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Alejandro Sanfeliciano , [Luis Angel Saúl](#) ^{*} , [Carlos Hurtado-Martínez](#) , [Luis Botella](#)

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Article

PH Space: A Mathematical Framework for Modeling Presence and Hierarchy in Psychological Change Through Fuzzy Cognitive Maps

Alejandro Sanfeliciano ¹, Luis Angel Saúl ^{1,*}, Carlos Hurtado-Martínez ¹ and Luis Botella ²

¹ Facultad de Psicología, Universidad Nacional de Educación a Distancia (UNED), Madrid, España

² FPCEE Blanquerna, Universidad Ramon Llull, Barcelona, España

* Correspondence: lasaul@psi.uned.es

Abstract: Understanding psychological change requires a quantitative framework capable of capturing the complex and dynamic relationships among personal constructs. Personal Construct Psychology emphasizes the hierarchical reorganization of bipolar constructs, yet existing qualitative methods inadequately model the reciprocal and graded influences involved in such change. This paper introduces the Presence–Hierarchy (PH) space, a centrality measure for constructs represented within Fuzzy Cognitive Maps (FCMs). FCMs model cognitive systems as directed, weighted graphs, allowing for nuanced analysis of construct interactions. The PH space operationalizes two orthogonal dimensions: *Presence*, representing the overall connectivity and activation of a construct, and *Hierarchy*, quantifying the directional asymmetry between influences exerted and received. By formalizing Hinkle’s hierarchical theory within a rigorous mathematical framework, the PH space enables precise identification of constructs that drive or resist transformation. This dual-dimensional model provides a structured method for analyzing personal construct systems, supporting both theoretical exploration and clinically relevant interpretations in the study of psychological change.

Keywords: Fuzzy Cognitive Maps (FCMs); graph theory; centrality measures; psychological change; personal construct psychology

1. Introduction

1.1. Psychological Change and Personal Construct Systems

Grasping psychological change requires a detailed understanding of the complex interplay of meanings individuals construct to interpret and anticipate their experiences. Personal Construct Psychology (PCP) [1] conceptualises this as an evolving system of bipolar constructs that mediate personal anticipation and interpretation of events. Kelly’s foundational model suggests psychological change as a systematic reorganisation of these personal construct systems, involving the adaptation of existing constructs or the introduction of novel ones [2,3].

While Kelly’s model provides a rich qualitative description, it lacks explicit quantification of reciprocal and dynamic influences among constructs. Hinkle [4] sought to address this gap by introducing the laddering technique, emphasizing hierarchical structures among constructs and proposing that constructs higher in this hierarchy (supraordinate constructs) exercise greater influence and display more resistance to change compared to subordinate constructs. However, the inherently qualitative and unidirectional assumptions of laddering limit its capacity to represent the complex, reciprocal, and graded relationships characteristic of actual cognitive systems [5,6].

1.2. Fuzzy Cognitive Maps as Quantitative Models

To systematically represent these intricate relational patterns, Kosko [7] introduced Fuzzy Cognitive Maps (FCMs), a graph-theoretical framework where constructs are represented as nodes interconnected by directed, weighted edges reflecting perceived influences [7,8]. FCMs capture feedback loops,

mutual influences, and graded relationships, offering a dynamic model of human cognition that aligns closely with PCP's anticipatory nature [9,10].

Clinical applications of FCMs, such as through structured elicitation methods like the Weighted Implication Grid (WimpGrid) [11], significantly enhance practitioners' ability to map cognitive and affective networks with precision. Such mappings reveal critical constructs that guide or resist psychological transformation, offering clinicians robust analytical tools to foster therapeutic insight and intervention [12].

1.3. Centrality Measures in Graph Theory

Despite these advances, quantitatively identifying critical nodes—constructs that significantly shape psychological transformation—remains challenging. Traditional centrality measures in network analysis, initially developed for social networks [13–16], have been adapted to characterize influential elements within complex psychological networks. However, the directional and weighted nature of FCMs necessitates refined analytical strategies to accurately capture construct influence. Measures such as in-degree, out-degree, and eigenvector centralities have been proposed and adapted specifically for directed and weighted contexts, enhancing the sensitivity of analyses to the nuanced cognitive dynamics underpinning psychological change [10,17].

To address these analytical needs explicitly, this study introduces the Presence-Hierarchy (PH) space, a geometrically grounded analytic framework specifically tailored for interpreting construct centrality within FCMs. The PH space operationalises two critical dimensions: *presence*, which quantifies the general connectivity and influence activity of a construct, and *hierarchy*, capturing the directional balance between influences exerted and received. By mathematically formalising the hierarchical differentiation originally proposed by Hinkle, the PH framework significantly advances the capacity to quantitatively analyse construct interrelations within personal meaning systems.

Employing the PH framework within FCMs allows researchers and practitioners to simultaneously identify constructs by their general network prominence (presence) and their hierarchical role within the cognitive system. This dual-dimensional perspective not only offers deeper insight into the mechanisms that drive psychological change, but also facilitates targeted and effective therapeutic strategies. Consequently, the integration of PCP, Hinkle's hierarchical insights, FCM, and the new PH framework presented here enriches both theoretical understanding and practical intervention methodologies, representing a significant advancement in the mathematical modelling of psychological transformation.

2. Mathematical Foundations

To analyse psychological change, it is essential to establish a formal mathematical framework that captures the relational structure of personal constructs and their dynamic implications. The following foundations define a graph-theoretic model of psychological meaning making, based on the representation of constructs as vectors in bounded spaces, the simulation of hypothetical transformations, and the computation of influence weights between constructs. This section explains how self-perception, ideal goals, and construct interactions can be encoded as algebraic objects and how these structures enable the construction of an FCM that reflects the individual's psychological topology. The psychological data required to instantiate this model are obtained through the WimpGrid interview protocol [11], a structured assessment methodology to generate personal construct systems. These mathematical definitions provide the basis for deriving a directed graph of meaning relations and for quantifying the centrality and hierarchical role of each construct in psychological change.

2.1. FCM of Psychological Change

To formalise the representation of psychological change, we begin by defining the algebraic objects that capture the structure of an individual's personal construct system. These objects serve as the foundation for constructing a FCM, where constructs are modeled as nodes, and perceived causal influences among them are represented as weighted directed edges. This mathematical framework

enables the translation of qualitative self-perception and ideal self-evaluation, as elicited through the WimpGrid interview [11], into a structured and analyzable model. In what follows, we introduce the definitions and axioms necessary to construct such a map, starting from the self and ideal-self vectors and culminating in a weight matrix that encodes the system of cognitive implications.

Definition 1. The Self-Now of the individual can be represented as a vector $\vec{s} \in [-1, 1]^n$, where n is the number of constructs elicited in the WimpGrid interview, and each component s_i corresponds to the individual's Self-Now score on construct i within their personal construct system.

Remark 1. The notation $[-1, 1]^n$ represents the n -dimensional space where each coordinate corresponds to a construct elicited. Each score is bounded within the interval $[-1, 1]$, ensuring a normalized representation of personal meanings. A value of -1 indicates complete alignment with the left pole of the construct, while a value of 1 represents complete alignment with the right pole. Intermediate values correspond to positions between the two poles of the construct, reflecting varying degrees of association.

Definition 2. The Ideal-Self of the individual can be defined as a vector $\vec{d} \in [-1, 1]^n$, where n is the number of constructs elicited in the WimpGrid interview, and each component d_i corresponds to the individual's Ideal Self score on construct i within their personal construct system.

Definition 3. The intensities of the hypothetical situations proposed during the interview can be represented as a vector $\vec{h} \in [-1, 1]^n$, where n is the number of constructs elicited in the WimpGrid interview, and each component h_i represents the intensity score assigned to a hypothetical situation associated with construct i .

Remark 2. Following the WimpGrid protocol [11], $\vec{h}$ is computed based on the Self-Now vector $\vec{s}$ and the Ideal-Self vector $\vec{d}$. Specifically, each h_i is determined as:

$$h_i = \begin{cases} \frac{-s_i}{|s_i|}, & \text{if } s_i \neq 0, \\ \frac{d_i}{|d_i|}, & \text{if } s_i = 0 \text{ and } d_i \neq 0, \\ 1, & \text{if } s_i = 0 \text{ and } d_i = 0. \end{cases} \quad (1)$$

Definition 4. The hypothetical states of the individual (referred to as Hypothetical-Selves) can be represented using a matrix $M \in [-1, 1]^{n \times n}$, where each entry m_{ij} ($i \neq j$) represents the hypothetical score assigned to construct i under the influence of hypothetical situation j .

Remark 3. The diagonal entries m_{ii} are arbitrarily defined as the components of the Self-Now vector $\vec{s}$ for the purposes of subsequent mathematical computations.

Axiom 1 (Linearity of Change). It is assumed that there exists a linear relationship between the proposed change in the hypothetical situation, Δs_i , and the declared change in the construct j as a result of the influence of the construct i , denoted by Δs_j . Specifically, this relationship is expressed as:

$$\Delta s_j = w_{ij} \Delta s_i \quad (2)$$

Remark 4. The linearity assumption encapsulated in Equation (2) implies that the influence of construct i on construct j can be fully captured through a single proportionality coefficient, w_{ij} . This assumption simplifies the system's complexity by reducing non-linear dynamics to a manageable linear framework, making it amenable to graph-theoretic and algebraic analysis.

Theorem 1. Given the current state vector $\vec{s}$, the hypothetical intensity vector $\vec{h}$, and the matrix of Hypothetical-Selves M , a weight matrix $W \in \mathbb{R}^{n \times n}$ can be computed. Each entry w_{ij} in W quantifies the sensitivity of construct j to changes in construct i and is defined as:

$$w_{ij} = \frac{m_{ji} - s_j}{h_i - s_i} \quad (3)$$

Proof. To compute the weight matrix W , consider the following reasoning:

1. The matrix M encodes the hypothetical selves, where m_{ji} represents the hypothetical score of construct j when construct i undergoes a hypothetical shift (h_i). Thus, m_{ji} reflects how construct j is influenced by changes in construct i .
2. The Self-Now vector $\vec{s}$ provides the baseline values for each construct, where s_j is the score for construct j in its present state. The difference $m_{ji} - s_j$ therefore measures the deviation of construct j under the hypothetical shift, h_i , associated with construct i from its baseline state.
3. The hypothetical intensity vector $\vec{h}$ captures the proposed changes in the interview. The difference $h_i - s_i$ quantifies the magnitude of the hypothetical change introduced to construct i .
4. From the Linearity of Change Axiom (Axiom 1), we assume that the relationship between the change in construct i and the corresponding change in construct j can be described as proportional. This proportionality is expressed by:

$$\Delta s_j = w_{ij} \Delta s_i,$$

where w_{ij} is the proportionality coefficient that quantifies the influence of construct i on construct j .

5. Applying this axiom to the hypothetical scenario, we relate the deviation of construct j ($m_{ji} - s_j$) to the magnitude of the shift introduced to construct i ($h_i - s_i$). Specifically, the weight w_{ij} is given by normalizing the deviation $\Delta s_j = m_{ji} - s_j$ with respect to the hypothetical intensity $\Delta s_i = h_i - s_i$:
6. Equation (3) quantifies the sensitivity of construct j to changes in construct i under the assumption of linearity. This process is applied to all pairs of constructs $i, j \in \{1, \dots, n\}$, resulting in the weight matrix W , which captures the pairwise relationships across the entire system.

□

Example 1. Given the score matrix in Figure 1, we can operationalize it as follows:

$$(X|Y) = \left(\begin{array}{ccccc|c} 3 & 3 & 1 & 4 & 3 & 7 \\ 4 & 4 & 2 & 4 & 2 & 7 \\ 4 & 7 & 4 & 7 & 4 & 4 \\ 5 & 3 & 3 & 3 & 3 & 6 \\ 5 & 6 & 5 & 5 & 5 & 7 \end{array} \right) \quad (4)$$

1							7
	Self - Totally Calm	Self - Totally Sporty	Self - Totally Busy	Self - Totally Hard Worker	Self - Totally Unhealthy	Ideal-Self	
	Anxious	3	3	1	4	3	7
	Calm						
	No Sporty	4	4	2	4	2	7
	Sporty						
Relaxed	4	7	4	7	4	4	Busy
Lazy	5	3	3	3	3	6	Hard Worker
Unhealthy	5	6	5	5	5	7	Healthy

Figure 1. WimpGrid Score Matrix (Scale 1–7), used to elicit and structure the individual’s personal construct system. Each row corresponds to a bipolar construct (e.g., Anxious–Calm), and each column represents a hypothetical self-state: the first five columns depict imagined selves resulting from targeted changes in each construct, while the sixth column represents the Ideal-Self. Cells are filled with self-assigned ratings from 1 to 7, where 1 denotes complete alignment with the left pole and 7 with the right pole. Blue cells indicate the participant’s current Self-Now scores; green cells indicate their Ideal-Self scores; yellow cells highlight constructs where the participant anticipates significant change if another construct is modified.

Applying a rescaling transformation from the interval [1,7] to [−1,1], we obtain a normalized matrix:

$$(M|\vec{d}) = \left(\begin{array}{ccccc|c} -.33 & -.33 & -1 & 0 & -.33 & 1 \\ 0 & 0 & -.67 & 0 & -.67 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ .33 & -.33 & -.33 & -.33 & -.33 & .67 \\ .33 & .67 & .33 & .33 & .33 & 1 \end{array} \right) \tag{5a}$$

Using the normalized matrix $(M|\vec{d})$ and applying Weight Matrix Calculation Theorem (1), we can calculate the self vector $\vec{s}$, the ideal vector $\vec{d}$, and the weight matrix W for the example:

$$\vec{s} = (-.33, 0, 0, -.33, .33) \tag{6a}$$

$$\vec{d} = (1, 1, 0, .67, 1) \tag{6b}$$

$$W = \left(\begin{array}{ccccc} 0 & 0 & 0 & .50 & 0 \\ 0 & 0 & 1 & 0 & .33 \\ -.67 & -.67 & 0 & 0 & 0 \\ .25 & 0 & .75 & 0 & 0 \\ 0 & .50 & 0 & 0 & 0 \end{array} \right) \tag{6c}$$

Definition 5. Given a weight matrix $W \in \mathbb{R}^{n \times n}$, a self vector $\vec{s} \in [-1,1]^n$, and an ideal vector $\vec{d} \in [-1,1]^n$, there exists a directed graph $G = (V, E, \psi, \phi)$ such that:

- $V = \{v_1, v_2, \dots, v_n\}$, where each vertex corresponds to a construct.
- $E \subseteq V \times V$, where $(v_i, v_j) \in E$ if and only if $w_{ij} \neq 0$.
- $\psi : V \rightarrow [-1,1]^2$ assigns attributes to vertices, where $\psi(v_i) = (s_i, d_i)$.

- $\varphi : E \rightarrow \mathbb{R}$ assigns weights to edges, where $\varphi((v_i, v_j)) = w_{ij}$.

Figure 2 illustrates the directed graph G constructed from the data presented in Example 1, displaying both the edge weights w_{ij} and the node attributes (s_i, d_i) associated with each construct.

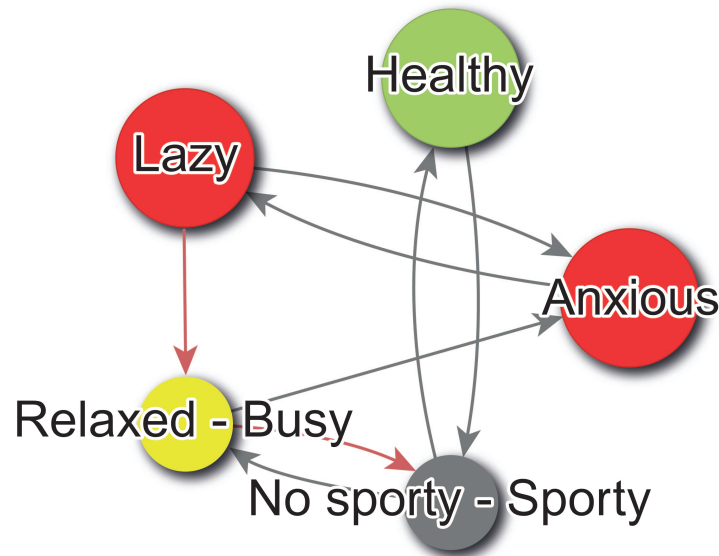


Figure 2. Directed graph representation of the personal construct system derived from the weight matrix W in Example 1. Each node corresponds to a bipolar construct, positioned and styled according to its psychological attributes. Node color encodes congruence with the Ideal-Self: green indicates full congruence ($s_i = d_i$), red indicates discrepancy ($s_i \neq d_i$), yellow marks undefined ideal values ($d_i = 0$), and grey indicates undefined self-perception ($s_i = 0$). Node size reflects the absolute self-score $|s_i|$, representing the construct's salience in current self-identity. Directed edges represent cognitive implications with black denoting a direct relationship ($w_{ij} > 0$) and red an inverse relationship ($w_{ij} < 0$). Edge thickness encodes the absolute weight $|w_{ij}|$, reflecting the strength of influence. This graph provides an interpretable structural summary of the FCM, supporting both visual intuition and algebraic analysis of psychological change dynamics. Graph generated using WimpTools [12].

The resulting graph G provides a visual and mathematical representation of the relationships between constructs. The directed edges encode the influences derived from W , while the node attributes $\psi(v_i)$ enrich the graph with the self and ideal scores, allowing for a comprehensive analysis of both structural relationships and individual goals.

2.2. PH Space

Definition 6. Let $v_i \in V$ be a vertex of the graph G , and let w_{ij} denote the weights associated with the edges of G . We define two measures for the vertex v_i : the in-degree K_i^- and the out-degree K_i^+ , as follows:

$$K_i^- = \sum_{j=1}^n |w_{ji}| \quad (7a)$$

$$K_i^+ = \sum_{j=1}^n |w_{ij}| \quad (7b)$$

Remark 5. The in-degree K_i^- and out-degree K_i^+ measures defined above are adapted from the graph-theoretic approach presented by Borgatti [18].

Axiom 2 (Vertex Hierarchy). Let $v_i \in V$ be a vertex in the graph G . The hierarchical state of v_i is determined by the relationship between its in-degree K_i^- and out-degree K_i^+ , and is defined as follows:

- v_i is said to be supraordinate if $K_i^+ > K_i^-$, meaning the vertex exerts more influence (outputs) than it receives (inputs).
- v_i is said to be subordinate if $K_i^+ < K_i^-$, meaning the vertex is more influenced by other vertices (inputs) than it influences them (outputs).
- v_i is said to be neutral if $K_i^+ = K_i^-$, meaning the vertex has an equal balance of influence received (inputs) and exerted (outputs).

Remark 6. The concept of hierarchy described here is inspired by the framework introduced by Hinkle [4], who suggested that constructs that exert greater influence on a system while being less influenced themselves are considered to be more supraordinate. This principle extends naturally to vertices in a graph, where the relationship between inputs and outputs reflects their hierarchical role within the structure.

Theorem 2. Let $G = (V, E)$ be a directed graph, where each vertex $v_i \in V$ has an in-degree K_i^- and an out-degree K_i^+ , to compute presence P_i and hierarchy H_i , we apply the following linear transformation to (K_i^-, K_i^+) :

$$PH_i = (P_i, H_i) = \begin{pmatrix} 0.5\sqrt{2} & 0.5\sqrt{2} \\ -0.5\sqrt{2} & 0.5\sqrt{2} \end{pmatrix} \begin{pmatrix} K_i^- \\ K_i^+ \end{pmatrix} \quad (8)$$

Remark 7. The Presence index P_i quantifies the total connectivity of vertex v_i , combining its in-degree (K_i^-) and out-degree (K_i^+). And Hierarchy index H_i quantifies the net difference between the in-degree and out-degree of vertex v_i .

Proof. Let $(K_i^-, K_i^+) \in \mathbb{R}^2$ represent the in-degree and out-degree of a vertex v_i in a directed graph $G = (V, E)$. The goal is to compute two indices, Presence index (P_i) and Hierarchy index (H_i), by transforming the original vector (K_i^-, K_i^+) into a new coordinate system. The transformation is defined by the matrix:

$$T = \begin{pmatrix} 0.5\sqrt{2} & 0.5\sqrt{2} \\ -0.5\sqrt{2} & 0.5\sqrt{2} \end{pmatrix} \quad (9)$$

This matrix corresponds to a rotation of the original coordinate system by 45° counterclockwise, scaled by $0.5\sqrt{2}$. Applying this transformation to the vector (K_i^-, K_i^+) , we compute the Equation (8):

$$\begin{pmatrix} P_i \\ H_i \end{pmatrix} = \begin{pmatrix} 0.5\sqrt{2} & 0.5\sqrt{2} \\ -0.5\sqrt{2} & 0.5\sqrt{2} \end{pmatrix} \begin{pmatrix} K_i^- \\ K_i^+ \end{pmatrix}$$

Performing the matrix multiplication, we obtain:

$$P_i = 0.5\sqrt{2}K_i^- + 0.5\sqrt{2}K_i^+ \quad (10a)$$

$$H_i = -0.5\sqrt{2}K_i^- + 0.5\sqrt{2}K_i^+ \quad (10b)$$

Factoring out $0.5\sqrt{2}$, these simplify to:

$$P_i = 0.5\sqrt{2}(K_i^- + K_i^+) \quad (11a)$$

$$H_i = 0.5\sqrt{2}(K_i^+ - K_i^-) \quad (11b)$$

Geometrically:

- Presence index (P_i) represents the projection of the vector (K_i^-, K_i^+) onto the axis defined by $y = x$, capturing the total connectivity of vertex v_i , i.e., $K_i^- + K_i^+$.

- Hierarchy index (H_i) represents the projection of the vector (K_i^-, K_i^+) onto the axis defined by $y = -x$, capturing the net difference between K_i^+ and K_i^- , i.e., $K_i^+ - K_i^-$.

Thus, the transformation rotates the space by 45° so that the new axes correspond to the desired components of Presence (P_i) and Hierarchy (H_i). The scaling factor $0.5\sqrt{2}$ ensures that the magnitudes of the transformed indices are appropriately normalized. $\square$

The PH space visualization (Figure 3) thus provides a concise and interpretable summary of the relational dynamics within personal construct systems. By clearly distinguishing constructs according to their centrality and hierarchical roles, this geometric framework not only facilitates theoretical understanding but also supports targeted clinical interventions. Clinicians can readily identify influential constructs that serve as critical leverage points for therapeutic change, enabling a systematic and strategically informed approach to psychological intervention.

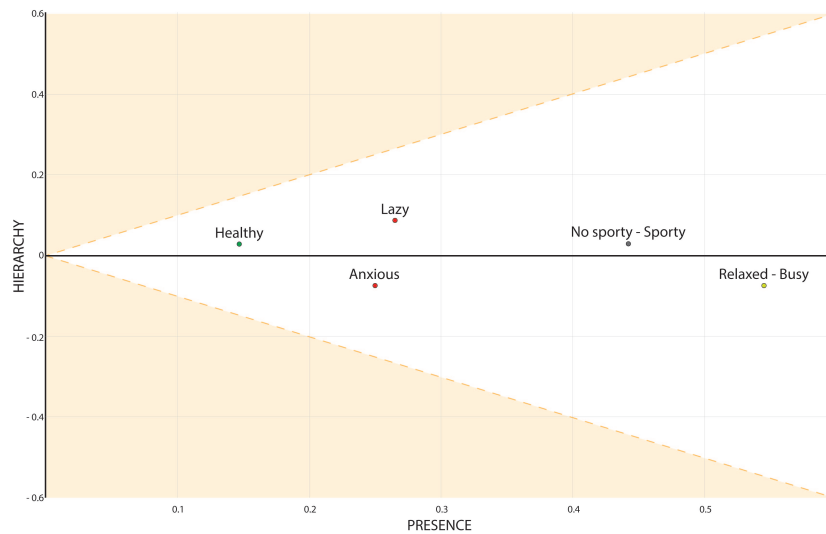


Figure 3. PH space representation of the psychological constructs from Example 1. Each construct is positioned according to its presence (overall connectivity) and hierarchy (directional balance of influence). Constructs toward the upper region (positive hierarchy) indicate supraordinate roles, exerting greater influence, while constructs in the lower region (negative hierarchy) indicate subordinate roles, being more influenced by others. Proximity to the right side signifies higher overall centrality and connectivity within the cognitive network.

2.3. Properties of the PH Space

This section presents a formal characterization of the PH space. The analysis is organized into two parts. First, we demonstrate that the PH space satisfies the defining properties of a two-dimensional convex cone in $\mathbb{R}^2$. Second, we examine the projection as a limiting representation of all possible weighted directed graphs of increasing order n , assuming continuous-valued edge weights in the range $[-1, 1] \subset \mathbb{R}$.

2.3.1. Bounding Theorem and Geometric Constraint

Theorem 3. All points (P, H) resulting from this transformation satisfy the following constraint:

$$P \geq |H| \quad \Leftrightarrow \quad -P \leq H \leq P \quad (12)$$

Proof. Non-negativity of P . Since $K^-, K^+ \geq 0$, it follows that:

$$P = \frac{\sqrt{2}}{2}(K^- + K^+) \geq 0$$

Bounding inequality $|H| \leq P$.

$$|H| = \left| \frac{\sqrt{2}}{2} (K^+ - K^-) \right| \quad (13)$$

$$= \frac{\sqrt{2}}{2} \cdot |K^+ - K^-| \quad (14)$$

$$\leq \frac{\sqrt{2}}{2} (K^+ + K^-) = P \quad (15)$$

Boundary cases. Equality holds in the following cases:

- If $K^- = 0$, i.e., the node is a pure source, then $H = P$.
- If $K^+ = 0$, i.e., the node is a pure sink, then $H = -P$. $\square$

2.3.2. Geometric Formulation

Any point (p, h) in the PH space results from a linear transformation applied to the normalized counts of in-degree and out-degree connections of a construct in an FCM. As shown above, each point satisfies:

$$P \geq 0 \quad \text{and} \quad -P \leq H \leq P \quad (16)$$

This inequality defines a conic region in $\mathbb{R}^2$, with vertex at the origin and bounded by the rays $(1, 1)$ and $(1, -1)$. At $P = 0$, hierarchy necessarily vanishes, yielding the origin. As P increases, the admissible range of H expands symmetrically.

Definition 7. The PH space is defined as the set:

$$C = \{(p, h) \in \mathbb{R}^2 \mid p \geq 0, |h| \leq p\}$$

This constraint arises naturally from the linear transformation applied to node in-strength and out-strength in a weighted directed graph. Since both quantities are defined as non-negative sums of edge weights, it follows that $P \geq 0$ and $|H| \leq P$ for all projected nodes. Therefore, the image of any fuzzy cognitive map under the PH transformation lies within C .

Theorem 4 (Convex Cone Structure). The set C defined in Definition 7 satisfies the axioms of a convex cone.

Proof. We prove that C satisfies the three fundamental properties:

1. **Closure under non-negative scalar multiplication.** Let $(p, h) \in C$ and $\alpha \geq 0$. Then:

$$|\alpha h| = \alpha |h| \leq \alpha p \Rightarrow (\alpha p, \alpha h) \in C$$

Figure 4 illustrates this property: scaling a vector in C preserves cone membership.

2. **Closure under vector addition.** Let $(p_1, h_1), (p_2, h_2) \in C$. Then:

$$|h_1 + h_2| \leq |h_1| + |h_2| \leq p_1 + p_2 \Rightarrow (p_1 + p_2, h_1 + h_2) \in C$$

Figure 5 exemplifies vector addition within C .

3. **Closure under convex combinations.** Let $0 \leq \lambda \leq 1$ and define:

$$(p, h) = \lambda(p_1, h_1) + (1 - \lambda)(p_2, h_2)$$

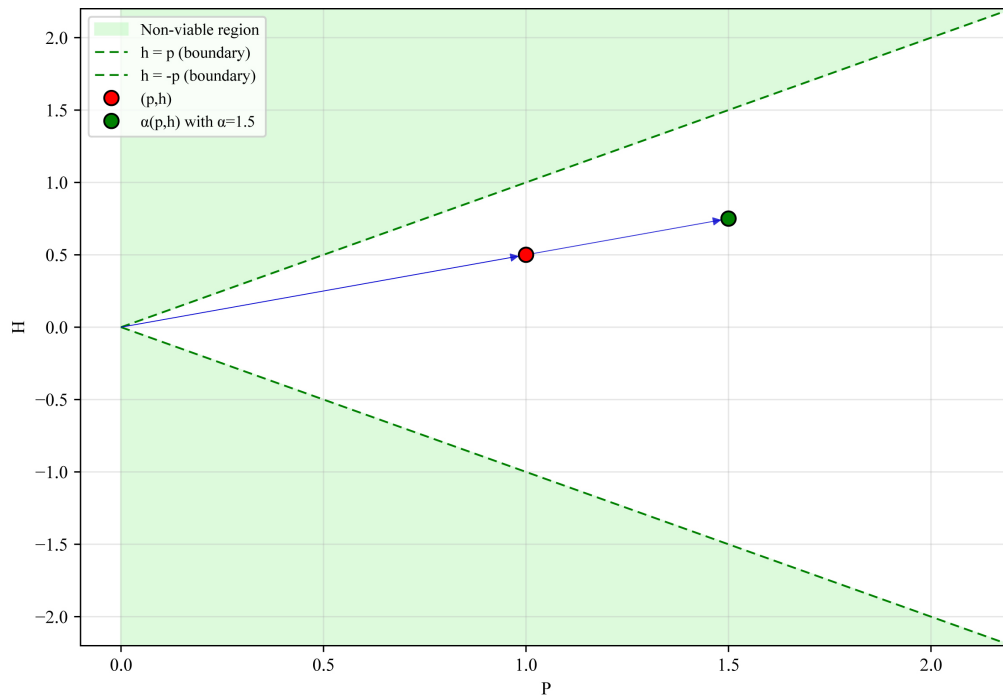


Figure 4. Closure under scalar multiplication (Theorem 4, Property 1)

Then:

$$\begin{aligned} |h| &\leq \lambda|h_1| + (1-\lambda)|h_2| \\ &\leq \lambda p_1 + (1-\lambda)p_2 = p \Rightarrow (p, h) \in C \end{aligned}$$

□

Figure 6 exemplifies convex combinations in $\mathcal{C}$.

2.3.3. Graph Projections and Asymptotic Behavior

Let $\mathcal{G}_n$ denote the set of all possible FCMs of order n , where each directed edge can take any value in $[-1, 1]$. Each node $c_i \in G \in \mathcal{G}_n$ maps to a pair (p_i, h_i) via the transformation described above.

Theorem 5 (Asymptotic Density). *For fixed n , the projection of $\mathcal{G}_n$ is a compact, continuous, non-convex subset of $\mathcal{C}$. As $n \rightarrow \infty$, the union of projections densely fills $\mathcal{C}$.*

Compactness and continuity. Since the projection is a continuous function of the compact set $[-1, 1]^{n(n-1)}$, its image is compact.

Non-convexity. Consider $n = 3$ and the points:

- $A = (\sqrt{2}, \sqrt{2})$ (source: $K^+ = 2, K^- = 0$)
- $C = (2\sqrt{2}, 0)$ (balanced: $K^+ = K^- = 2$)

Both are achievable. Their midpoint $M = \left(\frac{3\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$ requires $K^+ = 2, K^- = 1$. However, in any 3-node graph:

- Achieving A requires a node with $K^+ = 2$ (max out-strength).
- Achieving C requires another node with $K^+ = K^- = 2$ (max connections).

This exhausts all edge slots, leaving no capacity for a node with $K^- = 1$ at M . Thus, $M \notin \text{proj}(\mathcal{G}_3)$.

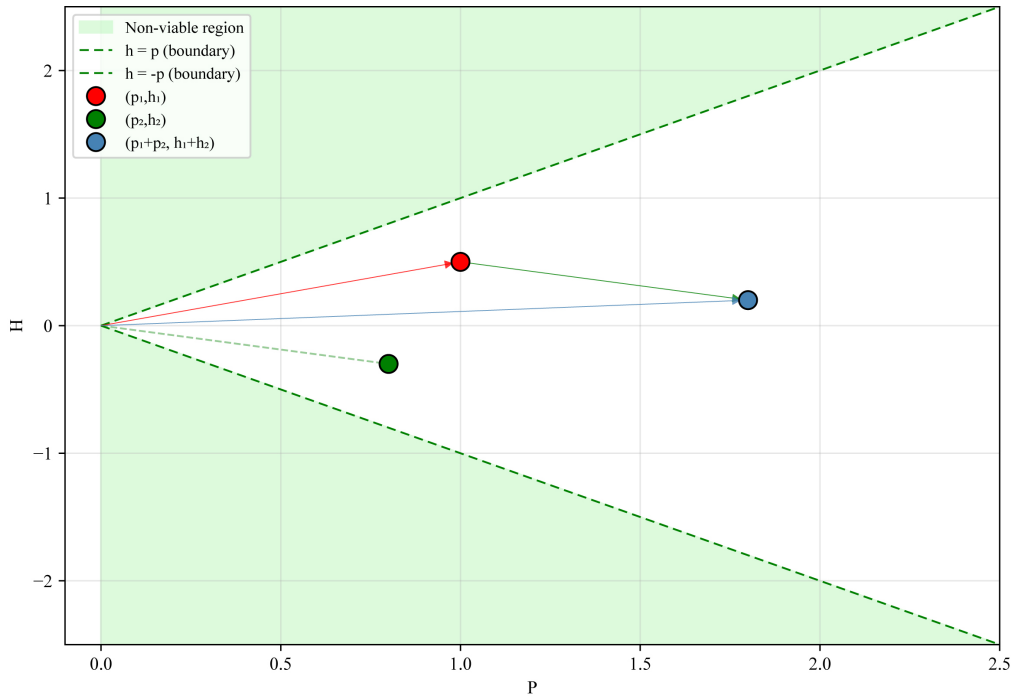


Figure 5. Exemplification of vector addition (Theorem 4, Property 2)

Asymptotic density. Let $C \subset \mathbb{R}^2$ be the PH space defined by $C = \{(p, h) \mid p \geq 0, |h| \leq p\}$. Then, for any $(p, h) \in C$ and any $\epsilon > 0$, there exists $n \in \mathbb{N}$ and a fuzzy cognitive map G_n of order n such that the projection of G_n contains a node (P, H) satisfying $\|(P, H) - (p, h)\| < \epsilon$.

Proof. Let $(p, h) \in C$ and $\epsilon > 0$ be arbitrary. Define:

$$K^+ = \frac{p+h}{\sqrt{2}}, \quad K^- = \frac{p-h}{\sqrt{2}}.$$

By construction, $K^+, K^- \geq 0$ due to $(p, h) \in C$. Our goal is to realize node strengths (K^+, K^-) within an additive error δ that guarantees $\|(P, H) - (p, h)\| < \epsilon$ in PH space.

Let $(p, h) \in C$ and $\epsilon > 0$ be arbitrary. Define

$$\delta = \frac{\epsilon}{\sqrt{2}}.$$

Then, if we construct a node with approximate in-strength and out-strength satisfying

$$|K_{approx}^+ - K^+| < \delta, \quad |K_{approx}^- - K^-| < \delta,$$

it follows by linearity of the PH transformation that

$$|P_{approx} - p| < \epsilon, \quad |H_{approx} - h| < \epsilon.$$

To achieve such an approximation, note that any desired value of K^+ can be represented as the sum of $\lfloor K^+ \rfloor$ outgoing connections with weight 1, plus a residual outgoing edge with weight $K^+ - \lfloor K^+ \rfloor$, provided this residual exceeds δ . The same construction applies symmetrically for K^- by selecting incoming edges in an analogous manner.

Define

$$s^+ = \lfloor K^+ \rfloor + 1, \quad s^- = \lfloor K^- \rfloor + 1,$$

and choose

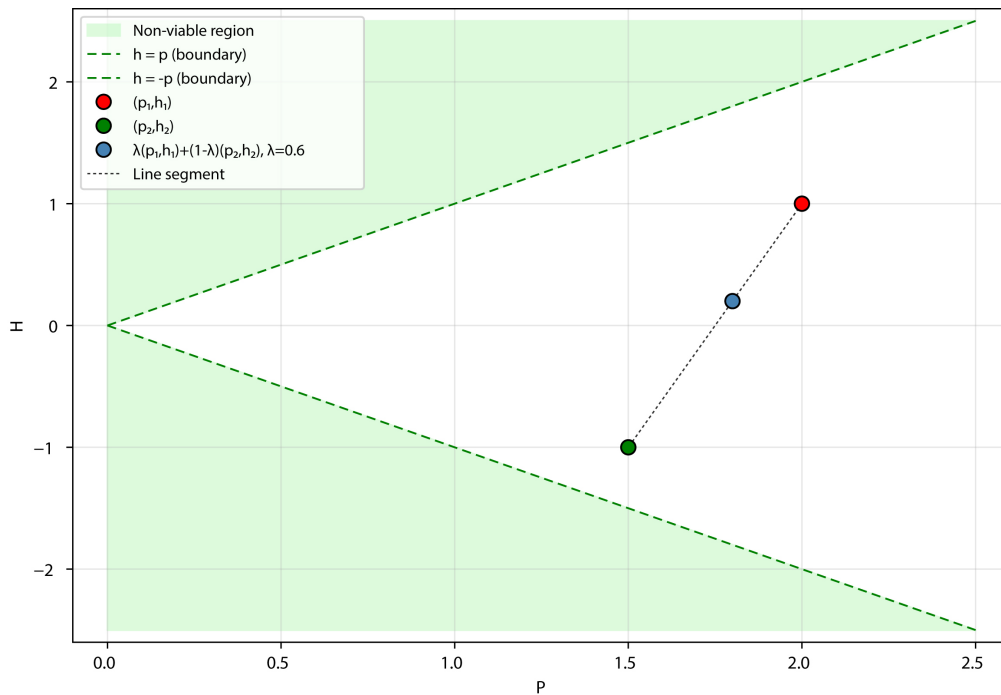


Figure 6. Convex combinations (Theorem 4, Property 3)

$$n \geq \max(s^+, s^-) + 1.$$

This choice of n ensures that there are enough nodes to provide the required number of incoming and outgoing connections without violating the no self-loop constraint inherent to the graph definition.

To complete the construction, assign all other edge weights in G_n to values strictly less than δ . This guarantees that their cumulative contribution to the in-strength or out-strength of the approximating node remains below δ and does not affect the targeted precision.

Thus, we have constructed a graph G_n containing a node with approximate strengths $(K_{approx}^+, K_{approx}^-)$ within δ of (K^+, K^-) , and consequently, a projection within ϵ of (p, h) in the PH space. Since (p, h) and ϵ were arbitrary, the density of projections in C follows as $n \rightarrow \infty$.

Hence, its projection (P, H) lies within distance ϵ of (p, h) in the PH space. Since (p, h) and ϵ were arbitrary, the union of projections over increasing n is dense in C .

□

2.3.4. Interpretational Implications

The PH space offers a faithful embedding of the topology and influence patterns of FCMs. Although it does not capture semantic distances to ideal constructs, it provides a scalable and rigorous geometric framework for examining the overall graph architecture.

In summary, the PH space is defined by the constraint $|H| \leq P$, forming a bidimensional convex cone in $\mathbb{R}^2$. This structure supports both the geometric interpretation of construct systems and their formal modeling as the number of nodes n increases, approaching a dense representation in the limit.

3. Discussion

3.1. Applications

The PH space, defined as a bidimensional convex and self-dual cone, admits several analytically grounded applications within the modeling of personal construct systems.

(a) Identification of hub constructs. Constructs located in the high- P region and exhibiting high Mahalanobis distance from the centroid can be interpreted as structural hubs within the cognitive system. To ensure interpretive validity, the analysis is restricted to constructs with $P > \bar{P}$,

and Mahalanobis distance is computed relative to the empirical covariance structure of this subset. Constructs exceeding a predefined χ^2 threshold (e.g., 95th percentile) are identified as hubs, reflecting both prominence and deviation from the system's central configuration.

(b) Dynamic trajectories and attractors. The convex geometry of the PH space allows for the analysis of simulated activation dynamics derived from fuzzy cognitive maps. Trajectories projected within the cone may exhibit convergence toward fixed points or bounded regions, interpretable as cognitive attractor states. These attractors may correspond to stable interpretative configurations, enabling formal study of equilibrium, rigidity, or response to perturbation in personal construct systems.

(c) Test-retest stability assessment. The geometric structure also supports the quantification of temporal stability across repeated assessments. Displacement of constructs in the PH space can be evaluated using Euclidean or Mahalanobis distances, allowing researchers to distinguish between stable core constructs and volatile peripheral ones. This approach provides an idiographic measure of consistency, sensitive to both structural position and system-wide distribution.

Each application benefits from the formal constraints of the PH space, enabling statistically consistent and geometrically interpretable operations within a reduced-dimensional framework.

3.2. Limitations and Future Research Directions

(a) Dimensionality reduction. The projection onto the PH plane reduces a potentially high-dimensional system of construct interrelations to two summary dimensions. While this facilitates visualization and analysis, it also entails a loss of information. Further investigation is required to assess which properties of the original system are preserved or distorted by the transformation, and whether the PH space can be extended to higher-dimensional analogues.

(b) Lack of evaluative distance representation. The PH projection reflects the structural connectivity of constructs within a fuzzy cognitive map, encoding their total influence (Presence) and directional balance (Hierarchy). However, it does not represent the evaluative proximity or distance of constructs relative to an ideal self or desired pole. This distinction is particularly relevant in clinical or developmental applications where the salience of a construct may depend not only on its centrality, but also on its alignment with aspirational goals. Future research could explore hybrid projections that integrate both structural and evaluative dimensions, or combine the PH space with discrepancy-based metrics.

(c) Empirical validation. Finally, while the PH space has demonstrated utility in exploratory analyses (e.g., identification of hubs or stability assessment), systematic empirical validation is still required. Future research should examine how constructs identified as central or stable in the PH space correspond to external psychological variables such as symptomatology, well-being, or therapeutic outcomes.

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Abbreviations

The following abbreviations are used in this manuscript:

FCM	Fuzzy Cognitive Map
PCP	Personal Construct Psychology
PH	Presence–Hierarchy
WimpGrid	Weighted Implication Grid

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