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Article

The Halo Effect and Quantum Vortices. Not So Dark with Alena Tensor

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Abstract

Alena Tensor is a recently discovered class of energy-momentum tensors that proposes a general equivalence of the curved path and geodesic for analyzed spacetimes which allows the analysis of physical systems in curvilinear (GR), classical and quantum descriptions. This paper demonstrates that extending the existing dust description to a form that provides a full matter energy-momentum tensor in GR, naturally leads to the development of a halo effect for continuum media. This result provides a good approximation of the galaxy rotation curve for approximately 100 analyzed objects from the SPARC catalog and allows for further adjustments dependent on anisotropy and energy flux. The same equations in flat spacetime allow for the inclusion of rotation-related effects in the quantum description, model quantum vortices and reproduce Mashhoon effect. This provides a physical interpretation of mass generation as an emergent property of the phase-spin equilibrium and enables a reconstruction of the Yukawa and Higgs mechanisms as consequence of the stability conditions of quantum vortices.

Keywords: unification; alena tensor; dark energy; dark matter

1. Introduction

Research on the dark sector has been ongoing for many years [1] and there is still no theoretical consensus [2] or convincing experimental evidence regarding its nature [3]. We have not found dark matter signals for the WIMP models [4], axions/ALPs [5], SIDM [6], despite new experimental approaches [7], experiments in underground detectors [3], LZ experiments [4], XENON-nT [8] or SuperCDMS [9], and the latest observational data (e.g., "Hubble tension", "Sigma-8 tension") make the issue of the dark sector even more puzzling [10]. The dominant theoretical model is still Λ CDM with dark matter haloes [11], although alternative theories such as MOND/relativistic generalizations [12,13], dark photons [14], TeVeS/related constructions [12], f(R) [15], black holes [16], or the recently popular "emergent/entropic gravity" [17] have achieved some success at selected scales. However, all these approaches experience ups and downs depending on the subsequent observational data [4,8,10,18]. There is also considered the possibility of hybrid models (e.g. a superfluid DM combining features of MOND and DM) [19], or even incorrect/weakened estimates of baryon masses and systematic errors in the M/L estimate [20], and the interaction of dark matter and energy [21] but so far this entire massive effort by the scientific community has not yielded a definitive conclusion. We have more certainty about dark energy because the universe is definitely expanding, and there are ample, multiple, independent confirmations, eg. from SN Ia, BAO, and CMB [11,22]. It is still uncertain whether the cosmological constant is indeed constant, although recent DESI (BAO) analyses indicate the possibility of $w(z)$ dynamics [23], and the Euclid mission is rapidly delivering high-quality weak lensing maps and surveys that will be crucial for measuring the dark energy equation of state [24].

The Alena Tensor is a relatively young field of research and has not had much relevance to the dark sector until now. Previous work has focused on developing a dual description for physical systems with matter and fields in which the metric tensor is not a feature of spacetime but only a method of describing it. Its aim was to provide a smooth transition between curvilinear description consistent with GR [25] and a flat (classical and quantum [26]) description for simple cases with dust

[27], which was analyzed mainly for physical systems with electromagnetic fields [28]. This paper will demonstrate that extending the Alena Tensor to the general case yields results, that provide a representation of the spherical dark matter halo and, naturally, provide an explanation for the cosmological constant. The obtained picture of GR equations leads to a potential explanation of the dark matter phenomenon based on the energy associated with the rotation itself and for about 100 diverse galaxies from the SPARC database [29] gives a fairly good approximation of the observed rotation curves.

In the first part of the paper, an introduction to the Alena Tensor model for dust will be presented, then this approach will be extended to the general form of the matter energy-momentum tensor, leading to the "dark matter halo effect". The quantum description resulting from the proposed approach will also be analyzed and tested to generate elementary particles. The results will be analyzed and discussed and shown to lead to the conclusions described in the abstract.

2. The Extended Alena Tensor approach

In this chapter, the conclusions reached so far regarding the Alena Tensor will be recalled, the notation will be introduced, and the reasoning from the previous articles will be generalized to all gauge fields and all forces acting in the physical system. The obtained results will be then used to formulate conclusions regarding the dark matter and compared with observational data. The author uses the Einstein summation convention, metric signature $(+, -, -, -)$ and commonly used notations.

2.1. Transforming Curved Path into Geodesic for Dust

As a first step, one may generalize the solution proposed in [27] in such a way that the forces resulting from all gauge fields are related to the metric tensor of curved spacetime.

One may begin the reasoning by introducing tensor $F^{\mu\nu\alpha\beta}$ defined in terms of the gauge field tensors $\mathbb{F}_A^{\mu\nu}$ for each gauge group A, and the stress-energy tensor $Y^{\mu\nu}$ for such generalized field

$$F^{\mu\nu\alpha\beta} \equiv \sum_A \mathbb{F}_A^{\mu\nu} \otimes \mathbb{F}_A^{\alpha\beta}; \quad Y^{\mu\nu} \equiv F^{\mu\alpha\nu\beta} g_{\alpha\beta} - g^{\mu\nu} \frac{1}{4} F^{\alpha\gamma\beta\delta} g_{\alpha\beta} g_{\gamma\delta} \quad (1)$$

and denote the invariant of this field as p_Λ . Following reasoning from [27] let this field invariant be defined dually as follows

$$p_\Lambda \equiv \frac{1}{4} F^{\alpha\gamma\beta\delta} g_{\alpha\beta} g_{\gamma\delta} \equiv p_o \mathbb{k}^2 \quad ; \quad \mathbb{k} \equiv \mathbb{k}_{\mu\nu} g^{\mu\nu} \quad (2)$$

where p_o is certain constant (or simply invariant, independent of the metric) and where $\mathbb{k}^{\mu\nu}$, as raised in [26], in this approach is a metric tensor describing a curved spacetime in which all motion occurs along geodesics. By making variation on $-p_\Lambda$ with respect to metric $g_{\mu\nu}$ (Hilbert's method) one obtains the energy-momentum tensor of the field from (1) expressed dually as

$$Y^{\mu\nu} = p_\Lambda \left(\frac{4}{\mathbb{k}} \mathbb{k}^{\mu\nu} - g^{\mu\nu} \right) \quad (3)$$

Such approach, exactly as shown in [27], establishes a relationship between the field and the metric tensor $\mathbb{k}^{\mu\nu}$ and in the spacetime considered as described by the metric tensor $g^{\mu\nu} \rightarrow \mathbb{k}^{\mu\nu}$, one obtains $\mathbb{k} = 4$ what yields that in curvilinear description of the system, energy-momentum tensor of the field $Y^{\mu\nu}$ vanishes, maintaining continuity of function. As shown in previous articles, this causes the presence of a field in curved spacetime to manifest itself solely through curvature, which replaces the four-force densities in flat spacetime.

In flat spacetime, one may assume that the equations of motion for the gauge fields A are satisfied, thus one obtains gauge four-currents $J_A^\nu \equiv D_\mu \mathbb{F}_A^{\mu\nu}$. Therefore, the total density of the Yang-Mills four-forces [30,31] f_{YM}^ν is

$$f_{YM}^\nu = \partial_\mu Y^{\mu\nu} = \sum_A J_A^\alpha \mathbb{F}_{A\alpha}^\nu \quad (4)$$

where the self-interactions of gauge fields in a non-Abelian theory are reducing. Then, following the reasoning presented in [27] one may define coefficient $\chi_m \equiv \frac{\varrho c^2}{p_\Lambda}$ where ϱ represents matter density and is associated with the translational current $\chi_m U^\mu$. In the Alena Tensor approach, the existence of matter is thus a manifestation of the existence of fields what ensures, that without fields matter does not exist.

This allows to define the Lagrangian $\mathcal{L}_{dust}$ and obtain from it the stress-energy tensor (Alena Tensor) for the system with dust $T_{dust}^{\mu\nu}$ by variation on the metric

$$\mathcal{L}_{dust} \equiv p_\Lambda (1 - \chi_m) \rightarrow T_{dust}^{\mu\nu} \equiv \varrho U^\mu U^\nu - (1 - \chi_m) Y^{\mu\nu} \quad (5)$$

In accordance with [28], assuming ϱ_0 as rest mass density, four-momentum density is defined as $\varrho U^\alpha \equiv \varrho_0 \gamma U^\alpha$ what takes into account motion and Lorentz contraction of the volume. Total translational four-force density acting on matter is therefore defined as

$$f^\nu \equiv \partial_\mu \varrho U^\mu U^\nu = \varrho U^\mu \partial_\mu U^\nu = \varrho \frac{dU^\nu}{d\tau}; \quad \partial_\mu \varrho U^\mu = 0 \quad (6)$$

As shown in [26], the above amendment introduces a natural property concerning curved spacetime, assuming that for dust, geodetic motion is expected

$$\partial_\alpha \varrho U^\alpha = 0 \rightarrow U^\alpha_{;\alpha} = -\frac{d\gamma}{dt} \rightarrow U^\alpha_{;\alpha} = 0 \quad (7)$$

$$U^\alpha U^\beta_{;\alpha} = 0 \rightarrow \frac{D U^\beta}{D \tau} = 0; \quad \left(\varrho U^\alpha U^\beta \right)_{;\alpha} = 0 \quad (8)$$

Now it can be noticed, that in flat spacetime the four-divergence of the above tensor $T_{dust}^{\mu\nu}$ can be interpreted as the density of the four-forces acting on matter f^ν reduced by the density of the field-related four-forces

$$f_{field}^\nu \equiv \partial_\mu [(1 - \chi_m) Y^{\mu\nu}] = (1 - \chi_m) f_{YM}^\nu + f_{gr}^\nu; \quad f_{gr}^\nu \equiv Y^{\mu\nu} \partial_\mu (1 - \chi_m) = -Y^{\mu\nu} \partial_\mu \chi_m \quad (9)$$

As shown in [26], f_{gr}^ν can be associated with the existence of gravity in the system, while $f_{rr}^\nu \equiv \chi_m f_{YM}^\nu$ behaves as a radiation-reaction force, reducing the value of forces due to the field and upholding the conservation of energy, ensuring that the increasing energy density associated with matter ϱc^2 does not exceed the total energy density p_Λ available in the system

$$\partial_\mu T_{dust}^{\mu\nu} = f^\nu - f_{field}^\nu = f^\nu - f_{YM}^\nu - f_{gr}^\nu + f_{rr}^\nu \quad (10)$$

where $\lim_{\chi_m \rightarrow 1} f_{field}^\nu = 0$. Presented approach also indicates the ansatz for the Kerr-Schild type metrics for curved spacetime

$$\mathbb{k}^{\mu\nu} \equiv \sum_i c_i l_i^\mu l_i^\nu + \frac{\mathbb{k}}{4} \eta^{\mu\nu}; \quad 0 = \eta_{\mu\nu} l_i^\mu l_i^\nu \quad (11)$$

where l_i^μ are null vectors and c_i are related coefficients.

In the next section, the above model will be expanded to include rotation-related components, which will prove crucial for describing dark sector phenomena and allow to obtain a description that agrees with the observational results.

2.2. Rotational Energy

It can be noticed that the radiation reaction force should take into account the total energy associated with the body, so in addition to the energy associated with the translational motion, it seems necessary to take into account the rotational energy.

One may thus introduce a projector $\Delta^{\mu\nu}$, flow vorticity tensor $\omega^{\mu\nu}$, positive coefficient χ_ω equal to the rotational energy $\mathcal{E}_{rot}$ up to p_Λ and some metric independent auxiliary α with the dimension of the square of time

$$\Delta^\mu{}_\nu \equiv g^\mu{}_\nu - \frac{1}{c^2} U^\mu U_\nu; \quad \omega_{\mu\nu} \equiv \Delta_\mu{}^\alpha \Delta_\nu{}^\beta \nabla_{[\alpha} U_{\beta]}; \quad \chi_\omega \equiv \frac{\alpha}{2} \omega^{\mu\nu} \omega_{\mu\nu}; \quad \mathcal{E}_{rot} \equiv p_\Lambda \chi_\omega \quad (12)$$

Defining Lagrangian density $\mathcal{L}_T$ for the whole system and tensor $S^{\lambda\mu\nu}$ to describe emerging boundary terms

$$\mathcal{L}_T = p_\Lambda (1 - \chi_\omega - \chi_m); \quad S^{\lambda\mu\nu} \equiv p_\Lambda \alpha \left(U^\lambda \omega^{\mu\nu} - U^{(\mu} \omega^{\nu)\lambda} \right) \quad (13)$$

and introducing $\Xi^{\mu\nu}$ tensor related to rotational properties of the system

$$-\Xi^{\mu\nu} \equiv \alpha p_\Lambda \left(\omega^{\mu\gamma} \omega^\nu{}_\gamma - \frac{1}{2} \Delta^{\mu\nu} \omega^{\alpha\beta} \omega_{\alpha\beta} \right) + \nabla_\lambda S^{\lambda(\mu\nu)} \quad (14)$$

one obtains Alena Tensor $T^{\mu\nu}$ for the system derived with help of variational method on $\mathcal{L}_T$ in the form

$$T^{\mu\nu} = \varrho U^\mu U^\nu - \Xi^{\mu\nu} - (1 - \chi_\omega - \chi_m) Y^{\mu\nu}; \quad T^{\mu\nu} g_{\mu\nu} = \varrho c^2 - \mathcal{E}_{rot} \quad (15)$$

Considering description in curved spacetime, described by the metric tensor $\mathbb{K}^{\mu\nu}$, the field tensor $Y^{\mu\nu}$ vanishes, the system tensor reduces to the form $T_{matt}^{\mu\nu} \equiv \varrho U^\mu U^\nu - \Xi^{\mu\nu}$ and its vanishing four-divergence means that any deviations from the geodesic motion with $a^\mu \equiv U^\nu \nabla_\nu U^\mu$ are compensated by rotation related forces. Using the standard kinematic decomposition one may calculate

$$\varepsilon \equiv \frac{1}{c^2} T_{matt}^{\mu\nu} U_\mu U_\nu = \varrho c^2 + 2\mathcal{E}_{rot}; \quad T_{matt}^{\mu\nu} U_\nu = \left(\varrho c^2 + 2\mathcal{E}_{rot} \right) U^\mu - \alpha p_\Lambda \left(a_\nu \omega^{\nu\mu} + \frac{c^2}{2} \Delta^\mu{}_\nu \nabla_\lambda \omega^{\lambda\nu} \right) \quad (16)$$

where the element in brackets in last equation represents in fact the purely spatial vorticity divergence. Assuming the classical definition of the energy flux q^α one also gets

$$q^\alpha \equiv \frac{1}{c^2} \Delta^\alpha{}_\mu T_{matt}^{\mu\nu} U_\nu = -\alpha p_\Lambda \left(a_\nu \omega^{\nu\mu} + \frac{c^2}{2} \Delta^\mu{}_\nu \nabla_\lambda \omega^{\lambda\nu} \right) \quad (17)$$

Introducing classical shear tensor $\sigma^{\mu\nu}$ and effective vortex stress tensor $\tau^{\mu\nu}$ as

$$\sigma^{\mu\nu} \equiv \Delta^{\mu\alpha} \Delta^{\nu\beta} \nabla_{(\alpha} U_{\beta)} - \frac{1}{3} \nabla_\alpha U^\alpha \Delta^{\mu\nu}; \quad \tau^{\mu\nu} \equiv \frac{p_\Lambda \alpha}{2} \left(\sigma^\mu{}_\lambda \omega^{\nu\lambda} + \sigma^\nu{}_\lambda \omega^{\mu\lambda} \right) \quad (18)$$

one may thus rewrite $T_{matt}^{\mu\nu}$ in curved spacetime as

$$T_{matt}^{\mu\nu} = \frac{\varepsilon}{c^2} U^\mu U^\nu + \frac{1}{c^2} (U^\mu q^\nu + U^\nu q^\mu) - \mathcal{E}_{rot} \Delta^{\mu\nu} - \tau^{\mu\nu} \quad (19)$$

One may notice, that the system has a built-in anisotropic stress described by $\tau^{\mu\nu}$, but its source is not viscosity, but the coupling between shear and vorticity (between flow deformation and local spin angular momentum).

Considered in flat spacetime such approach introduces additional four-force density f_{Ξ}^{ν} acting on matter and also changes f_{gr}^{ν} and the radiation reaction f_{rr}^{ν} to the form

$$f_{\Xi}^{\nu} \equiv \partial_{\mu} \Xi^{\mu\nu}; \quad f_{rr}^{\nu} = (\chi_m + \chi_{\omega}) f_{YM}^{\nu}; \quad f_{gr}^{\nu} = Y^{\mu\nu} \partial_{\mu} (1 - \chi_{\omega} - \chi_m) \quad (20)$$

One may now consider the impact of the above expansion of the Alena Tensor on quantum equations and GR equations.

2.2.1. Noether Tensor and Quantum Interpretation

Introducing for simplicity $\xi \equiv 1 - \chi_{\omega} - \chi_m$ one may in flat spacetime define Belinfante superpotential $Q^{\lambda\mu\nu}$ referring to (13) and improvement $I^{\mu\nu}$ resulting from the variable ξ as

$$Q^{\lambda\mu\nu} \equiv \frac{1}{2} (S^{\mu\nu\lambda} + S^{\nu\mu\lambda} - S^{\lambda\mu\nu}) \quad ; \quad I^{\mu\nu} \equiv \partial_{\lambda} Q^{\lambda\mu\nu} + \eta^{\mu\nu} (\partial_{\lambda} \ln \xi) C^{\lambda} - (\partial^{\mu} \ln \xi) C^{\nu} \quad (21)$$

where based on [32] approach, $I^{\mu\nu}$ takes into account virial currents, where

$$\mathbb{C}^{\mu} \equiv \Pi^{\mu\alpha} U_{\alpha} - \sum_A \mathbb{F}_A^{\mu\lambda} \mathbb{A}_{\lambda A} \quad ; \quad \Pi^{\mu\alpha} \equiv \frac{\partial \mathcal{E}_{rot}}{\partial (\partial_{\mu} U_{\alpha})} = \alpha p_{\Lambda} \Delta^{\alpha}_{\rho} \Delta_{\beta}^{\mu} \omega^{\rho\beta} \quad (22)$$

As will be shown, the combination $\mathbb{C}_{\mu}/\xi$ defines the total Noether current associated with the underlying symmetry. It remains gauge invariant and includes both matter and field contributions, ensuring a consistent formulation of the continuity equation.

It may be noticed, that $\partial_{\mu} I^{\mu\nu} = 0$ becomes zero on-shell, using the equations of motion and smooth boundary conditions. Since $Q^{\lambda\mu\nu}$ is antisymmetric in $\partial_{\mu} \partial_{\lambda}$ and partial derivatives commute in flat spacetime, therefore $\partial_{\mu} \partial_{\lambda} Q^{\lambda\mu\nu} = 0$. One may thus simplify $\partial_{\mu} I^{\mu\nu}$ to the form of

$$\partial_{\mu} I^{\mu\nu} = (\partial^{\nu} \ln \xi) \partial_{\mu} \left(\frac{\mathbb{C}^{\mu}}{\xi} \right) + (\partial_{\mu} \ln \xi) \left(\partial^{\mu} \left(\frac{\mathbb{C}^{\nu}}{\xi} \right) - \partial^{\nu} \left(\frac{\mathbb{C}^{\mu}}{\xi} \right) \right) \quad (23)$$

The equations of motion for U_{μ} and gauge fields $\mathbb{A}_{\mu A}$ derived from the Lagrangian $\mathcal{L}_T = \xi p_{\Lambda}$ imply the following two relations

$$\begin{aligned} \partial_{\mu} \left(\frac{\mathbb{C}^{\mu}}{\xi} \right) &= 0 \quad ; \quad \partial^{\nu} \mathbb{C}^{\lambda} - \partial^{\lambda} \mathbb{C}^{\nu} = (\partial^{\nu} \ln \xi) \mathbb{C}^{\lambda} - (\partial^{\lambda} \ln \xi) \mathbb{C}^{\nu} \\ &\rightarrow \partial^{\nu} \left(\frac{\mathbb{C}^{\lambda}}{\xi} \right) - \partial^{\lambda} \left(\frac{\mathbb{C}^{\nu}}{\xi} \right) = 0 \rightarrow \frac{\mathbb{C}^{\mu}}{\xi} \equiv \partial^{\mu} \frac{\Phi}{\kappa} \end{aligned} \quad (24)$$

The first relation is the Euler equation for the component U_{μ} (with $\Pi^{\mu\alpha}$ as the momentum conjugate to $\partial_{\mu} U_{\alpha}$) after taking into account the fact that $\mathcal{L}_T$ has variable ξ . The second relation follows from the gauge field equations $\partial_{\mu} (\xi \mathbb{F}_A^{\mu\nu}) = \mathbb{J}_A^{\nu}$ and from the definition of the "virial" part $\sum_A \mathbb{F}_A^{\mu\lambda} \mathbb{A}_{\lambda A}$ in $\mathbb{C}^{\mu}$, which ensures the proper gauge covariance.

The condition $\partial^{[\nu} (\mathbb{C}^{\lambda]}/\xi) = 0$ guarantees the local existence of a scalar potential Φ such that $\frac{\mathbb{C}^{\mu}}{\xi} = \partial^{\mu} \frac{\Phi}{\kappa}$, where κ was introduced to maintain dimensional consistency. In spaces with non-trivial topology this potential can be multi-valued, which corresponds to quantized vortices and circulations. In this representation $\mathbb{C}^{\mu}/\xi$ plays the role of an ordinary Noether current of Φ , since the above relations follow from the Euler-Lagrange equations for U_{μ} and $\mathbb{A}_{\mu A}$ derived from $\mathcal{L}_T$.

One may now introduce Noether Lagrangian density $\mathcal{L}_N$ as

$$\mathcal{L}_N[U, \mathbb{A}, \Phi] \equiv \mathcal{L}_T[U, \mathbb{A}] + \frac{\mathbb{C}^{\mu}}{\xi} [U, \mathbb{A}] \partial_{\mu} \Phi \quad \rightarrow \quad \Pi_{\Phi}^{\mu} = \frac{\partial \mathcal{L}_N}{\partial (\partial_{\mu} \Phi)} = \frac{\mathbb{C}^{\mu}}{\xi} \quad (25)$$

The field Φ is a local Noether phase (not a gauge phase) associated with the conserved total current $\mathbb{C}_{\mu}/\xi$. It introduces no new degrees of freedom but parametrizes the underlying $U(1)$ symmetry, so

that the continuity equation can be written in terms of the phase gradient $\partial_\mu \Phi$. From the above one obtains the Noether tensor in canonical form as

$$T_N^{\mu\nu} = \Pi^\mu_{\alpha(U)} \partial^\nu U^\alpha + \sum_A \Pi^{\mu A}_{\alpha(A)} \partial^\nu \mathbb{A}^{\alpha A} + \frac{\mathbb{C}^\mu}{\xi} \partial^\nu \Phi - \eta^{\mu\nu} \mathcal{L}_N \quad (26)$$

By the Rosenfeld-Belinfante identity, which equates the Hilbert energy-momentum tensor obtained by metric variation with the Belinfante-improved Noether tensor (on-shell), one obtains Alena Tensor as

$$T^{\mu\nu} = T_N^{\mu\nu} + I^{\mu\nu} = \varrho U^\mu U^\nu - \Xi^{\mu\nu} - \xi Y^{\mu\nu} \quad (27)$$

This construction yields a symmetric and conserved energy-momentum tensor, suitable for quantization. In the quantum description $T^{\mu\nu}$ is promoted to an operator whose expectation values reproduce the classical results. One may thus denote $D_\mu = \partial_\mu - i \sum_A g_A \mathbb{A}_{\mu A} T_A$ with g_A denoting the gauge couplings, $\mathbb{A}_{\mu A}$ the corresponding gauge fields, and T_A the generators of the gauge group in the representation of the matter field. In flat spacetime, one therefore obtains the Alena Tensor $T^{\mu\nu}$ as

$$T^{\mu\nu} = \frac{i}{4} \bar{\psi} \gamma^{(\mu} \overleftrightarrow{D}^{\nu)} \psi - \Xi^{\mu\nu} - \xi Y^{\mu\nu} \quad (28)$$

where $\overleftrightarrow{D}$ denotes the two-way derivative. It may be calculated, that the inclusion of the rotational energy term allows to construct below quantum-effective Lagrangian density, giving the expected structure of the energy-momentum tensor and providing a simple quantum interpretation

$$\begin{aligned} \mathcal{L}_{\text{eff}} = & \underbrace{\frac{\hbar c}{2} (\partial_\mu \rho)(\partial^\mu \rho) + \frac{\hbar c}{2} \rho^2 (\partial_\mu \phi)(\partial^\mu \phi)}_{\mathcal{L}_{\text{scalar (vortex)}}} + \underbrace{\bar{\psi} (i \hbar c \gamma^\mu D_\mu - mc^2) \psi}_{\mathcal{L}_\psi} \\ & + \underbrace{\frac{e^{-A(\rho)}}{2c^2 \kappa} \omega_{\mu\nu} \omega^{\mu\nu} - g \hbar \bar{\psi} \Sigma_{\mu\nu} \psi \omega^{\mu\nu}}_{\mathcal{L}_\omega + \mathcal{L}_{\text{spin-vortex}}} \end{aligned} \quad (29)$$

where

- $\phi(x)$ - vortex phase field (action phase). Its gradient $\partial_\mu \phi$ represents the generalized four-momentum flow associated with the vortex structure.
- $\rho(x)$ - amplitude of the complex condensate $\varphi = \hbar c \rho e^{i\phi}$. It determines the vortex core profile and sets the symmetry-breaking scale.
- $\omega_{\mu\nu}$ - vorticity tensor of the underlying medium. In this Lagrangian it is treated as an independent antisymmetric field capturing local rotational structure.
- $\Sigma_{\mu\nu}$ - spin generator in the fermionic representation $\Sigma_{\mu\nu} = \frac{i}{4} [\gamma_\mu, \gamma_\nu]$.
- $e^{A(\rho)}$ - plays the role of the a dimensionless state-dependent stiffness function, encoding the effective elastic response of the vortex condensate, where it is assumed for calculation simplicity $p_\Lambda \alpha = e^{-A(\rho)} / (c^2 \kappa)$
- g - dimensionless spin-vorticity coupling constant, determining the strength of the interaction between fermionic spin and the vortex background.

The Madelung decomposition $\varphi = \hbar c \rho e^{i\phi}$ introduces an effective phase field whose gradient reproduces the conserved current $\hbar c \rho^2 \partial^\mu \phi$ so that the hydrodynamic Noether current defined in the classical theory $\partial^\mu \frac{\Phi}{\kappa}$ is represented in the effective description by the amplitude-phase variables through the identification $\partial^\mu \Phi = \kappa \hbar c \rho^2 \partial^\mu \phi$. In this sense, ϕ does not introduce an additional dynamical degree of freedom, but parametrizes the same conserved current encoded in the underlying field texture, ensuring quantized circulation and capturing the spin-vorticity structure of the medium.

2.2.2. General Relativity Interpretation

One may now repeat the reasoning from [27] and define the generalized Ricci and Einstein tensors, where the $\tilde{\cdot}$ sign indicates normalization with the constant $\kappa/2 = \frac{4\pi G}{c^4}$, and additional tensor $\Theta^{\mu\nu}$ as

$$\tilde{R}^{\mu\nu} \equiv 2T_{matt}^{\mu\nu} + 2\chi_\omega Y^{\mu\nu} + (p_\Lambda + \mathcal{E}_{rot} - \varrho c^2)g^{\mu\nu} \quad (30)$$

$$\tilde{R} \equiv \tilde{R}^{\mu\nu} g_{\mu\nu} = 4p_\Lambda + 2\mathcal{E}_{rot} - 2\varrho c^2 \quad (31)$$

$$\begin{aligned} \tilde{G}^{\mu\nu} &\equiv \tilde{R}^{\mu\nu} - \frac{\tilde{R}}{2} \mathbb{K}^{\mu\nu} = \tilde{R}^{\mu\nu} - \frac{\tilde{R}}{2} g^{\mu\nu} - \frac{\tilde{R}}{2p_\Lambda} Y^{\mu\nu} \\ &= 2T_{matt}^{\mu\nu} - (2 - \chi_m - \chi_\omega) Y^{\mu\nu} - p_\Lambda g^{\mu\nu} \end{aligned} \quad (32)$$

$$\Theta^{\mu\nu} \equiv -(\chi_m + \chi_\omega) Y^{\mu\nu} \rightarrow \tilde{G}^{\mu\nu} + p_\Lambda g^{\mu\nu} = 2T^{\mu\nu} + \Theta^{\mu\nu} \quad (33)$$

where the last equality holds in any considered spacetime and for $G^{\mu\nu} = \frac{\kappa}{2} \tilde{G}^{\mu\nu}$, $\Lambda = \frac{\kappa}{2} p_\Lambda$ becomes the classical GR equation in curved spacetime (in curved spacetime $Y^{\mu\nu}$ and $\Theta^{\mu\nu}$ vanish).

The Lagrangian $\mathcal{L}_\Theta$ for $\Theta^{\mu\nu}$ may be obtained the same way as in [27] with the use of the interpolating path method $g^{\mu\nu}(\lambda) = (1 - \lambda)\mathbb{K}^{\mu\nu} + \lambda g^{\mu\nu}$. Using this method one obtains

$$\sqrt{-g} \mathcal{L}_\Theta \equiv \frac{1}{2} \int_0^1 d\lambda \sqrt{-g(\lambda)} \Theta_{\mu\nu} [g^{\mu\nu}(\lambda)] \partial_\lambda g^{\mu\nu}(\lambda) \quad (34)$$

Since the variation of the functional is located on the boundary $\lambda = 1$, thus $\Theta_{\mu\nu} = \frac{2}{\sqrt{-g}} \frac{\delta(\sqrt{-g} \mathcal{L}_\Theta)}{\delta g^{\mu\nu}}$.

In the last equation of (33) considered in curved spacetime ($g^{\mu\nu} \rightarrow \mathbb{K}^{\mu\nu}$) field invariant p_Λ acts as double the vacuum energy density (vacuum pressure). This equation may be derived from the Lagrangian density $\mathcal{L}_G$ in the form

$$\mathcal{L}_G \equiv 2\mathcal{L}_T + \mathcal{L}_\Theta = \mathcal{L}_{matt} + \frac{1}{2} \tilde{R} - 2\mathcal{E}_{rot}; \quad \mathcal{L}_{matt} \equiv \mathcal{L}_\Theta - p_\Lambda (\chi_m + \chi_\omega) \quad (35)$$

where the variation by Hilbert's method on $\mathcal{L}_{matt}$ gives $T_{matt}^{\mu\nu} \equiv \varrho U^\mu U^\nu - \Xi^{\mu\nu}$.

The equation (32) considered in curved spacetime simplifies to $\tilde{G}^{\mu\nu} = \tilde{R}^{\mu\nu} - \frac{\tilde{R}}{2} g^{\mu\nu}$ where $Y^{\mu\nu}$ and thus also $\Theta^{\mu\nu}$ vanishes. It is also worth noting that in flat spacetime the generalized Einstein tensor is associated with the four-divergence of $\Theta^{\mu\nu}$

$$\partial_\mu \tilde{G}^{\mu\nu} = \partial_\mu \Theta^{\mu\nu} = f_{gr}^\nu - f_{rr}^\nu \quad (36)$$

so the curvature it describes in curved spacetime replaces this four-force density, where f_{gr}^ν is related to gravity and f_{rr}^ν is the density of radiation-reaction four-force. The presence of the radiation-reaction force has already been discussed in previous works [25], and it now prevents the matter energy and rotational energy from increasing beyond the maximum energy density p_Λ available in the system.

It is worth noting that internal energy density and rotational energy, essentially exhaust the possible forms of energy that can be attributed to material bodies (other forms of energy, e.g., chemical energy, can be treated as their components) which could be present in the radiation reaction force. This means that the model proposed here seems complete (with the possible extension of α to a tensor form for more complex systems) and should allow for reproducing the results obtained from GR, as well as reproducing observational results that are inconsistent (such as the dark sector) with currently used interpretation of GR.

Alena Tensor approach therefore allows to look at Einstein's equations in a new light and analyze the possibilities of explaining the dark sector in a consistent mathematical framework that allows analysis in both flat and curved spacetime. Importantly, it is also possible to analyze the system using a quantum approach (in the description for flat spacetime) and to use standard tools of continuum mechanics for continuous media in flat and curved spacetime, where the description of the behavior of

matter has been separated into effects related to fundamental interactions f_{YM}^v , gravity and radiation reaction $f_{gr}^v - f_{rr}^v$, and forces related to the distribution of matter f_{Ξ}^v .

3. Results

The following section will present the results of applying the Alena Tensor model both to cosmological objects and to describe quantum vortices.

3.1. The Halo Effect

The obtained results de facto means, that Alena Tensor ensures correct operation of the standard continuum mechanics equations and GR equations (Euler equations, EOS, TOV, first integrals for rotating stars, etc.), with assumption that the energy density used in them is $\varepsilon = \varrho c^2 + 2\mathcal{E}_{rot}$ and pressure is equal to $\mathcal{E}_{rot}$. In the next steps, this approach will be analyzed to show that it leads to consistency with the observational data.

It's worth starting with a simple approximation. Denoting u_{rot} as rotational velocity and assuming

$$p_{\Lambda}\alpha = \frac{1}{c^2\kappa}; \quad \varrho_{rot} \equiv \frac{2\mathcal{E}_{rot}}{c^2} = \frac{\omega^{\alpha\beta}\omega_{\alpha\beta}}{8\pi G} \rightarrow \lim_{r \rightarrow \infty} \varrho_{rot} = \frac{u_{rot}^2}{4\pi G r^2} \quad (37)$$

one may notice, that $p_{\Lambda}\alpha$ in (14) plays the role of the density of the moment of inertia, while ϱ_{rot} increases the body's effective mass within its own frame. This would allow to consider galaxies as continuous media, where the effective mass M_{eff} and its density ϱ_{eff} responsible for gravity $\varrho_{eff} \equiv \frac{\varepsilon}{c^2} = \varrho + \varrho_{rot}$ from (16) increases with the galactic disk size and angular velocity, causing the halo effect.

For far regions, denoting M_b as baryonic mass, for spherical symmetry one obtains from Poisson's equation simple linear ODE in the Newtonian limit. In the far regions it could determine a constant rotation speed and might be used to measure of deviation from the vacuum solution.

Vacuum solution in curved spacetime from (15) and (16) yields

$$\tilde{R} = 0; \quad p_{\Lambda} = 2\varepsilon \rightarrow \varrho c^2 = -3\mathcal{E}_{rot}; \quad p_{\Lambda} = -2\mathcal{E}_{rot}; \quad \Lambda = -\kappa\mathcal{E}_{rot}; \quad 0 = \tau^{\mu\nu} = U^{(\mu}q^{\nu)}; \quad \mathcal{L}_T = 0 \quad (38)$$

This means that Keplerian profiles are still possible for systems that can be approximated by a vacuum solution or does not rotate ($\mathcal{E}_{rot} \approx 0$).

Going into a more detailed analysis, one may consider ($0 = \tau^{\mu\nu} = U^{(\mu}q^{\nu)}$) as a simple ideal fluid system with pressure $p \equiv \mathcal{E}_{rot}$ according to (16) and (19). In the GR equations one may consider above with an axisymmetric, spherical metric

$$\begin{aligned} ds^2 &= N^2 c^2 dt^2 - A^2 (dr^2 + r^2 d\theta^2) - B^2 r^2 \sin^2 \theta (d\phi - \omega dt)^2 \\ T_{ideal}^{\mu\nu} &\equiv \left(\varrho + \frac{2p}{c^2} \right) U^{\mu} U^{\nu} - p \Delta^{\mu\nu} \quad ; \quad \Omega \equiv \frac{d\phi}{dt} \end{aligned} \quad (39)$$

Analyzing Euler's energy and momentum equations in $\nabla_{\mu} T_{ideal}^{\mu\nu} = 0$ one may notice, that u_{ZAMO} is not a geodesic movement, and on the equator it takes the value

$$\frac{u_{ZAMO}^2}{c^2} = \frac{(Br)^2}{N^2} (\Omega - \omega)^2 = r \partial_r \ln N + \gamma_p^2 - 1 \quad ; \quad \gamma_p^2 \equiv 1 - r \frac{p'}{\varrho c^2 + 3p} \quad (40)$$

where γ_p coefficient determines the deviation from the geodesic. One may therefore define pressure according to conclusions from previous section as follows, what yields

$$\kappa p \equiv \frac{u_{ZAMO}^2}{r^2 c^2} \rightarrow \frac{dM_{eff}}{dr} = 4\pi r^2 \varrho(r) + \frac{u_{ZAMO}^2}{G} \rightarrow \frac{d}{dr} u_{ZAMO}^2 = 4\pi G r \varrho(r) \quad (41)$$

In obtained picture the velocity increase depends solely on the baryon mass distribution, while the flattening of the tail is maintained by the rotational energy. This precisely corresponds to the expected behavior of a dark matter "halo."

The introduction of q^μ and $\tau^{\mu\nu}$ into a system can be approximated by defining parameter $\chi(r)$ changing original p used for isotropic model

$$p_r \equiv \alpha p; p_\theta \equiv \beta p; p_\phi \equiv p \rightarrow \chi = \alpha + \beta; \quad 3p_\chi \rightarrow (1 + \chi)p; \quad 2p \rightarrow \chi p \quad (42)$$

In practice, even a constant χ should be sufficient for analyzing the fit of galaxy rotation curves. Using a constant χ also provides a simpler ODE and the ability to quickly perform preliminary fits of χ to observational data for large amounts of data.

$$\begin{aligned} \frac{dM_{eff}}{dr} &= 4\pi r^2 \varrho(r) + \frac{\chi}{2} \frac{u_{ZAMO}^2}{G} \rightarrow \frac{d}{dr} u_{ZAMO}^2 = 4\pi G r \varrho(r) + \left(\frac{\chi}{2} - 1\right) \frac{u_{ZAMO}^2}{r}; \\ \lim_{\varrho \rightarrow 0} u_{ZAMO}^2 &\propto r^{\frac{\chi}{2}-1} \end{aligned} \quad (43)$$

Below one may find the expected course of the rotation curves depending on the assumed constant χ . The calculations used an averaged Hernquist bulge baryon model ϱ_{bulge} and a "spherical proxy" of the exponential disk ϱ_{disc}

$$\varrho_{bulge} = \frac{M_{bulge}}{2\pi} \frac{a}{r(r+a)^3}; \quad M_{bulge} = 10^{10} M_\odot; \quad a = 1 \text{ kpc} \quad (44)$$

$$\varrho_{disc} = \frac{M_{disc}}{4\pi R_d^2} \frac{e^{-r/R_d}}{r}; \quad R_d = 3 \text{ kpc} \quad (45)$$

with total baryon density $\varrho_b(r) = \varrho_{bulge} + \varrho_{disc}$, standard G value, anisotropy and energy stream simulated by constant χ . As can be seen from the graph, the increasing anisotropy towards the outskirts of the galaxy $\chi(r=0) = 0$; $\lim_{r \rightarrow \infty} \chi(r) = \text{const}$ would allow the graph to align with the expected curve shapes for spiral galaxies.

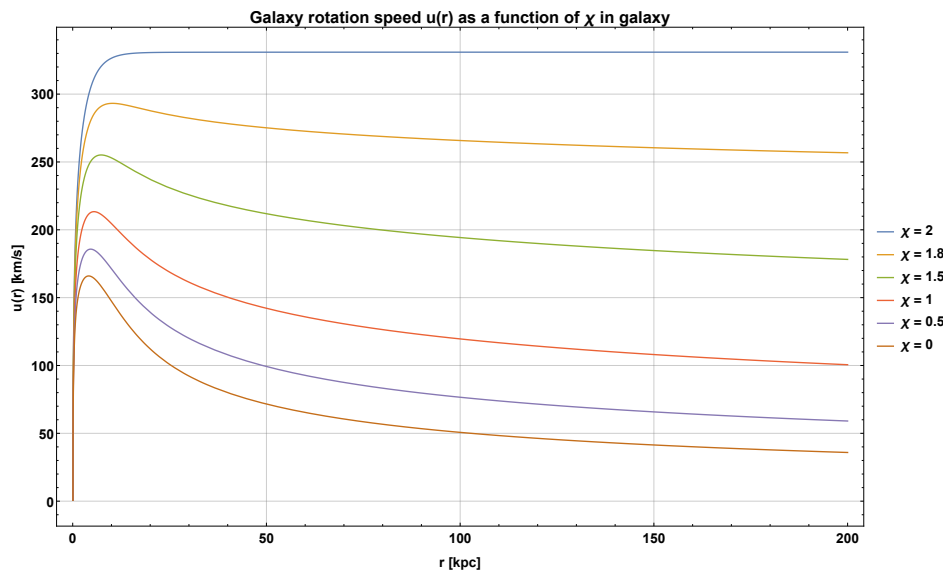


Figure 1. Approx. rotation curves in Alena Tensor model

As it appears, constant χ actually allows to tune the rotation velocity distribution for some part of galaxies what gives an overview of the method and may help in further analysis and tuning of $\chi(r)$ function to achieve full agreement with observations. The A presents the results of tuning the χ constant for about 100 galaxies from the SPARC catalog. The results seem encouraging, and worth

further analysis with the $\chi(r)$ function or with the full $T^{\mu\nu}$ tensor representation. For most galaxies the fit is very good, and for the rest it is obtainable by introducing a simple function $\chi(r)$ which provides $\chi(r) \approx 0$ for small r (chaotic motions in the center, no ordered rotation) to a stabilized rotation in the outskirts with a fixed χ .

The χ parameter was adjusted for each galaxy by iteratively solving the radial motion differential equation resulting from (43), with the condition of normalizing the rotation rate in the outer disk $\langle V_\chi \rangle_{\text{outer}} = \langle V_{\text{obs}} \rangle_{\text{outer}}$. The entire procedure, including reading the rotation data from the SPARC catalog, interpolating the baryonic component $V_{\text{bar}}(r)$, numerically solving the equation for $V_\chi(r, \chi)$ and saving the resulting plots in PDF format, was fully automated in the Mathematica script, and run in an environment with the SPARC source data. The script is included in the supplementary files.

3.2. Quantum Vortices and Elementary Particles

The obtained Alena Tensor equations allow for the analysis of vortices in a quantum model, it is therefore worth analyzing their consequences. In the next steps it will be shown that by treating the Lagrangian (29) as an effective field model and choosing an appropriate stiffness function $A(\rho)$, both Yukawa-type mass generation and Higgs-type potential structure can emerge naturally.

The vorticity tensor enters the Lagrangian (29) only through the elastic term and the spin-vorticity interaction. Its algebraic equation of motion yields

$$\frac{\partial \mathcal{L}_{\text{eff}}}{\partial \omega^{\mu\nu}} = 0 \quad \rightarrow \quad \omega_{\mu\nu} = c^2 \kappa e^{A(\rho)} g \hbar \bar{\psi} \Sigma_{\mu\nu} \psi \quad (46)$$

so that eliminating $\omega_{\mu\nu}$ produces the effective fermion term

$$\mathcal{L}_f \equiv -\frac{g^2 \hbar^2 c^2 \kappa e^{A(\rho)}}{2} (\bar{\psi} \Sigma^{\mu\nu} \psi) (\bar{\psi} \Sigma_{\mu\nu} \psi) = -\mathcal{E}_{\text{rot}} \quad (47)$$

Evaluated at the homogeneous equilibrium, this yields a dynamically generated fermion mass

$$m_{\text{eff}} = \frac{g^2 \hbar^2 \kappa e^{A(\rho)}}{2} \langle \Sigma \rangle \quad ; \quad \langle \Sigma \rangle \equiv \frac{\langle (\bar{\psi} \Sigma^{\mu\nu} \psi) (\bar{\psi} \Sigma_{\mu\nu} \psi) \rangle}{\langle \bar{\psi} \psi \rangle} \quad (48)$$

In this picture, the mass generation results from the equilibrium between the scalar amplitude ρ and the spin-vorticity background, rather than from a fundamental Yukawa coupling where the condensate $\langle \Sigma \rangle$ plays the role of an order parameter. For a stationary vortex configuration $\varphi = \rho e^{i\phi}$, the scalar sector of (29) leads to the standard energy functional

$$E[\rho, \phi] = \frac{\hbar c}{2} \int d^3x [(\nabla \rho)^2 + \rho^2 (\nabla \phi)^2] \quad (49)$$

Euler-Lagrange equations dedived from (29) are

$$\hbar c \partial_\mu \partial^\mu \rho - \left[\hbar c \rho (\partial_\mu \phi) (\partial^\mu \phi) - \frac{A'(\rho) e^{-A(\rho)}}{2c^2 \kappa} \omega_{\alpha\beta} \omega^{\alpha\beta} \right] = 0 \quad ; \quad \partial_\mu (\rho^2 \partial^\mu \phi) = 0 \quad (50)$$

and show, that the amplitude adjusts to the local phase gradient. In the presented model, ρ has the dimension of the inverse of the length while Standard Model vev v_{SM} is expressed as energy. Considering the equations in the limit $\rho = v$ one may thus assume that $\hbar c v = v_{\text{SM}} \sqrt{2}$. Since the effective Yukawa parameter can be extracted directly from (48) therefore in order to reproduce the Standard Model it should satisfy $m_{\text{eff}} c^2 = y_f \frac{\hbar c}{2} v$, thus

$$\frac{\hbar c}{2} y_f \equiv \frac{\partial m_{\text{eff}}(\rho) c^2}{\partial \rho} \Big|_{\rho=v} = m_{\text{eff}} c^2 A'(v) \quad \rightarrow \quad \frac{\hbar c y_f}{2 m_{\text{eff}} c^2} = \frac{1}{v} \quad \rightarrow \quad A'(v) = \frac{1}{v} \quad (51)$$

Assuming the stationary energy-minimizing configuration (static state)

$$\rho(x) = v = \text{const} \quad ; \quad \partial_i \rho = 0 \quad ; \quad \phi(x) = \mu c t \quad ; \quad \partial_i \phi = 0 \quad (52)$$

the Euler-Lagrange equation for $\rho = v$ reduces therefore to

$$\mathcal{E}_{\text{rot}}(v) = m_{\text{eff}} c^2 \langle \bar{\psi} \psi \rangle = \frac{e^{-A(v)}}{2c^2 \kappa} \omega_{\alpha\beta} \omega^{\alpha\beta} = \hbar c \frac{v \mu^2}{A'(v)} = \hbar c v^2 \mu^2 \quad (53)$$

Under suitable assumptions on the stiffness function $A(\rho)$ and on the spin-vorticity condensate encoded in $\omega_{\alpha\beta} \omega^{\alpha\beta}$, this relation plays the role of a gap equation, equivalent to the condition $\partial V_{\text{eff}}(\rho)/\partial \rho|_{\rho=v} = 0$ for an effective Higgs-like potential. The competition between $\mathcal{E}_{\text{rot}}$ and $\hbar c v^2 \mu^2$ might therefore reproduce the characteristic symmetry-breaking structure of the Higgs potential.

Assuming that the effective potential $V_{\text{eff}}(\rho)$ is indeed of Higgs type, the mass of the small radial (Higgs-like) excitation around the equilibrium value $\rho = v$, using emergent mass formula (48) and using the condensate identity (46), may be obtained in a compact expression

$$m_H^2 \equiv \frac{\hbar}{c^3} \left. \frac{\partial^2 V_{\text{eff}}(\rho)}{\partial \rho^2} \right|_{\rho=v} = \frac{\hbar}{c} m_{\text{eff}} \langle \bar{\psi} \psi \rangle \left[A''(v) + (A'(v))^2 - \frac{A'(v)}{v} \right] = \frac{\hbar}{c} m_{\text{eff}} \langle \bar{\psi} \psi \rangle A''(v) \quad (54)$$

Therefore (54) may be further simplified, imposing a specific form on the function $A(\rho)$ as below, for the preservation of the Yukawa condition and the Higgs-like potential

$$A(\rho) \equiv \ln \frac{\rho}{v} + \frac{\lambda}{2v^2} (\rho - v)^2 \quad \rightarrow \quad \mathcal{E}_{\text{rot}}(v) = m_{\text{eff}} c^2 \langle \bar{\psi} \psi \rangle = \frac{m_H^2 c^3}{\hbar} \frac{v^2}{\lambda - 1} \quad (55)$$

Since in the considered model, the relation between Standard Model vev v_{SM} and v is $\hbar c v = v_{\text{SM}} \sqrt{2}$, therefore introducing the Standard Model Higgs self-coupling as λ_{SM} and substituting $m_{\text{eff}} c^2$ from (51) one obtains

$$\hbar c v = v_{\text{SM}} \sqrt{2} = \frac{m_H c^2}{\sqrt{\lambda_{\text{SM}}}} \quad \rightarrow \quad \frac{y_f}{2} \cdot \langle \bar{\psi} \psi \rangle = \frac{\lambda_{\text{SM}}}{\lambda - 1} \cdot v^3 \quad (56)$$

Therefore, making the natural assumption $\langle \bar{\psi} \psi \rangle = v^3$, one may separate condensate from Yukawa parameter as follows

$$m_{\text{eff}} = m_H \frac{\sqrt{\lambda_{\text{SM}}}}{\lambda - 1} \quad (57)$$

replacing the Yukawa parameter y_f with the parameter λ describing the quantum vortex. In this way the above approach allows to model elementary particles as stable quantum vortices, reproducing the successful Yukawa and Higgs mechanism, where all stability requirements follow directly from the field equations and the structure of the effective potential.

In the resulting framework, elementary particles arise as stable, self-sustained vortices of the underlying vacuum field, whose rotational structure is captured by the antisymmetric vorticity tensor $\omega_{\mu\nu}$. This interpretation is fully consistent with the general paradigm of topological solitons, where localized and finite-energy excitations of a continuous field behave as particle-like objects, as in the Skyrme model [33], the Faddeev-Niemi hopfion model [34,35], and the superfluid-vacuum approach of Volovik [36]. In presented case, the spin-vorticity coupling and the rotational energy functional play the role of stabilizing terms, ensuring that the vortex configuration propagates as a massive fermionic quasiparticle, closely analogous to knotted vortical solitons in classical and quantum field theories [34,37].

One may notice, that the effective Lagrangian (29) naturally supports four distinct conserved charges, arising from its symmetries and from the structure of the vorticity sector

- *Phase (Noether) charge* $Q_\Phi = \int d^3x j^0 \quad ; \quad j^\mu \equiv \rho^2 \partial^\mu \phi$ originating from the global shift symmetry $\Phi \rightarrow \Phi + \text{const}$. It corresponds to the conserved circulation associated with the phase field.

- *Topological vortex number* $N_v = \frac{1}{2\pi} \oint_C \nabla \phi \cdot d\vec{\ell} \in \mathbb{Z}$ defined for static configurations with nontrivial winding of the phase ϕ around the vortex core. This integer counts the number of 2π windings.
- *Spin-vorticity charge* $Q_{sv} = \int d^3x \partial_\mu (g\hbar \omega^{\mu 0})$ where the vorticity tensor $\omega_{\mu\nu}$ satisfies the algebraic field equation $\omega_{\mu\nu} = c^2 \kappa e^{A(\rho)} g\hbar \psi \Sigma_{\mu\nu} \psi$. This charge reflects the conserved flow associated with the spin-vorticity coupling term $g\hbar \psi \Sigma_{\mu\nu} \psi \omega^{\mu\nu}$.
- *Hopf (linking) charge* $Q_H = \frac{1}{32\pi^2} \int d^3x \epsilon^{ijk} \mathcal{A}_i \mathcal{F}_{jk}$ defined when the dual vorticity vector $\omega_i \equiv \frac{1}{2} \epsilon_{ijk} \omega_{jk}$ is normalized to a unit field $\vec{n}(x) = \vec{\omega}/|\vec{\omega}|$, with $\mathcal{F}_{ij} = \partial_i \mathcal{A}_j - \partial_j \mathcal{A}_i$ denoting the pullback of the area form on S^2 . This integer-valued invariant characterizes the knotting and linking of vorticity lines.

The presence of these conserved quantities enables a deeper analysis of the model in the context of reproducing known structural features of elementary particles, including the possibility of identifying particle families with different topological sectors.

It may be also noticed, that assuming non-zero m , variation of (29) with respect to $\bar{\psi}$ gives the modified Dirac equation

$$\left(i\hbar c \gamma^\mu D_\mu - mc^2 - g\hbar \Sigma^{\mu\nu} \omega_{\mu\nu} \right) \psi = 0 \quad (58)$$

where the last term is a local spin-vorticity coupling formally analogous to the Mashhoon effect [38,39]. For the normalization of $\omega_{\mu\nu}$ assumed in (12) this correspondence fixes $g = \frac{1}{2}$. This thus allows the obtained model to be used to describe the atom and for other classical applications.

4. Discussion and Conclusions

It is worth discussing the conclusions of this article by dividing them into issues concerning GR/Cosmology and quantum issues.

4.1. Discussion and Conclusions Regarding GR and Cosmology

As seen in the above article, supplementing the Alena Tensor with the energy associated with the rotation of bodies naturally leads to the creation of halo effects, known from dark matter studies. Preliminary analysis allows for a fairly good match of this effect to observational results, although this obviously requires further development and verification for a larger number of cosmological objects. Importantly, the proposed approach does not require modification of the GR equations, but rather fits naturally into the applied GR equations and continuum mechanics. Since the source is described in this approach by $\varepsilon = \rho c^2 + 2\mathcal{E}_{rot}$, this means that the observed increase in effective mass also affects gravitational lensing to an extent precisely corresponding to the increase in effective mass by the energy associated with (in this case - rotational) "dark matter". This is precisely what is obtained in observations [40,41].

The proposed solution fits quite well with the research direction represented by [42–44] and also [45] (including baryotropy), who investigated anisotropic fluid in cosmology and its potential connections with the dark sector. However, it complements these studies with the natural halo effect resulting directly from the GR equations for the Alena Tensor. The proposed model also expands and, in a sense, substantiates the hypothesis posed by C. Rourke [46], complementing the research [47–50] with a justification for linking rotation with the halo effect. The idea that rotation-related effects can mimic dark matter is not new, but Alena Tensor gives it some additional structure, making it a direct consequence of a coherent mathematical model.

Importantly, the Alena Tensor also provides a natural interpretation of dark energy. The value of p_Λ is an invariant of the field tensor and becomes constant (or, at least, metric-independent invariant) in curvilinear description. In a sense, a nonzero value of Λ can therefore be interpreted as a scale of deviation from pure wave solutions, without matter (for example, for the electromagnetic field, $p_\Lambda = 0$ would mean that the electric and magnetic fields are equal, so the solutions must be pure electromagnetic waves). Since the value of p_Λ measured in flat spacetime is $p_\Lambda = p_o (\mathbb{k}_{\mu\nu} \eta^{\mu\nu})^2$, it is a measure of the "flatness" of spacetime, or more precisely, a measure of how much the metric tensor for the curvilinear description deviates from the Minkowski tensor. This interpretation seems

particularly interesting in the context of the works [51,52], because it strengthens and details the conclusions described therein, providing a geometric, anisotropic source that can be interpreted as a specific backreaction mechanism leading to acceleration.

This approach could be applied to many other continuous systems (e.g., stars or black holes) and seems worth to describe the extreme in which $\varrho = 0$ and all the energy in the system is rotational energy. Although at first glance it seems absurd, one may notice, that the source in vacuum solution (38) is indeed solely rotational energy. As shown in 3.2, replacing rotation with vorticity and treating it as a consequence of the circulating field creates the possibility to model elementary particles as quasi-stable systems of three-dimensional vortices.

4.2. Discussion and Conclusions Regarding Quantum Issues

In quantum theory, the same structure of the energy-momentum tensor gives rise to topological vortices whose stability determines states with interpretable mass. Obtained interpretation of the Higgs mechanism creates the possibility of further analysis of the properties of elementary particles depending on properties of the associated vortices, because in the obtained picture the particle masses result from the self-consistent spin condensation (gap-equation), not from the external Yukawa condition. Hence, the model admits particle-like, topologically stabilized vortices, naturally emerging from the improved Belinfante tensor structure.

It is thus possible, taking into account the conclusions from these analyses, to further develop the idea of rotation in quantum systems. In quantum field theory systems with significant local vorticity, the inclusion of a rotational-energy contribution in the energy-momentum tensor implies a new spin-vorticity coupling term, which may lead to measurable spin polarization of fermions as predicted e.g. in heavy-ion physics [53]. Such additional rotational energy terms effectively modify the Hamiltonian density and thus alter propagators or dispersion relations of excitations in the medium, analogous to how vorticity-induced spin currents appear in hydrodynamic and condensed-matter contexts as in [54]. Since in considered approach the modified tensor now couples orbital angular momentum and intrinsic spin via the rotational energy density, therefore the renormalization group flows and transport coefficients (like shear viscosity or spin diffusion) may pick up new contributions, offering a potential probe of vorticious quantum media such as the quark-gluon plasma [55].

The extended Alena Tensor naturally suggests a broad research programme connecting microscopic particle physics, topological soliton theory and relativistic hydrodynamics. The modified Dirac equation (58) exhibits a local coupling of the intrinsic spin to the vorticity tensor $\omega_{\mu\nu}$. In the nonrelativistic limit this reduces to a Mashhoon-like term $-\mathbf{\Omega} \cdot (\mathbf{L} + \mathbf{S})$, in agreement with earlier analyses of inertial couplings in the Dirac theory [56–58]. This provides a direct physical interpretation of the Alena vorticity field as an observable rotational background.

The emergence of spin-vorticity coupling suggests several measurable signatures. These include possible energy-level shifts in atomic systems, modifications of fermion propagators in rotating backgrounds, and a connection to the experimentally observed spin polarization in the quark-gluon plasma (QGP) [59,60]. Since heavy-ion experiments have already demonstrated substantial vorticity and spin alignment of Λ hyperons, the Alena Tensor offers a concrete framework to reinterpret such data and potentially extract constraints on the coupling parameter g .

The effective theory possesses four distinct conserved quantities: the phase charge Q_Φ , the vortex number N_v , the spin-vorticity charge Q_{sv} and the Hopf charge Q_H . These invariants classify vortex solutions according to their topology and encode information about stability and possible decay channels. Such a structure closely parallels the theory of topological solitons, including Skyrme-type models [33], Faddeev-Niemi hopfions [34] and modern knot-like soliton constructions [61,62]. In this context, the Alena Tensor model suggests a possible reinterpretation of elementary particles as quantized vortex excitations distinguished by their topological invariants. For instance, fermions may correspond to stable spin-carrying vortices, while different particle families could be captured by distinct Hopf sectors. This motivates the construction of fully three-dimensional vortex solutions with prescribed $(Q_\Phi, N_v, Q_{sv}, Q_H)$ and the analysis of their energetics and stability.

Eliminating the antisymmetric field $\omega_{\mu\nu}$ leads to an effective four-fermion interaction term and a gap equation for the modulus ρ . This mechanism resembles dynamical mass generation in Nambu-Jona-Lasinio models [63], composite Higgs scenarios [64] and superfluid-vacuum analogies [36]. In this picture, particle masses arise not from fundamental Yukawa couplings but from vortex stability and the curvature of the effective potential encoded in $A(\rho)$. The Higgs particle appears as a radial excitation of the condensate, and its mass follows from differentiating the effective potential. This approach suggests a unified mechanism for fermion and Higgs masses, and establishes a link between soliton stability and the observable mass spectrum. Determining the function $A(\rho)$ from either phenomenological constraints or first-principles requirements is therefore a key target for future investigations.

The three directions outlined above: spin-vorticity dynamics, topological classification of vortex excitations and dynamical mass generation, form a coherent research strategy in which microscopic and macroscopic aspects of the Alena Tensor are treated on equal footing. Future work may include, e.g.:

- computing atomic and astrophysical signatures of the modified Dirac equation,
- constructing fully nonlinear three-dimensional vortex solitons with conserved topological charges,
- deriving fermion mass spectra from vortex equilibrium and comparing with Standard Model data,
- exploring possible vortex-induced contributions to anisotropic stress in neutron stars [65,66],
- studying hydrodynamic transport and spin polarization in vortical media [67,68].

Equally interesting direction of further analysis could be e.g. the use of the possibilities of quantum description of the dark sector in the Alena Tensor model, for further development of works such as [69]. It also seems that describing matter (e.g. a neutron star, as in [70,71]) using the mechanism proposed here for GR, would be the simplest way to confirm or falsify the Alena Tensor, due to the high symmetry of such a solution. However, all these analyses deserve separate articles.

In conclusion, it remains an open question whether the Alena Tensor is a correct way to describe physical systems, but this paper shows that, beyond the compliances with available knowledge achieved so far, it naturally leads to the existence of halo effects, interpretation of dark energy and modeling quantum vortices. Taken together, these results indicate that the extended Alena Tensor offers a unified geometric and topological framework that connects elementary particles, relativistic fluids and large-scale astrophysical structures. This opens a way for theoretical and phenomenological studies, extending far beyond the cosmological applications emphasized in the present work.

The author hopes that the results obtained in this paper will facilitate further use and development of the discussed approach and, potentially, many similar concepts. It also seems that further analysis of Alena Tensor may provide useful descriptions of the transformation between curved and flat spacetime and bring new insights that will contribute to a better understanding of issues related to the broadly understood unification of physical theories.

5. Statements

All data that support the findings of this study are included within the article (and any supplementary files).

During the preparation of this work the author did not use generative AI or AI-assisted technologies, except for continuous learning.

Author did not receive support from any organization for the submitted work.

Author have no relevant financial or non-financial interests to disclose.

Appendix A. Results of Fitting the Constant χ

The charts are placed on the last pages of the document.



Figure A1. Rotation curves at constant χ for galaxies 1/4

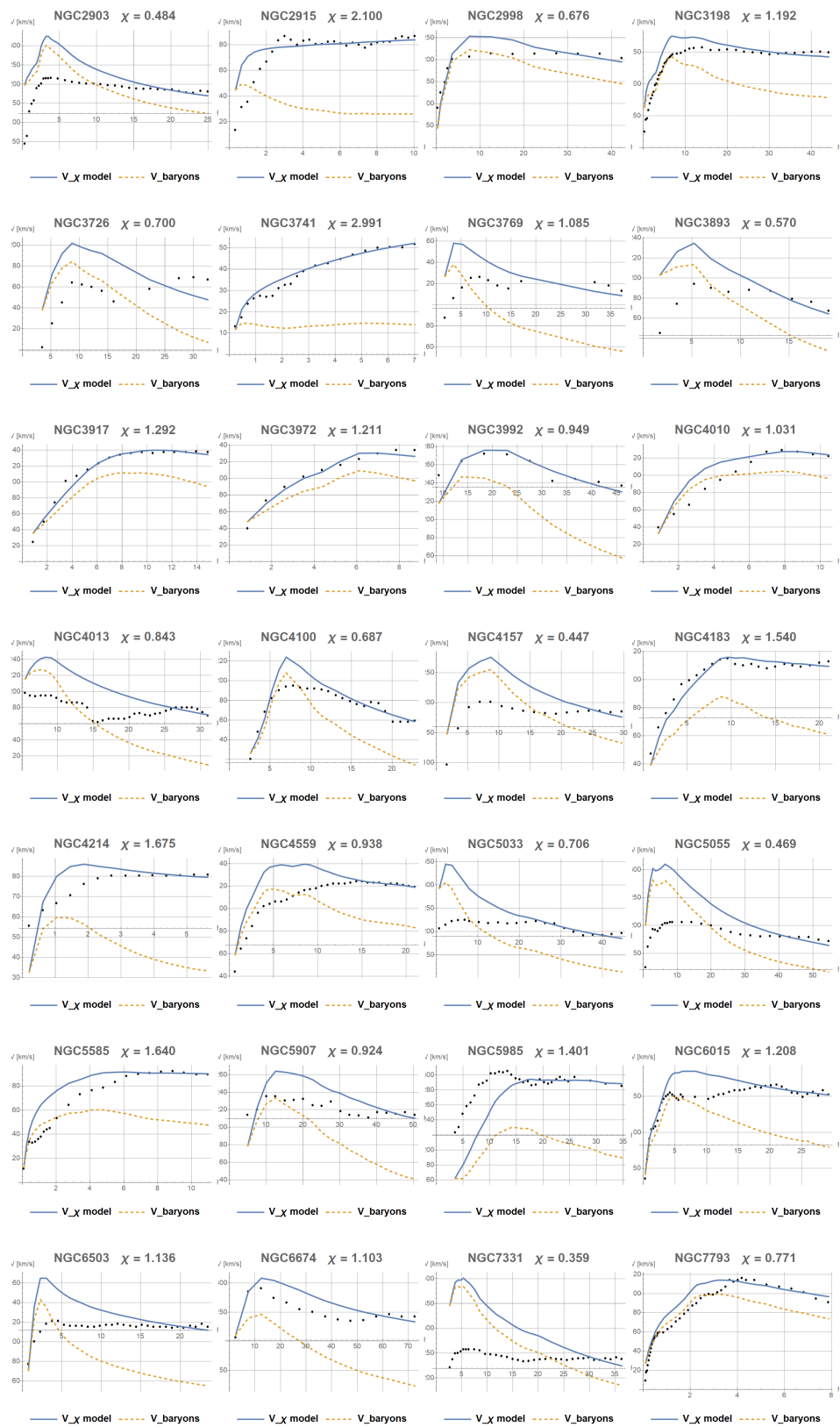


Figure A2. Rotation curves at constant χ for galaxies 2/4

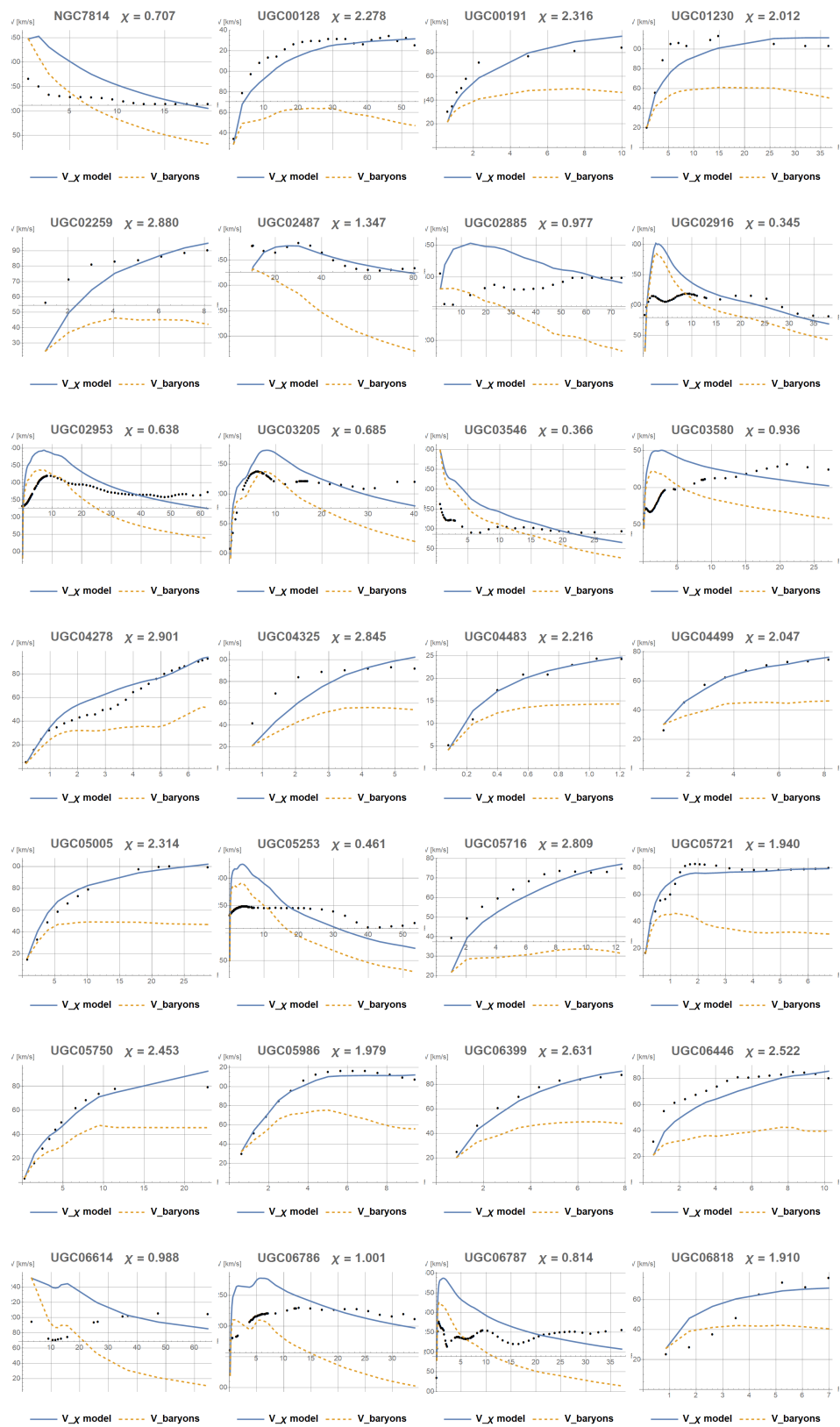


Figure A3. Rotation curves at constant χ for galaxies 3/4

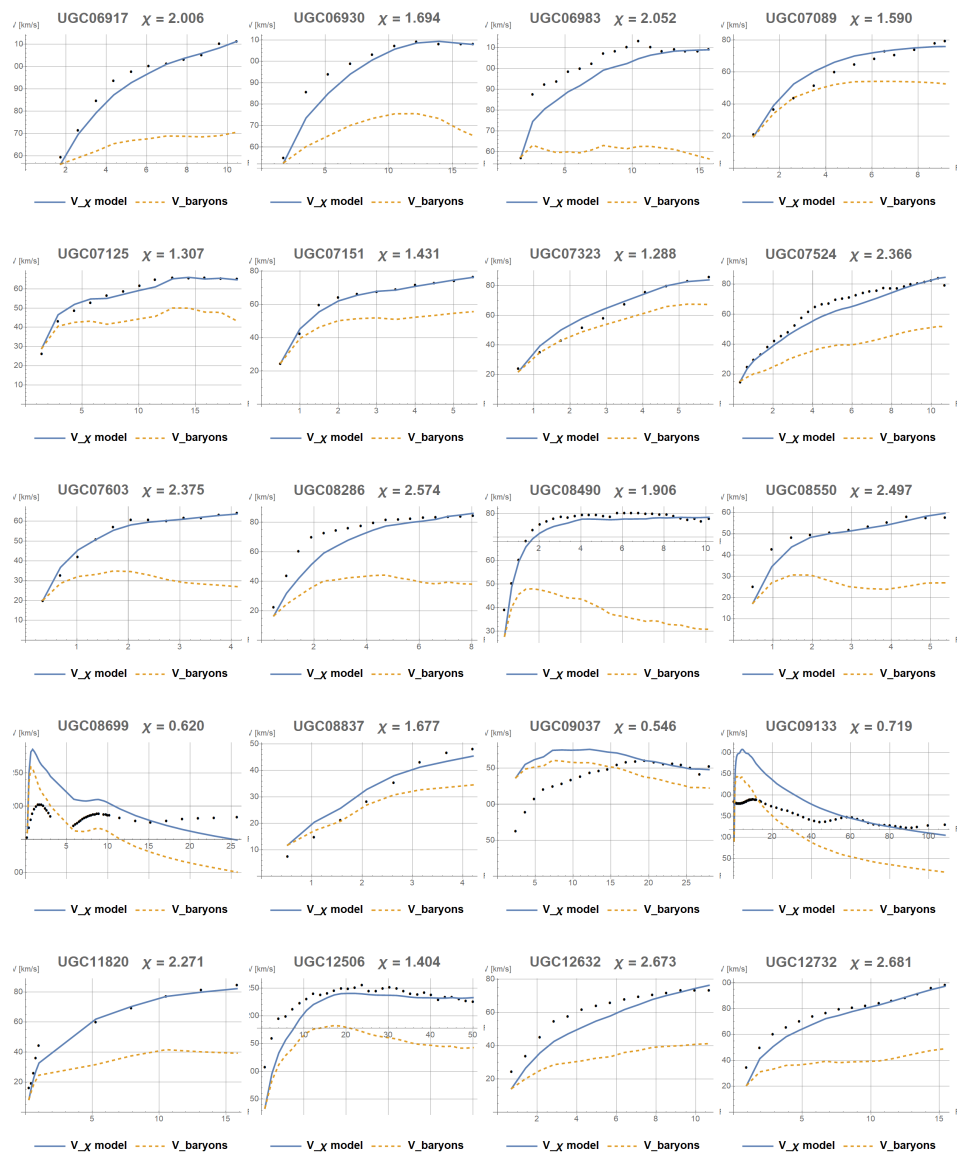


Figure A4. Rotation curves at constant χ for galaxies 4/4

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