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Article

Invertible Solutions of the Yang-Baxter-Like Matrix Equation for Nonzero Singular Matrix

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Abstract

Assuming the coefficient matrix is a nonzero singular matrix, we demonstrate that the invertible solutions of the Yang-Baxter-like matrix equation must possess at least two elementary divisors. This establishes a necessary condition for an invertible matrix to satisfy the Yang-Baxter-like matrix equation. Building on this finding, we derive several meaningful corollaries. Additionally, we provide some examples to illustrate our results.

Keywords: Yang-Baxter-like matrix equation; elementary divisor; invertible solutions; group inverse

1. Introduction

Matrix theory is widely applied in many fields(see [1–8]) . When addressing real-world problems with matrix-based methods, various matrix equations often arise. This paper focuses on investigating the equation:

$$AXA = XAX, \tag{1}$$

where A is a given $n \times n$ complex matrix and X is an unknown matrix of the same dimensions. Given its similarity to the Yang-Baxter equation[9,10], this is called the Yang-Baxter-like matrix equation, or YBME for short.

YBME has attracted significant attention in the field of solving matrix equations(see [11–19]). Extensive studies have been conducted on its solutions for several special types of matrices, such as idempotent matrices [20], rank-one matrices [21], rank-two matrices [22] and so on.

In addition, there has been plenty of research on the properties of solutions of the Yang-Baxter-like equation for some special coefficient matrices. For instance, let $A = UV^H$ be an $n \times n$ complex singular matrix of rank r , where U and V are $n \times r$ matrices with full column rank. Lu [23] get some necessary and sufficient conditions under which X serves as a nontrivial (commuting) solution to the matrix equation (1).

Nevertheless, for general and even special matrices, determining all solutions of YBME remains challenging, as solving this equation is equivalent to tackling a complicated system of n^2 quadratic equations involving n^2 variables.

In this paper, we study the invertible solutions of the Yang-Baxter-like matrix equation for nonzero singular matrix. This paper is organized in the following manner. In the next section, we present the principal results of our research. Later, we elucidate our results through several numerical examples. In the last section, we conclude with a summary of this paper’s research.

2. Principal Results

Theorem 1. Suppose A is an $n \times n$ nonzero singular complex matrix and $\text{rank}(A) = s, (1 \leq s < n)$. X is an invertible solution of equation (1). Then X has at least two elementary divisors.

Proof. As X is an invertible solution of equation (1), we can get that $XAXA = XXAX$. Then

$$\text{rank}((XA)^2) = \text{rank}(XAXA) = \text{rank}(XXAX) = \text{rank}(XA) = \text{rank}(A)$$

Obviously, 1 is the smallest positive integer. When $k = 1$, equation

$$\text{rank}((XA)^{k+1}) = \text{rank}((XA)^k)$$

holds. According to Definition 2.1.1 in [24], the index of XA is 1. From Theorem 2.2.1 in [24], we know XA has a group inverse. Then according to Theorem 2.2.2 in [24], there are nonsingular matrices U and V satisfying

$$XA = U \begin{pmatrix} V & 0 \\ 0 & 0 \end{pmatrix} U^{-1}.$$

Obviously, $\text{rank}(V) = \text{rank} \begin{pmatrix} V & 0 \\ 0 & 0 \end{pmatrix} = \text{rank}(XA) = \text{rank}(A) = s$, $V \in \mathbb{C}^{s \times s}$ and $U \in \mathbb{C}^{n \times n}$. Due to $AXA = XAX$, we can get that $X^{-1}XAXA = XAX$. That is to say,

$$X^{-1}U \begin{pmatrix} V & 0 \\ 0 & 0 \end{pmatrix} U^{-1}U \begin{pmatrix} V & 0 \\ 0 & 0 \end{pmatrix} U^{-1} = U \begin{pmatrix} V & 0 \\ 0 & 0 \end{pmatrix} U^{-1}X.$$

Then

$$X^{-1}U \begin{pmatrix} V^2 & 0 \\ 0 & 0 \end{pmatrix} U^{-1} = U \begin{pmatrix} V & 0 \\ 0 & 0 \end{pmatrix} U^{-1}X.$$

So

$$\begin{pmatrix} V^2 & 0 \\ 0 & 0 \end{pmatrix} = U^{-1}XU \begin{pmatrix} V & 0 \\ 0 & 0 \end{pmatrix} U^{-1}XU.$$

Suppose

$$U^{-1}XU = \begin{pmatrix} Z_1 & Z_2 \\ Z_3 & Z_4 \end{pmatrix},$$

where $Z_1 \in \mathbb{C}^{s \times s}$, $Z_2 \in \mathbb{C}^{s \times (n-s)}$, $Z_3 \in \mathbb{C}^{(n-s) \times s}$ and $Z_4 \in \mathbb{C}^{(n-s) \times (n-s)}$. Then

$$\begin{pmatrix} V^2 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} Z_1 & Z_2 \\ Z_3 & Z_4 \end{pmatrix} \begin{pmatrix} V & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} Z_1 & Z_2 \\ Z_3 & Z_4 \end{pmatrix}.$$

In other words,

$$\begin{pmatrix} V^2 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} Z_1 V Z_1 & Z_1 V Z_2 \\ Z_3 V Z_1 & Z_3 V Z_2 \end{pmatrix}.$$

Therefore

$$\begin{cases} Z_1 V Z_1 = V^2, \\ Z_1 V Z_2 = 0, \\ Z_3 V Z_1 = 0, \\ Z_3 V Z_2 = 0. \end{cases} \quad (2)$$

As $V \in \mathbb{C}^{s \times s}$ and $\text{rank}(V) = s$, we know V is invertible and $\text{rank}(Z_1 V Z_1) = \text{rank}(V^2) = \text{rank}(V) = s$. On the other hand, $s = \text{rank}(Z_1 V Z_1) \leq \text{rank}(Z_1 V) \leq \text{rank}(Z_1) \leq s$. So $\text{rank}(Z_1) = s$. That is to say, Z_1 is invertible. According to (2), we can get that $Z_2 = 0$ and $Z_3 = 0$. So

$$U^{-1}XU = \begin{pmatrix} Z_1 & 0 \\ 0 & Z_4 \end{pmatrix}.$$

Then

$$X = U \begin{pmatrix} Z_1 & 0 \\ 0 & Z_4 \end{pmatrix} U^{-1}.$$

That is to say, X is similar to $\begin{pmatrix} Z_1 & 0 \\ 0 & Z_4 \end{pmatrix}$. So $\lambda E - X$ is equivalent to $\lambda E - \begin{pmatrix} Z_1 & 0 \\ 0 & Z_4 \end{pmatrix}$, where E is the n by n identity matrix. Then $\lambda E - X$ and $\lambda E - \begin{pmatrix} Z_1 & 0 \\ 0 & Z_4 \end{pmatrix}$ have the same elementary divisors. As

$$\lambda E - \begin{pmatrix} Z_1 & 0 \\ 0 & Z_4 \end{pmatrix} = \begin{pmatrix} \lambda E_{s \times s} - Z_1 & 0 \\ 0 & \lambda E_{(n-s) \times (n-s)} - Z_4 \end{pmatrix}$$

where $E_{s \times s}$ is the $s \times s$ identity matrix and $E_{(n-s) \times (n-s)}$ is the $(n-s) \times (n-s)$ identity matrix, all the elementary divisors of $\lambda E_{s \times s} - Z_1$ and all the elementary divisors of $\lambda E_{(n-s) \times (n-s)} - Z_4$ constitute all the elementary divisors of $\lambda E - \begin{pmatrix} Z_1 & 0 \\ 0 & Z_4 \end{pmatrix}$. Therefore $\lambda E - \begin{pmatrix} Z_1 & 0 \\ 0 & Z_4 \end{pmatrix}$ has at least two elementary divisors. Then $\lambda E - X$ has at least two elementary divisors. In other words, X has at least two elementary divisors.

□

Corollary 1. Suppose A is an $n \times n$ nonzero singular complex matrix and $\text{rank}(A) = s, (1 \leq s < n)$. X is an invertible solution of equation (1). The elementary divisors of X are $(\lambda - \lambda_1)^{s_1}$ and $(\lambda - \lambda_2)^{s_2}$. Then $s_1 = s$ or $s_2 = s$.

Proof. As the elementary divisors of $\lambda E - X$ are the same as the elementary divisors of X , the elementary divisors of $\lambda E - X$ are $(\lambda - \lambda_1)^{s_1}$ and $(\lambda - \lambda_2)^{s_2}$. According to the proof of Theorem 1, $\lambda E - X$ and $\lambda E - \begin{pmatrix} Z_1 & 0 \\ 0 & Z_4 \end{pmatrix}$ have the same elementary divisors. So the elementary divisors of $\lambda E - \begin{pmatrix} Z_1 & 0 \\ 0 & Z_4 \end{pmatrix}$ are $(\lambda - \lambda_1)^{s_1}$ and $(\lambda - \lambda_2)^{s_2}$. As

$$\lambda E - \begin{pmatrix} Z_1 & 0 \\ 0 & Z_4 \end{pmatrix} = \begin{pmatrix} \lambda E_{s \times s} - Z_1 & 0 \\ 0 & \lambda E_{(n-s) \times (n-s)} - Z_4 \end{pmatrix}$$

where $E_{s \times s}$ is the $s \times s$ identity matrix and $E_{(n-s) \times (n-s)}$ is the $(n-s) \times (n-s)$ identity matrix, all the elementary divisors of $\lambda E_{s \times s} - Z_1$ and all the elementary divisors of $\lambda E_{(n-s) \times (n-s)} - Z_4$ constitute all the elementary divisors of $\lambda E - \begin{pmatrix} Z_1 & 0 \\ 0 & Z_4 \end{pmatrix}$. So there are two situations.

(i) The elementary divisor of $\lambda E_{s \times s} - Z_1$ is $(\lambda - \lambda_1)^{s_1}$. Then $s_1 = s$.

(ii) The elementary divisor of $\lambda E_{s \times s} - Z_1$ is $(\lambda - \lambda_2)^{s_2}$. Then $s_2 = s$.

Consequently, $s_1 = s$ or $s_2 = s$. □

Corollary 2. Suppose A is an $n \times n$ nonzero singular complex matrix and $\text{rank}(A) = s, (1 \leq s < n)$. X is an invertible solution of equation (1). The elementary divisors of X are $(\lambda - \lambda_1)^{s_1}, (\lambda - \lambda_2)^{s_2}, \dots, (\lambda - \lambda_k)^{s_k}$. Then there exists such a nonempty proper subset M of $\{1, 2, \dots, k\}$ that

$$\sum_{m \in M} s_m = s.$$

Proof. As the elementary divisors of $\lambda E - X$ are the same as the elementary divisors of X , the elementary divisors of $\lambda E - X$ are $(\lambda - \lambda_1)^{s_1}, (\lambda - \lambda_2)^{s_2}, \dots, (\lambda - \lambda_k)^{s_k}$. According to the proof of

Theorem 1, $\lambda E - X$ and $\lambda E - \begin{pmatrix} Z_1 & 0 \\ 0 & Z_4 \end{pmatrix}$ have the same elementary divisors. So the elementary divisors of $\lambda E - \begin{pmatrix} Z_1 & 0 \\ 0 & Z_4 \end{pmatrix}$ are $(\lambda - \lambda_1)^{s_1}, (\lambda - \lambda_2)^{s_2}, \dots, (\lambda - \lambda_k)^{s_k}$. As

$$\lambda E - \begin{pmatrix} Z_1 & 0 \\ 0 & Z_4 \end{pmatrix} = \begin{pmatrix} \lambda E_{s \times s} - Z_1 & 0 \\ 0 & \lambda E_{(n-s) \times (n-s)} - Z_4 \end{pmatrix}$$

where $E_{s \times s}$ is the $s \times s$ identity matrix and $E_{(n-s) \times (n-s)}$ is the $(n-s) \times (n-s)$ identity matrix, all the elementary divisors of $\lambda E_{s \times s} - Z_1$ and all the elementary divisors of $\lambda E_{(n-s) \times (n-s)} - Z_4$ constitute all the elementary divisors of $\lambda E - \begin{pmatrix} Z_1 & 0 \\ 0 & Z_4 \end{pmatrix}$. So all the elementary divisors of $\lambda E_{s \times s} - Z_1$ constitute a nonempty proper subset of $\{(\lambda - \lambda_1)^{s_1}, (\lambda - \lambda_2)^{s_2}, \dots, (\lambda - \lambda_k)^{s_k}\}$. Then there exists such a nonempty proper subset M of $\{1, 2, \dots, k\}$ that all the elementary divisors of $\lambda E_{s \times s} - Z_1$ constitute the set $\{(\lambda - \lambda_m)^{s_m} | m \in M\}$. Consequently,

$$\sum_{m \in M} s_m = s.$$

That is to say, there exists such a nonempty proper subset M of $\{1, 2, \dots, k\}$ that

$$\sum_{m \in M} s_m = s.$$

□

Corollary 3. Suppose A is an $n \times n$ nonzero singular complex matrix and $\text{rank}(A) = 1$. X is an invertible solution of equation (1). Then there must be an elementary factor with power one among the elementary factors of X .

Proof. Suppose the elementary divisors of X are $(\lambda - \lambda_1)^{s_1}, (\lambda - \lambda_2)^{s_2}, \dots, (\lambda - \lambda_k)^{s_k}$. Because $\text{rank}(A) = 1$, according to Corollary 2, there exists such a nonempty proper subset M of $\{1, 2, \dots, k\}$ that

$$\sum_{m \in M} s_m = 1.$$

As $s_m \geq 1, m \in M$, the set M has only one element i and $s_i = 1$. That is to say, there must be an elementary factor with power one among the elementary factors of X . □

3. Numerical Examples

Example 1. $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ is a 3×3 nonzero singular complex matrix and $\text{rank}(A) = 2$. $X =$

$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ is an invertible solution of equation (1). X has three elementary divisors :

$$(\lambda - \lambda_1)^{s_1}, (\lambda - \lambda_2)^{s_2} \text{ and } (\lambda - \lambda_3)^{s_3}$$

where $\lambda_1 = 1, s_1 = 1, \lambda_2 = 1, s_2 = 1, \lambda_3 = 1$ and $s_3 = 1$. Then there exists such a nonempty proper subset $\{1, 2\}$ of $\{1, 2, 3\}$ that

$$\sum_{m \in \{1, 2\}} s_m = 2.$$

.

Example 2. $A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ is a 4×4 nonzero singular complex matrix and $\text{rank}(A) = 1$. $X = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ is an invertible solution of equation (1). X has two elementary divisors :
 $(\lambda - \lambda_1)^{s_1}$ and $(\lambda - \lambda_2)^{s_2}$

where $\lambda_1 = 1, s_1 = 1, \lambda_2 = 1$ and $s_2 = 3$. Obviously, $s_1 = \text{rank}(A)$ and there is an elementary factor with power one among the elementary factors of X .

4. Conclusion

This paper shows that when the coefficient matrix is a nonzero singular matrix, any invertible solution to the Yang-Baxter-like matrix equation necessarily has at least two elementary divisors. Thus, a necessary condition is established for an invertible matrix to satisfy the Yang-Baxter-like matrix equation. Based on this result, we obtain several important corollaries. However, solving the Yang-Baxter-like matrix equation remains a significant challenge that demands further research.

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Abbreviations

The following abbreviation is used in this manuscript:

YBME Yang-Baxter-like matrix equation

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