypothesis of Theorem 57; "$\operatorname{Re} s > \frac{7}{8}$" is the half-plane announced in [8]. Rows 6–19 also hold under $\operatorname{Re} s > \frac{7}{8}$, in the form with $\frac{7}{8}$ in place of $\frac{1}{2}$. A superscript letter after "Lean" is a link to the separate repository of that result, which cites this paper and the parent one; the rows marked [f] or [g] are checked in that repository, which builds on the parent one, row 23 needs no hypothesis on zeros, and row 24 no hypothesis at all.

	Conjecture or statement	Result	Input	How it is proved	Lean declaration	Checked
1	Riemann hypothesis	Theorem 66	H	reflection of a zero by the functional equation; Γ has no zeros	`rh_of_H`	Lean
2	GRH on the critical line, primitive χ	Theorem 66	H	functional equation of $\Lambda(s,\bar{\chi})$	`critLine`	Lean
3	Hypothesis H from holomorphy of L'/L	Sec. 20	L'/L holomorphic	order of vanishing on a convex region	`hypothesisH_of_logDeriv`	Lean
4	RH and GRH from holomorphy of L'/L	Sec. 20	L'/L holomorphic	the two rows above	`riemannHypothesis_of_logDeriv`, `grh_of_logDeriv`	Lean
5	Energy bound $Q(N) \ll N^{1+\varepsilon}$	Corollary 22	H	Theorem 1 at $\theta = \frac{1}{2}$	–	text
6	Lindelöf hypothesis for ζ and $L(s,\chi)$	Corollary 33	H	Borel–Carathéodory, three circles, Phragmén–Lindelöf	`lindelof_of_H`, `lindelof_half`	Lean[i] (ζ at $\sigma = \frac{1}{2}$); Lean ($\operatorname{Re} s \geq \frac{1}{2} + w$); text
7	Prime number theorem with $x^{1/2+\varepsilon}$ (von Koch)	Corollary 34(i)	H	smoothing lemma with $-\zeta'/\zeta$	`psi_bound_of_H`, `theta_bound_of_H`	Lean[g] (ψ,θ); text
8	$M(x) \ll x^{1/2+\varepsilon}$ (Littlewood)	Corollary 34(ii)	H	smoothing lemma with $1/\zeta$	`littlewoodRH_of_H`	Lean[f]
9	$L(x) \ll x^{1/2+\varepsilon}$ (Liouville sums)	Corollary 34(ii)	H	smoothing lemma with $\zeta(2s)/\zeta(s)$; in Lean $\lambda = \mathbf{1}_{\square} * \mu$	`liouvilleBound_of_H`	Lean[f]
10	Primes in $(x, x + x^{1/2+\varepsilon}]$	Corollary 34(iv)	H	part (i) with $h = x^{1/2+\varepsilon}$	`prime_gap_of_H`	Lean[g]
11	Character sums $\ll q^{\varepsilon} x^{\theta+\varepsilon}$	Corollary 34(iii)	H	smoothing lemma with $L(s,\chi)$	`character_sum_bound_half`	Lean
12	Twisted prime sums	Corollary 34(iii)	H	smoothing lemma with $-L'/L$	`chi_psi_of_H`	Lean[j] ($q^{\varepsilon} x^{1/2+\varepsilon}$); text
13	Pólya–Vinogradov with $\log\log q$ (Montgomery–Vaughan)	Corollary 27	H	major and minor arcs, App. L	`polya_vinogradov`	Lean
14	Vinogradov's least non-residue conjecture	Corollary 36	H	character sum bound at $x = Cp^{\varepsilon}$	`least_nonresidue`	Lean[a]
15	Vinogradov's least primitive root conjecture	Corollary 37	H	Vinogradov's detector and prime sums	`least_prime_primitive_root`	Lean[b]
16	Erdős Problem 985 for large p	Corollary 37	H	$\ell \ll p^{1/2}$	`erdos_985`	Lean[b,c]
17	Vinogradov's character sum conjecture, every $\theta > \frac{1}{2}$	Corollary 38	H	$q^{w} \leq x^{w/\eta}$	`vinogradov_character_sum`	Lean
18	$L'/L(1,\chi) \ll \log\log q$ (Littlewood)	Corollary 39	H	Riesz mean split at $(\log q)^{A}$	`logDeriv_one_bound`	Lean[e]
19	Euler–Kronecker constants of $\mathbb{Q}(\zeta_q)$ (Ihara)	Corollary 40	H	orthogonality, Brun–Titchmarsh	`euler_kronecker_bound`	Lean[e]

continued on the next page

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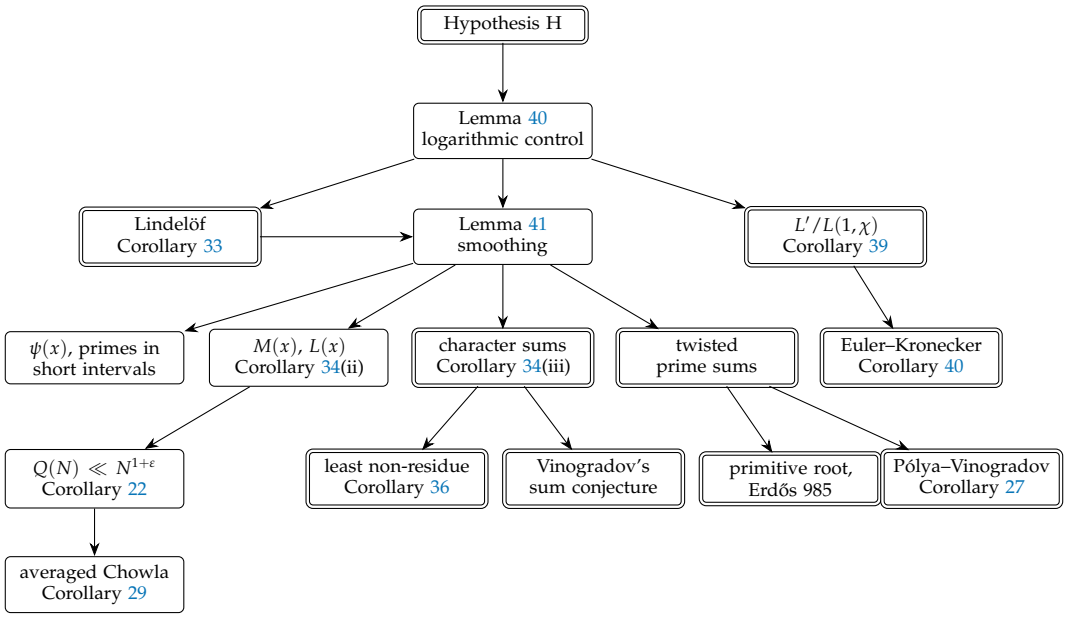

Figure 20. How the corollaries depend on Hypothesis H. Each arrow is a proof in Section 26 or Section 25. Double borders mark the statements checked in Lean [10], in the forms given in Table 20.

Table 20, continued

	Conjecture or statement	Result	Input	How it is proved	Lean declaration	Checked
20	Averaged two-point Chowla, power saving	Corollary 29	H	Fejér identity and Corollary 22	–	text
21	One-point Chowla, every shift, $\sum_{n\le N}\lambda(n+h)\ll (N+h)^{1/2+\varepsilon}$	Corollary 34(ii)	H	$L(N+h)-L(h)$	`shifted_sum_bound`, `chowla_one_of_H`	Lean[f]
22	Two-point Chowla averaged over $h\le N$, $\sum_{h\le N} C_h(N)\ll N^{3/2+\varepsilon}$	Corollary 30	H	exchange of sums, $L(n+H)-L(n)$	`two_point_shift_average`, `two_point_square_average`	Lean[f]
23	Chowla for almost all shifts, every order k, all sign patterns	Corollary 31	MRT [47]	pattern expansion, Markov	`almostAll_patterns`	Lean[h]
24	Both signs of $\lambda(n)\lambda(n+h)$ at every fixed shift, in explicit windows	Corollary 32	none	$\lambda(3^5)=-1$, $\lambda(2^8)=1$, walks of step h	`exists_neg_in_window`, `exists_pos_in_window`	Lean[h]
25	Pólya–Vinogradov without RH	Theorem 67	$\mathrm{Re}\, s>\frac{7}{8}$	as Corollary 27	–	text
26	$Q(N)\ll N^{7/4+\varepsilon}$ without RH	Corollary 28	$\mathrm{Re}\, s>\frac{7}{8}$	Theorem 8	–	text
27	Primes in progressions, $\varphi(q)\psi(x;q,a)=x+O(q^{1+\varepsilon}x^{1/2+\varepsilon})$; least prime $p\equiv a\ (q)$ is $\ll q^{2+\varepsilon}$	Corollary 35	H	orthogonality, $-L'/L$ smoothing	`psiAP_of_H`, `least_prime_of_H`	Lean[j]

What these methods do not give should also be said. The least non-residue bound $O((\log p)^2)$ of Ankeny and Bach, and with it a deterministic polynomial-time Miller primality test, needs the explicit formula with the zeros on the critical line and not only a zero-free half-plane. Artin's primitive root conjecture needs the Riemann hypothesis for Dedekind zeta functions of Kummer fields, which is not part of Hypothesis H. Chowla's conjecture for a single shift does not follow, as explained after Corollary 29.

Appendix A. The Architecture of the Announced Lean Proof

This appendix describes how the project [4] obtains Hypothesis H. It is a description, made by reading the Lean source, and not a proof: the argument is that of [4], it has not been read by experts,

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and it is verified only when the whole project compiles (Section 19). Names in typewriter type are Lean declarations of [4].[55]

Appendix A.1. The Top of the Chain

The final theorem `riemannHypothesis` is the step of Theorem 56(i) applied to `everyCharacter`, which is Hypothesis H with Mathlib's $L(s,\chi)$. That theorem is obtained as

$$\texttt{every_character_of_literature_of_vWin}(\mathcal{O},\ \mathrm{VWin}),$$

from two inputs, $\mathcal{O}$ and VWin, described below, and the input $\mathcal{O}$ is in turn derived from a single statement about Galois representations (Input A), from which VWin is also derived (Input B). Figure A1 shows the chain.

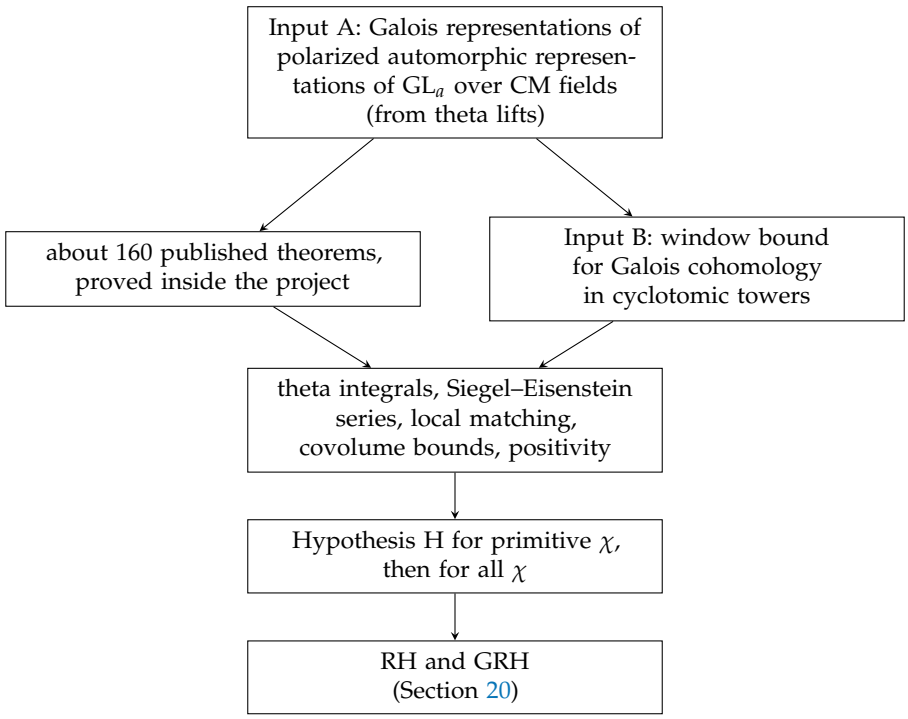

Figure A1. The logical structure of the announced Lean proof [4], as read from its source. Only the last arrow is checked independently in this paper (Table 19); the rest is verified when the whole project compiles.

Appendix A.2. Input A: Galois Representations

The statement `Row0b` asserts that for every CM number field E, every prime ℓ and every regular algebraic, polarized, cuspidal automorphic representation π of $\mathrm{GL}_a(\mathbb{A}_E)$ there is a continuous semisimple ℓ-adic representation $r_\ell(\pi)$ of the absolute Galois group of E, with the Hodge–Tate weights determined by π, de Rham and potentially semistable above ℓ, crystalline where π is unramified, and compatible with the local Langlands correspondence at every place away from ℓ.[56] The project proves it, as `galoisLocalGlobalStrong_recParameter`, from a theta-lift construction between definite unitary groups.

[55] We give the names so that a reader can find each step in the source. The project is organised into the libraries `Formalism`, `Development`, `GroundTheory`, `PositivityTheory`, `ReciprocityTheory`, `CompanionTheory`, `Lemmata` and `Supplements`, with 175,187 files in all.

[56] In the published literature this is the work of Harris and Taylor, Shin, Chenevier and Harris, Harris, Lan, Taylor and Thorne, Scholze, and Caraiani, among others. The project does not assume it.

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Appendix A.3. The Published Theorems

The structure $\mathcal{O}$ (`OpenLiteratureStatements`) names three statements: Caraiani's local–global compatibility, the existence of the local Langlands correspondence, and Galois representations for polarized pairs. All three are supplied from Input A. From $\mathcal{O}$ the project then proves, rather than assumes, about 160 published theorems that the positivity argument needs. They fall into five groups.

- *Local representation theory of* GL_n *and unitary groups:* the Bernstein–Zelevinsky classification, the Langlands classification and its multisegments, Casselman's and Borel's theorems, the Casselman–Wallach globalisation, discrete series, Howe duality and the Weil representation, and the local converse theorem.
- *Automorphic forms:* the cuspidal spectrum of GL_n, multiplicity one and the rigidity theorem of Jacquet and Shalika, rapid decay of cusp forms, the doubling method, the Siegel–Weil formula for unitary groups, convergence of theta integrals and of the Rallis inner product, and base change for definite unitary groups.
- *Arithmetic geometry:* toroidal compactifications, Arakelov heights, the arithmetic Hilbert–Samuel theorem, the Kawamata–Viehweg and Nadel vanishing theorems, and semistable reduction.
- *p-adic Hodge theory:* Fontaine's period ring B_{cris}, integral comparison in the form of Bhatt and Scholze, prismatic Dieudonné theory, Grothendieck–Messing deformation theory, and the local conditions of Bloch and Kato.
- *Class field theory and complex multiplication:* global Artin reciprocity, Chebotarev's theorem, the main theorem of complex multiplication, the theorems of Siegel–Klingen and of Deligne and Ribet, and the functional equations of Hecke and Tate.

Appendix A.4. Input B: the Window Bound

For a primitive character ψ the project considers subfields E of cyclotomic fields cut out by ψ, by a 2-power level 2^{e+2} and by two auxiliary primes $q \equiv 3 \pmod 8$ and $d \equiv 7 \pmod{16}$ in narrow ranges, $\log q \asymp Y^{\varepsilon/5}$ and $\log d \asymp Y^{\varepsilon/5}(\log Y)^2$. The statement VWin (`UpperTheory.Top.VWin`) says that, for every $\varepsilon < 1/52$, the lengths of certain Tate–Shafarevich-type Galois cohomology groups of these fields, twisted by e up to x^{401}, are finite, and that their sum weighted by $\log \ell$ over primes ℓ is at most $\exp(x^{\varepsilon}/2)$. The bound must hold for every choice of local data, so it is not satisfied trivially. The project derives it from Input A (`vwin_of_row`) by an automorphic window argument and a base-change step.

Appendix A.5. From the Inputs to the Half-Plane

The theorem `every_character_of_theta_integrals_local_matching` combines the two inputs through theta integrals on unitary groups, the continuation of Siegel–Eisenstein series and the local matching of their coefficients, bounds for covolumes and self-intersection volumes, and a positivity statement (`GlobalTheory.GLq.Green`) for regular characters. It gives Hypothesis H for primitive characters (`zero_free_all_of_green_v_win`), and a short reduction extends it to every character (`UpperTheory.Top.imprimitive`), since $L(s,\chi)$ and $L(s,\chi^*)$ differ by finitely many Euler factors with no zeros in $\mathrm{Re}\, s > 0$.

Appendix A.6. How It Meets the Energy

In the language of this paper the positivity step is where the proof uses structure that λ_S lacks (Theorem 3): theta series, the automorphy of $L(s,\chi)$, and the Galois representations of Input A. Section 19 describes the relation of this positivity to the energy as a positive Toeplitz form.

Appendix B. Calculations for the Liouville Energy

This appendix contains the calculations used in Sections 2–5: the window counts behind the Fejér identity, the algebra of Lemma 1, the Perron calculation for Theorem 6, the extremal values of the weight K, and the kernel estimates for Proposition 3.

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Appendix B.1. Window Counts

A *window* is a set $(t, t+H] \cap \mathbb{Z} = \{t+1, \dots, t+H\}$ with $t \in \mathbb{Z}$.

Lemma A1. *Let $N, H \geq 1$ be integers.*

(a) For integers m, n, the number of windows containing both m and n is $(H - |m-n|)_+$. In particular each integer lies in exactly H windows, and the windows meeting $[1, N]$ are those with $1 - H \leq t \leq N - 1$; there are $N + H - 1$ of them.

(b) Let $H \geq N$. The windows meeting $[1, N]$ are of three kinds:

- *$N - H \leq t \leq 0$: the window contains $[1, N]$; there are $H - N + 1$ such t;*
- *$1 - H \leq t \leq N - H - 1$: the window meets $[1, N]$ in the prefix $[1, j]$ with $j = t + H$, and j takes each value in $\{1, \dots, N-1\}$ exactly once;*
- *$1 \leq t \leq N - 1$: the window meets $[1, N]$ in the suffix $[t+1, N]$, and $j = t$ takes each value in $\{1, \dots, N-1\}$ exactly once.*

Proof. (a) The window $(t, t+H]$ contains m and n if and only if $t < \min(m,n)$ and $\max(m,n) \leq t + H$, that is, $\max(m,n) - H \leq t \leq \min(m,n) - 1$. The number of such integers t is $\min(m,n) - \max(m,n) + H = H - |m-n|$ if this is positive, and zero otherwise. With $m = n$ the count is H. The window meets $[1, N]$ if and only if $t + 1 \leq N$ and $t + H \geq 1$.

(b) The window contains $[1, N]$ if and only if $t \leq 0$ and $t + H \geq N$. If $t + H \leq N - 1$, then $t \leq N - H - 1 \leq -1$, so the window starts to the left of 1 and its intersection with $[1, N]$ is $[1, t+H]$; as t runs over $[1 - H, N - H - 1]$, $j = t + H$ runs over $[1, N-1]$. If $t \geq 1$, then $t + H \geq 1 + N > N$, and the intersection is $[t+1, N]$. The three ranges of t are disjoint and their union is $[1 - H, N - 1]$. □

The shift form. By Lemma A1(a),

$$H\,\mathcal{Q}_a(N,H) = \sum_{t \in \mathbb{Z}} \Big(\sum_{\substack{n \leq N \\ t < n \leq t+H}} a_n \Big)^2 = \sum_{m,n \leq N} a_m a_n \,(H - |m-n|)_+ .$$

The pairs with $m = n$ contribute $H \sum_{n \leq N} a_n^2$. The pairs with $|m-n| = h \geq 1$ occur only for $h \leq N - 1$, have a nonzero weight only for $h \leq H - 1$, and contribute $2(H-h) \sum_{n \leq N-h} a_n a_{n+h} = 2(H - h) C_{a,h}(N-h)$. Division by H gives (3).

The Fourier form. Expanding the square,

$$\Big| \sum_{j=1}^{H} \mathrm{e}(j\alpha) \Big|^2 = \sum_{j,k=1}^{H} \mathrm{e}\big((j-k)\alpha\big) = \sum_{|h| < H} (H - |h|)\, \mathrm{e}(h\alpha),$$

since $j - k = h$ has $H - |h|$ solutions. Hence $F_H(\alpha) = \sum_{|h|<H} (1 - |h|/H) \mathrm{e}(h\alpha)$ and $0 \leq F_H(\alpha) \leq F_H(0) = H$. Since $|A_N(\alpha)|^2 = \sum_{m,n \leq N} a_m a_n \mathrm{e}((m-n)\alpha)$ and $\int_0^1 \mathrm{e}(k\alpha)\, \mathrm{d}\alpha$ is 1 for $k = 0$ and 0 for other integers k,

$$\int_0^1 |A_N(\alpha)|^2 F_H(\alpha)\, \mathrm{d}\alpha = \sum_{m,n \leq N} a_m a_n \Big(1 - \frac{|m-n|}{H}\Big)_+ = \mathcal{Q}_a(N,H).$$

The boundary form. For $H \geq N$, Lemma A1(b) gives

$$H\,\mathcal{Q}_a(N,H) = (H - N + 1) L_a(N)^2 + \sum_{j=1}^{N-1} L_a(j)^2 + \sum_{j=1}^{N-1} \big(L_a(N) - L_a(j)\big)^2,$$

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which is (5). At $H = N$, write $x_j = L_a(j)$ and $y = L_a(N)$. Then $x_j^2 + (y - x_j)^2 = 2x_j^2 - 2x_j y + y^2 = 2(x_j - \frac{1}{2}y)^2 + \frac{1}{2}y^2$, so that

$$N\,Q_a(N) = y^2 + \frac{N-1}{2}\,y^2 + 2\sum_{j=1}^{N-1}\left(x_j - \tfrac{1}{2}y\right)^2 = \frac{N+1}{2}\,y^2 + 2\sum_{j=1}^{N-1}\left(x_j - \tfrac{1}{2}y\right)^2,$$

which is (6).

Appendix B.2. The Algebra of Lemma 1

Let a be real, $x_j = L_a(j)$, $y = L_a(N)$, $Q = Q_a(N)$, $V = V_a(N) = \sum_{j\le N} x_j^2$.

Proof of (11). Both terms of (6) are nonnegative, so

$$y^2 \le \frac{2N}{N+1}\,Q \le 2Q, \qquad \sum_{j=1}^{N-1}\left(x_j - \tfrac{1}{2}y\right)^2 \le \frac{N}{2}\,Q.$$

From $x_j = (x_j - \frac{1}{2}y) + \frac{1}{2}y$ and $(p+q)^2 \le 2p^2 + 2q^2$ we get $x_j^2 \le 2(x_j - \frac{1}{2}y)^2 + \frac{1}{2}y^2$. Therefore

$$V = \sum_{j=1}^{N-1} x_j^2 + y^2 \le 2\cdot\frac{N}{2}\,Q + \frac{N-1}{2}\,y^2 + y^2 \le NQ + (N-1)Q + 2Q = (2N+1)Q.$$

Proof of (12). Assume $|a_n| \le 1$. In (5) at $H = N$ use $(y - x_j)^2 \le 2y^2 + 2x_j^2$:

$$N\,Q \le y^2 + \sum_{j=1}^{N-1}\left(3x_j^2 + 2y^2\right) = (2N-1)y^2 + 3\sum_{j=1}^{N-1} x_j^2 \le 2N\,y^2 + 3V,$$

so $Q \le 2y^2 + 3V/N$. Let $m = |y|$; we may assume $m > 0$. Since $|x_N - x_j| \le N - j$, we have $|x_j| \ge m - (N-j) \ge m/2$ whenever $N - m/2 \le j \le N$. The number of integers in $[N - m/2, N]$ is $\lfloor m/2 \rfloor + 1 \ge m/2$, and they are positive because $m \le N$ gives $N - m/2 \ge N/2 > 0$. Hence

$$V \ge \frac{m}{2}\cdot\left(\frac{m}{2}\right)^2 = \frac{m^3}{8}, \qquad \text{that is,} \qquad y^2 = m^2 \le 4V^{2/3}.$$

Also $|x_j| \le j$ gives $V \le \sum_{j\le N} j^2 \le N^3$, so $N \ge V^{1/3}$ and $V/N \le V^{2/3}$. Altogether $Q \le 2\cdot 4V^{2/3} + 3V^{2/3} = 11V^{2/3}$.

Table A1 records the extreme values of the two ratios for $a = \lambda$ in a finite range; they show that neither constant is close to being attained by the Liouville function.

Table A1. The two inequalities of Lemma 1 for $a = \lambda$: largest observed ratios for $1 \le N \le 3\cdot 10^6$. Computed values, given as a check; the lemma is proved above for all N.

inequality	proved	largest ratio found (computed)
$V(N) \le (2N+1)Q_\lambda(N)$	$V/((2N+1)Q_\lambda) \le 1$	0.4125
$Q_\lambda(N) \le 11V(N)^{2/3}$	$Q_\lambda/(11V^{2/3}) \le 1$	0.1336

Appendix B.3. From a Zero-Free Half-Plane to Cancellation

This subsection proves the implication (i)⇒(ii) of Theorem 6. Throughout, $\frac{1}{2} \le \theta < 1$ and $\zeta(s) \ne 0$ for $\operatorname{Re} s > \theta$. The argument is the classical one for the consequences of the Riemann hypothesis [57, Sections 14.2 and 14.25], carried out to the right of the line $\operatorname{Re} s = \theta$. We write $s = \sigma + it$.

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Two classical facts are used. The first is the Borel–Carathéodory inequality [49, Section 6.1]: if g is holomorphic on a neighbourhood of the disc $|s-s_0| \le R$ and $0 < r < R$, then

$$\max_{|s-s_0|\le r} |g(s)| \le \frac{2r}{R-r} \max_{|s-s_0|=R} \operatorname{Re} g(s) + \frac{R+r}{R-r} |g(s_0)|. \tag{A1}$$

The second is Hadamard's three-circles theorem: if g is holomorphic on a neighbourhood of $|s-s_0| \le r_2$, $0 < r_0 < r_1 < r_2$, and $\mathcal{M}(r) = \max_{|s-s_0|=r} |g(s)|$, then

$$\mathcal{M}(r_1) \le \mathcal{M}(r_0)^{1-\alpha} \mathcal{M}(r_2)^{\alpha}, \qquad \alpha = \frac{\log(r_1/r_0)}{\log(r_2/r_0)} \in (0,1). \tag{A2}$$

Lemma A2 (Subpower growth of the reciprocal). *Let $\frac{1}{2} \le \theta < 1$ and suppose $\zeta(s) \ne 0$ for $\operatorname{Re} s > \theta$. For $0 < \delta < \frac{1}{4}$ and $\eta > 0$,*

$$\left| \frac{1}{\zeta(\sigma+it)} \right| \ll_{\theta,\delta,\eta} (1+|t|)^{\eta} \qquad (\theta+\delta \le \sigma \le 2),$$

where $1/\zeta$ takes the value 0 at $s = 1$.

Proof. *Bounded t.* The function $1/\zeta$ is continuous on the compact set $\theta+\delta \le \sigma \le 2$, $|t| \le 4$, because ζ has no zero there and its pole at $s = 1$ is a zero of $1/\zeta$. So $1/\zeta$ is bounded on this set.

Large t. Let $|t| \ge 4$, $s_0 = 2 + it$, and

$$R = 2 - \theta - \tfrac{1}{4}\delta, \qquad r_2 = 2 - \theta - \tfrac{1}{2}\delta, \qquad r_1 = 2 - \theta - \delta, \qquad r_0 = \tfrac{1}{2}.$$

Then $r_0 < r_1 < r_2 < R < \frac{3}{2}$. The closed disc $D_R = \{|s-s_0| \le R\}$ lies in the half-plane $\sigma \ge \theta + \frac{1}{4}\delta$, where ζ has no zero, and it does not contain $s = 1$, since $|s_0 - 1| \ge |t| \ge 4$. The same holds for a slightly larger open disc. On it there is a holomorphic branch $g = \log\zeta$, which we normalize by $g(s_0) = \sum_p \sum_{k\ge1} k^{-1} p^{-ks_0}$. On the half-plane $\sigma \ge \frac{3}{2}$, which contains the disc $|s-s_0| \le r_0$, this series represents g, and

$$|g(s)| \le \sum_p \sum_{k\ge1} \frac{p^{-3k/2}}{k} = \log\zeta(\tfrac{3}{2}) \qquad (|s-s_0| \le r_0). \tag{A3}$$

For $\sigma > 0$, $s \ne 1$, partial summation gives [57, Section 2.1]

$$\zeta(s) = \frac{s}{s-1} - s\int_1^{\infty} \{x\} x^{-s-1}\,dx,$$

so that $|\zeta(s)| \le |s|/|s-1| + |s|/\sigma$. On D_R we have $\sigma \ge \frac{1}{2}$, $|s-1| \ge 1$ and $|s| \le 4 + |t|$, hence $|\zeta(s)| \le 3(4+|t|)$ and

$$\max_{s\in D_R} \operatorname{Re} g(s) = \max_{s\in D_R} \log|\zeta(s)| \le \log\bigl(12 + 3|t|\bigr).$$

By (A1) with $r = r_2$, $R - r_2 = \frac{1}{4}\delta$, and (A3) at s_0,

$$\mathcal{M}(r_2) \le \frac{8r_2}{\delta} \log(12+3|t|) + \frac{4(R+r_2)}{\delta} \log\zeta(\tfrac{3}{2}) \le C_1 \log(2+|t|),$$

with $C_1 = C_1(\delta)$. By (A2) and (A3),

$$\mathcal{M}(r_1) \le \bigl(\log\zeta(\tfrac{3}{2})\bigr)^{1-\alpha} \bigl(C_1 \log(2+|t|)\bigr)^{\alpha} \le C_2 \bigl(\log(2+|t|)\bigr)^{\alpha},$$

where $\alpha = \log(2r_1)/\log(2r_2) < 1$ depends only on θ and δ. By the maximum principle the same bound holds on the whole disc $|s-s_0| \le r_1$, which contains the segment $\theta+\delta \le \sigma \le 2$ at height t. There

$$\left|\frac{1}{\zeta(s)}\right| = e^{-\operatorname{Re} g(s)} \le \exp\bigl(C_2(\log(2+|t|))^{\alpha}\bigr) \le C_3(1+|t|)^{\eta},$$

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because $C_2 u^\alpha \le \eta u + C_4$ for all $u \ge 0$, with $C_4 = C_4(C_2, \alpha, \eta)$. $\square$

The truncated Perron formula is used in the following form [49, Section 5.1]; compare [57, Lemma 3.12]. Let $\sum_n a_n n^{-s}$ converge absolutely for $\sigma > 1$ to $D(s)$. For $c > 1$, $T \ge 1$ and $x \ge 1$ not an integer,

$$\sum_{n \le x} a_n = \frac{1}{2\pi i} \int_{c-iT}^{c+iT} D(s) \frac{x^s}{s}\, \mathrm{d}s + O\Big(\sum_{n \ge 1} |a_n| \Big(\frac{x}{n}\Big)^c \min\Big\{1, \frac{1}{T|\log(x/n)|}\Big\}\Big), \tag{A4}$$

with an absolute implied constant.

Proposition A1 (Perron calculation)**.** *Let $\frac12 \le \theta < 1$ and suppose $\zeta(s) \ne 0$ for $\operatorname{Re} s > \theta$. Then $L(x) \ll_{\theta,\varepsilon} x^{\theta+\varepsilon}$ for every $\varepsilon > 0$ and all $x \ge 1$.*

Proof. Fix $0 < \varepsilon < 1$, and choose

$$0 < \delta < \min\big(\tfrac14, \tfrac12 \varepsilon\big), \qquad \sigma_0 = \theta + \delta, \qquad 0 < \eta < \tfrac14 \varepsilon.$$

Let $x = k + \frac12$ with an integer $k \ge 3$, and put $c = 1 + 1/\log x$ and $T = x^2$. Then $1 < c < 2$ and $x^c = ex$. Let $D(s) = \zeta(2s)/\zeta(s)$.

The integrand. On the half-plane $\sigma \ge \sigma_0$, $|\zeta(2s)| \le \zeta(2\sigma_0)$, a constant depending on θ and δ, because $2\sigma_0 > 1$. The function D is holomorphic on $\sigma > \theta$: the only pole of $\zeta(2s)$ is at $s = \frac12 < \sigma_0$, and ζ has no zero. By Lemma A2,

$$|D(\sigma + it)| \le C_5 (1 + |t|)^\eta \qquad (\sigma_0 \le \sigma \le 2), \tag{A5}$$

with $C_5 = C_5(\theta, \delta, \eta)$.

The Perron error. In (A4) with $a_n = \lambda(n)$ we have $|a_n| = 1$. For $x/2 < n < 2x$, $(x/n)^c \le 2^c \le 4$ and

$$\Big|\log \frac{x}{n}\Big| = \Big|\int_n^x \frac{\mathrm{d}v}{v}\Big| \ge \frac{|x-n|}{\max(x,n)} \ge \frac{|x-n|}{2x}.$$

Since x is a half-integer, the numbers $|x-n|$ for n in this range are of the form $j + \frac12$ with $0 \le j \le x$, each value occurring at most twice. These n contribute at most

$$4 \cdot \frac{2x}{T} \cdot 2 \sum_{0 \le j \le x} \frac{1}{j + 1/2} \le \frac{16x}{T}(3 + \log x).$$

For $n \le x/2$ or $n \ge 2x$ we have $|\log(x/n)| \ge \log 2$, and these n contribute at most

$$\frac{x^c}{T \log 2} \sum_{n \ge 1} n^{-c} = \frac{ex\,\zeta(c)}{T \log 2} \le \frac{ex(1 + \log x)}{T \log 2},$$

since $\zeta(c) \le 1 + 1/(c-1)$. With $T = x^2$ the error term in (A4) is $O(x^{-1} \log x) = O(1)$.

The contour shift. The function $D(s) x^s / s$ is holomorphic on a neighbourhood of the rectangle with vertices $\sigma_0 \pm iT$, $c \pm iT$. By Cauchy's theorem the integral over $[c - iT, c + iT]$ equals the integral over $[\sigma_0 - iT, \sigma_0 + iT]$ plus the integrals over the two horizontal sides. On the horizontal sides $|s| \ge T$, and by (A5)

$$\Big|\int_{\sigma_0 \pm iT}^{c \pm iT} D(s) \frac{x^s}{s}\, \mathrm{d}s\Big| \le 2\, C_5 (1 + T)^\eta \frac{x^c}{T} \le C_6\, x^{1 + 2\eta - 2} \le C_6.$$

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On the left side $|s| \ge \max(\frac{1}{2}, |t|) \ge \frac{1}{4}(1+|t|)$, so

$$\Big|\int_{\sigma_0-iT}^{\sigma_0+iT} D(s)\frac{x^s}{s}\,\mathrm{d}s\Big| \le 4\,C_5\,x^{\sigma_0}\int_{-T}^{T}(1+|t|)^{\eta-1}\,\mathrm{d}t \le \frac{8C_5}{\eta}\,x^{\sigma_0}(1+T)^{\eta} \le C_7\,x^{\theta+\delta+2\eta}.$$

Since $\delta + 2\eta < \varepsilon$, we obtain $|L(x)| \le C_8\,x^{\theta+\varepsilon}$ for half-integers $x \ge \frac{7}{2}$, with $C_8 = C_8(\theta,\varepsilon)$.

All x. For real $x \ge 3$, $L(x) = L(\lfloor x\rfloor + \frac{1}{2})$ and $\lfloor x\rfloor + \frac{1}{2} \le 2x$, so $|L(x)| \le 2C_8 x^{\theta+\varepsilon}$. For $1 \le x < 3$, $|L(x)| \le 3$. For $\varepsilon \ge 1$ the bound is trivial. □

Appendix B.4. The Weight K and the Gram Formulas

Let $\operatorname{Re} z, \operatorname{Re} z' > -\frac{1}{2}$. From $w_z(u) = (u^z, 1-u^z)$ and $\int_0^1 u^p\,\mathrm{d}u = 1/(1+p)$ for $\operatorname{Re} p > -1$,

$$\langle w_z, w_{z'}\rangle = \int_0^1 \Big(u^{z+\overline{z'}} + (1-u^z)(1-u^{\overline{z'}})\Big)\,\mathrm{d}u = 1 - \frac{1}{1+z} - \frac{1}{1+\overline{z'}} + \frac{2}{1+z+\overline{z'}}. \tag{A6}$$

With $z = z' = s = \sigma + i\tau$ this gives

$$K(s) = 1 - 2\,\operatorname{Re}\frac{1}{1+s} + \frac{2}{1+2\sigma} = 1 + \frac{2}{1+2\sigma} - \frac{2(1+\sigma)}{(1+\sigma)^2+\tau^2},$$

which is (18). Since $\overline{w_z} = w_{\bar z}$, we have $\operatorname{Re} w_z = \frac{1}{2}(w_z + w_{\bar z})$, and (A6) applied to the four pairs gives

$$\langle \operatorname{Re} w_z, \operatorname{Re} w_{z'}\rangle = \frac{1}{4}\sum_{\zeta_1\in\{z,\bar z\}}\sum_{\zeta_2\in\{z',\overline{z'}\}}\Big(1 - \frac{1}{1+\zeta_1} - \frac{1}{1+\overline{\zeta_2}} + \frac{2}{1+\zeta_1+\overline{\zeta_2}}\Big) = 1 - \operatorname{Re}\frac{1}{1+z} - \operatorname{Re}\frac{1}{1+z'} + \operatorname{Re}\frac{1}{1+z+z'} + \operatorname{Re}\frac{1}{1+z+\overline{z'}},$$

which is (34). Two special cases are used in Corollary 10 with $J = 1$ and $\rho = \frac{1}{2} + i\gamma$:

$$\|\operatorname{Re} w_\rho\|^2 = \frac{3}{2} + \operatorname{Re}\frac{1}{1+2\rho} - 2\,\operatorname{Re}\frac{1}{1+\rho},$$
$$\langle w_{1/2}, \operatorname{Re} w_\rho\rangle = \frac{1}{3} - \operatorname{Re}\frac{1}{1+\rho} + 2\,\operatorname{Re}\frac{1}{3/2+\rho}.$$

Proof of (19). Let $\sigma \ge \frac{1}{2}$. The subtracted term in (18) is positive and $2/(1+2\sigma) \le 1$, so $K(s) < 2$. Since $K(\frac{1}{2}+i\tau) \to 2$ as $|\tau| \to \infty$, the value 2 is the supremum, and it is not attained. For fixed σ the subtracted term is largest at $\tau = 0$, where

$$K(\sigma) = 1 + \frac{2}{1+2\sigma} - \frac{2}{1+\sigma} = 1 - \varphi(\sigma), \qquad \varphi(\sigma) = \frac{2\sigma}{(1+2\sigma)(1+\sigma)}.$$

Dividing numerator and denominator by σ, $\varphi(\sigma) = 2/(\sigma^{-1} + 3 + 2\sigma)$. By the inequality of the arithmetic and geometric means, $\sigma^{-1} + 2\sigma \ge 2\sqrt{2}$, with equality only at $\sigma = 1/\sqrt{2}$. Hence

$$\varphi(\sigma) \le \frac{2}{3+2\sqrt{2}} = 6 - 4\sqrt{2}, \qquad K(s) \ge K(\sigma) \ge 4\sqrt{2} - 5 = 0.65685\ldots,$$

with equality only for $\tau = 0$ and $\sigma = 1/\sqrt{2}$. Finally $K(\frac{1}{2}) = 1 + 1 - \frac{4}{3} = \frac{2}{3}$, and for $\sigma = \frac{1}{2}$ the formula (18) reads $K(\frac{1}{2}+i\tau) = 2 - 2\,\operatorname{Re}(\frac{3}{2}+i\tau)^{-1}$.

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Appendix B.5. Kernel Estimates for Proposition 3

We use the following standard bound [57, Section 5.1]: for all real $\kappa_1 < \kappa_2$ there is $A_0 = A_0(\kappa_1, \kappa_2)$ such that

$$|(s-1)\zeta(s)| \ll_{\kappa_1,\kappa_2} (1+|t|)^{A_0} \qquad (\kappa_1 \le \sigma \le \kappa_2). \tag{A7}$$

Let $\rho_0 = \beta_0 + i\gamma_0$, ν, W, w, G and $I(Y)$ be as in Proposition 3 and its proof. When $\rho_0 = \frac{1}{2}$ we put $\nu = 0$ in the formulas for W, so that in both cases

$$W(s) = \frac{s(s-1)\zeta(s)}{(s-\rho_0)^\nu}\, e^{(s-\rho_0)^2}.$$

Lemma A3. *The statements* (K1)–(K4) *in the proof of Proposition 3 hold.*

Proof. *The function W.* The function $(s-1)\zeta(s)$ is entire, and when $\zeta(\rho_0) = 0$ it has a zero of order ν at $\rho_0 \neq 1$. So W is entire. For $s = \sigma + it$,

$$\left|e^{(s-\rho_0)^2}\right| = e^{(\sigma-\beta_0)^2-(t-\gamma_0)^2}.$$

Let $\kappa_1 < \kappa_2$. For $|s-\rho_0| \ge 1$ the denominator $(s-\rho_0)^\nu$ has modulus at least 1, and on $|s-\rho_0| \le 1$ the entire function W is bounded. With (A7) this gives

$$|W(\sigma+it)| \le C(\kappa_1,\kappa_2)\, e^{-t^2/2} \qquad (\kappa_1 \le \sigma \le \kappa_2,\ t \in \mathbb{R}), \tag{A8}$$

because $(1+|t|)^{A_0+1} e^{-(t-\gamma_0)^2} \ll e^{-t^2/2}$.

(K1). By (A8) the integral defining $w(u)$ converges absolutely for every κ. For $\kappa_1 < \kappa_2$, Cauchy's theorem on the rectangle with vertices $\kappa_1 \pm iT$, $\kappa_2 \pm iT$, together with (A8) on the horizontal sides as $T \to \infty$, shows that the integrals over $\operatorname{Re} s = \kappa_1$ and $\operatorname{Re} s = \kappa_2$ agree. On the line $\operatorname{Re} s = \kappa$ we have $|u^{-s}| = u^{-\kappa}$, hence

$$|w(u)| \le \frac{u^{-\kappa}}{2\pi} \int_{-\infty}^{\infty} |W(\kappa+it)|\, \mathrm{d}t.$$

Taking $\kappa = A$ and $\kappa = -A$ gives $|w(u)| \le C_A \min(u^A, u^{-A})$. With $A = 1$, $|w(u)|^2 u^{-2} \le C_1^2 \min(1, u^{-4})$, which is integrable on $(0,\infty)$; so $C_w < \infty$.

(K2). Take $\kappa = 2$ in the definition of w. Since $|L(x)| \le x$,

$$\int_1^\infty \int_{-\infty}^{\infty} |L(x)|\,|W(2+it)| \Big(\frac{x}{Y}\Big)^{-2} \mathrm{d}t\, \frac{\mathrm{d}x}{x} \le Y^2 \int_1^\infty x^{-2}\,\mathrm{d}x \int_{-\infty}^{\infty} |W(2+it)|\,\mathrm{d}t < \infty.$$

By Fubini's theorem and (14),

$$\begin{aligned} I(Y) &= \frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} W(s)\, Y^s \Big(\int_1^\infty L(x) x^{-s-1}\,\mathrm{d}x \Big)\,\mathrm{d}s \\ &= \frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} W(s) M(s)\, Y^s\,\mathrm{d}s. \end{aligned}$$

(K3). By the definitions,

$$G(s) = W(s)M(s) = \frac{s(s-1)\zeta(s)}{(s-\rho_0)^\nu}\, e^{(s-\rho_0)^2} \cdot \frac{\zeta(2s)}{s\,\zeta(s)} = \frac{(s-1)\zeta(2s)}{(s-\rho_0)^\nu}\, e^{(s-\rho_0)^2},$$

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first where $\zeta(s) \neq 0$ and then as an identity of meromorphic functions. The function $(s-1)\zeta(2s)$ is holomorphic on $\mathbb{C}$ except for a simple pole at $s = \frac{1}{2}$. If $\zeta(\rho_0) = 0$ with $\beta_0 > \frac{1}{2}$, the only pole of G in $\operatorname{Re} s > \frac{1}{2}$ is at ρ_0, and it has order exactly ν, since the numerator

$$h(s) = (s-1)\zeta(2s)e^{(s-\rho_0)^2}$$

satisfies $h(\rho_0) = (\rho_0 - 1)\zeta(2\rho_0) \neq 0$: indeed $\operatorname{Re}(2\rho_0) > 1$ and $\rho_0 \neq 1$. If $\rho_0 = \frac{1}{2}$, then $G = h$ has the single simple pole $s = \frac{1}{2}$, with

$$\lim_{s\to 1/2} (s - \tfrac{1}{2})\, G(s) = (\tfrac{1}{2} - 1) \cdot \tfrac{1}{2} \cdot e^0 = -\tfrac{1}{4}, \tag{A9}$$

because $\zeta(2s) \sim 1/(2s-1)$ as $s \to \frac{1}{2}$.

(K4), the case $\zeta(\rho_0) = 0$. Let $b = \frac{1}{2}(\frac{1}{2} + \beta_0)$, so that $\frac{1}{2} < b < \beta_0$. On the strip $b \le \sigma \le 2$ we have $|\zeta(2s)| \le \zeta(2b)$, and therefore, for $|s - \rho_0| \ge \frac{1}{2}(\beta_0 - b)$,

$$|G(\sigma + it)| \le C(1 + |t|)\, e^{(2-\beta_0)^2 + \beta_0^2}\, e^{-(t-\gamma_0)^2} \qquad (b \le \sigma \le 2). \tag{A10}$$

By the residue theorem on the rectangle with vertices $b \pm iT$, $2 \pm iT$, with $T \to \infty$, and by (K2),

$$I(Y) = \operatorname*{Res}_{s=\rho_0}\bigl(G(s)Y^s\bigr) + \frac{1}{2\pi i}\int_{b-i\infty}^{b+i\infty} G(s)Y^s\,\mathrm{d}s.$$

By (A10) the last integral is at most $C_b Y^b$ in modulus. Since $G(s)Y^s = h(s)Y^s(s-\rho_0)^{-\nu}$ and $\frac{\mathrm{d}^k}{\mathrm{d}s^k}Y^s = (\log Y)^k Y^s$, Leibniz's rule gives

$$\begin{aligned}\operatorname*{Res}_{s=\rho_0}\bigl(G(s)Y^s\bigr) &= \frac{1}{(\nu-1)!}\frac{\mathrm{d}^{\nu-1}}{\mathrm{d}s^{\nu-1}}\bigl[h(s)Y^s\bigr]_{s=\rho_0} \\ &= Y^{\rho_0}\sum_{k=0}^{\nu-1}\frac{h^{(\nu-1-k)}(\rho_0)}{k!\,(\nu-1-k)!}(\log Y)^k.\end{aligned}$$

The term $k = \nu - 1$ is $h(\rho_0)(\log Y)^{\nu-1}Y^{\rho_0}/(\nu-1)!$, and the other terms are smaller by a factor $O(1/\log Y)$. Hence there is Y_1 such that, for $Y \ge Y_1$,

$$\Bigl|\operatorname*{Res}_{s=\rho_0}\bigl(G(s)Y^s\bigr)\Bigr| \ge \frac{|h(\rho_0)|}{2\,(\nu-1)!}\,(\log Y)^{\nu-1}\,Y^{\beta_0} \ge \frac{|h(\rho_0)|}{2\,(\nu-1)!}\,Y^{\beta_0},$$

the last step for $Y \ge e$. Since $b < \beta_0$, the term $C_b Y^b$ is at most half of this for large Y. This gives (K4) with $\kappa_0 = |h(\rho_0)|/(4(\nu-1)!)$.

(K4), the case $\rho_0 = \frac{1}{2}$. Here $G(s) = (s-1)\zeta(2s)e^{(s-1/2)^2}$. By (A7) applied to $\zeta(2s)$, the function $(2s-1)\zeta(2s)$ is bounded by a power of $1+|t|$ on the strip $\frac{1}{4} \le \sigma \le 2$, so $|G(\sigma+it)| \ll e^{-t^2/2}$ there for $|s - \frac{1}{2}| \ge \frac{1}{8}$. The residue theorem on the rectangle with vertices $\frac{1}{4} \pm iT$, $2 \pm iT$, and (A9), give

$$I(Y) = -\frac{Y^{1/2}}{4} + \frac{1}{2\pi i}\int_{1/4-i\infty}^{1/4+i\infty} G(s)Y^s\,\mathrm{d}s = -\frac{Y^{1/2}}{4} + O\bigl(Y^{1/4}\bigr).$$

So $|I(Y)| \ge \frac{1}{8}Y^{1/2}$ for large Y, which is (K4) with $\beta_0 = \frac{1}{2}$ and $\kappa_0 = \frac{1}{8}$. □

Remark A1. *The Gaussian factor in W serves only to make all contour integrals absolutely convergent; any entire factor of rapid decay in vertical strips that does not vanish at ρ_0 would do. The factor $s(s-1)\zeta(s)(s-\rho_0)^{-\nu}$ is the essential one. Table A2 lists what each factor of W removes from $M(s) = \zeta(2s)/(s\zeta(s))$.*

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Table A2. The factors of the kernel W in (25) and their effect on $G = WM$.

factor of $W(s)$	effect on $G(s) = W(s)\,\zeta(2s)/(s\,\zeta(s))$
s	cancels the factor $1/s$ of M
$(s-1)\zeta(s)$	entire; cancels $1/\zeta(s)$, hence all poles of M at zeros of ζ
$(s-\rho_0)^{-\nu}$	restores the pole at the chosen zero ρ_0, with its full order ν
$e^{(s-\rho_0)^2}$	decay $e^{-(t-\gamma_0)^2}$ on vertical lines; equal to 1 at ρ_0

Appendix C. Calculations for the Obstructions

This appendix contains the computations referred to in Sections 6–11. Numerical values appear only in tables and serve as checks; no proof depends on them.

Appendix C.1. Rounding Controls

The discrepancy identity (58). List the primes above P as $p_1 < p_2 < \cdots$ and put $S_0 = 0$, $S_i = S(p_i) = S_{i-1} + q_{p_i}$. By (57), $I_{p_i} = \lfloor S_i \rfloor - \lfloor S_{i-1} \rfloor$, and since $0 < q_{p_i} < 1$ this difference is 0 or 1. Summing over $i \le n$ gives $\sum_{i\le n} I_{p_i} = \lfloor S_n \rfloor$, hence

$$\sum_{i\le n} (I_{p_i} - q_{p_i}) = \lfloor S_n \rfloor - S_n \in (-1, 0].$$

Both sides of (58) are constant between consecutive primes, which gives the identity for all real $t \ge P$.

The centred series (60). For $T > P$ and $\operatorname{Re} s > 0$, Abel summation gives

$$\sum_{P<p\le T} (I_p - q_p) p^{-s} = E(T)T^{-s} + s\int_P^T E(t)t^{-s-1}\,\mathrm{d}t.$$

Since $|E| \le 1$, the first term tends to zero and the integral converges absolutely as $T \to \infty$, uniformly for $\operatorname{Re} s \ge \sigma_1 > 0$ and $|s|$ bounded. This proves (60), the holomorphy of A on $\operatorname{Re} s > 0$, and the bound $|A(s)| \le |s| P^{-\operatorname{Re} s}/\operatorname{Re} s$.

The logarithmic factorization (61). For $\operatorname{Re} s > 1$ all the following series converge absolutely. The Euler products give

$$\log G_P(s) = \sum_{p\in\mathcal{S}_P} \bigl(\log(1+p^{-s}) - \log(1-p^{-s})\bigr) = 2\sum_{p>P} I_p \sum_{k\ \mathrm{odd}} \frac{p^{-ks}}{k},$$
$$\log \zeta(s+1-\beta) = \sum_p \sum_{k\ge 1} \frac{p^{-k(s+1-\beta)}}{k}.$$

In the difference, the terms with $k = 1$ are

$$2\sum_{p>P} I_p p^{-s} - \sum_p p^{-(s+1-\beta)} = 2\sum_{p>P} (I_p - q_p)p^{-s} - \sum_{p\le P} p^{-(s+1-\beta)},$$

because $2q_p p^{-s} = p^{-(s+1-\beta)}$. The first sum is $2A(s)$. The terms with $k \ge 3$ in $\log G_P$ and $k \ge 2$ in $\log\zeta(s+1-\beta)$ are the two double series of (61). For $\operatorname{Re} s \ge \sigma > 1/2$ they are bounded in absolute value by

$$\frac{2}{3}\sum_p \frac{p^{-3\sigma}}{1-p^{-2\sigma}} \le \frac{4}{3}\sum_p p^{-3\sigma}, \qquad \frac{1}{2}\sum_p \frac{p^{-2(\sigma+1-\beta)}}{1-p^{-(\sigma+1-\beta)}} \le 2\sum_p p^{-2(\sigma+1-\beta)},$$

using $p^{-2\sigma} \le 1/2$ and $p^{-(\sigma+1-\beta)} \le 2^{-1/2}$. Both majorants converge, since $3\sigma > 3/2$ and $2(\sigma+1-\beta) > 3 - 2\beta > 1$.

Table A3 shows the first steps of the rounding and compares the counting function of $\mathcal{S}_P$ with $S(x)$ and with the leading term $x^\beta/(2\beta\log x)$; the convergence to the latter is slow, as for the prime number theorem itself.

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Table A3. The rounding (57) for $\beta = 3/4$, $P = 2$. Left: the first fourteen primes. Right: the counting function of $\mathcal{S}_P$.

p	q_p	$S(p)$	I_p	$E(p)$
3	0.3799	0.3799	0	−0.3799
5	0.3344	0.7143	0	−0.7143
7	0.3074	1.0217	1	−0.0217
11	0.2746	1.2962	0	−0.2962
13	0.2633	1.5596	0	−0.5596
17	0.2462	1.8058	0	−0.8058
19	0.2395	2.0453	1	−0.0453
23	0.2283	2.2736	0	−0.2736
29	0.2155	2.4891	0	−0.4891
31	0.2119	2.7010	0	−0.7010
37	0.2027	2.9037	0	−0.9037
41	0.1976	3.1013	1	−0.1013
43	0.1953	3.2965	0	−0.2965
47	0.1910	3.4875	0	−0.4875

x	$\pi_{\mathcal{S}_P}(x)$	$S(x)$	$\dfrac{x^\beta}{2\beta \log x}$
10^3	20	20.95	17.16
10^4	86	86.26	72.38
10^5	375	375.12	325.63
10^6	1711	1711.82	1525.96

Appendix C.2. Euler-factor Bounds

Three elementary inequalities enter Lemma 12, Theorem 28 and Table 10.

Exponential bounds. For complex w, $|e^w - 1| \le \sum_{j\ge1} |w|^j/j! = e^{|w|} - 1$. For $0 \le x < 1$, comparing $\sum_{j\ge1} x^j/j!$ with $\sum_{j\ge1} x^j$ gives $e^x - 1 \le x/(1-x)$.

A rational bound for C_r. For $0 < r < 1/2$, comparison with the integral gives $\zeta(2-2r) \le 1 + \int_1^\infty t^{-(2-2r)}\,\mathrm{d}t = (2-2r)/(1-2r)$, hence

$$C_r = \frac{\zeta(2-2r)}{(1-2r)^2} \le \frac{2-2r}{(1-2r)^3}.$$

The rows of Table 10. Let $d = 1 - r - \beta$ and $P = 2^e$ with ed an integer. Then $\Delta(P, 1-r) \le 2P^{-d}/d = 2^{1-ed}/d$, and by (63) the largest of the differences $|\mathcal{J}_P^{(k)}(1) - \mathcal{J}_\lambda^{(k)}(1)|$, $k \le K$, is at most $M\Delta/(1-\Delta)$, where $M = C_r \max_{k\le K} k!\, r^{-k}$. All quantities in Table A4 are rational numbers, and the last column is rounded upwards.

Table A4. Rational data for Table 10.

β	r	d	K	$C_r \le$	$\max_{k\le K} k!\, r^{-k}$	$M \le$	$\Delta \le$	bound
3/4	1/8	1/8	3	112/27	3072	114688/9	2^{-36}	$1.9\cdot10^{-7}$
3/4	1/8	1/8	0	112/27	1	112/27	2^{-16}	$6.4\cdot10^{-5}$
3/4	1/8	1/8	6	112/27	188743680	2348810240/3	2^{-56}	$1.1\cdot10^{-8}$
5/8	1/4	1/8	3	12	384	4608	2^{-36}	$6.8\cdot10^{-8}$
7/8	1/16	1/16	3	960/343	24576	23592960/343	2^{-55}	$2.0\cdot10^{-12}$

For instance, in the first row $P = 2^{320}$, $P^{-d} = 2^{-40}$, $\Delta \le 2\cdot 8\cdot 2^{-40} = 2^{-36}$, and $(114688/9)\cdot 2^{-36}/(1-2^{-36}) < 1.9\cdot10^{-7}$.

Appendix C.3. The Vandermonde System

In the proof of Theorem 30 the moments satisfy

$$|t_j| \le \sum_{n\ge2} n^{\beta-2}(\log n)^j < \infty, \qquad \int_1^\infty x^{\beta-2}(\log x)^j\,\mathrm{d}x = \frac{j!}{(1-\beta)^{j+1}},$$

the integral indicating their size. Write $x_m = b_m 2^{-m}$ and $s_j = -t_j/(-\log 2)^j$. The system (65) becomes

$$\sum_{m=0}^{k-1} x_m\, m^j = s_j \qquad (0 \le j < k),$$

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with the convention $0^0 = 1$. Its matrix $(m^j)_{0\le j,m<k}$ is a Vandermonde matrix with nodes $0, 1, \dots, k-1$ and determinant $\prod_{0\le m<n<k}(n-m) = \prod_{n=1}^{k-1} n! \neq 0$. To invert it, let

$$\ell_m(x) = \prod_{\substack{0\le n<k \\ n\neq m}} \frac{x-n}{m-n} = \sum_{j=0}^{k-1} \ell_{m,j} x^j$$

be the Lagrange polynomials of the nodes. Since $\sum_m m^i \ell_m(x)$ is the polynomial of degree less than k that interpolates x^i at the nodes, it equals x^i for $0 \le i < k$; comparing coefficients, $\sum_m m^i \ell_{m,j} = \delta_{ij}$. Hence

$$x_m = \sum_{j=0}^{k-1} \ell_{m,j} s_j, \qquad b_m = 2^m \sum_{j=0}^{k-1} \ell_{m,j} \frac{-t_j}{(-\log 2)^j} \tag{A11}$$

is the unique solution. The coefficients $\ell_{m,j}$ are rational; Table A5 lists them for $k \le 4$. For $k = 1$ the system reads $b_0 = -t_0$, and the sequence is $a_1 = -\varepsilon_0 t_0$, $a_n = \varepsilon_0 n^{\beta-1}\cos(\gamma \log n)$ for $n \ge 2$, with $\varepsilon_0 = 1/(1+|t_0|)$. For $k = 2$, $b_0 = -t_0 - t_1/\log 2$ and $b_1 = 2t_1/\log 2$.

Table A5. Coefficients $\ell_{m,j}$ of the Lagrange polynomials on the nodes $0, \dots, k-1$.

k	m	$\ell_{m,0}$	$\ell_{m,1}$	$\ell_{m,2}$	$\ell_{m,3}$
2	0	1	−1		
2	1	0	1		
3	0	1	−3/2	1/2	
3	1	0	2	−1	
3	2	0	−1/2	1/2	
4	0	1	−11/6	1	−1/6
4	1	0	3	−5/2	1/2
4	2	0	−3/2	2	−1/2
4	3	0	1/3	−1/2	1/6

Appendix C.4. Rudin–Shapiro Blocks

Let $p^{(m)}, q^{(m)} \in \{\pm 1\}^{2^m}$ be the coefficient vectors of P_m, Q_m. The recursion (52) reads

$$p^{(m+1)} = (p^{(m)}, q^{(m)}), \qquad q^{(m+1)} = (p^{(m)}, -q^{(m)}) \tag{A12}$$

as concatenations. An induction on $m - j$ shows that every aligned block of length 2^j inside $p^{(m)}$ or $q^{(m)}$ is one of $\pm p^{(j)}, \pm q^{(j)}$, and the same holds in the infinite sequence r, whose first 2^m terms are $p^{(m)}$.

The complementary identity. For complex u, v, $|u+v|^2 + |u-v|^2 = 2|u|^2 + 2|v|^2$. With $u = P_m(z)$, $v = z^{2^m} Q_m(z)$ and $|z| = 1$ this gives $|P_{m+1}|^2 + |Q_{m+1}|^2 = 2(|P_m|^2 + |Q_m|^2)$, and (53) follows from $|P_0|^2 + |Q_0|^2 = 2$. In particular $|P_m(z)|, |Q_m(z)| \le 2^{(m+1)/2}$ on the unit circle. Writing $P_m(z)\overline{P_m(z)} = \sum_{|h|<2^m} c_P(h) z^h$ with $c_P(h) = \sum_t p_t^{(m)} p_{t+h}^{(m)}$ for $h \ge 0$, and similarly for Q_m, the identity (53) is equivalent to

$$c_P(h) + c_Q(h) = 0 \qquad (1 \le h < 2^m). \tag{A13}$$

The sup norm. Let $N \ge 1$ and $2^J \le N < 2^{J+1}$. The binary expansion $N = \sum_{j\in\mathcal{B}} 2^j$, read from the largest exponent downwards, cuts $[1, N]$ into consecutive aligned blocks, one of each length 2^j, $j \in \mathcal{B}$. The exponential sum over an aligned block of length 2^j is a unimodular multiple of $\pm P_j(\mathrm{e}(\alpha))$ or $\pm Q_j(\mathrm{e}(\alpha))$. Therefore

$$\Big|\sum_{n\le N} r_n \mathrm{e}(n\alpha)\Big| \le \sum_{j=0}^{J} 2^{(j+1)/2} < \frac{2\cdot 2^{J/2}}{\sqrt{2}-1} = (2+2\sqrt{2})\,2^{J/2} \le (2+2\sqrt{2})\sqrt{N}.$$

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Fixed shifts. Let $h, Y \geq 1$. If $h > Y$ then $|C_{r,h}(Y)| \leq Y < \sqrt{hY}$. Let $h \leq Y$, choose k with $\sqrt{hY} \leq 2^k < 2\sqrt{hY}$ and put $D = 2^k$, so that $h \leq D$. Let $g = \lfloor Y/(2D) \rfloor$ and cut $[1, 2Dg]$ into the aligned blocks $\Lambda_i = (2D(i-1), 2Di]$, $1 \leq i \leq g$. The remaining starting points $n \in (2Dg, Y]$ number fewer than $2D$. Fix i and let Λ_i' and Λ_i'' be the two halves of Λ_i. By (A12) and the block structure, r restricted to Λ_i' is $\pm p^{(k)}$ and r restricted to Λ_i'' is $\pm q^{(k)}$. Split the $n \in \Lambda_i$ into four classes:

- $n, n+h \in \Lambda_i'$: these contribute $c_P(h)$, the signs cancelling in the product;
- $n, n+h \in \Lambda_i''$: these contribute $c_Q(h)$;
- $n \in \Lambda_i'$, $n+h \in \Lambda_i''$: exactly h values of n;
- $n \in \Lambda_i''$, $n+h \notin \Lambda_i$: exactly h values of n.

Since $h \leq D$ there are no other cases. By (A13) with $m = k$ the first two classes cancel when $h < D$, and both are empty when $h = D$. So $|\sum_{n \in \Lambda_i} r_n r_{n+h}| \leq 2h$, and

$$|C_{r,h}(Y)| \leq 2hg + 2D \leq \frac{hY}{D} + 2D \leq \sqrt{hY} + 4\sqrt{hY} = 5\sqrt{hY}.$$

The repeated block. Applying (A12) twice,

$$p^{(m+2)} = (p^{(m)},\, q^{(m)},\, p^{(m)},\, -q^{(m)}).$$

For $H = 2^m$ and $1 \leq n \leq H$ this gives $r_{n+2H} = r_n$ and $r_{n+3H} = -r_{n+H}$. Hence

$$C_{r,2H}(H) = \sum_{n=1}^{H} r_n^2 = H, \qquad C_{r,2H}(2H) = H + \sum_{n=H+1}^{2H} r_n r_{n+2H} = 0.$$

The endpoint $Y = H$ satisfies $Y \leq N - h$ for $N = 3H$, $h = 2H$, so $C^*_{r,2H}(3H) \geq H$ and $\mathbb{E}_r(3H, H) \geq H^2$.

Table A6 lists the first terms and numerical checks of the three estimates.

Table A6. Rudin–Shapiro data. Top: $r_1, \ldots, r_{16}$, grouped into the blocks $p^{(2)}, q^{(2)}, p^{(2)}, -q^{(2)}$. Bottom: computed values of $\sup_\alpha |\sum_{n \leq N} r_n e(n\alpha)|/\sqrt{N}$ (bound $2 + 2\sqrt{2} < 4.83$) and of $\max |C_{r,h}(Y)|/\sqrt{hY}$ over $1 \leq h, Y \leq 2048$ (bound 5).

$p^{(2)}$	$q^{(2)}$	$p^{(2)}$	$-q^{(2)}$
$+\,+\,+\,-$	$+\,+\,-\,+$	$+\,+\,+\,-$	$-\,-\,+\,-$

N	10	100	1000	4096	5000
$\sup_\alpha \lvert A_N(\alpha)\rvert/\sqrt{N}$	1.90	1.92	1.63	1.41	2.06

$\max_{h,Y \leq 2048} \lvert C_{r,h}(Y)\rvert/\sqrt{hY}$	1.414
attained at (h, Y)	$(1, 2)$

Appendix C.5. The Legendre Kernel

Define Λ_j by Rodrigues' formula

$$\Lambda_j(t) = \frac{1}{2^j j!} \frac{\mathrm{d}^j}{\mathrm{d}t^j} (t^2 - 1)^j.$$

It is a polynomial of degree j with leading coefficient $\ell_j = (2j)!/(2^j (j!)^2)$, and it has the parity of j.

Orthogonality and norm. The function $(t^2 - 1)^j$ vanishes to order j at ± 1. For a polynomial g, integrating by parts j times therefore gives

$$\int_{-1}^{1} \Lambda_j(t) g(t)\, \mathrm{d}t = \frac{(-1)^j}{2^j j!} \int_{-1}^{1} (t^2 - 1)^j g^{(j)}(t)\, \mathrm{d}t,$$

which vanishes if $\deg g < j$. Taking $g = \Lambda_j$, whose j-th derivative is $j!\, \ell_j$,

$$\int_{-1}^{1} \Lambda_j^2 = \frac{\ell_j}{2^j} \int_{-1}^{1} (1 - t^2)^j\, \mathrm{d}t.$$

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Put $W_j = \int_{-1}^{1}(1-t^2)^j\,\mathrm{d}t$. One integration by parts gives $W_j = 2j\int_{-1}^{1} t^2(1-t^2)^{j-1}\,\mathrm{d}t = 2j(W_{j-1}-W_j)$, so $W_j = \frac{2j}{2j+1}W_{j-1}$ and $W_j = 2^{2j+1}(j!)^2/(2j+1)!$. Hence $\int_{-1}^{1}\Lambda_j^2 = 2/(2j+1)$. By Leibniz' rule the j-th derivative of $(t-1)^j(t+1)^j$ at $t=1$ is $j!\,2^j$, so $\Lambda_j(1)=1$.

Values at the origin. By parity $\Lambda_j(0)=0$ for odd j. For $j=2i$ the coefficient of t^{2i} in $(t^2-1)^{2i}$ is $(-1)^i\binom{2i}{i}$, so

$$\Lambda_{2i}(0) = \frac{(2i)!\,(-1)^i\binom{2i}{i}}{2^{2i}(2i)!} = (-1)^i\binom{2i}{i}4^{-i}.$$

Let $w_i = (\binom{2i}{i}4^{-i})^2$. Then $w_0=1$ and $w_{i+1} = w_i(2i+1)^2/(2i+2)^2$. We claim $w_i \le 1/(i+1)$. This holds for $i=0$, and if it holds for i then

$$w_{i+1} \le \frac{(2i+1)^2}{4(i+1)^3} \le \frac{1}{i+2},$$

because $(2i+1)^2(i+2) = 4i^3+12i^2+9i+2 \le 4i^3+12i^2+12i+4 = 4(i+1)^3$.

The kernel. By the substitution $t=4u$, the functions

$$\phi_j(u) = \sqrt{2(2j+1)}\,\Lambda_j(4u) \qquad (0\le j\le m)$$

form an orthonormal basis of the space of polynomials of degree at most m in $L^2[-1/4,1/4]$, and $K_m(v,u) = \sum_{j\le m}\phi_j(v)\phi_j(u)$. For $p=\sum_j c_j\phi_j$,

$$\int_{-1/4}^{1/4} K_m(0,u)p(u)\,\mathrm{d}u = \sum_j c_j\phi_j(0) = p(0),$$

$$|p(0)|^2 \le \sum_j |c_j|^2 \sum_j \phi_j(0)^2 = K_m(0,0)\int_{-1/4}^{1/4}|p|^2,$$

and $\int_{-1/4}^{1/4}K_m(0,u)^2\,\mathrm{d}u = K_m(0,0)$. This is (68). Finally

$$K_m(0,0) = 2\sum_{i\le m/2}(4i+1)w_i \le 2\sum_{i=0}^{\lfloor m/2\rfloor}\frac{4i+1}{i+1} \le 8\Big(\Big\lfloor\frac{m}{2}\Big\rfloor+1\Big) \le 4(m+2),$$

which is (69). Table A7 compares the two sides; the true growth is $K_m(0,0)\sim 4m/\pi$.

Table A7. The diagonal value $K_m(0,0)$ and the bound (69).

m	0	2	4	8	16	32	64
$K_m(0,0)$	2.000	4.500	7.031	12.112	22.291	42.658	83.400
$4(m+2)$	8	16	24	40	72	136	264

Appendix C.6. Elementary Prime Sums

We use Chebyshev's bounds in the form $\pi(t) \le 6t/\log t$ for $t\ge 2$ and $\vartheta(t) = \sum_{p\le t}\log p \le \pi(t)\log t \le 6t$ [3].

The sum $\sum_{p\le y}p^{-1/2}$. For $y\ge 4$, partial summation gives

$$\sum_{p\le y}p^{-1/2} = \frac{\pi(y)}{\sqrt{y}} + \frac{1}{2}\int_2^y \frac{\pi(t)}{t^{3/2}}\,\mathrm{d}t \le \frac{6\sqrt{y}}{\log y} + 3\int_2^y \frac{\mathrm{d}t}{\sqrt{t}\,\log t}.$$

On $[2,\sqrt{y}]$ use $\log t\ge\log 2$, and on $[\sqrt{y},y]$ use $\log t\ge\frac{1}{2}\log y$:

$$\int_2^y \frac{\mathrm{d}t}{\sqrt{t}\log t} \le \frac{2y^{1/4}}{\log 2} + \frac{4\sqrt{y}}{\log y}.$$

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Hence $\sum_{p\le y} p^{-1/2} \le 18\sqrt{y}/\log y + 9y^{1/4}$. For $0 < u \le 2^{-1/2}$, $\log(1+u) \le u$ and $-\log(1-u) \le u/(1-u)$ give $\log\frac{1+u}{1-u} \le \frac{2u}{1-u} < 7u$. Therefore

$$-\log c_y(1/2) = \sum_{p\le y} \log\frac{1+p^{-1/2}}{1-p^{-1/2}} \le \frac{126\sqrt{y}}{\log y} + 63y^{1/4},$$

and for $y = (\log X)^2$ the right side is $63\log X/\log\log X + 63\sqrt{\log X} = o(\log X)$.

The sum $P_u = \sum_{p\le u}(\log p)p^{-\sigma}$, $0 < \sigma < 1$. Partial summation with $\vartheta(t) \le 6t$ gives

$$P_u = \vartheta(u)u^{-\sigma} + \sigma\int_2^u \vartheta(t)t^{-\sigma-1}\,\mathrm{d}t \le 6u^{1-\sigma} + \frac{6\sigma}{1-\sigma}u^{1-\sigma} = \frac{6u^{1-\sigma}}{1-\sigma}.$$

This is the bound used in Proposition 20, where also

$$\sum_{p\le R}(\log p)p^{-\sigma}\log\frac{R}{p} = \sum_{p\le R}(\log p)p^{-\sigma}\int_p^R\frac{\mathrm{d}u}{u} = \int_2^R P_u\frac{\mathrm{d}u}{u} \le \frac{6R^{1-\sigma}}{(1-\sigma)^2}.$$

With the prime number theorem $\vartheta(t) \sim t$ in place of Chebyshev's bound the same partial summation gives $P_u \sim u^{1-\sigma}/(1-\sigma)$, as used in Theorem 33.

Appendix D. Growth in a Zero-Free Half-Plane

This appendix proves Lemma 17. The method is that of [57, §14.2]; the point to check is the uniformity in the character.

Let χ be primitive of conductor $q \ge 1$ and suppose $L(s,\chi)$ has no zero in $\operatorname{Re} s > \theta$, where $1/2 \le \theta < 1$. Let $0 < \delta \le 1/4$ and let t be real, with $|t| \ge 4$ if $q = 1$. Put $s_0 = 2 + it$ and

$$r_0 = \tfrac{1}{2}, \qquad r_1 = 2 - \theta - \delta, \qquad r_2 = 2 - \theta - \tfrac{\delta}{2}, \qquad R = 2 - \theta - \tfrac{\delta}{4},$$

so that $r_0 < r_1 < r_2 < R \le 3/2$. Every point $\sigma + it$ with $\theta + \delta \le \sigma \le 2$ lies in the closed disc of radius r_1 about s_0.

The logarithm. The closed disc $|s - s_0| \le R$ lies in $\operatorname{Re} s \ge \theta + \delta/4$, where $L(s,\chi)$ has no zero. It does not contain the point $s = 1$ when $q = 1$, because then $|\operatorname{Im} s| \ge 4 - R > 1$. So there is a holomorphic branch $G(s) = \log L(s,\chi)$ on a neighbourhood of this disc, equal to $\sum_p\sum_{k\ge1}\chi(p)^k p^{-ks}/k$ for $\operatorname{Re} s > 1$. On the disc of radius r_0 we have $\operatorname{Re} s \ge 3/2$ and therefore $|G(s)| \le \log\zeta(3/2) < 1$.

An upper bound for $\operatorname{Re} G$. If $q > 1$, the sums of χ over intervals are bounded by q in modulus, and partial summation gives, for $\operatorname{Re} s > 0$,

$$L(s,\chi) = s\int_1^\infty \Big(\sum_{n\le x}\chi(n)\Big)x^{-s-1}\,\mathrm{d}x, \qquad |L(s,\chi)| \le \frac{q|s|}{\operatorname{Re} s}.$$

If $q = 1$, the formula $\zeta(s) = s/(s-1) - s\int_1^\infty\{x\}x^{-s-1}\,\mathrm{d}x$ gives $|\zeta(s)| \le |s|/|s-1| + |s|/\operatorname{Re} s$. On the disc of radius R we have $|s| \le 4 + |t|$, $\operatorname{Re} s \ge 1/2$, and $|s-1| \ge 1$ when $q = 1$. In both cases

$$\operatorname{Re} G(s) = \log|L(s,\chi)| \le \Lambda := \log\big(6q(2+|t|)\big) \qquad (|s - s_0| \le R),$$

and $\Lambda \ge \log 12 > 1$.

Borel–Carathéodory. For a function G holomorphic on $|s - s_0| \le R$ and $0 < r < R$,

$$\max_{|s-s_0|\le r}|G(s)| \le \frac{2r}{R-r}\max_{|s-s_0|=R}\operatorname{Re} G(s) + \frac{R+r}{R-r}|G(s_0)|$$

[49, §6.1]. With $r = r_2$, $R - r_2 = \delta/4$ and $|G(s_0)| < 1 < \Lambda$, this gives $\max_{|s-s_0|\le r_2}|G| \le (24/\delta)\Lambda$.

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Three circles. Let $\mathsf{M}(r) = \max_{|s-s_0|=r} |G(s)|$. Hadamard's theorem states that $\log \mathsf{M}(r)$ is a convex function of $\log r$; hence

$$\mathsf{M}(r_1) \le \mathsf{M}(r_0)^{1-\tau}\mathsf{M}(r_2)^{\tau}, \qquad \tau = \frac{\log(r_1/r_0)}{\log(r_2/r_0)} \in (0,1),$$

where τ depends only on θ and δ. With $\mathsf{M}(r_0) < 1$ we get $|G(s)| \le (24\Lambda/\delta)^{\tau}$ on the disc of radius r_1, by the maximum principle. Since $\tau < 1$, for every $\kappa > 0$ there is $C(\theta,\delta,\kappa)$ with $(24\Lambda/\delta)^{\tau} \le \kappa\Lambda + C(\theta,\delta,\kappa)$. Therefore, for $\theta + \delta \le \sigma \le 2$,

$$\big|\log|L(\sigma+it,\chi)|\big| \le |G(\sigma+it)| \le \kappa \log\big(6q(2+|t|)\big) + C(\theta,\delta,\kappa),$$

which is the two-sided bound of the lemma. No constant depends on χ or q.

Small $|t|$ for $q = 1$. The reciprocal $1/\zeta$, extended by 0 at $s = 1$, is continuous on the compact set $\theta + \delta \le \sigma \le 2$, $|t| \le 4$, which contains no zero of ζ. It is therefore bounded there. This completes the proof.

Appendix E. Contour Estimates for the Smoothed Sum

This appendix proves the estimates (84), (85) and (86) used for Theorem 35. The notation is that of the proof: $T = \log X \ge 3$, $\omega = c_1/\ell(T) \le \omega_0 \le 1/4$, and $\Phi(s) = \Gamma(s)\zeta(2s)/\zeta(s)$.

Appendix E.1. The Gamma Function on Vertical Lines

Lemma A4 (Elementary Gamma bounds)**.** *For real t and $0 < \sigma \le 2$,*

$$|\Gamma(\sigma+it)| \le \Gamma(\sigma)\,|\Gamma(2+it)|, \qquad |\Gamma(2+it)|^2 = \frac{(1+t^2)\,\pi t}{\sinh \pi t} \le 2\pi(1+|t|)^3 \mathrm{e}^{-\pi|t|}.$$

Consequently, for $1/2 \le \sigma \le 2$ and all real t,

$$|\Gamma(\sigma+it)| \le 5\,(1+|t|)^{3/2}\mathrm{e}^{-\pi|t|/2}.$$

Proof. The Weierstrass product $1/\Gamma(s) = s\mathrm{e}^{\gamma s}\prod_{n\ge1}(1+s/n)\mathrm{e}^{-s/n}$ [49, Appendix C] gives, for $\sigma > 0$,

$$\left|\frac{\Gamma(\sigma)}{\Gamma(\sigma+it)}\right|^2 = \prod_{k\ge0}\Big(1+\frac{t^2}{(\sigma+k)^2}\Big).$$

Each factor decreases as σ increases, so $|\Gamma(\sigma+it)|/\Gamma(\sigma)$ is nondecreasing in σ; comparing with $\sigma = 2$, where $\Gamma(2) = 1$, gives the first inequality. The reflection formula gives $|\Gamma(1+it)|^2 = \Gamma(1+it)\Gamma(1-it) = \pi t/\sinh \pi t$, and $|\Gamma(2+it)|^2 = (1+t^2)|\Gamma(1+it)|^2$. For $x \ge 0$ the inequality $\mathrm{e}^{2x} \ge 1+x$ is equivalent to $x/\sinh x \le 2(1+x)\mathrm{e}^{-x}$. With $x = \pi|t|$, and $1+t^2 \le (1+|t|)^2$, $1+\pi|t| \le \pi(1+|t|)$, this gives the bound for $|\Gamma(2+it)|^2$. Finally $\Gamma(\sigma) \le \Gamma(1/2) = \sqrt{\pi}$ on $[1/2,2]$, and $\sqrt{\pi}\cdot\sqrt{2\pi} = 4.44\ldots < 5$. □

The exponent $3/2$ is not optimal for $\sigma < 2$ but is harmless here. The decay $\mathrm{e}^{-\pi|t|/2}$ is the source of the saving: with $T = \log X$,

$$\mathrm{e}^{-\pi T/2} = X^{-\pi/2}, \qquad 2 - \frac{\pi}{2} = 0.42920\ldots < \frac{1}{2}.$$

Appendix E.2. The Region $|T| \le 3$

The rectangle $\{1-\omega_0 \le \sigma \le 2,\ |t| \le 3\}$ is compact and contains no zero of ζ, by the choice of ω_0. The function $1/\zeta$, with the value 0 at $s = 1$, is continuous on it. Let K_0 be its maximum there.

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For $|t| \geq 3$ and $\sigma \geq 1 - c_1/\ell(|t|)$, Lemma 13 and $\ell(|t|) \leq \log|t|$ give $|1/\zeta(\sigma+it)| \leq C_1 \log|t|$. Since $\omega \leq \omega_0$ and $1-\omega \geq 1 - c_1/\ell(|t|)$ for $3 \leq |t| \leq T$, we have, with $C_2 = \max(K_0, C_1)$,

$$\left|\frac{1}{\zeta(\sigma+it)}\right| \leq C_2 \log(3+|t|) \qquad (1-\omega \leq \sigma \leq 2,\ |t| \leq T). \tag{A14}$$

In the same region $\mathrm{Re}(2s) \geq 2 - 2\omega \geq 3/2$, so $|\zeta(2s)| \leq \zeta(3/2)$.

Appendix E.3. The Three Estimates

The half-lines. On $\mathrm{Re}\, s = 2$ we have $|\zeta(4+2it)| \leq \zeta(4)$ and $|1/\zeta(2+it)| \leq \sum_n \mu(n)^2 n^{-2} = \zeta(2)/\zeta(4)$, so $|\zeta(2s)/\zeta(s)| \leq \zeta(2)$. By Lemma A4,

$$\int_{\mathrm{Re}\, s=2,\ |t|\geq T} |\Phi(s) X^s|\,|\,\mathrm{d}s| \leq 2\sqrt{2\pi}\,\zeta(2) X^2 \int_T^\infty (1+t)^{3/2} \mathrm{e}^{-\pi t/2}\,\mathrm{d}t.$$

Writing $t = T + u$ and using $1 + T + u \leq (1+T)(1+u)$,

$$\int_T^\infty (1+t)^{3/2} \mathrm{e}^{-\pi t/2}\,\mathrm{d}t \leq (1+T)^{3/2} \mathrm{e}^{-\pi T/2} \int_0^\infty (1+u)^{3/2} \mathrm{e}^{-\pi u/2}\,\mathrm{d}u.$$

The last integral is an absolute constant, and $\mathrm{e}^{-\pi T/2} = X^{-\pi/2}$. This proves (84).

The horizontal segments. For $s = \sigma \pm iT$ with $1-\omega \leq \sigma \leq 2$, Lemma A4, the bound $|\zeta(2s)| \leq \zeta(3/2)$ and Lemma 13 at height T give

$$|\Phi(\sigma \pm iT)| \leq 5(1+T)^{3/2} X^{-\pi/2}\, \zeta(3/2)\, C_1 \ell(T).$$

The segment has length at most $5/4$ and $X^\sigma \leq X^2$ on it. Since $(1+T)^{3/2}\ell(T) \ll T^{3/2}\log T \ll T^2$, this proves (85).

The left side. For $s = 1 - \omega + it$ with $|t| \leq T$, Lemma A4 and (A14) give

$$|\Phi(1-\omega+it)| \leq 5\,\zeta(3/2)\, C_2\, (1+|t|)^{3/2} \log(3+|t|)\, \mathrm{e}^{-\pi|t|/2},$$

whose integral over the real line is an absolute constant. As $|X^s| = X^{1-\omega}$, this proves (86).

The three estimates give $B(X) \ll X^{1-\omega} + X^{2-\pi/2}(\log X)^2$, and

$$X^{-\omega} = \exp\Big(-\frac{c_1 \log X}{(\log\log X)^{2/3} (\log\log\log X)^{1/3}}\Big),$$

which tends to zero more slowly than any negative power of X. The second term is smaller than the first for large X, and Theorem 35 follows.

Appendix F. Further Proofs for the Smoothed Sum and Its Twists

Proof of Proposition 24. If the limit superior is infinite there is nothing to prove. Otherwise let C exceed it, so that $\mathcal{H}_B(X) \leq CX$ for $X \geq X_1$. For $\kappa > 0$, Tonelli's theorem and the substitution $X = Y/t$ give

$$\int_0^\infty \mathcal{H}_B(X) X^{-1-\kappa} \frac{\mathrm{d}X}{X} = r(1+\kappa) P(\kappa),$$

where

$$r(s) = \int_{1/2}^2 t^s \frac{\mathrm{d}t}{t} = \frac{2^s - 2^{-s}}{s}, \qquad P(\kappa) = \int_0^\infty |B(Y)|^2 Y^{-1-\kappa} \frac{\mathrm{d}Y}{Y}.$$

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The part of the left side over $X \le X_1$ is bounded uniformly for $0 < \kappa \le 1$, by (80), and the part over $X \ge X_1$ is at most C/κ. Hence $P(\kappa) \le (C/\kappa + O(1))/r(1+\kappa)$. For $\delta > 0$ the Cauchy–Schwarz inequality gives

$$\left| \int_1^\infty B(Y) Y^{-1/2-\delta} \frac{\mathrm{d}Y}{Y} \right|^2 \le \int_1^\infty |B(Y)|^2 Y^{-1-\delta} \frac{\mathrm{d}Y}{Y} \int_1^\infty Y^{-\delta} \frac{\mathrm{d}Y}{Y} \le \frac{P(\delta)}{\delta}.$$

The same inequality with absolute values inside shows that $\int_1^\infty B(Y) Y^{-s} \, \mathrm{d}Y/Y$ converges absolutely for $\operatorname{Re} s > 1/2$. It is therefore holomorphic there and, by (81), equals $\Phi(s)$ minus an entire function. Multiplying by δ^2 and letting $\delta \downarrow 0$, the residue (82) gives

$$c_0^2 \le \liminf_{\delta \downarrow 0} \delta \, P(\delta) \le \frac{C}{r(1)} = \frac{2C}{3}.$$

Let C decrease to the limit superior. □

Proof of Theorem 47. Let r be the limit superior. Every primitive χ carries a positive weight, so $B_\chi(X) \ll_{\chi,\varepsilon} X^{r/2+\varepsilon}$. As in (81),

$$\int_0^\infty B_\chi(X) X^{-s} \frac{\mathrm{d}X}{X} = \Gamma(s) \frac{L(2s, \chi^2)}{L(s,\chi)} \qquad (\operatorname{Re} s > 1),$$

and the integral is holomorphic in $\operatorname{Re} s > r/2$. A zero ρ of $L(s,\chi)$ with $\operatorname{Re}\rho > 1/2$ is a pole of the right side, because $\Gamma(\rho) L(2\rho, \chi^2) \ne 0$; hence $\operatorname{Re}\rho \le r/2$. For $q = 1$ the pole at $s = 1/2$ gives $r \ge 1$. Thus $r \ge 2\Theta_D$.

If $\Theta_D = 1$ the bound $A_*(X) \le \zeta(2) X^2$ gives $r \le 2$. Let $\Theta_D < 1$ and fix $\sigma \in (\Theta_D, 1)$ and $0 < \kappa < 1/2$. Lemma 17, with $\theta = \Theta_D$ and $\delta = \min(\sigma - \Theta_D, 1/4)$, gives

$$\left| \frac{1}{L(u+it,\chi)} \right| \le C'(\sigma,\kappa) \big(q(2+|t|)\big)^\kappa \qquad (\sigma \le u \le 2, \ t \in \mathbb{R})$$

for every primitive χ, and the constant is the same for all characters and all conductors. This uniformity is essential, since the bound is summed over q below. The analogue of (83) is

$$B_\chi(X) = \frac{1}{2\pi i} \int_{\operatorname{Re} s = 2} \Gamma(s) \frac{L(2s,\chi^2)}{L(s,\chi)} X^s \, \mathrm{d}s.$$

The integrand is holomorphic in the strip $\sigma \le \operatorname{Re} s \le 2$, where $|L(2s,\chi^2)| \le \zeta(2\sigma)$. By Lemma A4, $|\Gamma(u+it)| \le 5(1+|t|)^{3/2} \mathrm{e}^{-\pi|t|/2}$ in the strip, so the horizontal integrals tend to zero and the line may be moved to $\operatorname{Re} s = \sigma$:

$$|B_\chi(X)| \le \frac{5\,\zeta(2\sigma) C'(\sigma,\kappa)}{2\pi} q^\kappa X^\sigma \int_{-\infty}^\infty (1+|t|)^{3/2} (2+|t|)^\kappa \mathrm{e}^{-\pi|t|/2} \, \mathrm{d}t.$$

This is (103). Since there are at most $\varphi(q)$ primitive characters modulo q,

$$A_*(X) \le C(\sigma,\kappa)^2 X^{2\sigma} \sum_{q \ge 1} q^{-2+2\kappa} \ll_{\sigma,\kappa} X^{2\sigma}.$$

Letting $\sigma \downarrow \Theta_D$ gives $r \le 2\Theta_D$. □

Appendix G. The Plancherel Computation at the Exponent One

This appendix supplies the details of the proof of Theorem 42. Write $\varepsilon(u) = \exp(-c_0' \eta(u))$ for $u \ge 3$, so that $|\mathcal{M}(u)| \le C_W u \, \varepsilon(u)$ by (97), and let $h(v) = v^{3/5} (\log v)^{-1/5}$, so that $\eta(u) = h(\log u)$. The function h is increasing for $v \ge \mathrm{e}^{1/3}$, and $h(v) \ge \sqrt{v}$ for all $v > 1$, because $\sqrt{v} \ge \log v$.

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Appendix G.1. The Tail of the M Öbius Series on the Line

For a continuously differentiable ϕ and $3 \le M < Y$, partial summation gives

$$\sum_{M<n\le Y} \mu(n)\phi(n) = \mathcal{M}(Y)\phi(Y) - \mathcal{M}(M)\phi(M) - \int_M^Y \mathcal{M}(u)\phi'(u)\,\mathrm{d}u.$$

Take $\phi(u) = u^{-s}$ with $\operatorname{Re} s \ge 1$. Then $|\phi(u)| \le 1/u$ and $|\phi'(u)| \le |s|u^{-2}$, and letting $Y \to \infty$,

$$|R_M(s)| \le C_W\,\varepsilon(M) + C_W|s| \int_M^\infty \varepsilon(u)\,\frac{\mathrm{d}u}{u} \qquad (\operatorname{Re} s \ge 1). \tag{A15}$$

The integral converges, as shown below, so the series $\sum_n \mu(n)n^{-s}$ converges uniformly on compact subsets of $\operatorname{Re} s \ge 1$. Its sum is continuous there and equals $1/\zeta(s)$ for $\operatorname{Re} s > 1$; hence it equals $1/\zeta(s)$ on the line $\operatorname{Re} s = 1$ as well, with the value 0 at $s = 1$. With $\phi(u) = u^{-s}\log u$, for which $|\phi'(u)| \le (|s|\log u + 1)u^{-2}$, the same computation bounds the termwise derivative:

$$|R'_M(s)| \le C_W\,\varepsilon(M)\log M + C_W \int_M^\infty \varepsilon(u)\big(|s|\log u + 1\big)\,\frac{\mathrm{d}u}{u} \qquad (\operatorname{Re} s \ge 1).$$

The differentiated series converges uniformly on compact subsets of the line, so $t \mapsto R_M(1+it)$ is continuously differentiable with derivative $iR'_M(1+it)$. Put

$$\varepsilon_1(M) = \varepsilon(M)\log M + \int_M^\infty \varepsilon(u)(1+\log u)\,\frac{\mathrm{d}u}{u}.$$

Then $|R_M(1+it)| + |R'_M(1+it)| \le 4C_W(1+|t|)\varepsilon_1(M)$, which is the first part of (98).

For the second part let $M \ge 5$, so that $\log M \ge \mathrm{e}^{1/3}$. The substitution $v = \log u$ and the inequalities $h(v) \ge h(\log M)$ and $h(v) \ge \sqrt{v}$ for $v \ge \log M$ give

$$\begin{aligned}\int_M^\infty \varepsilon(u)(1+\log u)\,\frac{\mathrm{d}u}{u} &= \int_{\log M}^\infty \mathrm{e}^{-c_0' h(v)}(1+v)\,\mathrm{d}v \\ &\le \mathrm{e}^{-c_0'\eta(M)/2} \int_0^\infty \mathrm{e}^{-c_0'\sqrt{v}/2}(1+v)\,\mathrm{d}v.\end{aligned}$$

Also $\varepsilon(M)\log M = \exp(-c_0'\eta(M) + \log\log M) \ll \exp(-c_0'\eta(M)/2)$, because $\eta(M) \ge \sqrt{\log M}$. Hence $\varepsilon_1(M) \ll \exp(-c_1'\eta(M))$ with $c_1' = c_0'/2$.

Appendix G.2. The Three Ranges

On $\operatorname{Re} s = 1$, $s \ne 1$, we have $\sum_{n\le M} \mu(n)n^{-s} = 1/\zeta(s) - R_M(s)$, and at $s = 1$ the sum equals $-R_M(1)$. By (89) with $w_n = 1$,

$$U_M(s) = \frac{1 - \zeta(s)\big(1/\zeta(s) - R_M(s) + R_M(1)\big)}{s} = \frac{\zeta(s)\big(R_M(s) - R_M(1)\big)}{s}.$$

Write $s = 1 + it$ and $\varepsilon_1 = \varepsilon_1(M)$.

For $0 < |t| \le 1$: $|R_M(1+it) - R_M(1)| \le |t| \max_{|\tau|\le 1} |R'_M(1+i\tau)| \le 8C_W|t|\varepsilon_1$, and $|t\zeta(1+it)|$ is bounded on this range, so $|U_M(1+it)| \ll \varepsilon_1$.

For $1 \le |t| \le T$: $|R_M(1+it) - R_M(1)| \ll (1+|t|)\varepsilon_1$, $|\zeta(1+it)| \ll \log(2+|t|)$ and $|s| \ge |t|$, so $|U_M(1+it)| \ll \varepsilon_1 \log(2+|t|)$.

For $|t| > T \ge 1$: from the first form of U_M and $|\sum_{n\le M} \mu(n)(n^{-s} - n^{-1})| \le 2(1+\log M)$,

$$|U_M(1+it)| \le \frac{1 + 2(1+\log M)|\zeta(1+it)|}{|t|} \ll \frac{(\log M)\log(2+|t|)}{|t|}.$$

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The function $u_M^\sharp$ is bounded by $1+M$, so Lemma 16 applies with $\sigma = 1$:

$$\begin{aligned}\|u_M^\sharp\|_1^2 &= \frac{1}{2\pi}\int_{-\infty}^{\infty} |U_M(1+it)|^2\,\mathrm{d}t \\ &\ll \varepsilon_1^2 + \varepsilon_1^2\int_1^T \log^2(2+t)\,\mathrm{d}t + (\log M)^2\int_T^\infty \frac{\log^2(2+t)}{t^2}\,\mathrm{d}t.\end{aligned}$$

The two integrals are $O(T\log^2(2+T))$ and $O(\log^2(2+T)/T)$. With $T = 1/\varepsilon_1$,

$$\|u_M^\sharp\|_1^2 \ll \varepsilon_1\,(\log M)^2\log^2(2/\varepsilon_1) \ll \exp\bigl(-c_1'\eta(M)\bigr)(\log M)^4,$$

since $\log(2/\varepsilon_1) \ll \eta(M) \le \log M$. As $(\log M)^4 = \exp(4\log\log M)$ and $\eta(M) \ge \sqrt{\log M}$, the right side is $O(\exp(-c\,\eta(M)))$ for any fixed $c < c_1'$. This assumes $M \ge 5$ and $\varepsilon_1(M) \le 1$, which holds for $M \ge M_0$ with an absolute M_0; for $3 \le M < M_0$ the bound holds after adjusting the implied constant, since $\|u_M^\sharp\|_1 \le 1+M$.

Appendix H. Proofs for the Reciprocal-Dilate Problem

This appendix contains the proofs deferred from Section 13. The notation is that of the corresponding statements.

Appendix H.1. Fixed Tapers

Proof of Theorem 43. Extend w by 0 to $t > 1$ and put $f = 1 - w$ on $(0,\infty)$. Then f is bounded and of bounded variation, $f(0+) = 0$ and $f(t) = 1$ for $t > 1$. For real $X \ge 1$ define U_X by the same formula as U_M, and let $D_{X,w}(1) = \sum_{n\le X}\mu(n)w(n/X)/n$, so that $|D_{X,w}(1)| \ll_w \log(2X)$.

Rescaling. By (88) and the identity $\sum_{n\le y}\mu(n)\lfloor y/n\rfloor = 1$ for $y \ge 1$,

$$U_X(1/y) = \lfloor y\rfloor D_{X,w}(1) + \sum_{n\le y}\mu(n)f(n/X)\Bigl\lfloor\frac{y}{n}\Bigr\rfloor \qquad (y \ge 1). \tag{A16}$$

Real parameters. Let $m = \lfloor X\rfloor$. For $n \le m$ the point n/X lies in $[(n-1)/m, n/m]$, so $\sum_{n\le m}|w(n/X) - w(n/m)| \le \mathrm{Var}(w)$ and $|U_X - U_m| \le \mathrm{Var}(w)$ on $(0,1)$. At a fixed $x = 1/y \in (0,1)$ and for $K > y$, the terms with $n \le K$ tend to zero as $m \to \infty$ because $w(0+)$ exists, and the terms with $n > K$ contribute at most $y\,\mathrm{Var}(w)/K$, since $g_n(1/y) = \lfloor y\rfloor/n$ for $n > y$. Hence $U_X - U_m \to 0$ pointwise, and by dominated convergence $\|U_X - U_{\lfloor X\rfloor}\|_{1/2} \to 0$. Because $\beta \ge 1/2$, this gives $X^{1/2-\beta}\|U_X\|_{1/2} \le m^{1/2-\beta}(\|U_m\|_{1/2} + o(1))$, so the limit superior over real X does not exceed the limit superior over integers. It suffices to prove the assertions for real $X \to \infty$.

A test function. Let $W \in C_c^\infty(0,\infty)$ be supported in $[a,b]$ with $\int vW(v)\,\mathrm{d}v = 0$, and put

$$\begin{aligned}k_W(t) &= f(t)\int_0^\infty W(v)\Bigl\lfloor\frac{v}{t}\Bigr\rfloor\,\mathrm{d}v = -f(t)\int_0^\infty W(v)\Bigl\{\frac{v}{t}\Bigr\}\,\mathrm{d}v,\\ H(X) &= \sum_{n\ge1}\mu(n)k_W(n/X).\end{aligned}$$

The function k_W is bounded, vanishes for $t > b$, and tends to 0 as $t \to 0$; so $\widehat{k}_W(s) = \int_0^\infty k_W(t)t^{s-1}\,\mathrm{d}t$ is holomorphic in $\mathrm{Re}\,s > 0$, and $H(X)$ is a finite sum that vanishes for $X < 1/b$. For $X \ge 1/a$, integrate (A16) with $y = Xv$ against $W(v)$. The first term contributes $D_{X,w}(1)\int\lfloor Xv\rfloor W(v)\,\mathrm{d}v = -D_{X,w}(1)\int\{Xv\}W(v)\,\mathrm{d}v = O_W(\log 2X)$, and the second contributes $H(X)$. On the other side, the substitution $x = 1/(Xv)$ and the Cauchy–Schwarz inequality give

$$\Bigl|\int_0^\infty U_X\Bigl(\frac{1}{Xv}\Bigr)W(v)\,\mathrm{d}v\Bigr| \le \sqrt{X}\,\|vW\|_{L^2}\,\|U_X\|_{1/2}.$$

Therefore

$$|H(X)| \le \sqrt{X}\,\|vW\|_{L^2}\,\|U_X\|_{1/2} + O_W(\log 2X). \tag{A17}$$

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For $\operatorname{Re} s > 1$, termwise integration gives

$$\int_0^\infty H(X) X^{-s-1}\,\mathrm{d}X = \frac{\widehat{k}_W(s)}{\zeta(s)}. \tag{A18}$$

Detection of the zero. We claim that W can be chosen with $\widehat{k}_W(\rho) \neq 0$. By Fubini's theorem,

$$\widehat{k}_W(\rho) = \int_0^\infty W(v) J_\rho(v)\,\mathrm{d}v, \qquad J_\rho(v) = -\int_0^\infty f(t)\Big\{\frac{v}{t}\Big\} t^{\rho-1}\,\mathrm{d}t.$$

The integral for J_ρ converges absolutely because $0 < \beta < 1$; the substitution $t = v/u$ shows $|J_\rho(v)| \ll v^\beta$; and J_ρ is continuous by dominated convergence. If every admissible W annihilated J_ρ, then $J_\rho(v) = \kappa v$ for a constant κ, and the bound v^β with $\beta < 1$ would force $J_\rho \equiv 0$. Put $F(r) = f(\mathrm{e}^r)$ and $k_\rho(u) = \{\mathrm{e}^u\}\mathrm{e}^{-\rho u}$. Then $J_\rho(\mathrm{e}^r) = -\mathrm{e}^{\rho r}(k_\rho * F)(r)$. The kernel k_ρ is integrable, and by the computation in the proof of Lemma 15 its Fourier transform is

$$\int_{-\infty}^\infty k_\rho(u)\mathrm{e}^{-i\xi u}\,\mathrm{d}u = -\frac{\zeta(\rho + i\xi)}{\rho + i\xi},$$

whose real zeros are isolated. The function F is bounded and of bounded variation, with $F(-\infty) = 0$ and $F(+\infty) = 1$; its derivative is a finite complex measure ν of total mass 1. If $k_\rho * F = 0$, then $k_\rho * \nu = 0$, and taking Fourier transforms gives $\widehat{k}_\rho\,\widehat{\nu} = 0$.[57] But $\widehat{\nu}$ is continuous with $\widehat{\nu}(0) = 1$, so $\widehat{k}_\rho$ would vanish on a neighbourhood of 0. This contradiction proves the claim.

Conclusion. Fix such a W. The right side of (A18) is meromorphic in $\operatorname{Re} s > 0$ and has at ρ a pole of the same order as the zero. Suppose $\limsup X^{1/2-\beta}\|U_X\|_{1/2} = 0$. By (A17), $H(X) = o(X^\beta)$, so the integral in (A18) is holomorphic in $\operatorname{Re} s > \beta$, the identity persists there, and

$$\varepsilon \int_0^\infty |H(X)| X^{-\beta-\varepsilon-1}\,\mathrm{d}X \longrightarrow 0 \qquad (\varepsilon \downarrow 0),$$

as one sees by splitting the integral at a large X_1. This is incompatible with a pole at ρ. If the limit superior is finite, then $H(X) = O(X^\beta)$ and the integral at $\rho + \varepsilon$ is $O(1/\varepsilon)$, which excludes a pole of order two or more. □

Appendix H.2. The Harmonic Cost

Proof of Theorem 44. By (88), $U(1/y)$ is constant on $k \le y < k+1$ for each $k \ge 1$; call the value V_k and set $V_0 = 0$. With $C = \sum_{n\ge2} a_n/n$,

$$V_k = 1 + kC - \sum_{n\ge2} a_n \Big\lfloor \frac{k}{n} \Big\rfloor \qquad (k \ge 1),$$
$$\Delta V_r := V_r - V_{r-1} = C + \mathbf{1}_{r=1} - \sum_{n\mid r,\ n\ge2} a_n \qquad (r \ge 1).$$

Möbius inversion gives $a_n - \mu(n) = -\sum_{r\mid n} \mu(n/r)\Delta V_r$ for every $n \ge 2$. Also

$$d^2 = \int_1^\infty |U(1/y)|^2 \frac{\mathrm{d}y}{y^2} = \sum_{k\ge1} \frac{|V_k|^2}{k(k+1)},$$

whence, for $R \ge 1$,

$$\sum_{r\le R} \frac{|\Delta V_r|^2}{r} \le 2\sum_{r\le R} \frac{|V_r|^2}{r} + 2\sum_{r\le R} \frac{|V_{r-1}|^2}{r} \le 2(R+1)d^2 + 2(R-1)d^2 = 4Rd^2.$$

[57] The differentiation is carried out after pairing with a smooth compactly supported test function; Fubini's theorem applies because $k_\rho \in L^1$, F is bounded and ν is finite.

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Writing $n = mr$ and applying Minkowski's inequality to the sum over m,

$$\Big(\sum_{2\le n\le K}\frac{|a_n-\mu(n)|^2}{n}\Big)^{1/2}\le\sum_{m\le K}\frac{1}{\sqrt{m}}\Big(\sum_{r\le K/m}\frac{|\Delta V_r|^2}{r}\Big)^{1/2}\le 2d\sqrt{K}\sum_{m\le K}\frac{1}{m}.$$

The triangle inequality in the norm $(\sum_{2\le n\le K}|\cdot|^2/n)^{1/2}$ gives the first assertion.

Let $c = 6/\pi^2$ and $Q_2(t) = \sum_{n\le t}\mu(n)^2$. From $\mu(n)^2 = \sum_{r^2\mid n}\mu(r)$ one gets $Q_2(t) = \sum_{r\le\sqrt{t}}\mu(r)\lfloor t/r^2\rfloor$, and bounding the fractional parts by $\sqrt{t}$ and the tail $t\sum_{r>\sqrt{t}}r^{-2}$ by $2\sqrt{t}$ gives $|Q_2(t)-ct|\le 3\sqrt{t}$ for $t\ge 1$. Partial summation then yields

$$\sum_{2\le n\le K}\frac{\mu(n)^2}{n} = \frac{Q_2(K)}{K} - 1 + \int_1^K\frac{Q_2(t)}{t^2}\,\mathrm{d}t \ge c\log K - 7.$$

Let $\Lambda = \log(1/d)\ge 4$ and $K = \lfloor e^{2\Lambda}/\Lambda^4\rfloor \ge 11$. Then $\log K\ge 2\Lambda - 4\log\Lambda - \log 2$ and

$$e := 2d\sqrt{K}\sum_{m\le K}\frac{1}{m}\le\frac{2(1+2\Lambda)}{\Lambda^2}\le\frac{4.5}{\Lambda}.$$

With $A = (c\log K - 7)_+\le 2c\Lambda$, the first assertion and $(\sqrt{A}-e)_+^2\ge A - 2e\sqrt{A}$ give

$$\sum_{n\ge 2}\frac{|a_n|^2}{n}\ge 2c\Lambda - 4c\log\Lambda - c\log 2 - 7 - \frac{9\sqrt{2c}}{\sqrt{\Lambda}}.$$

Since $c\log 2 < 0.43$ and $9\sqrt{2c}/\sqrt{\Lambda} < 4.97$ for $\Lambda\ge 4$, the constant terms sum to less than 12.4, which proves (99); the asymptotic statement follows. □

Appendix H.3. The Logarithmic Taper

Proof of Proposition 26. For $y>0$, $\frac{1}{2\pi i}\int_{\mathrm{Re}\,w=2}y^w\,\mathrm{d}w/w^2 = (\log y)_+$, so with $F = 1/\zeta$,

$$D_M^{\log}(s) = \frac{1}{2\pi i\,\log M}\int_{\mathrm{Re}\,w=2}\frac{M^w}{w^2}F(s+w)\,\mathrm{d}w.$$

Moving the line to $\mathrm{Re}\,w = -\delta$ as in the proof of Theorem 37, the double pole at $w=0$ contributes $F(s)\log M + F'(s)$, and for $\mathrm{Re}\,s=\sigma$

$$D_M^{\log}(s) = F(s) + \frac{F'(s)}{\log M} + O_{\sigma,\theta,\delta,\kappa}\Big(\frac{M^{-\delta}(1+|t|)^{\kappa}}{\log M}\Big).$$

At $s=1$, where $F(1)=0$ and $F'(1)=1$, the same computation gives $D_M^{\log}(1) = (1+O(M^{-\delta}))/\log M$. Inserting both in (89) and using $-\zeta F' = \zeta'/\zeta$, the transform of $(\log M)u_M^{\log}$ on $\mathrm{Re}\,s=\sigma$ equals $(\zeta(s)+\zeta'(s)/\zeta(s))/s$ up to an error $O(M^{-\delta}|\zeta(s)|(1+|t|)^{\kappa}/|s|)$. By Lemma 17 and Lemma 16 with $\kappa < 1/4$, the functions $(\log M)u_M^{\log}$ converge in $\|\cdot\|_\sigma$, with error $O(M^{-\delta})$, to a function with that transform. By (100) and $(\log M)D_M^{\log}(1)\to 1$ they converge pointwise on $(0,1)$ to $E(1/x)$. A norm limit and a pointwise limit agree almost everywhere. □

Appendix H.4. Growth Forced by an Off-Line Zero

Proof of Proposition 27. Put $\alpha = 2\beta - 1$ and suppose $I(y)\le cy^{\alpha}$ for $y\ge y_0$. For $0<\varepsilon\le 1$, integration by parts gives

$$\int_1^\infty y^{-\alpha-\varepsilon}\,\mathrm{d}I(y) = (\alpha+\varepsilon)\int_1^\infty I(y)y^{-\alpha-\varepsilon-1}\,\mathrm{d}y\le O(1) + \frac{c(\alpha+\varepsilon)}{\varepsilon}.$$

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Let $F_E(s) = \int_1^\infty E(y) y^{-s-1}\,\mathrm{d}y$. For $\operatorname{Re} s > 1$ it equals $(\zeta(s) + \zeta'(s)/\zeta(s))/s$. By the Cauchy–Schwarz inequality,

$$\Big(\int_1^\infty |E(y)| y^{-\beta-\varepsilon-1}\,\mathrm{d}y\Big)^2 \le \int_1^\infty y^{-\alpha-\varepsilon}\,\mathrm{d}I(y) \int_1^\infty y^{-1-\varepsilon}\,\mathrm{d}y = \frac{1}{\varepsilon}\int_1^\infty y^{-\alpha-\varepsilon}\,\mathrm{d}I(y).$$

Hence F_E is holomorphic in $\operatorname{Re} s > \beta$, where it coincides with its continuation, and $|F_E(\rho+\varepsilon)|^2 \le \varepsilon^{-1}(O(1) + c(\alpha+\varepsilon)/\varepsilon)$. The residue of ζ'/ζ at ρ is m, so $F_E(\rho+\varepsilon) \sim m/(\rho\varepsilon)$. Letting $\varepsilon \downarrow 0$ gives $m^2/|\rho|^2 \le c\alpha$. This holds for every c exceeding the limit superior, which proves the inequality.

If the interior norms are bounded, then $I(M) \ll (\log M)^2$ by (101), and by monotonicity $I(Y) \ll \log^2(2Y)$; the lower bound excludes a zero with $\beta > 1/2$. Under the Riemann hypothesis, Cramér's estimate $\int_Y^{2Y} |\psi(y) - y|^2\,\mathrm{d}y \ll Y^2$ [49, Ch. 13] gives $I(M) \ll \log M$. □

Appendix H.5. Ordinary Dilates

Proof of Proposition 28. The function $\{x\} - 1/2$ has Fourier coefficients $-1/(2\pi i\ell)$ for $\ell \neq 0$, and Parseval's identity gives

$$\int_0^1 \{jx\}\{kx\}\,\mathrm{d}x = \frac{1}{4} + \frac{\gcd(j,k)^2}{12\,jk}.$$

Put $t = \sum_k c_k$ and $u_d = \sum_{d\mid k\le M} c_k/k$; the map $(c_k) \mapsto (u_d)$ is triangular and invertible. Since $\sum_{d\mid k} \varphi(d) = k$ and $\sum_{d\mid k} J_2(d) = k^2$, we have $t = \sum_d \varphi(d) u_d$ and

$$\Big\|1 - \sum_{k\le M} c_k\{kx\}\Big\|^2 = \Big|1 - \frac{t}{2}\Big|^2 + \frac{1}{12}\sum_{d\le M} J_2(d)|u_d|^2.$$

For fixed t, the Cauchy–Schwarz inequality shows that the minimum of the last sum is $|t|^2/S_M$, attained at $u_d = t\varphi(d)/(S_M J_2(d))$. The resulting quadratic in t is minimal at $t = 6S_M/(1+3S_M)$, with value $1/(1+3S_M)$. Finally $a(d) = \varphi(d)^2/J_2(d)$ is multiplicative with $a(p^k) = (p-1)/(p+1)$ for $k \ge 1$, so $a = 1 * b$ with b supported on squarefree integers and $b(p) = -2/(p+1)$. Since $\sum_d |b(d)|/d < \infty$, $S_M = \sum_d b(d)\lfloor M/d\rfloor \sim M\sum_d b(d)/d$, which is the stated product. □

Appendix I. Short Windows

Proof of Theorem 49. We use [46, Theorem 1] for the multiplicative function λ: there are absolute constants $C, C' > 1$ such that, for $2 \le h \le X$ and $\delta > 0$,

$$\Big|\frac{1}{h}\sum_{x\le n\le x+h} \lambda(n) - m_X\Big| \le \delta + C'\frac{\log\log h}{\log h}, \qquad m_X = \frac{1}{X}\sum_{X\le n\le 2X}\lambda(n), \tag{A19}$$

for all integers $x \in [X, 2X]$ outside a set of cardinality at most

$$CX\left(\frac{(\log h)^{1/3}}{\delta^2 h^{\delta/25}} + \frac{1}{\delta^2(\log X)^{1/50}}\right).$$

Let $16 \le h \le X$ and take $\delta = \max\big(125\log\log h/\log h,\ (\log X)^{-1/200}\big)$. Then $h^{-\delta/25} \le (\log h)^{-5}$ and $\delta^{-2} \le \min\big((\log h)^2, (\log X)^{1/100}\big)$, so the exceptional set has at most $CX((\log h)^{-8/3} + (\log X)^{-1/100})$ elements. On it the left side of (A19) is at most 4. Hence

$$\frac{1}{X}\sum_{X\le x\le 2X}\Big|\frac{1}{h}\sum_{x\le n\le x+h}\lambda(n) - m_X\Big|^2 \ll \Big(\frac{\log\log h}{\log h}\Big)^2 + (\log X)^{-1/100}. \tag{A20}$$

By Theorem 48, $m_X \ll 1/\log X$, so (A20) holds with m_X removed.

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Recall that $\mathcal{Q}_\lambda(N,H) = H^{-1}\sum_t |S(t)|^2$ with $S(t) = \sum_{t<n\le t+H,\,1\le n\le N}\lambda(n)$. Put $T_0 = \max(H, \lceil\sqrt{N}\,\rceil)$. For $T_0 \le t \le N-H$ the window lies inside $[1,N]$, and with $x = t+1$ the sum in (A19) with $h = H$ equals $S(t) + \lambda(t+H+1)$. Cover the range of x by dyadic blocks $[X,2X]$ with $X = 2^k T_0 \le N$. Each has $H \le X$ and $\log X \ge \frac{1}{2}\log N$, and the blocks have total length at most $2N$. From (A20),

$$\frac{1}{H}\sum_{T_0\le t\le N-H}|S(t)|^2 \ll NH\left(\left(\frac{\log\log H}{\log H}\right)^2 + (\log N)^{-1/100}\right) + \frac{N}{H},$$

and the last term is smaller than the first.

The remaining windows are those with $-H < t < T_0$ or $N-H < t < N$; there are at most $3H + \sqrt{N} + 1$ of them. Each satisfies $|S(t)| \le \min(H, 2\max_{j\le N}|L(j)|) \le \min(H, C''N/\log N)$ by Theorem 48. If $H \le \sqrt{N}$ their contribution to $\mathcal{Q}_\lambda(N,H)$ is at most $H^{-1}\cdot 5\sqrt{N}\cdot H^2 = 5\sqrt{N}\,H$. If $H > \sqrt{N}$ it is at most $5\min(H, C''N/\log N)^2 \le 5C''NH/\log N$. Both are within the stated bound.

The trivial bound follows from $|S(t)| \le H$ and the count of at most $N+H-1 \le 2N$ windows. □

Appendix J. The Explicit Constants

This appendix gives the details of the explicit part of Theorem 48.

Appendix J.1. The Decomposition

Every n is uniquely $n = d^2m$ with m squarefree, and $\lambda(n) = \mu(m)$ is then one term of $\sum_{d^2|n}\mu(n/d^2)$; the other terms vanish because n/d'^2 is squarefree only for $d' = d$. Thus $\lambda(n) = \sum_{d^2|n}\mu(n/d^2)$, and summing over $n \le x$ gives (105).

Let $y_0 = 2\,160\,535$ and split the sum at $d = \sqrt{x/y_0}$.

Large arguments. If $x/d^2 \ge y_0$, then (106) gives $|\mathcal{M}(x/d^2)| \le x/(4345\,d^2)$. The sum over all such d is at most

$$\frac{\zeta(2)}{4345}\,x = \frac{x}{2641.44\ldots} < \frac{x}{2641}.$$

Small arguments. If $1 \le x/d^2 < y_0$, the finite verification $|\mathcal{M}(y)| \le \sqrt{y}$ for $1 \le y \le 3\cdot 10^6$ gives $|\mathcal{M}(x/d^2)| \le \sqrt{x}/d$. Let $z = \sqrt{x/y_0}$ and $w = \sqrt{x}$, and let d_1 be the least integer greater than z. If $z \ge 1$, then

$$\sum_{z<d\le w}\frac{1}{d} \le \frac{1}{d_1} + \int_{d_1}^{w}\frac{\mathrm{d}u}{u} \le 1 + \log\frac{w}{z} = 1 + \tfrac{1}{2}\log y_0.$$

If $z < 1$, then $x < y_0$ and $\sum_{d\le w} 1/d \le 1 + \log w \le 1 + \frac{1}{2}\log y_0$. Numerically $1 + \frac{1}{2}\log y_0 = 8.29293\ldots < 8.3$.

Adding the two parts, $|L(x)| \le x/2641 + 8.3\sqrt{x}$ for every $x \ge 1$. By (6), $Q_\lambda(N) \le 5\max_{j\le N}|L(j)|^2$, and the majorant is increasing in j, which gives the bound for $Q_\lambda(N)$.

Appendix J.2. The Finite Verification

Table A8 summarises a sieve computation of μ and λ up to $3\cdot 10^6$. It confirms $|\mathcal{M}(y)| \le \sqrt{y}$ on $[1, 3\cdot 10^6]$, with equality only at $y = 1$. It also shows that the threshold in (106) cannot be lowered: $\mathcal{M}(y_0 - 1) = -498$, whereas $(y_0-1)/4345 = 497.246\ldots$; and on $[y_0, 3\cdot 10^6]$ the largest value of $4345\,|\mathcal{M}(y)|/y$ is $0.9995\ldots$

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Table A8. Finite checks for Theorem 48. Left: the maximum of $|\mathcal{M}(y)|/\sqrt{y}$ over the integers y in each range, and the point where it is attained. Right: $L(x)$ against the explicit majorant $x/2641 + 8.3\sqrt{x}$.

Range of y	Maximum	at y	x	$L(x)$	Majorant
$[2, 10^3]$	0.8944	5	10^2	-2	83.0
$[10^3, 10^4]$	0.4722	2803	10^3	-14	262.8
$[10^4, 10^5]$	0.4630	24 185	10^4	-94	833.8
$[10^5, 10^6]$	0.4378	300 551	10^5	-288	2662.6
$[10^6, 3 \cdot 10^6]$	0.4182	1 066 854	10^6	-530	8678.6
			$3 \cdot 10^6$	-942	15 512.0

Over $1 \le x \le 3 \cdot 10^6$ the largest value of $|L(x)|/(x/2641 + 8.3\sqrt{x})$ is $0.1580\ldots$, attained at $x = 96\,862$, and $\max_{x \le 3\cdot 10^6} |L(x)| = 1803$. The explicit inequality is thus a safe majorant and not an approximation. Its only inputs are (106) and the finite range of $\mathcal{M}$; no zero-free region enters.

Appendix J.3. Constants Used in Part C

Table A9 lists the numerical constants that occur in Sections 12–15.

Table A9. Numerical constants.

Constant	Value	Where used
$\zeta(1/2)$	$-1.46035\ldots$	(82)
$c_0 = -\sqrt{\pi}/(2\zeta(1/2))$	$0.60685\ldots$	Proposition 23
$\frac{3}{2}c_0^2$	$0.55241\ldots$	Proposition 24
$2 - \pi/2$	$0.42920\ldots$	Theorem 35
$\sqrt{\pi} \cdot \sqrt{2\pi}$	$4.44288\ldots$	Lemma A4
$12/\pi^2$	$1.21585\ldots$	Theorem 44
$\zeta(2)/4345$	$1/2641.44\ldots$	Theorem 48
$1 + \frac{1}{2}\log 2\,160\,535$	$8.29293\ldots$	Theorem 48
$\zeta(2)/(2\zeta(3))$	$0.68421\ldots$	Section 14, diagonal

Appendix K. Elementary Prime Bounds and Exact Checks

Appendix K.1. Chebyshev-type Bounds

Sums over primes in this paper are estimated with the following elementary bounds, which need neither the prime number theorem nor any zero-free region. Let $\vartheta(t) = \sum_{p \le t} \log p$ and let $\pi(t)$ count the primes up to t.

Lemma A5. *For $t \ge 2$ and $y \ge 2$,*

$$\vartheta(t) \le 4(\log 2)\, t, \qquad \pi(t) \le \frac{7t}{\log t},$$

$$\sum_{p \le y} \frac{1}{\sqrt{p}} \le \frac{36\sqrt{y}}{\log y}, \qquad \sum_{p \le y} \frac{\log p}{p} \le 3(1 + \log y).$$

Proof. The product of the primes in $(n, 2n]$ divides $\binom{2n}{n} \le 4^n$, so $\vartheta(2n) - \vartheta(n) \le 2n \log 2$. Summing over $n = 2^j$ gives $\vartheta(2^k) \le 2^{k+1} \log 2$, and for $2^{k-1} < t \le 2^k$ this yields $\vartheta(t) \le 4(\log 2)t$.

Primes in $(\sqrt{t}, t]$ have $\log p \ge \frac{1}{2}\log t$, so $\pi(t) \le \sqrt{t} + 2\vartheta(t)/\log t$. Since $\log t \le \sqrt{t}$, the first term is at most $t/\log t$, and $1 + 8\log 2 < 7$.

By partial summation,

$$\sum_{p \le y} \frac{1}{\sqrt{p}} = \frac{\pi(y)}{\sqrt{y}} + \frac{1}{2}\int_2^y \frac{\pi(t)}{t^{3/2}}\,\mathrm{d}t \le \frac{7\sqrt{y}}{\log y} + \frac{7}{2}\int_2^y \frac{\mathrm{d}t}{\sqrt{t}\log t}.$$

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On $[2,\sqrt{y}]$ the integrand is at most $1/(\sqrt{t}\log 2)$, and on $[\sqrt{y},y]$ it is at most $2/(\sqrt{t}\log y)$; the integral is at most $2y^{1/4}/\log 2 + 4\sqrt{y}/\log y$. Since $y^{1/4}\log y \le (4/\mathrm{e})\sqrt{y}$, the total is at most $(7+14+7\cdot 4/(\mathrm{e}\log 2))\sqrt{y}/\log y < 36\sqrt{y}/\log y$.

Finally $\sum_{p\le y}(\log p)/p = \vartheta(y)/y + \int_2^y \vartheta(t)t^{-2}\,\mathrm{d}t \le 4(\log 2)(1+\log y)$, and $4\log 2 < 3$. □

Appendix K.2. Exact Checks of the Sign of B

The values $B(1) > 0$ and $B(3) < 0$ quoted after Proposition 23 follow from a finite sum and a geometric tail. For every $K \ge 1$ and $X > 0$,

$$\left| B(X) - \sum_{n\le K} \lambda(n)\mathrm{e}^{-n/X} \right| \le \frac{\mathrm{e}^{-(K+1)/X}}{1-\mathrm{e}^{-1/X}}.$$

With $\lambda(1),\dots,\lambda(9) = 1,-1,-1,1,-1,1,-1,-1,1$,

$$\sum_{n\le 9} \lambda(n)\mathrm{e}^{-n} = 0.195689\ldots, \qquad \frac{\mathrm{e}^{-10}}{1-\mathrm{e}^{-1}} < 7.2\cdot 10^{-5},$$

so $0.1956 < B(1) < 0.1958$. Similarly

$$\sum_{n\le 40} \lambda(n)\mathrm{e}^{-n/3} = -0.079206\ldots, \qquad \frac{\mathrm{e}^{-41/3}}{1-\mathrm{e}^{-1/3}} < 4.1\cdot 10^{-6},$$

so $-0.0793 < B(3) < -0.0792$. Table A10 records these values. The sum B therefore takes both signs at small scales, and any statement about its sign can only be an eventual one.

Table A10. Certified values of B at two small scales.

X	Terms K	Partial sum	Tail bound
1	9	0.195689	$7.2\cdot 10^{-5}$
3	40	−0.079206	$4.1\cdot 10^{-6}$

Appendix L. Proof of the P ólya–Vinogradov Bound

This appendix proves Corollary 27 in full. The argument is that of Montgomery and Vaughan [7], organised as a transfer principle: it needs the zeros only through zero-free rectangles for the twists $\chi\psi$ by characters ψ of small modulus, and it keeps track of the width $1-\theta$ of the rectangle. Hypothesis H supplies the rectangles with $\theta = 1/2$.

Throughout, p denotes a prime, Λ is the von Mangoldt function, $e(x) = e^{2\pi i x}$, $\|x\|$ is the distance from x to the nearest integer, and $P(n)$ is the largest prime factor of $n \ge 2$, with $P(1) = 1$. For a Dirichlet character ξ put

$$\psi(x,\xi) = \sum_{n\le x} \xi(n)\Lambda(n), \qquad \vartheta(x,\xi) = \sum_{p\le x} \xi(p)\log p.$$

The letters c, C, with or without subscripts, denote positive absolute constants, and $A \ll B$ means $|A| \le CB$ with such a constant. A zero of an L-function always means a zero in the strip $0 \le \mathrm{Re}\, s \le 1$, written $\rho = \beta + i\gamma$.

Character sums through Gauss sums

Lemma A6. *Let χ be primitive modulo $q \ge 3$, $H = \lfloor q/2 \rfloor$, and let $B \ge 0$ be such that*

$$\left| \sum_{m\le H} \frac{\bar\chi(m)e(m\beta)}{m} \right| \le B \qquad \text{for every } \beta \in \mathbb{R}.$$

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Then $M(\chi) \le \sqrt{q}\,(B+2)$.

Proof. Let $\tau = \sum_{a \bmod q} \bar\chi(a) e(a/q)$. For primitive $\bar\chi$ one has $\sum_a \bar\chi(a) e(an/q) = \chi(n)\tau$ for every integer n and $|\tau| = \sqrt{q}$ [32, Ch. 9]. It suffices to bound $\sum_{n \le T} \chi(n)$ for integers $T \ge 1$. Summing the first identity over $1 \le n \le T$ and writing the residues as $-q/2 < a \le q/2$,

$$\tau \sum_{n \le T} \chi(n) = \sum_{0 < |a| \le q/2} \bar\chi(a)\big(e(aT/q) - 1\big)\, g(a/q), \qquad g(\vartheta) = \frac{1}{1 - e(-\vartheta)},$$

since the geometric sum is $\sum_{n=1}^{T} e(n\vartheta) = e(\vartheta)\frac{e(T\vartheta)-1}{e(\vartheta)-1} = \frac{e(T\vartheta)-1}{1-e(-\vartheta)}$, and $a = 0$ contributes nothing because $\bar\chi(0) = 0$. With $x = 2\pi\vartheta$, $0 < |x| \le \pi$, one has $g(\vartheta) = \frac12 + \frac{i}{2}\cot\frac{x}{2}$, so

$$g(\vartheta) = \frac{1}{2\pi i \vartheta} + r(\vartheta), \qquad r(\vartheta) = \frac12 + \frac{i}{2}\Big(\cot\frac{x}{2} - \frac{2}{x}\Big).$$

On $0 < |y| \le \pi/2$ the odd function $\cot y - 1/y$ has modulus at most $2/\pi$, its value at $|y| = \pi/2$, because it is decreasing on $(0, \pi/2]$ and tends to 0 at 0. Hence $|r(\vartheta)| \le (1/4 + 1/\pi^2)^{1/2} < 0.6$, and the terms with r contribute at most $2 \cdot 0.6 \cdot q$. In the remaining terms put $a = m$ or $a = -m$ with $1 \le m \le H$ (when q is even the residue $q/2$ is not prime to q and has $\bar\chi(q/2) = 0$). They equal

$$\frac{q}{2\pi i} \sum_{m \le H} \frac{\bar\chi(m)}{m}\Big(e\Big(\frac{mT}{q}\Big) - 1 - \bar\chi(-1)\Big(e\Big(-\frac{mT}{q}\Big) - 1\Big)\Big),$$

a combination with coefficients of modulus 1 of four sums of the kind in the hypothesis, so they contribute at most $4qB/(2\pi)$. Dividing by $|\tau| = \sqrt{q}$ gives $|\sum_{n \le T} \chi(n)| \le \sqrt{q}\,(2B/\pi + 1.2)$. □

Lemma A7. *Let χ be primitive modulo q, let $1 \le r < q$ and let ψ be a character modulo r. Then $\chi\psi$, as a character modulo qr, is nonprincipal. If χ' is another primitive character modulo q with $\chi'\psi = \chi\psi$, then $\chi' = \chi$.*

Proof. Let $d = (q, r)$. Every n prime to q with $n \equiv 1 \pmod d$ has, by the Chinese remainder theorem, a representative $n' \equiv n \pmod q$ with $n' \equiv 1 \pmod r$, and n' is prime to qr. If $\chi\psi$ were principal, then $\chi(n) = \chi(n')\psi(n') = 1$ for all such n, so χ would be trivial on the kernel of $(\mathbb{Z}/q\mathbb{Z})^\times \to (\mathbb{Z}/d\mathbb{Z})^\times$ and hence induced by a character modulo $d \le r < q$, which is impossible for a primitive character. If $\chi'\psi = \chi\psi$, then χ' and χ agree at every integer prime to qr; every residue class prime to q contains such an integer, again by the Chinese remainder theorem, so $\chi' = \chi$. □

Zeros and the explicit formula

We use the following facts for a primitive character ξ modulo $k \ge 3$; all are in [32], Chapters 16 and 19.

(Z1) For every real t the number of zeros of $L(s, \xi)$ with $|\gamma - t| \le 1$ is at most $c_8 \log(k(|t| + 2))$.
(Z2) For $-1 \le \sigma \le 2$ and $s = \sigma + it$,

$$\frac{L'}{L}(s, \xi) = \sum_{|\gamma - t| < 1} \frac{1}{s - \rho} + O\big(\log(k(|t| + 2))\big).$$

(Z3) For $x \ge 2$ and $2 \le T \le x$,

$$\psi(x, \xi) = -\sum_{|\gamma| < T} \frac{x^\rho}{\rho} - (1 - a)\log x - b(\xi) + \sum_{m \ge 1} \frac{x^{a-2m}}{2m - a} + O\Big(\frac{x \log^2(kx)}{T} + x^{1/4}\log x\Big),$$

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where $a \in \{0,1\}$ is given by $\xi(-1) = (-1)^a$ and $b(\xi)$ is a constant with

$$b(\xi) = -\sum_{|\gamma|<1} \frac{1}{\rho} + O(\log k).$$

Exponential sums with multiplicative coefficients

The minor arcs are handled by an estimate of Montgomery and Vaughan [7, Corollary 1]: there is an absolute constant c_{10} such that, if f is multiplicative with $|f(n)| \le 1$, a, r are coprime with $|\alpha - a/r| \le r^{-2}$, and $2 \le R \le r \le N/R$, then

$$\Big|\sum_{n\le N} f(n) e(n\alpha)\Big| \le c_{10}\Big(\frac{N}{\log N} + \frac{N(\log R)^{3/2}}{R^{1/2}}\Big). \tag{A21}$$

Two elementary bounds

Lemma A8. *There is an absolute constant c_{11} such that for $Y \ge 2$ and every $H \ge 1$*

$$\sum_{\substack{n\le H \\ P(n)\le Y}} \frac{1}{n} \le \prod_{p\le Y}\Big(1-\frac{1}{p}\Big)^{-1} \le c_{11}\log Y.$$

Proof. Expanding each factor as a geometric series shows that the product is the sum of $1/n$ over all n with $P(n) \le Y$; the second inequality is Mertens' theorem. □

Lemma A9. *For $3/4 \le u \le 2$ and real t, $|\Gamma(u+it)| \le 15(1+t^2)^{-1}$. For $-1/4 \le u \le 2$ and $|t| \ge 1$, $|\Gamma(u+it)| \le c_{12}|t|^{3/2}e^{-\pi|t|/2}$.*

Proof. The product formula gives $|\Gamma(u+it)|^2 = \Gamma(u)^2 \prod_{n\ge 0}\big(1 + t^2/(u+n)^2\big)^{-1}$. Keeping the factors $n = 1, 2$ and using $u+1 \le 3$, $u+2 \le 4$,

$$|\Gamma(u+it)| \le \Gamma(u)\Big(1+\frac{t^2}{9}\Big)^{-1/2}\Big(1+\frac{t^2}{16}\Big)^{-1/2} \le \Gamma(u)\Big(1+\frac{t^2}{12}\Big)^{-1} \le \frac{12\,\Gamma(u)}{12+t^2},$$

by the inequality of arithmetic and geometric means, and $\Gamma(u) \le \Gamma(3/4) < 1.23$ on $[3/4, 2]$; so the left side is at most $14.8/(12+t^2) \le 15/(1+t^2)$. The second is Stirling's formula, uniform for u in a compact interval and $|t| \ge 1$ [12, §4.42]. □

Appendix L.1. Prime Sums from a Zero-Free Rectangle

In this section $1/2 \le \theta < 1$ is fixed and ξ is a nonprincipal character modulo $k \ge 3$. The constant c_{13} below does not depend on θ; this is what keeps the final constant of order $(1-\theta)^{-1}$.

Lemma A10. *Let $x \ge 2$ and suppose that $L(s,\xi) \ne 0$ for $\operatorname{Re} s > \theta$, $|\operatorname{Im} s| \le x$. Then*

$$|\psi(x,\xi)| \le c_{13}\, x^{\theta}\log^2(kx) \qquad \text{and} \qquad |\vartheta(x,\xi)| \le c_{13}\, x^{\theta}\log^2(kx).$$

Proof. Let ξ^* be the primitive character modulo $k^* \mid k$ that induces ξ; then $k^* \ge 3$, since ξ is nonprincipal and there is no primitive character modulo 2. The two characters differ only at integers sharing a prime factor with k, so

$$|\psi(x,\xi) - \psi(x,\xi^*)| \le \sum_{p\mid k}\sum_{p^m\le x} \log p \le \omega(k)\log x \le \frac{\log k \log x}{\log 2},$$

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and $L(s,\xi)$, $L(s,\xi^*)$ have the same zeros with $\operatorname{Re} s > 0$. Apply (Z3) to ξ^* with $T = x$. Every zero with $|\gamma| < x$ has $\beta \le \theta$ by hypothesis. The zeros with $|\gamma| < 1$ are combined with $b(\xi^*)$:

$$-\sum_{|\gamma|<1} \frac{x^\rho}{\rho} - b(\xi^*) = -\sum_{|\gamma|<1} \frac{x^\rho - 1}{\rho} + O(\log k),$$

$$\Big|\frac{x^\rho - 1}{\rho}\Big| = \Big|\int_0^{\log x} e^{\rho u}\,du\Big| \le x^\beta \log x \le x^\theta \log x,$$

and by (Z1) there are at most $c_8 \log(2k)$ of them. The zeros with $j \le |\gamma| < j+1$, $1 \le j < x$, number at most $2c_8 \log(k(j+3))$ and contribute at most x^θ/j each, so together at most $2c_8 x^\theta \log(k(x+3)) \sum_{j \le x} 1/j \ll x^\theta \log^2(kx)$. The terms $(1-a)\log x$ and $\sum_m x^{a-2m}/(2m-a) \le \sum_m x^{1-2m} \le 1$ are smaller, and the error term of (Z3) is $O(\log^2(kx) + x^{1/4}\log x)$. Since $\theta \ge 1/2$, all of this is $O(x^\theta \log^2(kx))$. Finally $\psi(x,\xi) - \vartheta(x,\xi)$ is a sum over prime powers $p^m \le x$ with $m \ge 2$, of modulus at most $\sum_{2 \le m \le \log x/\log 2} \vartheta(x^{1/m}, \xi_0) \ll \sqrt{x}\log x$ by Chebyshev's bound. □

Lemma A11. *Let $2 \le A' < B' \le x$, let $\gamma \in \mathbb{R}$, and suppose that $L(s,\xi) \ne 0$ for $\operatorname{Re} s > \theta$, $|\operatorname{Im} s| \le x$. Then*

$$\Big|\sum_{A' < p \le B'} \xi(p) e(p\gamma)\Big| \le c_{13} \frac{(3 + 2\pi|\gamma|B')\, B'^\theta \log^2(kB')}{\log A'}.$$

Proof. Write $V = \max_{2 \le u \le B'} |\vartheta(u,\xi)|$, which is at most $c_{13} B'^\theta \log^2(kB')$ by Lemma A10. By partial summation,

$$\sum_{A' < p \le B'} \xi(p) e(p\gamma) = \Big[\vartheta(u,\xi)\frac{e(u\gamma)}{\log u}\Big]_{A'}^{B'} - \int_{A'}^{B'} \vartheta(u,\xi)\frac{d}{du}\Big(\frac{e(u\gamma)}{\log u}\Big)\,du.$$

The derivative has modulus at most $2\pi|\gamma|/\log u + 1/(u\log^2 u)$, whose integral over $[A', B']$ is at most $(2\pi|\gamma|B' + 1)/\log A'$. The boundary terms are at most $2V/\log A'$. □

The next lemma is the major-arc estimate. Here and later χ is primitive modulo $q \ge 3$.

Lemma A12. *Let $1 \le r < q$, $b \in \mathbb{Z}$, $\beta \in \mathbb{R}$, let $Y \ge \max(2, r)$ and let $t \le q$ be a positive integer. Suppose that $L(s, \chi\psi) \ne 0$ for $\operatorname{Re} s > \theta$, $|\operatorname{Im} s| \le q$*

$$\Big|\sum_{\substack{n \le t \\ P(n) > Y}} \chi(n)\, e\Big(n\Big(\frac{b}{r} + \beta\Big)\Big)\Big| \le c_{13} \frac{r\,(3 + 2\pi|\beta|t)\log^2(qrt)}{(1-\theta)\log Y} \cdot \frac{t}{Y^{1-\theta}}.$$

Proof. Each $n \le t$ with $P(n) > Y$ factors uniquely as $n = pm$ with $p = P(n)$; then $p > Y$, $p \ge P(m)$ and $m \le t/p < t/Y$. Conversely, for $m < t/Y$ the admissible p are the primes in $(A_m, t/m]$ with $A_m = \max(Y, P(m) - 1)$. Since $p > Y \ge r$, p is prime to r, and on the reduced residues modulo r

$$e\Big(\frac{umb}{r}\Big) = \sum_{\psi \bmod r} c_\psi(m)\psi(u), \qquad c_\psi(m) = \frac{1}{\varphi(r)} \sum_{\substack{u \bmod r \\ (u,r)=1}} \bar\psi(u) e\Big(\frac{umb}{r}\Big), \quad |c_\psi(m)| \le 1.$$

Hence the sum to be estimated equals

$$\sum_{m < t/Y} \chi(m) \sum_{\psi \bmod r} c_\psi(m) \sum_{A_m < p \le t/m} (\chi\psi)(p)\, e(pm\beta).$$

By Lemma A7 each $\chi\psi$ is a nonprincipal character modulo qr, and by hypothesis it has no zero with $\operatorname{Re} s > \theta$, $|\operatorname{Im} s| \le q$. Lemma A11 with $k = qr$, $x = q$, $\gamma = m\beta$, $A' = A_m \ge Y$ and $B' = t/m$ bounds the

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inner sum by, for every character ψ modulo r. Then $c_{13}(3+2\pi|\beta|t)(t/m)^{\theta}\log^2(qrt)/\log Y$. There are $\varphi(r)\le r$ characters ψ, and

$$\sum_{m<t/Y} m^{-\theta} \le \int_0^{t/Y} u^{-\theta}\,du = \frac{(t/Y)^{1-\theta}}{1-\theta}.$$

Multiplying gives the bound, since $t^{\theta}(t/Y)^{1-\theta} = tY^{\theta-1}$. □

Remark A2. *The factor $(1-\theta)^{-1}$ in Lemma A12 is harmless: it is cancelled in the next section by choosing $\log Y$ of size $(1-\theta)^{-1}\log\log q$. That choice is what produces the factor $(1-\theta)^{-1}$ of Theorem A1, through the smooth numbers below Y.*

Appendix L.2. The Transfer Theorem

Theorem A1 (Transfer)**.** *There is an absolute constant c_4 with the following property. Let χ be a primitive character modulo $q\ge 3$, let $1/2\le\theta<1$, put $R=\lceil\log q\rceil^{16}$, and suppose that for every $r\le R$ and every character ψ modulo r*

$$L(s,\chi\psi)\neq 0 \qquad \text{for } \operatorname{Re} s>\theta,\ |\operatorname{Im} s|\le q. \tag{A22}$$

Then $M(\chi)\le c_4(1-\theta)^{-1}\sqrt{q}\log\log q$.

Let χ be primitive modulo $q\ge 3$ and $1/2\le\theta<1$. Put

$$L=\log q, \qquad M=\lceil L\rceil, \qquad R=M^{16}, \qquad A=\left\lceil\frac{40}{1-\theta}\right\rceil, \qquad Y=M^{A}.$$

Let $L_0\ge e^{e}$ be an absolute constant, large enough for the finitely many numerical inequalities used below, among them $R<q$ and $72c_{13}M^{-6}(1+L)+c_{10}(16\log M)^{3/2}M^{-8}(1+L)\le 1$ for $L\ge L_0$.

Proposition A2. *Suppose that $L\ge L_0$ and that (A22) holds for every $r\le R$ and every character ψ modulo r. Then for every integer $1\le H\le q$ and every $\beta\in\mathbb{R}$*

$$\Big|\sum_{m\le H}\frac{\chi(m)e(m\beta)}{m}\Big| \le \frac{c_{14}}{1-\theta}\log\log q.$$

Proof. Split the sum according to $P(m)\le Y$ or $P(m)>Y$. The first part is at most $c_{11}\log Y$ by Lemma A8. For the second let

$$\mathcal{A}(t)=\sum_{\substack{n\le t\\ P(n)>Y}}\chi(n)e(n\beta), \qquad w(t)=\frac{1}{t}-\frac{1}{t+1}.$$

Partial summation and $|\mathcal{A}(H)|\le H$ give

$$\Big|\sum_{\substack{m\le H\\ P(m)>Y}}\frac{\chi(m)e(m\beta)}{m}\Big| \le 1+\sum_{1\le t<H}|\mathcal{A}(t)|\,w(t). \tag{A23}$$

Since $A\ge 80$, $Y\ge R$, and $\mathcal{A}(t)=0$ for $t\le Y$. Fix an integer t with $Y<t<H$ and put $Q=\lfloor t/R\rfloor\ge 1$. By Dirichlet's approximation theorem there are coprime integers a,r with $1\le r\le Q$ and $|\beta-a/r|\le 1/(r(Q+1))$. As $Q+1>t/R$, this gives $|\beta-a/r|\,t\le R$.

Major arcs, $r\le R$. Here $r\le R<q$ and $Y\ge\max(2,r)$, and the hypothesis of Lemma A12 is (A22) for this r. That lemma, with $b=a$ and $\beta-a/r$ in place of β, gives

$$|\mathcal{A}(t)| \le c_{13}\,\frac{R\,(3+2\pi R)\log^2(qRt)}{(1-\theta)\log Y}\cdot\frac{t}{Y^{1-\theta}}.$$

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Now $3+2\pi R \le 8M^{16}$, $\log(qRt) \le 2L + 16\log M \le 3M$, $Y^{1-\theta} = M^{A(1-\theta)} \ge M^{40}$ and $(1-\theta)\log Y = (1-\theta)A\log M \ge 40\log M \ge 1$. Hence $|\mathcal{A}(t)| \le 72c_{13}\,tM^{-6}$.

Minor arcs, $r > R$. Here $R < r \le Q \le t/R$ and $|\beta - a/r| \le 1/(r(Q+1)) \le r^{-2}$, so (A21) with $f = \chi$ and $N = t$ applies to the complete sum $\sum_{n\le t}\chi(n)e(n\beta)$. Removing the $\Psi(t,Y) = \#\{n \le t : P(n) \le Y\}$ terms with $P(n) \le Y$,

$$|\mathcal{A}(t)| \le c_{10}\Big(\frac{t}{\log t} + t\varrho\Big) + \Psi(t,Y), \qquad \varrho = \frac{(\log R)^{3/2}}{R^{1/2}} = \frac{(16\log M)^{3/2}}{M^8}.$$

Summation over t. Since $t\,w(t) \le 1/t$, the two cases together give

$$\sum_{1\le t<H} |\mathcal{A}(t)|\,w(t) \le \big(72c_{13}M^{-6} + c_{10}\varrho\big)\sum_{t\le q}\frac{1}{t} + c_{10}\sum_{2\le t\le q}\frac{1}{t\log t} + \sum_{1\le t<H}\Psi(t,Y)\,w(t).$$

The first term is at most 1 by the choice of L_0, since $\sum_{t\le q} 1/t \le 1 + L$. The second is at most $c_{10}(\log L + 2)$. In the third, each $n \le H-1$ with $P(n) \le Y$ is counted by $\Psi(t,Y)$ for $n \le t < H$, so the term equals $\sum_{n<H,\,P(n)\le Y}(1/n - 1/H) \le c_{11}\log Y$. With (A23) and the smooth part,

$$\Big|\sum_{m\le H}\frac{\chi(m)e(m\beta)}{m}\Big| \le 2c_{11}\log Y + 2 + c_{10}(\log L + 2).$$

Finally $\log Y = A\log M \le \big(\frac{40}{1-\theta} + 1\big)\log(2L) \le \frac{82}{1-\theta}\log L$, because $L \ge e^e$. The proposition follows with $c_{14} = 164c_{11} + 3c_{10} + 2$, using $\log L = \log\log q \ge 1$. □

Figure 14 shows the dissection.

Proof of Theorem A1. If $L < L_0$, then every sum $\sum_{m\le H}\bar\chi(m)e(m\beta)/m$ with $H \le q$ has modulus at most $1 + \log q < 1 + L_0$, and Lemma A6 gives $M(\chi) \le (3+L_0)\sqrt{q} \le \frac{3+L_0}{\log\log 3}\sqrt{q}\log\log q$ without any hypothesis. Let $L \ge L_0$. The conjugate character $\bar\chi$ is primitive modulo q and satisfies (A22) for every $r \le R$ and ψ mod r, because the zeros of $L(s,\bar\chi\psi)$ are the complex conjugates of those of $L(s,\chi\bar\psi)$ and the rectangle is symmetric about the real axis. Proposition A2 for $\bar\chi$ with $H = \lfloor q/2\rfloor$ and Lemma A6 give

$$M(\chi) \le \sqrt{q}\Big(\frac{c_{14}}{1-\theta}\log\log q + 2\Big) \le \frac{c_{14}+2}{1-\theta}\sqrt{q}\log\log q.$$

Theorem A1 holds with $c_4 = \max\big(c_{14}+2, (3+L_0)/\log\log 3\big)$. □

Appendix M. Machine Verification

The finite identities and the numerical constants of the paper are checked twice, once by a Lean 4 project over Mathlib and once by three Julia programs. Both are available at github.com/anaxia-research/Openproblemsforfieldsmedal1. No proof in the text uses them; they guard against slips in algebra and in arithmetic. The computer-assisted step in the proof of Corollary 13 is described in Subsection M.3.

Appendix M.1. The Lean Project

The project builds with Lean 4.35 and the Mathlib release pinned in its manifest. It contains no unproved statement and adds no axiom: for every theorem in Table A11 the command `#print axioms` reports only `propext`, `Classical.choice` and `Quot.sound`. The statements are about real numbers and finite sums, so that each one can be compared with the displayed formula it formalises.[58]

[58] The analytic arguments (contour shifts, Mellin inversion, the kernel of Pintz) are not formalised. The project checks every step that reduces to a finite identity or a numerical inequality.

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Table A11. Theorems of the Lean project. The left column gives the Lean name, the middle column the statement in the notation of the paper, and the right column the place where it is used. All twenty-four theorems build without `sorry`.

Lean name	Statement	Used in
Energy identities and constants		
`boundary_term`	$x^2 + (y-x)^2 = 2(x - \frac{1}{2}y)^2 + \frac{1}{2}y^2$	(6)
`completion`	the completion identity at $H = N$	(6)
`half_sq_le_energy`	$\frac{1}{2}L_a(N)^2 \le Q_a(N)$	(7)
`energy_nonneg`	$Q_a(N) \ge 0$, with equality only if all $L_a(j) = 0$	Proposition 2
`gram_counting`	$K^2M \le KM^2 + 16BK(K-1)$, $K = 2M \Rightarrow M < 32B$	Theorem 25
`plancherel_weight_lower`	$K(\sigma + i\tau) > 0.65$ for $\sigma \ge 0$	(19)
`plancherel_weight_min`	$K(\sigma + i\tau) \ge 4\sqrt{2} - 5$ for $\sigma \ge 0$	(19)
`zeta_two_constant`	$\zeta(2)/4345 < 1/2641$	Theorem 48
`log_threshold`	$1 + \frac{1}{2}\log 2160535 < 8.3$	Theorem 48
`shift_correlation_le`	$\left\lvert\sum_{n \le Y} a_n a_{n+h}\right\rvert \le Y$ for $\lvert a_n\rvert \le 1$	Sec. 7
`oscillation_constant`	$2.3723^2/2 > 2.8139$	Theorem 10
`ten_thirds_count`	$N(N-1) + N^2 + \frac{4}{3}N^2 \le \frac{10}{3}N^2$	Proposition 7
`positions_exponents`	the two exponent comparisons in the proof	Proposition 4
Gram matrices of families		
`diagonal_sq_le_frobenius`, `trace_sq_le_dimension_mul_frobenius`	$(\operatorname{tr} A)^2 \le m \operatorname{tr}(A^{\mathsf{T}}A)$ for $m \times m$ real A	Theorem 25
`frobenius_sum`, `family_rank_lower`, `family_energy_upper`	expansion of $\operatorname{tr}((\sum_i B_i)^2)$ and its two bounds	Theorem 25
`gram_rank_obstruction`, `twice_dimension_bound`	the counting inequality for K nearly orthogonal Gram matrices	Theorem 25
`averagedGram_diagonal`, `averagedGram_entry_sq_le_one`, `averagedGram_trace_and_self`, `averaged_sign_rank_obstruction`	averaged Gram matrices of sign vectors: unit diagonal, entries in $[-1, 1]$, trace M, and the rank obstruction	Theorem 25

Appendix M.2. The Julia Programs

The programs use exact integer and rational arithmetic, and for the transcendental constants the ball arithmetic of Arb [42], so that every reported value comes with a rigorous enclosure. Table A12 lists the checks; all of them pass.

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Table A12. Checks made by the Julia programs. Ranges are complete: every integer in the stated range is tested. Enclosures are certified intervals, written with their midpoints rounded to five digits.

Program	Check	Result
`finite_identities`	window energy equals the prefix form, all 8190 sign strings of lengths 1 to 12	exact equality
	completion identity, 2000 random prefix vectors	exact equality
	$L(x) = \sum_{d \le \sqrt{x}} M(x/d^2)$, $1 \le x \le 1024$	exact equality
	Abel-jet constants $112/27$ and $114688/9$ of Table A4	exact
	Rudin–Shapiro complementarity, $m \le 10$, and the block pattern	exact
`energy_checks`	$\frac{1}{2}L(N)^2 \le Q(N) \le 5 \max_{j \le N} L(j)^2$, all $N \le 3 \cdot 10^7$	no violation
	$\lvert M(y)\rvert \le \sqrt{y}$, $1 \le y \le 3 \cdot 10^7$	no violation
	$\lvert M(y)\rvert \le y/4345$, $2160535 \le y \le 3 \cdot 10^7$	max ratio 0.9995
	$\lvert L(x)\rvert \le x/2641 + 8.3\sqrt{x}$, $1 \le x \le 3 \cdot 10^7$	max ratio 0.158
	monotone core, (11) and (12), $N \le 3 \cdot 10^7$	max ratios 0.41, 0.13
	$\min K = 4\sqrt{2} - 5$ on a grid in $\sigma \ge \frac{1}{2}$	0.65685
	largest value of $Q(N)/N$, $N \le 3 \cdot 10^7$	1.1534 at $N = 12\,897\,104$
	largest value of $\lvert L(x)\rvert/\sqrt{x}$, $x \le 3 \cdot 10^7$	1.3583 at the same point
	mean of $Q(N)/N^2$ against $1/\log$, $10^4 < N \le 3 \cdot 10^7$	0.4540
`pole_certificates`	1000 disjoint boxes isolating the first zeros on $\operatorname{Re} s = \frac{1}{2}$	certified
	right side of (30), $n = 1000$	$[0.45770 \pm 10^{-34}]$
	right side of (33), $J = 1$, $m = 100$	$[0.65142 \pm 10^{-35}]$
	right side of (33), $J = 2$, $m = 30$ (3721 frequencies)	$[0.72446 \pm 10^{-35}]$

Appendix M.3. The Lattice Certificate

The certificate of Corollary 13 is computed with the FLINT library and with Arb. The zeros $\gamma_1, \dots, \gamma_{8100}$ are enclosed at 11264 bits by Arb's certified zero routine, and the residues A_γ at 192 bits. The reduction of the lattice Λ_0 by FLINT's LLL routine, with the last coordinate fed in layers of 400 to 500 bits, takes about 100 minutes of one core; the reduced matrix U is stored. The check itself takes about four minutes: $\det U = \pm 1$, the Gram–Schmidt norms from the fraction-free LU decomposition of the integer Gram matrix, and the enumeration for all 7650 target ordinates, which visits 249 337 nodes and at most 48 for one target. The reduction software is not trusted: the basis property and every inequality are checked in exact integer arithmetic or in ball arithmetic. The programs and the matrix U are in the repository.

Appendix M.4. Numerical Illustration

Table A13 and Figure A2 show the quantities of the paper for N and x up to 10^7 and $3 \cdot 10^6$, and X up to $6 \cdot 10^4$. For $N \le 3 \cdot 10^7$ the ratio $Q(N)/N$ has largest value 1.1534, attained at $N = 12\,897\,104$, where $L(N)/\sqrt{N} = -1.3583$; this is consistent with the order $N^{1+\varepsilon}$ that the Riemann hypothesis predicts, and its logarithmic mean 0.4540 is close to the constant $S > 0.4579$ of Theorem 14. The modified function f that is $+1$ on about $x^{3/4}$ primes has $\log \lvert L_f(x)\rvert / \log x$ close to $3/4$, as Theorem 20 requires; and $B(X)/\sqrt{X}$ approaches the value $-c_0 = -0.60685$ of the pole at $s = 1/2$ (Proposition 23).[59]

[59] Finite data of this kind cannot decide the Riemann hypothesis: by Platt and Trudgian [54] a zero off the line would have height above $3 \cdot 10^{12}$, and its effect in this range would be invisible.

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Table A13. Values of L, of the energy Q and of the sparse flip f with $+1$ on $\lfloor 2^{3j/4} \rfloor$ primes of each interval $(2^j, 2^{j+1}]$, and of the smoothed sum B. Computed exactly, except B, which is rounded to four digits.

$N = x$	$L(N)$	$Q(N)$	$Q(N)/N$	$\dfrac{\log Q(N)}{\log N}$	$L_f(x)$	$\dfrac{\log \lvert L_f(x) \rvert}{\log x}$
10^2	-2	26.12	0.2612	0.7085	—	—
10^3	-14	261.35	0.2614	0.8057	52	0.575
10^4	-94	5783.54	0.5784	0.9405	594	0.694
10^5	-288	56316.88	0.5632	0.9501	5126	0.742
10^6	-530	382792.54	0.3828	0.9305	30750	0.748
10^7	-842	3835484.17	0.3835	0.9405	252510	0.772
X	10^3	10^4	$3 \cdot 10^4$	$6 \cdot 10^4$		
$B(X)/\sqrt{X}$	-0.5752	-0.5969	-0.6011	-0.6028		

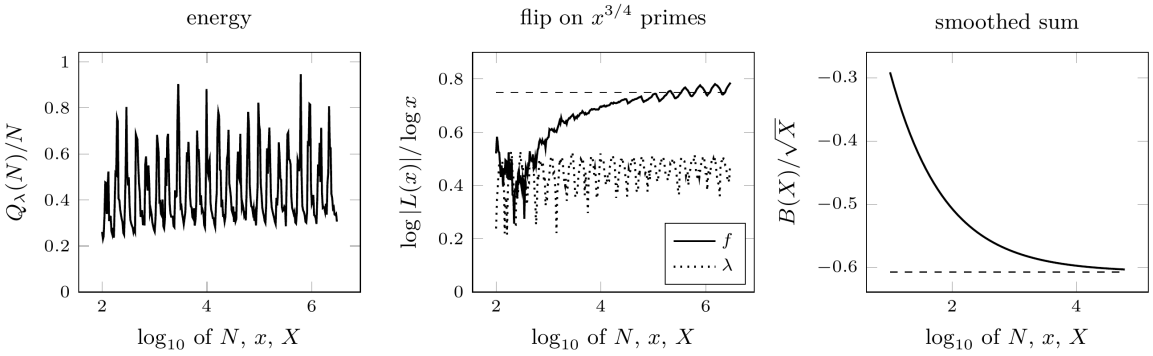

Figure A2. Left: the ratio $Q(N)/N$ at 300 logarithmically spaced $N \le 3 \cdot 10^6$. Middle: the exponent $\log|L(x)|/\log x$ for the sparse flip f on about $x^{3/4}$ primes (solid) and for λ (dotted); the dashed line is $3/4$. Right: $B(X)/\sqrt{X}$, with the dashed line at $-c_0 = -0.60685$.

Open Gluon and Anaxia

Two tools built by D. B. stand behind the way this project was organised. Open Gluon is a workspace he assembled and maintains, which keeps pinned copies of the mathematical libraries, the source record of the project and the checks in one reproducible place. Anaxia, the engine of the eBLLM project, is his construction engine: it stores each construction as a route with its input, its output, the obligations that must be discharged and a test that makes the route fail when an obligation is skipped, and it certifies finite statements in exact rational arithmetic. He wrote both and developed them from the constructions of his earlier papers [25–27]. Three of their constructions were used here. The first is the reduction of every target bound to a typed statement about the rightmost zero, which showed that each route proposed for the critical energy bound was equivalent to a zero-free half-plane and led to the form of Theorem 6 and of the zero-one law. The second is the perturbation test on the signs of λ, which flips the values on a prescribed set of primes and records whether the energy exponent moves; it produced the experiment in the middle panel of Figure A2 and the threshold of Corollary 18. The third is exact certification: the Julia programs recompute the constants of the paper in exact rational arithmetic wherever the value is rational, and in certified ball arithmetic otherwise. The engine proposes routes and tests them; every statement of this paper is proved in the text.

Author Contributions: Deep Bhattacharjee, who leads the project, posed the problem in the form treated here, carried out the research and wrote the paper. The proofs are his: the energy identities and the equivalence theorem, the zero-one law, the sensitivity theorem and its threshold, the Mellin and Nyman–Beurling results, the explicit bounds and the appendices. He made all the computations: the Lean 4 project of Table A11, the Julia programs of Table A12 with their exact and ball-arithmetic evaluations, the numerical tables and the figures; and he brought the work to its present state. Ushashi Bhattacharya read the manuscript and verified its arguments, its constants and its references. Both authors approved the final version.

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