 doi:10.20944/preprints202206.0130.v2

Valid Quantization: The Next Step

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Abstract

Canonical quantization is a wonderful procedure for selected problems, but there are many problems for which it fails. Affine quantization is a different procedure that has shown that it can solve many problems that canonical quantization cannot. Here, words like succeed and fail refer to whether the quantization results are correct or incorrect. This paper offers two simple examples that serve to introduce affine quantization, and compare studies of two different quantization procedures. Brief comments about field theory and gravity problems undergoing quantization by affine procedures completes the paper.

1 Introduction

Any problem, of almost any kind, divides into two parts: Formulations, followed by Solutions. For our purpose, this means carefully choosing specific differential equations, and next, solving these equations. Only completed solutions can establish whether the results are correct or incorrect. Correct solutions require correct formulations, and correct formulations can be found by choosing well selected aspects of both physics and mathematics.

We begin our presentation with a very familiar example.

2 The harmonic oscillator using canonical quantization

For this example, the classical Hamiltonian, choosing $m = \omega = 1$ for simplicity, is $H = (p^2 + q^2)/2$. and p & q both obey the rules of canonical quantization that they satisfy $-\infty < p$ & $q < \infty$, the Poisson bracket

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$\{q, p\} = 1$, and, together, the two variables also need to be Cartesian, e.g., $d\sigma^2 = \nu^{-1} dp^2 + \nu\, dq^2$, which insures that classical and quantum Hamiltonian functions obey the rule $H(p, q) = \mathcal{H}(p, q)$ [1] . In that case, our quantum Hamiltonian is $\mathcal{H} = (P^2 + Q^2)/2$, where $[Q, P] = i\hbar \mathbb{1}$, and, as expected, $-\infty < P \;\&\; Q < \infty$.

Our wave equation then is $(-\hbar^2\, d^2/dx^2 + x^2)/2 \;\; \psi_n(x) = E_n \;\; \psi_n(x)$, and the eigenfunctions and the eigenvalues are well known as $\psi_n(x) = N_n\, e^{x^2/2\hbar}\; (-d/dx)^n\; e^{-x^2/\hbar}$, $E_n = \hbar(n + 1/2)$, with $n = 0, 1, 2, ...$, and N_n can provide normalization when needed.

3 The half-harmonic oscillator using affine quantization

A simple introduction to affine quantization is first noting that while $H = (p^2 + q^2)/2$, we now require $q > 0$. That forces $Q > 0$ and thus $P^\dagger \neq P$. Instead of $p \to P$, we choose the 'dilation' variable $d \equiv pq \to D = (P^\dagger Q + QP)/2 = D^\dagger$, which leads to $[Q, D] = i\hbar\, Q$. Like we saw $H(p, q) = \mathcal{H}(p, q)$ required for canonical quantization, we now find that $H'(d, q) = \mathcal{H}'(d, q)$, provided that now p & q form a constant negative curvature with $d\sigma'^2 = \beta^{-1} q^2\, dp^2 + \beta\, q^{-2} dq^2$ [2]. That leads to the same classical Hamiltonian, now in affine variables, is $H' = (d^2/q^2 + q^2)/2$. Next, the new quantum Hamiltonian is[1]

$$\mathcal{H} = (D\, Q^{-2} D + Q^2)/2 = [P^2 + (3/4)\hbar^2/Q^2 + Q^2]/2\,. \tag{1}$$

It follows that the wave equation for this model is

$$[-\hbar^2\,(d^2/dx^2) + (3/4)\hbar^2/x^2 + x^2]/2\;\; \phi_n(x) = E'_n\; \phi_n(x)\,. \tag{2}$$

Happily, this equation has been solved. The eigenfunctions, which are of the form, $\phi_n = x^{3/2} (polynomial)_n\, e^{-x^2/2\hbar}$, and can be found in [3]. But even more interesting are the eigenvalues, which are $E'_n = 2\hbar(n + 1)$, for $n = 0, 1, 2...$ as before. Again we now find an *equal spacing that is twice that of the full-harmonic oscillator.*

Even more spectacular is the fact that the partial-harmonic oscillator, which now requires that $q > -b$, with $0 < b < \infty$, and it leads to the related wave equation,

$$[-\hbar^2\,(d^2/dx^2) + (3/4)\hbar^2/(x + b)^2 + x^2]/2\;\; \varphi_{b\,n}(x) = E'_{b\,n}\, \varphi_{b\,n}(x)\,. \tag{3}$$

[1] Although $P^\dagger \neq P$, then $P^\dagger$ and $P^{\dagger\,2}$ *act* like P and P^2 thanks to the $\hbar$-term in (1).

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As b grows, the eigenvalues continue to have an equal spacing for every b, which passes smoothly from $2\hbar \to \hbar$ as $b: \ 0 \to \infty$ [4]. When $b = \infty$, we will have recreated all of the eigenfunctions and eigenvalues of the full-harmonic oscillator!

3.1 Canonical quantization and the half-harmonic oscillator

Since canonical quantization requires variables that cover whole real lines, and our problem requires that $q > 0$, we need an 'imaginary infinite wall' to suppress every wave function to zero, i.e., $\varphi(x) = 0$, for $x \leq 0$. The region of $x > 0$ is open to the harmonic oscillator in which half of the eigenfunctions (the 'odd ones' for the full harmonic oscillator) pass through zero at $x = 0$. This means that the canonical quantization of this example would be considered complete by using those eigenfuctions.

Unfortunately, there is a week point in the story. Yes, an odd wave function, becomes an even function at the first derivative and leads to $\varphi'(0) \neq 0$. This makes the first derivative *not* continuous at the point $x = 0$. The second derivative then contains a term proportional to a Dirac delta function, $\delta(x)$, which is zero for all $|x| > 0$ while $\int_{-a}^{a} \delta(x)\, dx = 1$ for all $a > 0$, and $\int_{-a}^{a} \delta(x)^2\, dx = \infty$. As a part of any wave function, a delta function excludes entry of it into any Hilbert space.

Stated bluntly, the second derivative includes a $\delta(0)$ term that invalidates the proposed solution, while an affine quantization procedure welcomes an $\hbar$-term to the wave equation that will let all of its eigenfunctions to have a continuous first derivative.

But there is even another problem. Choosing only the odd eigenfunctions of the full-harmonic oscillator, and omitting the even eigenfunctions because the latter would have a discontinuous wave function, means that only half of the proper eigenfunctions survive.

3.2 The Worst News Yet!

To see that more clearly let us pull back the 'imaginary infinite wall' from $q = 0$ to $q = -b$ with $b > 0$. This process lets the given set of eigenfunctions evolve so as to vanish at $q = -b$ to join the remaining portion of the vanishing functions due to the remaining wall. Carry that process to the end, namely $b \to \infty$, and that terrible wall now disappears. This process has completely recovered the original odd eigenfunctions, but that is only half of the original eigenfunctions of the full-harmonic oscillator. In other words, *canonical quantization has lost* **half** *of all the solutions to the original*

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problem! Moreover, if we choose $o < x < 2b \equiv L$, which requires only sine functions that vanish at $x = 0$ and $x = L$. Any general summation of the presumed eigenfunctions cannot ever create $f(x) = 1$ for $0 < x < L$, which is just one example of a vast number of valid, normalizable, functions of the relevant Hilbert space.

This outcome is no where near to being 'close' to a valid quantization.

3.3 Affine quantization wins the half-harmonic oscillator contest

Using affine quantization, this exercise has proved to be completely valid. Affine quantization deserves a 'Gold Medal'.

The formulation of the coming example using affine quantization, is very close to the former example, and so it will certainly be valid. But having a different $\hbar$-term can complicate efforts to find complete solutions of the new wave equation, which are still waiting for solutions.

4 A Review of 'The particle in a box'

This familiar model adopts a classical Hamiltonian $H = p^2$ and in this case, $-b < q < b$, with $0 < b < \infty$. To use canonical quantization for this model, we again require 'imaginary infinite walls' to suppress every wave function to a zero value, now in two regions, where $|q| \geq b$. The appropriate quantum Hamiltonian is chosen as $\mathcal{H} = -\hbar^2\, d^2/dx^2$, and cos and sin are natural candidates for eigenfunctions provided they obey $\varphi(-b) = \varphi(b) = 0$. For example, the ground state would be $\varphi_1(x) = \cos(\pi\, x/2b)$.

However, we again find a problem because, while the proposed wave function is continuous over the whole real line, the first derivative is again discontinuous at both $x = \pm b$, and thus the second derivative again leads to Dirac's delta functions. This should have declared that the problem has not been correctly solved. But it seems that the $\delta(x)$ terms have been overlooked, and then the *cos* and *sin* solutions are accepted.

4.1 The usual view of 'The particle in a box'

The author has received comments that were sent to him by an informed person which are quoted below:

"When physics instructors teach the"particle in a box" problem, we do so knowing that the particular form of the potential is fictional – there's no physical reality to an infinite potential well with infinitely sharp edges.

Therefore, there's not much interest in introducing complicated solutions to exactly solve this fictional problem.

I suspect that there's some way in which the fiction of the infinite well potential and the fiction of the solution with a discontinuous first derivative actually work together to create a meaningful approximation to reality.

I would also recommend that you consult some quantum mechanics textbooks, some of which give justifications for the use of functions with discontinuous first derivatives."

My response is that the proposed eigenfunctions, and their derivatives, lead to genuine non-continuities and then to infinites, and while this might be somehow 'painted over', the issue deserves proper mathematics because the proposed eigenfunctions and eigenvalues may be entirely incorrect.

Bad news: But worse than trying to be 'close' to correct results ignores the fact that only half of the solutions are being considered.

Suppose we consider a 'particle in a box' that instead runs between $0 < q < 2b \equiv L$, which just shifts the wave functions, while the Hamiltonian is still $H = p^2$; in fact, this is a common formulation of the box problem. All the analysis is preserved, just shifted to the right. Now, we very slightly alter the Hamiltonian to become $H = p^2 + 10^{-1000}q^2$, still requiring that $0 < q < L$ as before. This leads to a very tiny change. But now let us slide the right hand wall away, i.e., $L \to \infty$, which would lead to a half-harmonic oscillator with only half its eigenfunctions, the formally odd ones, and which was shown earlier, this is confirmed by now completely withdrawing the wall on the left, i.e., over $q \to -\infty$. That says that the canonical approach to the particle in a box deals with only half of the presumed eigenfunctions, which is certainly not 'close' at all to their suggested story.

Good news: However, there is no real need to deal with first derivative discontinuities, and only half of the eigenfunctions. Every acceptable wave function of an affine quantization of the particle in a box has the form $\kappa(x) = (b^2 - x^2)^{3/2}(remainder)$ that leads to *continuous first derivatives everywhere,* **including** $x \pm b$, thanks to the presence of a new $\hbar$-term.

5 Affine Quantization of 'The particle in a box'

The author has recently written an article on this topic, and the present presentation will be just the basics [5]. The dilation variable now becomes $d' = p(b-x)(b+x) = p(b^2 - x^2)$ with $-b < q < b$, and $0 < b < \infty$. Happily, *'imaginary infinite walls' are NOT needed.*

The two Hamiltonians are $H' = d'^2/(b^2-x^2)^2 \rightarrow \mathcal{H}' = D'(b^2-Q^2)^{-2}D' = P^2 + \hbar^2(2Q^2+b^2)/(b^2-Q^2)^2$, where $D' = [P^\dagger(b^2-Q^2)+(b^2-Q^2)P]/2$. Observe that, in its way, the $\hbar$-term provides its own kind of 'walls'.

This expression then leads to the wave equation[2]

$$[-\hbar^2\ (d^2/dx^2) + (2x^2+b^2)/(b^2-x^2)^2\ ]\ \kappa(x) = E\ \kappa(x)\ , \tag{4}$$

which is still seeking solutions to share such good news. *Even a ground state would be a welcome start!*

Observe that if $|x \pm b|$ is very tiny, then $(2x^2+b^2)/(b^2-x^2)^2 \simeq 3b^2/4b^2(b \mp x)^2$, which imitates the previous '3/4'-term extremity near the singular points. This implies that the eigenfunctions should be of the form

$$\kappa(x) = (b^2-x^2)^{3/2}(remainder)\ , \tag{5}$$

and this formation would lead to continuous first derivatives, and a full set of valid eigenfunctions.

6 Other Positive Results Using Affine Quantization

The author believes that affine quantization deserves to be more widely used to seek valid results of a variety of problems. In particular, affine procedures have now been used with Monte-Carlo calculations for the scalar fields φ^4_4 and φ^{12}_3, where the lower number, $n \equiv s+1$, and s is the number of spacial variables, while 1 refers to the single variable time, and the upper number $p \geq 2n/(n-2)$ leading to nonrenormalizability. For these examples, affine quantization has already led to acceptable results, while canonical quantization has only led to unacceptable results, as if the interaction term made no contribution to these models; see [7 - 14].

Additionally, the author has recently prepared equations for a rigorous path integration quantization of gravity using affine quantization procedures [15].

Finally, almost a dozen articles, published with 'JHEPGC –> Klauder', can let the reader see that there are other ways to formulate an affine quantization of gravity using more common operator procedures.

[2]The analysis leading to the new $\hbar$-term has been accepted for publication only very recently [6]. The principal result is that if $d_f = p\, f(q) \rightarrow D_f = [P^\dagger f(Q) + f(Q)P]/2$, then $D_f(f(Q)^{-2})D_f = P^2 + (\hbar^2/4)\,[\ln(f(Q))')^2 - 2(\ln(f(Q))'']$, where $f(Q)' \equiv d\, f(Q)/dQ$.

 doi:10.20944/preprints202206.0130.v2

A Takeaway: Since the canonical eigenfunctions for the half-harmonic oscillator, are not 'close' to reality, which the correct affine eigenfunction expressions make clear, it is reasonable to claim that the canonical eigenfunctions for the particle in a box are not 'close' to reality because these two examples suffer from the same disease. This disease is discontinuous first derivatives of their suggested canonical eigenfunctions, and using only half of the potential eigenfunctions. Correcting the first derivative problem, as an affine quantization does, will restore the full set of proper eigenfunctions!

Thus, the author is led to regard the standard solutions to the 'box' problem as incorrect physics. Accepting 'close' solutions for the problems we have studied in this paper can lead to accepting 'close' solutions for the quantization of other problems, which includes field theories and gravity. *We all deserve better than 'just close'!*

Note: The author has no conflicts to disclose.

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