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Article

Asymptotic Renormalization: Complete Quantum Electrodynamics

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Abstract

Wick ordering is frequently invoked in canonical quantum field theory as a device to control vacuum fluctuations and to simplify perturbation theory. In four-dimensional QED, however, the ultraviolet divergences of physical relevance are not removed by normal ordering, and additional pathologies occur for local composite operators, notably in current commutators and gauge invariance constraints. We show that in gauge-invariant entire-function UV completions, ultraviolet finiteness and quasi-locality are achieved at the level of propagators and observables. In this setting the operational role of Wick ordering as a divergence subtraction is eliminated where loop integrals are absolutely convergent, and regulated composite operators are defined by covariant smearing through an entire function of the d'Alembertian. Wick ordering remains only as an optional finite convention relative to a chosen state, rather than a structural necessity.

Keywords: ED; locality; microcausality

1. Introduction

Relativistic quantum field theory is built from operator-valued distributions. As a consequence, products of fields at coincident spacetime points are generically ill-defined without further prescription. Historically, two intertwined issues motivated the introduction and widespread use of Wick normal ordering as the subtraction of vacuum contractions in canonical quantization and the organization of perturbation theory through Wick's theorem [1]. In gauge theories such as QED, additional constraints arise from gauge invariance and current conservation, making the definition of composite operators and their commutators conceptually delicate [2].

The purpose of this paper is twofold. First, to give a precise account of what Wick ordering does and does not accomplish in QED. The central point is that in four-dimensional QED the ultraviolet divergences that require renormalization arise in loop corrections to the electron self-energy, photon vacuum polarization, and the vertex function. These divergences are not removed by normal ordering, which primarily subtracts vacuum expectation values in a chosen Fock representation. Second, we show that in gauge-invariant entire-function UV completions, ultraviolet finiteness and quasi-locality are obtained by construction. In such theories the need for Wick ordering as a divergence-cleanup mechanism disappears as loops are exponentially damped after Wick rotation, and composite operators are replaced by regulated quasi-local observables defined via an entire function of the covariant Laplace–Beltrami operator [3–5].

A key historical motivation for understanding the limitations of formal operator manipulations in QED appears in the analysis of vacuum expectation values of Heisenberg-picture currents. In particular, Moffat developed a method of regularizing vacuum expectation values so as to maintain gauge invariance constraints in current commutators [6]. That work illustrates that gauge invariance is not automatically preserved by naive operator rearrangements, including normal ordering, when contact terms and light-cone singularities are present.

We work throughout in $3 + 1$ dimensions with Minkowski metric $\eta_{\mu\nu} = \text{diag}(+1, -1, -1, -1)$ and set $\hbar = c = 1$. Greek indices μ, ν, ρ, σ run over $0, 1, 2, 3$, and repeated indices are summed. For a

four-vector $p = (p^0, \mathbf{p})$ we write $p^2 = \eta_{\mu\nu} p^\mu p^\nu = (p^0)^2 - \mathbf{p}^2$. Euclidean momenta are denoted by p_E with $p_E^2 = \sum_{j=1}^4 (p_E^j)^2$.

2. Wick Ordering

Let $\Phi(x)$ be a free scalar field and $\psi(x)$ a free Dirac field on Minkowski space. In the standard Fock representation one writes:

$$\Phi(x) = \Phi^{(+)}(x) + \Phi^{(-)}(x), \quad (1)$$

where $\Phi^{(+)}$ contains annihilation operators and $\Phi^{(-)}$ contains creation operators. Similarly, $\psi(x)$ and $\bar{\psi}(x) = \psi^\dagger(x)\gamma^0$ admit decompositions into positive- and negative-frequency parts. The vacuum vector is denoted by $|0\rangle$ and is annihilated by all annihilation operators.

Definition 1 (Wick ordering). *Given a polynomial $\mathcal{P}$ in creation and annihilation operators, its normal-ordered version $:\mathcal{P}:$ is obtained by moving all creation operators to the left of all annihilation operators, using canonical (anti)commutation relations only to effect the reordering. For fermionic operators, each swap introduces the usual sign.*

Normal ordering depends on the choice of decomposition into creation and annihilation operators and therefore depends on the chosen state, the vacuum and representation. Normal ordering is characterized by the vacuum expectation property:

$$\langle 0 | : \mathcal{P} : | 0 \rangle = 0, \quad (2)$$

for any polynomial $\mathcal{P}$ that is a finite sum of normally ordered monomials with at least one creation or annihilation operator. In particular, for a free scalar field has the identity:

$$:\Phi^2(x): = \Phi^2(x) - \langle 0 | \Phi^2(x) | 0 \rangle, \quad (3)$$

interpreted as an equality of operator-valued distributions after smearing with test functions.

A central utility of normal ordering is Wick's theorem, which organizes time-ordered products into normal-ordered products plus contractions. For a free bosonic field:

$$T\{\Phi(x)\Phi(y)\} = :\Phi(x)\Phi(y): + \overline{\Phi(x)\Phi(y)}, \quad (4)$$

where the contraction is the Feynman propagator:

$$\overline{\Phi(x)\Phi(y)} = \langle 0 | T\{\Phi(x)\Phi(y)\} | 0 \rangle \equiv i\Delta_F(x-y). \quad (5)$$

For fermions and gauge fields the statement is analogous, with appropriate sign rules and tensor structure [7,8].

3. Why Wick Ordering Does Not Renormalize QED

The QED Lagrangian density is:

$$\mathcal{L}_{\text{QED}} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\psi}(i\gamma^\mu D_\mu - m)\psi, \quad (6)$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, $D_\mu = \partial_\mu + ieA_\mu$, e is the electromagnetic coupling, and m is the electron mass. The interaction density is:

$$\mathcal{L}_{\text{int}} = -e\bar{\psi}\gamma^\mu\psi A_\mu. \quad (7)$$

In four dimensions the ultraviolet divergences that are physically relevant arise from loop integrals. The primitive divergent one-particle-irreducible graphs are the electron self-energy $\Sigma(p)$, the vacuum

polarization tensor $\Pi_{\mu\nu}(q)$, and the vertex function $\Gamma^\mu(p, p')$. Renormalization introduces counterterms:

$$\mathcal{L}_{\text{ct}} = \delta_Z \bar{\psi} i \not{\partial} \psi - \delta_m m \bar{\psi} \psi - \frac{\delta_3}{4} F_{\mu\nu} F^{\mu\nu} + \delta_1 e \bar{\psi} \gamma^\mu \psi A_\mu, \quad (8)$$

together with a gauge-fixing term. Ward identities enforce relations among these parameters, such as $\delta_1 = \delta_Z$ in standard schemes [8].

Normal ordering of $\mathcal{L}_{\text{int}}$ does not remove the ultraviolet divergences in Σ , $\Pi_{\mu\nu}$, or Γ^μ , because those divergences arise from the short-distance structure of time-ordered products and loop momentum integrals, not from vacuum expectation values of monomials in the free vacuum. Normal ordering can remove certain vacuum bubble contributions in canonical computations, but vacuum bubbles are removed more fundamentally by the normalized generating functional definition, as discussed below.

In the path integral formulation one defines the generating functional:

$$Z[J] = \frac{\int \mathcal{D}\Phi \exp(iS[\Phi] + i \int d^4x J(x)\Phi(x))}{\int \mathcal{D}\Phi \exp(iS[\Phi])}. \quad (9)$$

The denominator removes vacuum bubble diagrams by construction. This eliminates the need to impose normal ordering for the sole purpose of eliminating disconnected vacuum contributions. In local QED, the ultraviolet divergences in connected correlators remain and require renormalization.

4. Composite Operators, Current Commutators, and Gauge Invariance Pathologies

The conserved Noether current is:

$$j^\mu(x) = \bar{\psi}(x) \gamma^\mu \psi(x), \quad (10)$$

$$\partial_\mu j^\mu(x) = 0, \quad (11)$$

as an operator identity in the classical theory and as a renormalized operator statement in the quantum theory. The product of fields at the same point, such as $\bar{\psi}(x) \gamma^\mu \psi(x)$, is not a well-defined operator without renormalization. Normal ordering $:j^\mu(x):$ is a prescription relative to a chosen vacuum, but it does not in itself guarantee a gauge-invariant definition of the composite operator in an interacting theory.

Moffat analyzed the gauge invariance constraints arising in vacuum expectation values of Heisenberg-picture currents and showed that naive manipulations lead to paradoxes associated with contact terms and light-cone singularities [6]. In particular, enforcing transversality in vacuum polarization can be reduced to a condition on equal-time commutators of currents. The schematic structure of the problematic term takes the form:

$$\langle 0 | [j^0(x'), j^k(x)] | 0 \rangle \propto \partial_k \delta^{(3)}(\mathbf{x}' - \mathbf{x}) \int_0^\infty \rho(-a) da, \quad (12)$$

where ρ is a spectral density arising from a Källén–Lehmann representation [10]. If ρ is strictly nonnegative and not identically zero, the integral is nonvanishing and produces a contact term incompatible with the desired gauge invariance constraint unless additional regularization conditions are imposed [6].

This example exhibits a general principle, that gauge invariance and current algebra constraints depend sensitively on the short-distance definition of composite operators, and these issues are not resolved by normal ordering alone.

5. Gauge-Invariant Entire-Function UV Completions

We now introduce the class of nonlocal UV-complete theories we consider. The central ingredient is an entire function F of the covariant d'Alembertian, the Laplace–Beltrami operator. On a curved background with metric $g_{\mu\nu}$, define:

$$\square_g \equiv g^{\mu\nu} \nabla_\mu \nabla_\nu, \quad (13)$$

where ∇_μ is the Levi–Civita covariant derivative for $g_{\mu\nu}$. On flat space, $\square = \eta^{\mu\nu} \partial_\mu \partial_\nu$. Let $M_* > 0$ denote the intrinsic UV scale of nonlocality.

Definition 2 (Entire-function form factor). *An entire-function form factor is a function $F : \mathbb{C} \rightarrow \mathbb{C}$ that is entire and satisfies $F(0) = 1$. We assume in addition that F has no zeros on the real axis and admits Euclidean damping, so there exist constants $c > 0$ and $R > 0$ such that for all $p_E \in \mathbb{R}^4$ with $p_E^2 \geq R$,*

$$\left| F\left(\frac{p_E^2}{M_*^2}\right) \right| \leq \exp\left(-c \frac{p_E^2}{M_*^2}\right). \quad (14)$$

The gauge-invariant covariant implementation uses the background-field method. In that setting, the regulator is introduced as an entire function of the covariant operator acting on fluctuations, preserving background gauge invariance and the associated Ward or Slavnov–Taylor identities [3].

Expanding around a trivial flat background, plane waves diagonalize the d'Alembertian, so in momentum space the form factor becomes multiplicative [3]. For a scalar field, the regulated propagator takes the schematic form:

$$\Delta_F(p) = \frac{i F(-p^2/M_*^2)}{p^2 - m^2 + i\epsilon}. \quad (15)$$

After Wick rotation $p^0 \mapsto ip_E^4$, one obtains Euclidean damping in loop integrals. For fermions and gauge fields the statement is analogous, with the same ultraviolet damping factor multiplying the standard free propagators in an appropriate gauge-covariant construction.

6. UV Finiteness as a Theorem

We formalize the ultraviolet finiteness mechanism through Theorem [1]

Theorem 1 (UV finiteness from entire-function damping). *Consider a perturbative expansion in which each internal propagator carries a multiplicative entire-function factor $F(p^2/M_*^2)$ satisfying the Euclidean damping condition (14). Assume the vertices are polynomial in momenta and that the Euclidean integrand for an L -loop amplitude is a rational function of loop momenta times the product of such damping factors. Then every fixed-order Euclidean loop integral is absolutely convergent.*

Let $k = (k_1, \dots, k_L)$ denote the set of Euclidean loop momenta $k_\ell \in \mathbb{R}^4$. The integrand has the form:

$$\mathcal{I}(k) = \frac{P(k)}{Q(k)} \prod_{r=1}^I F\left(\frac{\ell_r(k)^2}{M_*^2}\right), \quad (16)$$

where P and Q are polynomials, I is the number of internal lines, and each $\ell_r(k)$ is an affine linear combination of loop momenta and external momenta corresponding to the momentum flowing through line r .

For large $|k|$ the rational prefactor satisfies a bound of the form:

$$\left| \frac{P(k)}{Q(k)} \right| \leq C(1 + |k|)^N \quad (17)$$

for some constants $C > 0$ and integer $N \geq 0$, where $|k|^2 = \sum_{\ell=1}^L k_\ell^2$. By the damping assumption (14), for sufficiently large $|k|$ each $F(\ell_r(k)^2/M_*^2)$ is bounded by an exponential:

$$\left| F\left(\frac{\ell_r(k)^2}{M_*^2}\right) \right| \leq \exp\left(-c \frac{\ell_r(k)^2}{M_*^2}\right). \quad (18)$$

Since the set of affine maps ℓ_r spans the loop momenta in any connected diagram, there exists $c' > 0$ such that:

$$\sum_{r=1}^I \ell_r(k)^2 \geq c' |k|^2 - C' \quad (19)$$

for some constant $C' \geq 0$. Therefore, for $|k|$ sufficiently large:

$$|\mathcal{I}(k)| \leq C'' (1 + |k|)^N \exp\left(-c'' \frac{|k|^2}{M_*^2}\right) \quad (20)$$

with constants $C'', c'' > 0$. The right-hand side is integrable over $\mathbb{R}^{4L}$, proving absolute convergence.

This theorem makes precise the statement that in entire-function UV completions ultraviolet divergences do not occur order-by-order in perturbation theory, because exponential damping dominates any polynomial growth from vertices and numerator algebra. This is the primary sense in which Wick ordering is rendered unnecessary as an ultraviolet subtraction device.

7. Quasi-Local Observables and the Replacement of Pointlike Composites

The same entire-function structure that yields UV damping also gives a canonical replacement for pointlike composite operators. For a local operator-valued distribution $O(x)$ define the regulated observable:

$$O^{(F)}(x) = F(\square/M_*^2) O(x). \quad (21)$$

On flat space, $F(\square/M_*^2)$ may be represented by an integral kernel K_F :

$$O^{(F)}(x) = \int d^4y K_F(x-y) O(y), \quad (22)$$

where K_F is smooth and rapidly decaying when F satisfies Euclidean damping and is of appropriate exponential type. The precise functional-analytic conditions are discussed in the general development of asymptotic microcausality [4].

Given a test function $f \in \mathcal{S}(\mathbb{R}^4)$, define the smeared operator:

$$O^{(F)}(f) = \int d^4x f(x) O^{(F)}(x) \quad (23)$$

$$= \int d^4x d^4y f(x) K_F(x-y) O(y). \quad (24)$$

In this framework, physically accessible observables are elements of a quasi-local algebra generated by such regulated smearings.

Let $f, g \in \mathcal{S}(\mathbb{R}^4)$ have spacelike separated supports with minimal invariant separation $\rho > 0$ between $\text{supp } f$ and $\text{supp } g$. The asymptotic microcausality theorem proved in [4] implies an operator norm bound on commutators of regulated observables.

Theorem 2 (Asymptotic microcausality bound [4]). *Let O_1, O_2 be local observables in a causal Wightman QFT and let $O_1^{(F)}(f), O_2^{(F)}(g)$ be their regulated smearings as above with an entire-function regulator F satisfying the hypotheses of [4]. Then for every integer $N \geq 0$ there exist constants $C_N > 0$ and $\alpha > 0$ such that:*

$$\left\| [O_1^{(F)}(f), O_2^{(F)}(g)] \right\| \leq C_N (1 + M_*\rho)^{-N} \exp(-\alpha M_*\rho). \quad (25)$$

In local QED, one confronts distributional contact terms in products and commutators at coincident or lightlike separated points. The bound (25) shows that in the regulated theory these singular structures are replaced by quasi-local commutators that are exponentially suppressed at spacelike separation. This provides a structural replacement for the role normal ordering is sometimes expected to play in taming operator-product singularities, while respecting the gauge-covariant implementation of the regulator.

8. The Status of Wick Ordering

The analysis above implies two principal eliminations of necessity. First, ultraviolet divergences in loop integrals are absent order-by-order due to Theorem 1. Thus Wick ordering is not needed to render perturbation theory finite. Second, the ill-defined nature of pointlike composites is replaced by the regulated observables $O^{(F)}$, whose products and commutators are controlled by smooth kernels and by the bound (25). In this sense, the regulated theory replaces the question of defining $:O(x)^2:$ by the question of defining $O^{(F)}(x)O^{(F)}(x)$, which is well-defined as an operator after smearing and remains quasi-local.

Although the ultraviolet role of Wick ordering is eliminated, normal ordering remains available as a finite convention relative to a chosen state. Given any quasifree state ω with two-point function $C_\omega(x, y)$, one can define Wick ordering with respect to ω by subtracting contractions determined by C_ω . This is a finite prescription in a UV-complete theory, and it may be useful for selecting a vacuum energy normalization in a given representation.

In a gravity-coupled theory, the absolute vacuum energy is not a purely conventional quantity as it is tied to the effective cosmological constant sector. UV finiteness ensures that vacuum contributions can be finite, but it does not uniquely determine their physical value. A matching condition is still required.

QED possesses infrared phenomena associated with massless gauge bosons and charged sectors, including soft-photon clouds and the infraparticle nature of charged states. These are infrared issues, not ultraviolet divergences, and they are not resolved by Wick ordering. Entire-function regulators are ultraviolet completions and do not in themselves eliminate infrared dressing requirements.

9. Connection to Operational Locality

In UV-complete nonlocal theories, the physically meaningful observables belong to quasi-local algebras generated by regulated smearings with intrinsic resolution scale $\ell_* \sim M_*^{-1}$. This perspective unifies operational notions of localization and clarifies the sense in which sharp projectors associated with strict pointlike localization are not fundamental observables. This operational viewpoint has been developed in the context of particle localization, reconciling Newton–Wigner and Foldy–Wouthuysen localization in regulated theories [5]. In the present context, the same quasi-local structure removes the temptation to interpret normal ordering as a fundamental cure for singular operator products: the regulated algebra is the correct arena in which locality and gauge invariance constraints are to be formulated.

10. Conclusions

We have given a rigorous account of Wick ordering in QED and explained why it does not constitute the renormalization mechanism responsible for ultraviolet consistency in four dimensions. We emphasized historical and structural pathologies for composite currents and their commutators, illustrating that gauge invariance constraints may fail under naive operator manipulations and require consistent regularization. We then showed that in gauge-invariant entire-function UV completions, loop integrals are absolutely convergent and observables are naturally quasi-local through covariant smearing. In this setting Wick ordering is no longer needed as a divergence subtraction tool. It remains only as an optional, representation-dependent finite convention relative to a chosen state.

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