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[Marius Ungarish](#) *

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Article

Box Model for Confined Power-Law Viscous Gravity Currents Including Surface Tension Effects

Marius Ungarish

Technion - Israel Institute of Technology, Department of Computer Science, Technion, Israel; unga@cs.technion.ac.il

Abstract

We consider the flow of a viscous fluid (power-law non-Newtonian) injected into a gap of height H between two horizontal plates. When the viscosity of the ambient (displaced) fluid is negligible, the injected fluid forms a tail-slug in contact with both plates connected (at a moving grounding line) to a leading gravity current (GC) whose interface does not touch the top of the gap. Surface tension menisci may appear at the grounding line and nose of the GC. Such systems, of interest in the injection molding industry, have been investigated recently in the framework of the lubrication theory (Hutchinson et al. 2023, Hutchinson 2024, Ungarish 2025) for the volume $\mathcal{V} = qt^\alpha$ (q and α are positive constants and t is time). Similarity appears for certain values of α . The similarity solution of the lubrication model requires manipulations and numerical calculations which obscure the underlying mechanisms and defy reliable interpretation, because the flow is dependent on four coupled parameters: viscosity exponent n , and J , σ , σ_N (the height ratio of the unconfined GC, grounding line meniscus and nose meniscus to H , respectively). Here we present a significantly simpler box-model analysis, which provides straightforward insights and facilitates the quantitative predictions. Comparisons with the rigorous lubrication-model solution and with previously-published data demonstrate that the box model provides a reliable physical description of the system, and a fairly accurate prediction of the propagation, for a wide range of parameters.

Keywords: gravity current; box model; confined flow; surface tension; non-Newtonian

1. Introduction

Gravity current (GC) is a generic name for the buoyancy-driven flow of a fluid of one density, ρ , into an ambient fluid of a different density, ρ_a , mostly in horizontal direction x (to be distinguished from the mostly vertical buoyancy-driven flows called plumes); see [11] and the references therein. The interpretation of the driving buoyancy mechanism is as follows: the hydrostatic pressure fields $p_j \propto -\rho_j g z$ produce a horizontal pressure gradient $\propto g' = |\rho/\rho_a - 1|g$, where g is the gravitational acceleration, z is the vertical upward coordinate, j denotes the ambient and current, and g' is the reduced gravity. The buoyancy is balanced by inertial or viscous effects. Here we focus attention on the so called viscous GC, dominated by a buoyancy–viscous dynamic balance, relevant to flows at a small Reynolds number. Viscous GCs have numerous applications in nature and industry. The systems of interest belong to various prototypes, such as Newtonian or non-Newtonian fluids, two-dimensional (2D) or cylindrical axisymmetric (AXI) propagation, fixed or time varying (influxed) volume, liquid or porous medium. An important distinction is between unconfined and confined (gap) domain into which the GC propagates. Geostrophic and environmental GCs are often unconfined (e.g., spread of lava or oilspills), and have received significant attention. The confined GC occur often in a gap where one viscous fluid displaces another viscous fluid particular in the context of porous layers (e.g., [2,4,10,14] and the review [15]).

Recent investigations addressed the flow of confined viscous GCs, relevant to injection molding (see [5]), see Figure 1(a). Consider the two dimensional (2D) propagation of a viscous fluid injected into a small gap of height H between two horizontal plates. This is a simplification of a rectangular

channel of width $\gg H$. Assume that the ambient fluid, displaced from the gap by the injected fluid, is less dense and significantly less viscous than the injected one (e.g., oil injected into air). In this case, the following type of flow may appear: the dense fluid forms a slug which fills the gap in $0 < x \leq x_G(t)$ while the fluid ahead of the slug, in $x_G(t) < x \leq x_N(t)$, forms a viscous GC on the bottom, while the interface is detached from the gap. The subscripts G and N denote the “grounding line” and the nose of the GC. Moreover, when the injected volume is $\mathcal{V} = qt^\alpha$, such flows may be self-similar, in the sense that the GC elongates like Kt^β while the slug elongates as $y_G Kt^\beta$, where q, α, K, β, y_G are positive constants and t is time. (The details of the source at $x = 0$ are outside the scope of this study. In the laboratory and industry, the influx is supplied by a pump or similar mechanical device, which provides the volume rate and the pressure necessary for the flow in the $x > 0$ domain. There is evidence that such systems work well, i.e., [8].)

The stringent task is to determine the values of α, β, y_G and K for a given physical system. Moreover, it is important to suggest a convenient scaling, to determine the governing dimensionless parameters, and predict the trends of the flow upon the variation of the input conditions. In particular, it is clear that for a sufficiently large H , the confined GC can be considered unconfined; a sharp criterion for this transition is of interest. The need for an efficient prediction method motivated our study.

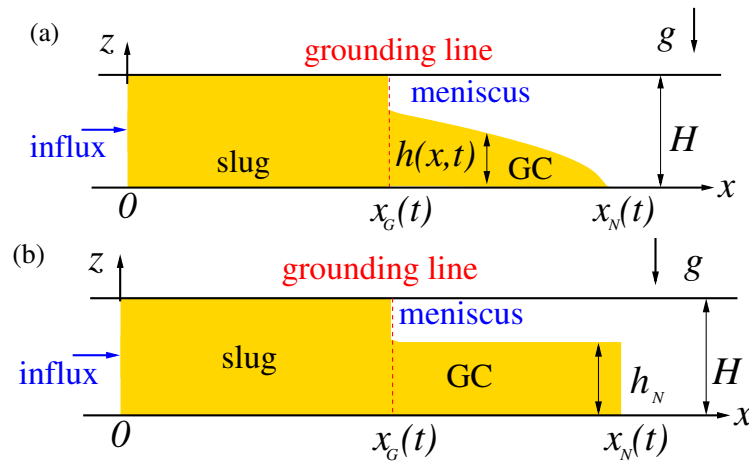


Figure 1. Sketch of the confined system: (a) lubrication model for the GC with inclined interface, and (b) box model with horizontal interface h_N . The height of the meniscus at the grounding line is σH . The volume (per unit width) is $\mathcal{V} = qt^\alpha$. In the self similar flow, $x_G = y_G Kt^\beta$, $x_N = Kt^\beta$, and y_G depends on the parameter J . In the axisymmetric geometry, r replaces x and $\mathcal{V}$ is per radian. α, β, K, y_G are constants.

[8] considered the confined axisymmetric (AXI) flow of a Newtonian fluid (theoretically and experimentally). They used a lubrication-theory model (with no surface tension effects) to show that a constant influx, $\alpha = 1$, produces self-similar propagation with power $\beta = 1/2$, and the only governing dimensionless parameter is $J = l/H$, where l is the typical height of the corresponding unconfined GC. The associated experiments (using golden syrup in a gap of about 1 cm filled initially with air), supported the theory (with some small discrepancies). [13] (referred to below as U25) extended the theory to cover also two-dimensional (2D) and non-Newtonian power-law fluids with exponent n (the Newtonian fluid is recovered for $n = 1$). The lubrication-theory framework indicates that a self-similar flow develops for select values of α , determined by n ; the power β is equal to α in 2D case and $\alpha/2$ in AXI case. The detached interface of the GC displays an inclined profile, see Figure 1(a), is defined by a second-order ordinary differential equation subjected to boundary conditions at the nose and at the grounding line, which in general must be solved numerically. When surface-tension effects are discarded, for a given fluid (specified by the viscosity exponent n) the scaled solution depends on only one dimensionless parameter, J . The theory provides a convenient scaling of variables, and a sharp prediction of the threshold value $J = J_0$ below which the confinement is irrelevant (the GC with the select α propagates on the bottom without touching the top of the gap).

The paper [7] revisits the system of [8], and demonstrates that the discrepancies between theory and experiment can be attributed (mainly) to surface-tension (Γ) effects between the current and the ambient fluids at the grounding line. The Young-Laplace formulation predicts the existence of a meniscus at the grounding line of height of the order of the capillary length $(\Gamma/\rho g')^{1/2}$, but the details and boundary conditions are complicated and hence an accurate solution of the meniscus in practical systems is not feasible. However, by modelling this meniscus as a vertical jump of height σH , see Figure 1, (where $\sigma \lesssim 0.5$ is a constant) the surface-tension effect was added conveniently to the original lubrication-model. The main point is that the similarity behaviour is maintained, with unchanged α and β . The dependencies between J , y_G and K are affected by σ . The theoretical solutions with small σ show better agreement with experiments than the original $\sigma = 0$ predictions. Although the exact value of σ is not known a-priori, this model provides valuable insights and can be applied with tentative theoretical or empirically estimated values. The study of [7] suggests that $\sigma = (\Gamma/\rho g')^{1/2}/H$ is a fair approximation. This promising extension of the original lubrication model similarity solution was derived and tested in [7] only for the a Newtonian fluid in AXI system ($n = 1, \alpha = 1, \beta = 1/2$). Here we show that the $\sigma > 0$ contribution can be implemented conveniently also in the theory of confined GCs of power-law fluids in both 2D and AXI systems.

The advantage of these published theories, based on the lubrication simplification, is the rigor of the governing equations and of the mathematical solution. The similarity solution predicts analytically the time behaviour, but the spatial profiles must be compiled numerically. The numerical task is straightforward. The disadvantage of this solution is that the needed mathematical manipulations are cumbersome and the results lack analytical description. This precludes straightforward extraction of trends and obscures the physical insights. This difficulty is exacerbated for the confined flow of power-law fluids in presence of surface-tension effects, which is dependent on three major parameters, J, σ, n , and the geometry (2D and AXI). (An additional parameter, σ_N , associated with the nose meniscus, will be introduced later.) For example, the value J_0 (for a given σ) is obtained from the numerical integration of a second-order ODE; a change of σ requires a new integration. This motivated the search for a simpler mathematical model for the same physical problem.

A simpler model is expected to be beneficial for both research and applications. For this objective, in this paper, we develop and test a box-model solution. The box-model approximation makes bold assumptions about the details of the local behaviour of the flow field, and applies integral balances of volume and momentum on the entire mass (the box) of the GC. Box models have been used successfully for influxed viscous GCs, but, to our knowledge, only for unconfined systems (e.g., [1,3,6,9,11,12]). A widely-used simplification (also implemented here) is that the interface of the GC is horizontal, see Figure 1(b); this eliminates the calculation of the profile of the inclined interface, which is the major challenge in the lubrication model.

The present work is a significant extension of the improved box-model presented in [12]. The previous model considers GCs in a very deep environment, i.e., the height of the container, H , is much larger than the height h of the current. In this configuration, called “unconfined,” the GC which propagates on the bottom is not influenced by the top boundary. In the present system, the GC is confined by the top. Moreover, here we take into account the surface-tension effect, because there is evidence ([7]) that the meniscus about the grounding line in the confined flow affects the motion (see Figure 1). In other words, the present box model incorporates two novel physical effects: confinement and surface tension. The present paper demonstrates that the box-model for the self-similar confined flow provides explicit simple and insightful results for both 2D and AXI systems with Newtonian and non-Newtonian power-law fluids, including surface tension effects.

The surface tension is expected to generate, in addition to the meniscus at the grounding line, also a meniscus at the nose, i.e., a modification of the tip of the GC at the bottom position $x_N(t)$ in Figure 1(a). This effect can be incorporated in both the lubrication and box models with an additional small parameter σ_N . However, the typical influence of this meniscus is significantly smaller than that of the

grounding-line meniscus, and hence the main analysis and discussion of the paper ignore this detail. The quantitative justification will be given in Appendix A.

The structure of the paper is as follows. The box-model governing equations are developed, and some useful analytical results are derived for the general system (including surface tension), for the 2D and AXI in § 2.2 and § 2.3, respectively. Results for the special (basic state) with excluded surface tension, $\sigma = 0$, are presented in § 3. At the end of each section we perform stringent quantitative comparisons with the more rigorous lubrication-model solution and show that there is good agreement for a wide range of parameters. A brief comparison with published data is discussed in § 4. Concluding remarks are given in § 5. The effect of the nose meniscus (not included in the main text) is estimated in Appendix A. The method of solution of the lubrication model, which is compared with the box-model predictions, is briefly presented in Appendix B.

2. Formulation and Analysis

2.1. The Bottom-Shear Approximation

An essential component of the approximate model is a simple formula which connects the depth-averaged velocity of the GC with the resulting shear at the bottom. The analysis can be performed in a convenient compact form for both Newtonian and non-Newtonian (power-law) fluids. We use dimensional variables unless stated otherwise.

The dynamic and kinematic viscosities of the current are given by

$$\mu = m \left| \frac{\partial u}{\partial z} \right|^{n-1}, \quad \nu = m/\rho. \quad (1)$$

where m is the consistency index and n is the behavior index (or exponent). A fluid is shear-thinning (pseudoplastic) if $n < 1$, shear-thickening (dilatant) if $n > 1$, and Newtonian if $n = 1$ (in this case, $m = \mu$, the standard dynamic viscosity of the fluid and ν is the standard kinematic viscosity coefficient). The dimension of ν is $\text{cm}^2 \text{s}^{n-2}$.

The flow of the GC is as sketched in Figure 1(a). Assume a 2D flow in x direction with velocity $u(x, z, t)$, and let $z = h(x, t)$ be the height of the interface above the bottom. The surface tension effects are confined to the small meniscus about x_G , and assumed negligible over the rest of the interface (because the capillary length is of the order of H , and the interface is much larger). The pressure in the thin-layer GC is hydrostatic, and hence the intrinsic driving force is given by $-g'(\partial h / \partial x)$, referred to as the buoyancy. The driving force is balanced by the shear, as expressed by the lubrication-simplification momentum equation

$$0 = -g' \frac{\partial h}{\partial x} + \nu \frac{\partial}{\partial z} \left[\left| \frac{\partial u}{\partial z} \right|^{n-1} \frac{\partial u}{\partial z} \right]. \quad (2)$$

This equation is integrated twice with respect to z to obtain $u(x, z, t)$. The constants of integration are determined by the boundary condition: (a) no slip, $u = 0$ at the bottom $z = 0$, and (b) no shear, $(\partial u / \partial z) = 0$, at the free interface $z = h$. We obtain

$$u(x, z, t) = \frac{n}{n+1} \mathcal{G}^{1/n} \left(-\frac{\partial h}{\partial x} \right)^{1/n} h^{(n+1)/n} \left[1 - \left(1 - \frac{z}{h} \right)^{(n+1)/n} \right], \quad (3)$$

where

$$\mathcal{G} = g' / \nu. \quad (4)$$

The depth-averaged velocity is

$$\bar{u} = \bar{u}(x, t) = \frac{1}{h} \int_0^h u(x, z, t) dz = \frac{n}{2n+1} \mathcal{G}^{1/n} \left(-\frac{\partial h}{\partial x} \right)^{1/n} h^{(n+1)/n}. \quad (5)$$

The shear stress at the bottom, $\mu(\partial u / \partial x)$ at $z = 0$, can then be expressed as

$$\tau = \tau(x, t) = \rho \nu \gamma^n \left(\frac{\bar{u}}{h} \right)^n, \quad \text{where } \gamma = 2 + 1/n. \quad (6)$$

In the subsequent analysis *we drop the bar notation from the depth-averaged u* . The box model adopts the result (6) as an approximation for the calculation of the total shear force which balances the total buoyancy on the GC; the local balances are not satisfied because $\partial h / \partial x = 0$ in Figure 1(b). We also note that the results (1)–(6) carry over to the axisymmetric system upon changing x to the radial coordinate r .

2.2. Two Dimensional Confined Flow

The basic simplification of the box model is that the interface of the GC is horizontal, i.e., $h(x, t) = h_N(t)$, see Figure 1(b). In the present case, the confinement requires that the interface is attached to the upper boundary by a meniscus of constant height at the moving grounding line $x_G(t)$. This is modeled by the following assumptions: the meniscus at the grounding line is a vertical jump of constant height σH , and hence the horizontal interface is time-independent at the position $z = h = h_N = (1 - \sigma)H$. (The x -dimension of the meniscus, $\sim H$, is negligible because the GC is assumed to be a thin layer, $H/x_N \ll 1$.) Note that $u(x, t)$ of the box model represents the depth-averaged velocity. For definiteness, we assume $\sigma \leq 0.5$ which means that the meniscus may be large, but not dominant.

We are interested in similarity propagation of the form

$$x_N(t) = Kt^\beta, \quad u_N = \beta t^{\beta-1}, \quad x_G(t) = y_G x_N(t) = y_G Kt^\beta, \quad (7)$$

where K , β and y_G are constants ($K, \beta > 0$, $0 < y_G < 1$). The reason for this choice is that when the volume $\sim t^\alpha$, then x_N and x_G are expected to behave like t to some power, say t^β and t^ω . As t increases, if $\omega < \beta$ then $x_G/x_N \rightarrow 0$ (i.e., unconfined GC) and if $\omega > \beta$ then $x_G/x_N \rightarrow 1$ (slug flow). Consequently, $\omega = \beta$ and the form $x_G = y_G x_N(t)$, with a constant $y_G \in (0, 1)$, correspond to the confinement problem of interest here, which displays a clear-cut grounding line during a long propagation time and distance. We also note that the form (7) is compatible with the similarity solution of the lubrication model; see Appendix B.

The balances are performed per unit width. The volume of the injected fluid is qt^α .

Consider the kinematic balances. The constant $h = h_N = (1 - \sigma)H$ has two important implications. First, total volume conservation

$$x_G H + (x_N - x_G) h_N = qt^\alpha, \quad (8)$$

upon use of (7), yields

$$\alpha = \beta, \quad (9)$$

$$K = \frac{q}{H[1 - \sigma(1 - y_G)]}. \quad (10)$$

Second, the continuity equation $(uh)_x = 0$ imposes $u(x, t) = u_N(t)$ in the GC.

The dynamic balance is between the forces acting on the entire GC in the box from x_G to x_N .¹ This is expressed as

$$F_V = F_B, \quad (11)$$

where the buoyancy force is given by

$$F_B = (1/2)\rho g' h_N^2, \quad (12)$$

¹ The force balance for the slug is unimportant in the present analysis. We assume that the source (pump) at $x = 0$ provides the pressure necessary for sustaining the flow of the influxed fluid in the gap $x > 0$.

and the viscous force due to the shear at the bottom is approximated by

$$F_V = \int_{x_G}^{x_N} \tau dx = \rho \nu \gamma^n \left(\frac{u_N}{h_N} \right)^n x_N (1 - y_G). \quad (13)$$

We used (6) and $u = u_N$ (justified above). The surface tension contributes a hindering (drag) force on the slug of dense fluid in the domain $x < x_G$, which is balanced by the buoyancy over the meniscus in the gap $h_N < z < H$. The GC in the domain $x > x_G$ is not directly affected by this force; however, the presence of the meniscus reduces F_B of the GC because $h_N = (1 - \sigma)H$; see (12).

In the assumed flow F_B is a constant. The force balance requires the same property for F_V . Inspection of (13) shows that this can be satisfied when

$$u_N^n x_N = \beta^n K^{n+1} t^{(n+1)\beta-n} = \beta^n \frac{q^{n+1}}{H^{n+1} [1 - \sigma(1 - y_G)]^{n+1}} t^{(n+1)\beta-n} = \text{const.} \quad (14)$$

in which (7) and (10) were used. This yields

$$\beta = n / (n + 1). \quad (15)$$

By combining equations (11)–(15), setting $h_N = (1 - \sigma)H$, and some algebra we obtain

$$\frac{1 - y_G}{[1 - \sigma(1 - y_G)]^{n+1}} = (1 - \sigma)^{n+2} \phi J^{-(2n+3)}, \quad (16)$$

or

$$J = \left[(1 - \sigma)^{n+2} \phi \frac{[1 - \sigma(1 - y_G)]^{n+1}}{1 - y_G} \right]^{1/(2n+3)}, \quad (17)$$

where

$$\phi = \phi(n) = \frac{1}{2} \left(\frac{n+1}{n} \right)^n, \quad (18)$$

$$J = \frac{l}{H}, \quad l = \left[\gamma^n q^{n+1} \mathcal{G}^{-1} \right]^{1/(2n+3)}. \quad (19)$$

The dimensionless parameter J is evidently the ratio of two lengths: l , the typical thickness of the unconstrained GC (with the corresponding influx conditions) and H , the height of the gap. The coefficient ϕ is approximately 1 (increases from 0.7 to 1.3 for $n \in (0.2 - 10)$).

Equation (16) (or (17)) is a useful result of the box-model analysis. This equation expresses in a quite explicit dimensionless form the connections between the governing parameter J , grounding line position y_G , meniscus height σ , and viscosity law n . We recall that $0 \leq y_G \leq 1$, and $0 \leq \sigma \lesssim 0.5$, while n is typically of the order of unity. We also keep in mind that $J \propto q/H$. Some parametric trends of the physical system of the confined GC can be inferred by inspection of these equations, as follows. (1) In general, a larger J corresponds to a larger y_G . In other words, a stronger flux (or smaller gap) generate a longer filling slug. (2) For a fixed y_G , as the meniscus height σ increases, J decreases.

It is convenient to define J_0 and $J_{0.9}$ which correspond to $y_G = 0$ (the transition to unconfined flow) and $y_G = 0.9$ (the confined slug occupies 90 % of the distance of propagation). The values are directly provided by (17). In particular

$$J_0 = (1 - \sigma) \phi^{1/(2n+3)}; \quad J_{0.9} = \left[10(1 - \sigma)^{n+2} (1 - 0.1\sigma)^{n+1} \phi \right]^{1/(2n+3)}. \quad (20)$$

The presence of surface-tension effects represented by $\sigma > 0$ may reduce significantly the transition value J_0 , and also the value of $J_{0.9}$. This could be expected, because the meniscus reduces the effective gap.

We now estimate the effect of the grounding-line meniscus on the major behaviour of the confined 2D GC. First, we consider a fixed J and estimate the change Δy_G of y_G when σ increases from 0 (position y_G). The leading term of an expansion for small σ of (16) gives

$$\Delta y_G = (1 - y_G)[2n + 3 - (n + 1)y_G] \sigma \quad (\text{fixed } J). \quad (21)$$

For $y_G < 0.5$, this is a very significant increase, $\sim 2\sigma$.

Second, we consider a fixed y_G , and estimate the required change of J when σ increases from zero. Equation (16) indicates that for keeping y_G fixed, J^{2n+3} must decrease at least as $(1 - \sigma)^{n+2}$. An expansion for small σ yields

$$\frac{\Delta J}{J} = - \left[1 - \frac{n+1}{2n+3} y_G \right] \sigma \quad (\text{fixed } y_G). \quad (22)$$

In particular, we note that J_0 is expected to decrease from about 1 to $1 - \sigma$. Again, the interpretation is that the meniscus reduces the effective gap into which the GC is injected. For a larger σ , a smaller flux (smaller J) preserves the position y_G .

The coefficients of σ in (22) and (21) are plotted in Figure 2. The box model predicts significant influence of σ on the behaviour of the confined GC. The effect is most pronounced for small y_G and larger n .

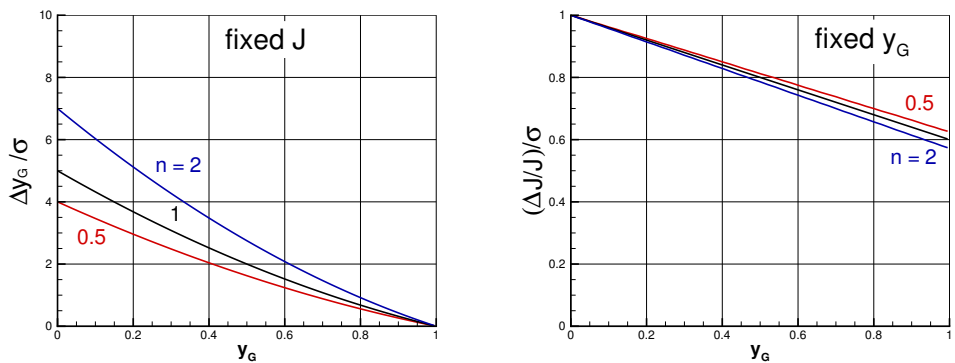


Figure 2. 2D case. The slopes $\Delta y_B / \sigma$ and $(\Delta J / J) / \sigma$ as functions of y_G for various n .

We switch to dimensionless variables denoted by an upper $\sim$. The scales for length, speed and time are H, U, T , as follows

$$U = \gamma^{-1} \mathcal{G}^{1/n} H^{(n+1)/n}, \quad T = H/U. \quad (23)$$

The coefficient K , see (10), is scaled with $HT^{-\beta}$. Now the propagation reads

$$\tilde{x}_N = \tilde{K} \tilde{t}^\beta, \quad \tilde{K} = \frac{J^{(2n+3)/(n+1)}}{1 - \sigma(1 - y_G)}. \quad (24)$$

The interpretation of $\tilde{K}$ requires some care: y_G in the denominator is provided by (16), not an independent variable.

The box model solution has been completed. We determined that the similarity 2D flow exists for $\alpha = \beta = n/(n+1)$; this is identical with the prediction of the lubrication model (see Appendix B). For a given fluid (i.e., a fixed n) the only free parameters, which govern the values of y_G and $\tilde{K}$, are J and σ . Qualitatively, this is also in agreement with the lubrication model.

A quantitative comparison between the box and lubrication models is shown in Figures 3 and 4. We see that there is good agreement between the predictions of y_G as a function of J and σ , for all tested values of n . For a given J , the box model predicts a smaller y_G than the lubrication model. This

could be anticipated because the constant h profile of the box produces a smaller shear per unit length, and hence requires a larger length $1 - y_G$ for balancing the buoyancy driving force. Figure 4 shows a remarkably good agreement between the models concerning the prediction of $\tilde{K}$. The curves of $\tilde{K}$ vs. J of the box and lubrication models, for a given n , practically coincide; this holds for both $\sigma = 0$ and $\sigma = 0.5$. This demonstrates that the box model captures well the dominant mechanisms of the flow, and gives confidence into the present results.

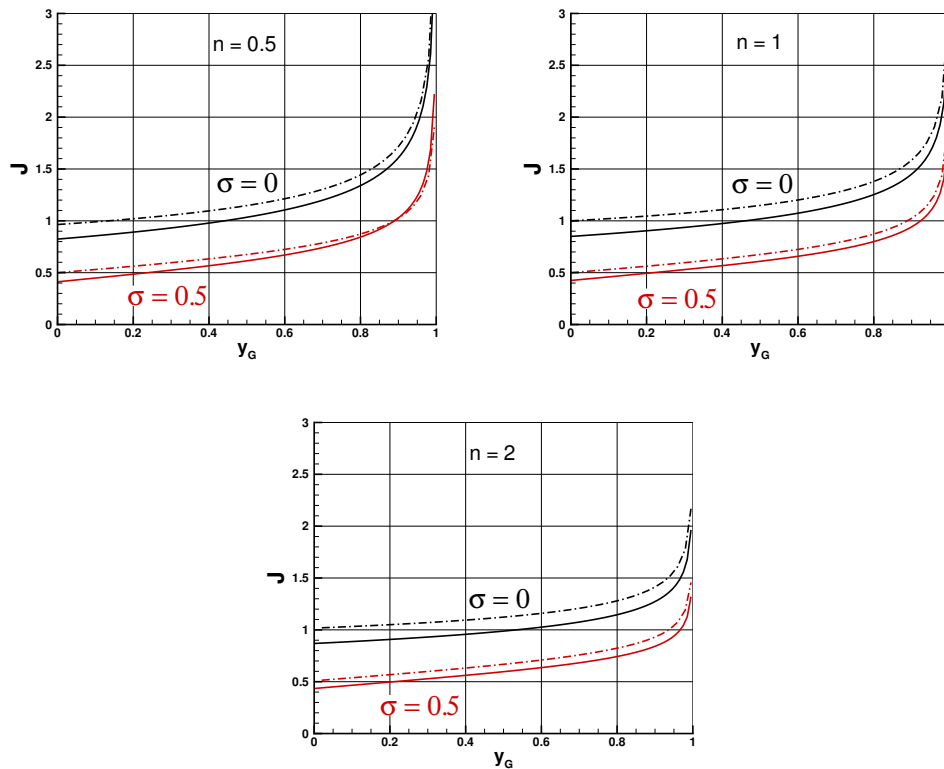


Figure 3. Box model (dash-dot line) and lubrication model (solid line) results J vs. y_G for $\sigma = 0$ (black lines) and $\sigma = 0.5$ (red lines) for 2D system. The frames are for $n = 0.5, 1, 2$.

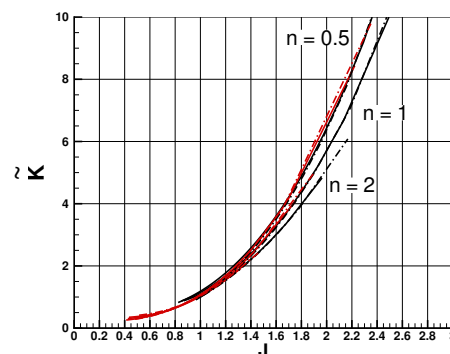


Figure 4. $\tilde{K}$ vs. J for the systems of previous figure.

2.3. Axisymmetric Confined Flow

We use dimensional variables unless stated otherwise. Again, the box model see Figure 1(b), subject to the confinement condition, assumes $h(r, t) = h_N = (1 - \sigma)H$. We shall keep the constant h_N

in our balances for a while, and at a later stage we shall impose $h_N = (1 - \sigma)H$. We are interested in similarity propagation of the form

$$r_N(t) = Kt^\beta, \quad u_N = \beta t^{\beta-1}, \quad r_G(t) = y_G r_N(t) = y_G Kt^\beta. \quad (25)$$

The balances are formulated per radian (i.e., we omit the coefficient 2π). The volume of the injected fluid is qt^α . We assume a source at $r = 0$ which is non-physical; a correction will be suggested later.

Consider the kinematic balances. First, total volume conservation

$$(1/2) \left[r_G^2 H + (r_N^2 - r_G^2) h_N \right] = qt^\alpha, \quad (26)$$

upon use of (25), yields

$$\alpha = 2\beta, \quad (27)$$

$$K = \left[\frac{2q}{H[1 - \sigma(1 - y_G^2)]} \right]^{1/2}. \quad (28)$$

Second, the continuity equation $(ruh)_r = 0$ shows that in the GC

$$u(r, t) = u_N(t) r_N(t) / r. \quad (29)$$

The dynamic balance is again given by (11). The buoyancy force is

$$F_B = (1/2) \rho g' h_N^2 r_N. \quad (30)$$

The viscous force due to the shear at the bottom is expressed as

$$F_V = \int_{r_G}^{r_N} \tau r dr = \rho \nu \gamma^n \left(\frac{u_N}{h_N} \right)^n r_N^2 \int_{y_G}^1 y^{1-n} dy, \quad (31)$$

where $y = r/r_N$; we used (6) and (29).

Comparing F_V with F_B we deduce that compatibility requires

$$u_N^n r_N = \beta^n K^{n+1} t^{(n+1)\beta-n} = \beta^n \left[\frac{2q}{H[1 - \sigma(1 - y_G^2)]} \right]^{(n+1)/2} t^{(n+1)\beta-n} = \text{const.} \quad (32)$$

where (25) and (28) were used This yields

$$\beta = n/(n+1). \quad (33)$$

The force balance in view of the results (30)–(33), after some algebra, yields

$$\frac{1 - y_G^{2-n}}{[1 - \sigma(1 - y_G^2)]^{(n+1)/2}} = (1 - \sigma)^{n+2} (2 - n) \Psi J^{-(3n+5)/2} \quad (n \neq 2), \quad (34)$$

$$\ln y_G = -(1 - \sigma)^4 [1 - \sigma(1 - y_G^2)]^{3/2} \left(\frac{9}{16\sqrt{2}} J^{-11/2} \right) \quad (n = 2), \quad (35)$$

where

$$\psi = \psi(n) = 2^{-(n+1)/2} \frac{1}{2} \left(\frac{n+1}{n} \right)^n = 2^{-(n+1)/2} \phi, \quad (36)$$

$$J = \frac{l}{H}, \quad l = \left[\gamma^n q^{(n+1)/2} \mathcal{G}^{-1} \right]^{2/(3n+5)}. \quad (37)$$

The set (34)–(35) is continuous about $n = 2$. To show this, we divide (34) by $2 - n = \epsilon$, and note that $\lim_{\epsilon \rightarrow 0} (1 - y_G^\epsilon)/\epsilon = -\ln y_G$.

The dimensionless parameter J is, again a ratio of two lengths, with the same physical interpretation as in the 2D case.

Equations (34) and (35) can be inverted to obtain explicit formulas for J as a function of y_G . However, here the transition to the unconfined GC is not straightforward at $y_G = 0$ because the theoretical u is unbounded at the axis, see (29). The same difficulty was noted in the lubrication model U25, and the remedy is to define the transition (and hence J_0) at some small but finite $y_0 = 0.05$; this mimics a source of finite radius which is unavoidable in practical systems.

We now estimate the effect of the grounding-line meniscus on the confined AXI GC. An expansion of of (34) in powers of σ yields, to leading order, the following changes from the basic flow ($\sigma = 0$):

$$\Delta y_G = y_G^{n-1} \left\{ \frac{1 - y_G^{2-n}}{2 - n} \right\} \frac{1}{2} [3n + 5 - (n + 1)y_G^2] \sigma \quad (\text{fixed } J). \quad (38)$$

The formula is valid for $n \neq 2$; for $n = 2$ the term in the curly brackets is replaced by $-\ln y_G$.

$$\frac{\Delta J}{J} = - \left[1 - \frac{n + 1}{3n + 5} y_G^2 \right] \sigma \quad (\text{fixed } y_G). \quad (39)$$

We note that for $y_G < 0.5$, the relative change of J needed to keep a fixed y_G is close to $-\sigma$, like in the 2D system. The change of y_G with σ is more complex in the diverging AXI flow. The coefficients of σ in (39) and (38) are plotted in Figure 5. The box model predicts significant influence of σ on the behaviour of the confined AXI GC. The effect is most pronounced for small y_G . For a fixed J and small y_G the axisymmetric system is very sensitive to surface-tension effects. This can be attributed to strong changes of the flow field near the axis.

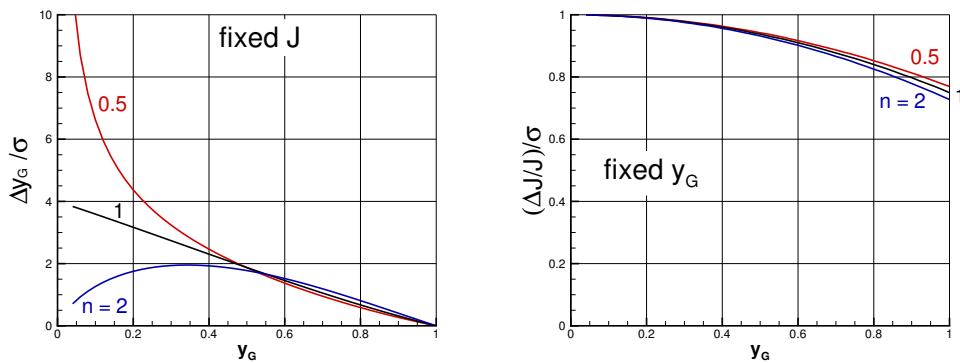


Figure 5. AXI case. The slopes $\Delta y_B/\sigma$ and $(\Delta J/J)/\sigma$ as functions of y_G for various n .

We switch to dimensionless variables denoted by an upper $\sim$. The scales for length, speed and time are H, U, T , as for the 2D case, see (23), and the coefficient K , see (28), is scaled again with $HT^{-\beta}$. Now the propagation reads

$$\tilde{r}_N = \tilde{K} \tilde{t}^\beta, \quad \tilde{K} = \frac{\sqrt{2} J^{(3n+5)/(2(n+1))}}{[1 - \sigma(1 - y_G^2)]^{1/2}}. \quad (40)$$

Again, the interpretation of $\tilde{K}$ requires some care: y_G in the denominator is provided by (34), not an independent variable.

The box model solution has been completed. We determined that the similarity AXI flow exists for $\alpha = 2n/(n + 1) = 2\beta$, identical with the prediction of the lubrication-model. The only free parameters, which governs the values of y_G and $\tilde{K}$, are J and σ (for a given fluid, i.e., a fixed n). Qualitatively, this is in agreement with the lubrication model. A quantitative comparison between the models is show in

Figures 6 and 7. There is fair agreement in general, and the differences could be anticipated with the same arguments presented for the 2D flow. Again, there is a remarkably good agreement between the models concerning the value of $\tilde{K}$.

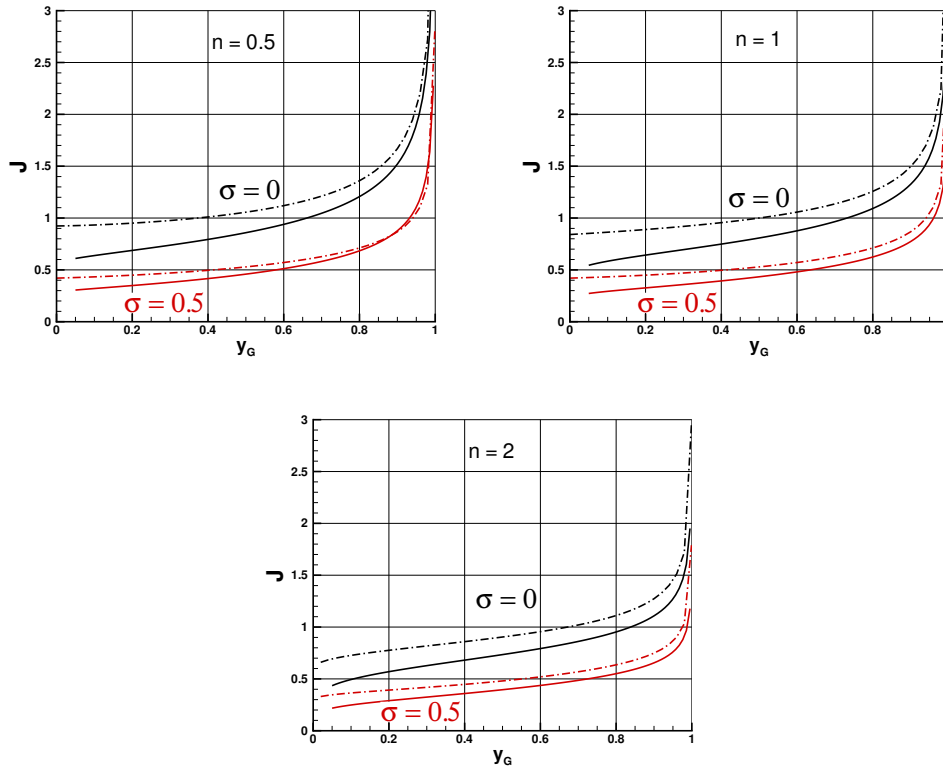


Figure 6. Box model (dash-dot line) and lubrication model (solid line) results J vs. y_G for $\sigma = 0$ (black lines) and $\sigma = 0.5$ (red lines) for AXI system. The frames are for $n = 0.5, 1, 2$.

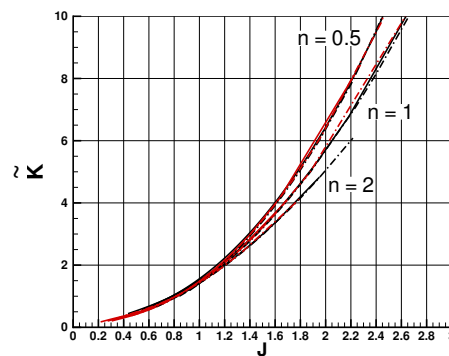


Figure 7. $\tilde{K}$ vs. J for the systems of previous figure.

3. Results for $\sigma = 0$

The $\sigma = 0$ (no surface tension effects) system merits a detailed discussion for several reasons. First, the formulas are amenable for simplifications which enhance the understanding of the confined GC flow. Second, systems with small (negligible) surface tension may be of interest in experiments and application. Third, the $\sigma = 0$ solution is a valuable reference (base) for the $\sigma > 0$ behaviour. We have developed simple formulas for estimating the change from this base when $\sigma > 0$ is present.

The $\sigma = 0$ system can be regarded as the limit $\sigma \rightarrow 0$ of the analysis and results presented in the previous sections. The interface of the GC in the $x > x_G(t)$ domain, see Figure 1(b), is approximated

by a horizontal line $z = h_N = H^-$, i.e., very near to, but detached from, the top boundary. The tiny gap, σH , is close to zero at the grounding line $x_G(t)$. The flow is governed by the parameter J , see (19) and (37). The values of α and β are as before.

3.1. 2D Confined Flow for $\sigma = 0$

A good insight into the confinement effect is provided by equation (16) with $\sigma = 0$, i.e.,

$$y_G = 1 - \phi J^{-(2n+3)}, \quad (41)$$

which predicts the relative position y_G of the grounding line for a given J (fixed n). Note that $J \propto q$. For $J > 1$ the second term in the RHS of (41) decays faster than J^{-3} . For a large J , y_G is close to 1, i.e., almost all the injected fluid propagates as a slug along the gap. As J decreases, y_G becomes small, i.e., a significant part of the injected fluid is a GC over the bottom with a free interface below the top of the gap. Further decrease of J brings y_G to zero (the slug domain disappears at $J = J_0$), and then to negative values. The non-physical $y_G < 0$ indicates the failure of the confinement assumption; in other words, the influx conditions produce a GC with $h_N < H$ which propagates over the bottom of the gap without touching the top. Equation (41) also clarifies that the decay $J^{-(2n+3)}$ becomes faster as n increases, i.e., for a given J the grounding y_G is closer to 1 for larger n . The shear increases with n , and hence a shorter distance $1 - y_G$ is needed for counteracting the buoyancy force.

Equation (41) can be rewritten as

$$J = [\phi / (1 - y_G)]^{1/(2n+3)}, \quad (0 \leq y_G < 1). \quad (42)$$

Since $\phi \approx 1$, it is evident that, for $y_G = 0$, J_0 is very close to 1 in general. This confirms the claim that l , see (19), is the thickness of the unconfined GC.

The dimensionless propagation, see (24) and (42), reads now

$$\tilde{x}_N = \tilde{K} \tilde{t}^\beta, \quad \tilde{K} = J^{(2n+3)/(n+1)} = \left(\frac{\phi}{1 - y_G} \right)^{1/(n+1)}. \quad (43)$$

A quantitative comparison between the box and lubrication models with $\sigma = 0$ is shown in Figures 8 and 9. We see that there is fair quantitative agreement between the predictions. The differences could be anticipated because the constant h profile of the box produces a smaller shear per unit length, and can accommodate the same volume over a shorter x_N . Consequently, for a fixed J (i.e., fixed q) the box model is expected to predict a larger y_G and smaller $\tilde{K}$ than the lubrication model (for the same n). The comparisons confirm these expectations. Moreover, the differences of $\tilde{K}$ for a given J are surprisingly small: Figure 9 shows that the lines of $\tilde{K}$ vs. J almost coincide for $J > 1.3$ (the agreement increases with n). This demonstrates that the box model captures well the dominant mechanisms of the flow, and gives confidence into the present results.

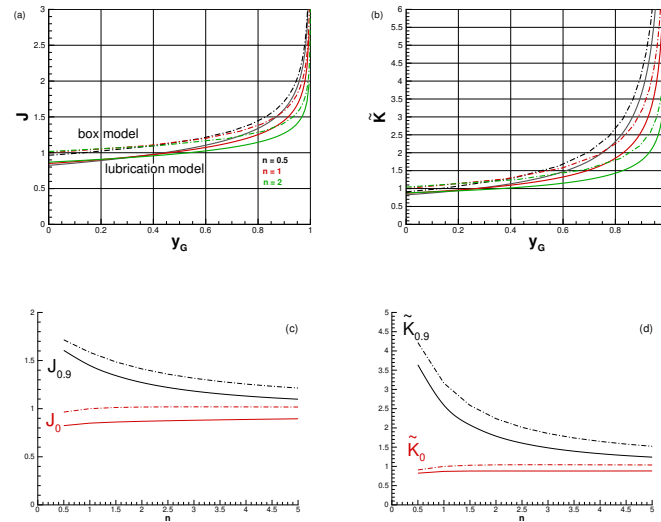


Figure 8. Box model (dash-dot line) and lubrication model (solid line) results for 2D system for $\sigma = 0$. (a,b) J and $\tilde{K}$ as functions of y_G for various n (0.5, black; 1, red, 2, green). (c,d) $J_0, J_{0.9}, \tilde{K}_0, \tilde{K}_{0.9}$ as functions of n .

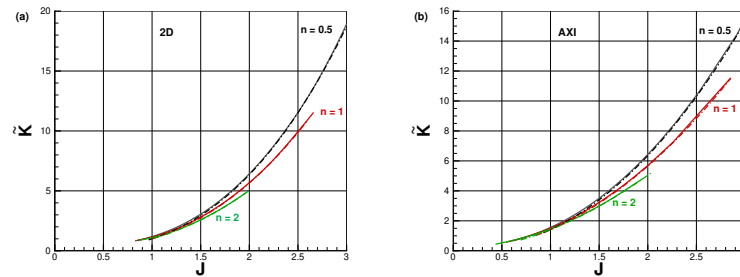


Figure 9. $\tilde{K}$ vs. J for $\sigma = 0$, various n . Box model (dash-dot line) and lubrication model (solid line). (a) 2D and (b) AXI system. The red and green lines end at the value of J for which $y_G = 0.995$.

3.2. AXI Confined Flow for $\sigma = 0$

For obtaining a good insight into the confinement effect we rearrange (34) for $\sigma = 0$, and use an expansion in powers of the type $(1 + \epsilon)^p \approx 1 + p\epsilon$, where $\epsilon = \Psi J^{-(3n+5)/2}$. We obtain

$$y_G \approx 1 - \Psi J^{-(3n+5)/2} = 1 - 2^{-(n+1)/2} \phi J^{-(3n+5)/2}. \quad (44)$$

This equation is exact for $n = 1$ and in general a good approximation for $J > 1$. We note that (44) is quite similar with the 2D counterpart (41); here, again, $J \propto q$, and the decay with J of the last RHS term is also strong, at least as $J^{-5/2}$. The arguments presented for the 2D confinement effect can be repeated here. For a large J (strong influx), y_G is close to 1, and as J decreases (weak influx) y_G tends to 0. However, in the AXI system $r = 0$ is a singular point, and hence the transition to unconfined flow is assumed to occur at a small but finite $y_G \approx 0.05$. The typical value of J_0 for $y_0 = 0.05$ is 0.7 for a fairly large range on n , see Figure 10. This supports the claim that l given by (37) is the typical thickness of the unconfined GC. For $J < J_0$ the GC can be considered unconfined.

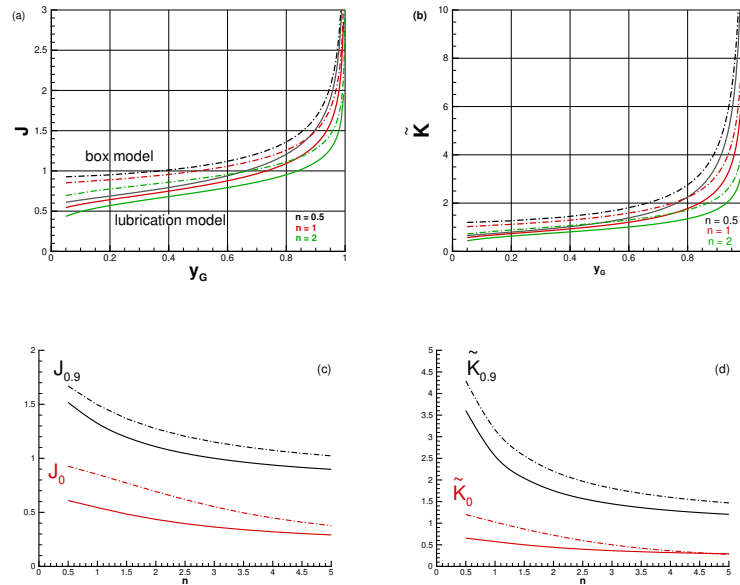


Figure 10. Box model (dash-dot line) and lubrication model (solid line) results for AXI system for $\sigma = 0$. (a,b) J and $\tilde{K}$ as functions of y_G for various n . (c,d) J_0 , $J_{0.9}$, $\tilde{K}_0$, $\tilde{K}_{0.9}$ as functions of n .

The dimensionless propagation, see (40) with $\sigma = 0$, reads

$$\tilde{r}_N = \tilde{K} \tilde{t}^\beta, \quad \tilde{K} = \sqrt{2} J^{(3n+5)/(2(n+1))}. \quad (45)$$

A quantitative comparison between the models with $\sigma = 0$ is shown in Figures 9(b) and 10. There is fair agreement in general, and the differences could be anticipated with the same arguments presented for the 2D flow. Again, the differences of $\tilde{K}$ for a given J are surprisingly small: Figure 9(b) shows that the lines of $\tilde{K}$ vs. J almost coincide for $J > 1.2$ (the agreement increases with n).

4. Comparison with Data

A useful test for the validity of the box model is provided by a comparison of the present predictions with the experimental data presented in the papers of [8] and [7]. The laboratory system is AXI, the GC fluid is Newtonian ($n = 1$) golden syrup, the ambient fluid is air (hence $g' = g$), the gaps H are 0.71, 1.07, and 1.48 cm. The viscous fluid was influxed at a constant rate ($\alpha = 1$) by a piston pump via a hole of 5.5 cm diameter in the bottom plate and spread out in a quite axisymmetric pattern (the maximum r_N was 27 cm). For sufficiently large influx rates, a clear separation between the slug and the GC, i.e. a clear grounding line, was observed. The propagation of the nose $r_N(t)$ and grounding line $r_G(t)$ (when present) were recorded for various constant volume fluxes. The absence of the grounding line indicates a free (unconfined) GC.

[8] and [7] use the same scaling as here for length, velocity and time, see (23), but employ the notations η_N and η_G , as follows

$$\tilde{r}_N = \eta_N \tilde{t}^{1/2}; \quad \tilde{r}_G = \eta_G \tilde{t}^{1/2}. \quad (46)$$

In terms of our scaled solution, see (40), the correspondence is: $\beta = 1/2$, $\eta_N = \tilde{K}$, $\eta_G/\eta_N = y_G$.

The definition of J here ((37)) for $n = 1$ recovers the definition of J in these papers.

The box model predicts that for $n = 1$ a self-similar propagation of the type (46) appears in the AXI system for $\alpha = 1$. The experiments confirmed this prediction: the measured r_N and r_G fitted very well that pattern. This allowed the calculation of the experimental η_N and η_G (the estimated error is 11%; the error of the experimental $y_G = \eta_G/\eta_N$ is 22 %).

Table 1 of [8] provides values of the experimental η_N and η_G as functions of J for three different gaps, H . [7], using available experimental values of the surface tension Γ , pointed out that the three

different gaps H also imply three different values of the parameter $S = (\Gamma/\rho g)^{1/2}/H = 0.16, 0.22, 0.34$. The suggestion is to use the approximation $\sigma = S$ in the models. Figure 4 of [7] shows the experimental points η_G and η_N vs. J (symbols), and the corresponding predictions (curves) of the lubrication model which takes into account the surface tension effect.

The box-model predictions for the tested flows are straightforward. For the AXI system with $n = 1$, we obtain $\alpha = 1, \beta = 1/2$, (see (27)) and (33)). For a given σ , the algebraic equation (34) and (40) provide values of $\tilde{K}$ and y_G for various J of interest. Using these results and the conversion $\eta_N = \tilde{K}, \eta_G = y_G \tilde{K}$, we present in Figure 11 a similar plot to Fig. 4 of [7]. The lines are box-model predictions. There is a fair agreement of the predictions with the data. We see that the model predicts correctly the behaviors of η_N and η_G . There is strong decay of η_N as J decreases from 1.2 to 0.5 (approximately). However, the effect of σ on this variable is remarkably small: the theoretical lines for the three values of σ almost overlap, and the open symbols for the three values of σ collapse on a smooth (imaginary) line. η_G decreases faster than η_N as J decreases. η_G increases with σ , as seen in the prediction lines and in the filled symbols. (For $J < 0.5$ the comparison with data is irrelevant because the GC is unconfined.)

The box model provides the physical interpretation of these trends. The parameter J is proportional to q/H . When the influx rate decreases and/or the gap increases, the parameter J decreases, and in the physical system the coefficients η_G and η_N of the propagation of the fronts, see (46), are expected to decrease due to volume conservation. A larger σ cause a smaller effective thickness h_N , see Figure 1(b), and a smaller buoyancy force $\sim h_N^2$. This allows a reduction of the bottom drag of the GC which is $\sim (\eta_N - \eta_G)$. Consequently, at a fixed J , a larger η_G appears for a larger σ . The behaviour of η_N is subtle. For a fixed J (fixed influx) a larger σ renders a smaller effective thickness h_N ; this would suggest an increase of η_N to accommodate the volume. Here enters the second front. The larger σ increases η_G , which means an increase of the radius of the slug. The larger slug absorbs a larger portion of the influxed fluid, and counteracts the expected increase of η_N . The resulting flow displays a value of η_N which is little affected by the variations of σ in these data.

The quantitative discrepancies between the data and the theoretical predictions observed in our Figure 11 are, roughly, of the same magnitude for the present box model and the lubrication model used by [7] (Fig. 4 in that paper). It is not possible to determine how much of the discrepancy is attributable to the box-model simplifications, because of the restricted data set, experimental errors, and uncertainty of the parameter σ . Taking into account the fact that the box model predicts correctly the values of α and β of the experimental system, the overall predictions of the box model in this test seem to capture well the physical balances of the flow under consideration. In our opinion, this comparison provides support to the box-model as a predictive tool for realistic systems, and motivates further tests in other systems, in particular power-law fluids.

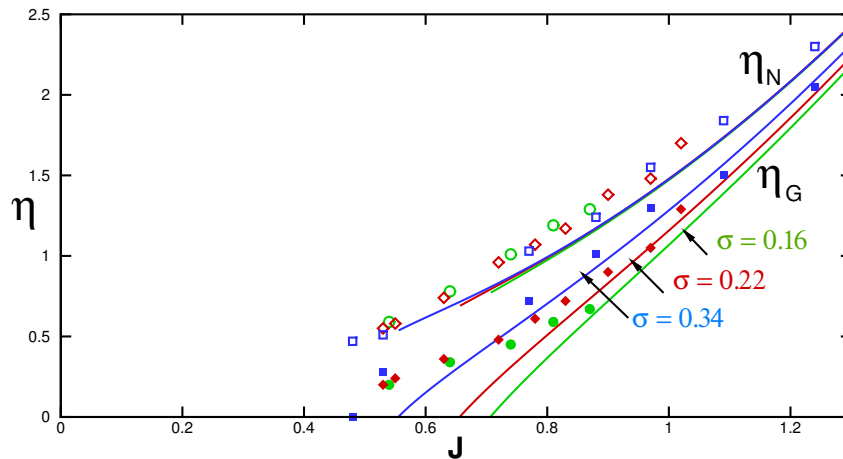


Figure 11. $\eta_N = \tilde{K}$ and $\eta_G = y_G \tilde{K}$ as functions of J for $\sigma = 0.16, 0.22, 0.34$ (green, red, blue). Box model (lines) and data of [7] (open symbols for η_N , filled symbols for η_G). (For $J < 0.5$ the GCs are typically unconfined, and outside the scope of our model.)

5. Conclusions

We presented a box-model analysis of the flow of a viscous fluid injected into a horizontal gap of height H in rectangular 2D and cylindrical AXI geometries, for a power-law non-Newtonian fluid ($\mu \sim |\partial u / \partial z|^{n-1}$). When the displaced (ambient) fluid is less dense and significantly less viscous than the injected one (a typical occurrence in molding industry), and the influx is sufficiently strong, the leading part of the flow is a GC while the tail fills the gap. We focused attention on the influxed volume $\mathcal{V} = qt^\alpha$. We showed that for certain values of α (depending on n) the flow is self-similar: the propagation of the nose and of the grounding line are like Kt^β and $y_G Kt^\beta$. In general, $\beta = \alpha$ in 2D and $\beta = \alpha/2$ in AXI. We recall that the unconfined GCs display self-similar propagation for any $\alpha \geq 0$. The confinement restricts the similarity flow to $\alpha = n/(n+1)$ and $2n/(n+1)$ in 2D and AXI flows, respectively. We obtained explicit formulas for the stringent features of the flow field, like $y_G, \tilde{K}, J_0, J_{0.9}$. All the results are in good agreement with the more rigorous lubrication-model solution (which does not assume a horizontal interface). In particular, we note identical prediction for the time-power coefficients α and β , and a surprisingly small discrepancy for the dimensionless prefactor $\tilde{K}$.

In general, the flow under consideration depends on four parameters: J, n, σ and σ_N . The box-model demonstrates that the dependencies are strongly coupled and non-linear, and hence a simple model is helpful for understanding the physical influence of each of the parameters.

The box model reveals that the main surface tension effects, $\propto \sigma$, are: (a) for a fixed J , the position y_G increases; and (b) for maintaining a fixed y_G , J must be reduced. The effects of a nose meniscus are in the same direction, but $\propto \sigma_N^2$.

The restriction of the solution to special values of the influxed volume coefficient α reproduces a physical condition, not a mathematical simplification. We argue that the similarity behaviour corresponding to this α is essential in the present problem; influx with a different α is bound to produce a very different flow pattern. The present α keeps the grounding line between the inlet and the nose ($0 < y_G < 1$). In this case the GC elongates, but maintains a constant average thickness. When α is reduced, the GC tends to become thinner with t , the grounding line moves toward the inlet, and an unconfined GC appears on the bottom of the gap. When α is increased, the GC tends to become thicker with t , the grounding line moves toward the nose, and all the injected fluid behaves like a slug in contact with both boundaries of the gap. It is remarkable that the similarity solution is robust: for a given geometry, the time powers α and β depend on n , and are unaffected by the presence of surface tension effects represented by σ and σ_N . This interesting prediction has been verified experimentally for the AXI system and $n = 1$ ([7]), and further tests will be beneficial.

We recall that the accurate value of the surface-tension parameter σ is not available for practical systems. The comparisons of predictions with data, performed by [7] and here in § 4, suggests that $\sigma = (\Gamma/\rho g')^{1/2}/H$ is a fair approximation. This is a good starting point for the practical use of the present models. However, we must keep in mind that the data set of [7] covers only AXI flow of a Newtonian fluid. For the assessment and improvement of the extended theory, a much wider experimental study is necessary. We hope that this will be reported in the near future.

The advantage of the box model is in the simplicity of the derivation and the predictions. We straightforwardly gain the insight that the kinematic (volume continuity) balances determine the relation between α and β , and the coefficient K ; the dynamic force balance determine the values of β and y_G . The interconnections between the dimensionless variables $y_G, \tilde{K}, J, \sigma, \sigma_N, n$ are expressed in explicit formulas, without any adjustable constants. The description of the flow has been reduced to a set of basic algebraic-substitution formulas. The influence of the surface tension meniscus at the grounding line (parameter σ) is well elucidated. The box model also clarifies that the effect of the nose meniscus (parameter σ_N) is typically negligible. These simplifications are expected to facilitate the analysis and design of various applications. Needless to say, the same information can be extracted from the more rigorous lubrication model, but this requires a more significant effort of programming, data processing, and interpretation.

We emphasize that, in general, the box model is an unreliable prediction tool, unless subjected to stringent verification against some more accurate results over a significant range of parameters. The present box model can be recommended because it has passed such tests (without using any adjustable constants), and we have reliable information about the expected error concerning the sign (under/over prediction) and order of magnitude. However, we keep in mind that this paper compares mostly between two theories based on simplifications (such as thin layer, hydrostatic pressure, sharp meniscus, well-controlled influx). The convincing assessment of the models requires critical comparison with experimental data. (The comparison with the data of [7] is in § 4 is encouraging, but covers a restricted domain, in particular $n = 1$.) In view of the large number of involved parameters, this is a challenging experimental task. We are confident that the present quite sharp theoretical predictions will facilitate the experimental verification.

Appendix A. The Effect of the Nose Meniscus

We assume that the height of the nose meniscus is $h_M = \sigma_N h_N$. The surface tension of this meniscus yields a drag force on the GC. We estimate this force, F_M , from the observation that the meniscus truncates the sharp tip of the GC shown in Figure 1(a). The lubrication balance in this region (before truncation, with no surface tension) is

$$\rho g' \frac{\partial h}{\partial x} = \frac{\partial \tau}{\partial z}, \quad (\text{A1})$$

where τ is the viscous stress in the fluid. Suppose that the truncation is from position x_N where $h = 0$ to position x_M where $h = h_M$. Integration of (A1) with respect to z and then x (with the appropriate boundary conditions) yields the connection between the height h_M and the friction force of the tip section $[x_M, x_N]$

$$\frac{1}{2} \rho g' h_M^2 = F_M. \quad (\text{A2})$$

After that truncation, point M is considered to be the nose (N) affected by the meniscus. We return to the box model. Letting $h_M = \sigma_N h_N$, we therefore estimate that the drag force of the nose meniscus is $(1/2) \rho g' (\sigma h_N)^2$ (for 2D). This force is conveniently subtracted from the buoyancy force of the original box model. Consequently, we estimate that the buoyancy available for counteracting the bottom shear is reduced to $\hat{F}_B = (1/2) \rho g' h_N (1 - \sigma_N^2)$ in 2D (multiplied by r_N in AXI).

The incorporation of this effect in the foregoing analysis requires a simple modification: use $F_V = \hat{F}_B$ instead of $F_V = F_B$. The box model indicates that the effect of σ_N (the nose meniscus) is (a) in

the same direction as that of σ , but (b) much smaller than that of σ (the grounding line meniscus). The details are as follows.

We note that now the RHS of (16) and (34), which govern the position y_G , will be multiplied by $(1 - \sigma_N^2)$. Using these balances and an expansions in powers of σ_N , we estimate the changes of the basic flow ($\sigma = \sigma_N = 0$) due to a small σ_N . We obtain:

For 2D.

$$\Delta y_G = (1 - y_G) \sigma_N^2, \quad (\text{fixed } J). \quad (\text{A3})$$

$$\frac{\Delta J}{J} = -\frac{1}{2n+3} \sigma_N^2 \quad (\text{fixed } y_G). \quad (\text{A4})$$

For AXI.

$$\Delta y_G = y_G^{n-1} \left\{ \frac{1 - y_G^{2-n}}{2-n} \right\} \sigma_N^2 \quad (\text{fixed } J), \quad (\text{A5})$$

$$\frac{\Delta J}{J} = -\frac{1}{3n+5} \sigma_N^2. \quad (\text{fixed } y_G). \quad (\text{A6})$$

(For $n = 2$ the term in the curly brackets of (A5) is replaced by $-\ln y_G$).

Suppose that the surface tension imposes some small $\sigma \approx \sigma_N > 0$. Comparison of (A5)–(A6) with (21), (22), (38) and (39) demonstrates that the contribution of the grounding-line meniscus (represented by the parameter σ) is consistently much more significant. These box-model estimates are in agreement with the lubrication-model tests of [7]. We have also performed lubrication-model tests with $\sigma_N > 0$ (details not shown) and detected small variations.

The conclusion is that for the confined GC the dominant surface tension effect is at the grounding line.

Appendix B. The Lubrication Model

The lubrication model for power-law viscous GCs with no surface tension ($\sigma = \sigma_N = 0$) in 2D and AXI geometries is presented in U25. Equations of that paper will be denoted by the addition of U in the number. Here we briefly describe the modifications when non-zero σ and/or σ_N are present. Physically, the menisci at the grounding line and/or nose are assumed to be vertical jumps of the interface which do not affect the mechanics of the GC. The governing equations of the GC are unchanged, but the boundary conditions at y_G and/or at $y = 1$ (the nose) must be modified.

Consider the GC of Figure 1(a). We use dimensionless variables scaled according to (23) (see also (4.6U)), but in this appendix we drop the upper $\sim$ (in particular, K here corresponds to $\tilde{K}$ in the main text). The dimensionless height of the gap is 1.

The continuity equations are

$$\frac{\partial h}{\partial t} + \frac{\partial}{\partial x}(hu) = 0 \quad (2D), \quad (\text{A7a})$$

$$\frac{\partial h}{\partial t} + \frac{1}{r} \frac{\partial}{\partial r}(ru) = 0 \quad (\text{AXI}). \quad (\text{A7b})$$

Let ξ denote x or r . In the GC, the depth-averaged velocity is $u = h^{(n+1)/n} (-\partial h / \partial \xi)^{1/n}$.

Let $\sigma_N = 0$. The boundary conditions at the nose ξ_N are $h_N = 0$ while $u_N = d\xi_N/dt$. These conditions are not affected by the meniscus at the grounding line.

For 2D. The grounding line poses two conditions for the GC at $x_G(t)+$. First, the obvious $h_G^+ = (1 - \sigma)$. Second, the total volume conservation

$$\mathcal{V} = \int_0^{x_N(t)} h(x, t) dx = x_G(t) + \int_{x_G(t)+}^{x_N(t)} h(x, t) dx = qt^\alpha. \quad (\text{A8})$$

We apply t derivative to the balance (A8), use (A7) to eliminate $\partial h / \partial t$, and recall $h_G^+ = (1 - \sigma)$, $h_N = 0$. We obtain the flux condition at $x_G +$

$$\sigma \frac{dx_G}{dt} + (hu)_G = \alpha q t^{\alpha-1}. \quad (\text{A9})$$

For AXI. The boundary conditions at the grounding line $r_G(t)$ is the obvious $h_G^+ = (1 - \sigma)$ and the total volume conservation (per radian) which reads

$$\mathcal{V} = \frac{1}{2} r_G^2 + \int_{r_G(t)+}^{r_N(t)} h(r, t) r dr = q t^\alpha. \quad (\text{A10})$$

We apply t derivative to (A10) and use (A7) to eliminate $\partial h / \partial t$. We obtain the flux condition at $r_G +$

$$\sigma r_G \frac{dr_G}{dt} + (rhu)_G = \alpha q t^{\alpha-1}. \quad (\text{A11})$$

The governing lubrication-model equations for the GC, (4.6U)–(4.9U) and (4.11U)–(4.12U), are not affected by the presence of the meniscus. The only difference is the change of the boundary condition at $x_G +$ (or $r_G +$) when $\sigma > 0$: instead of (4.10aU) and (4.10bU) we should impose (A9) and (A11).

The lubrication model postulates the similarity solution

$$x_N(t) \text{ (or } r_N(t)) = K t^\beta, \quad (\text{A12a})$$

$$h(y, t) = h(y) = K^{(n+1)/(n+2)} \lambda(y), \quad \bar{u}(y, t) = [\lambda^{n+1} (-\lambda')]^{1/n} K t^{\beta-1}, \quad (\text{A12b})$$

where y is the reduced length coordinate, ξ / ξ_N . It can be verified that the new flux conditions are compatible with these similarity assumptions when $\beta = \alpha$ in 2D case (or $\beta = \alpha / 2$ in AXI case).

The conclusion is that the similarity solution of the lubrication model for the confined GC in the $\sigma = 0$ system developed in U25 can be extended to the system with $\sigma > 0$ (at this stage, without a nose meniscus). For a given n , the values of the time-powers α and β are unchanged, and in full agreement with the box-model predictions ((9) and (15) for 2D, (27) and (33) for AXI). The definition and meaning of the parameter J is unchanged, in full agreement with the box model. The new parameter σ will affect the values of J and K for a given y_G .

The height profile of the interface of the GC can be calculated by the same method as described in U25 (in particular using (4.16U)–(4.21U)). To be specific, the major equation is

$$\left[y^{k-2} \lambda^{(2n+1)/n} (-\lambda')^{1/n} \right]' - \beta y^{k-1} \lambda' = 0, \quad (\text{A13})$$

subject to the values of λ and λ' at $y = 1 - \Delta$, for a small Δ , obtained from the approximation

$$\lambda(y) = [(n+2)\beta^n(1-y)]^{1/(n+2)} = \left[(n+2) \left(\frac{n}{n+1} \right)^n (1-y) \right]^{1/(n+2)} \quad (y \rightarrow 1) \quad (\text{A14})$$

In other words, when $\sigma \geq 0$, $\sigma_N = 0$, for a given n and geometry, $\lambda(y)$ is a universal profile for $0 < y < 1$. We obtain $\lambda(y)$ and $\lambda'(y)$ by numerical integration of (A13). This part of the solution process is independent of the value of σ , and thus exactly as described in U25.

The calculation of J and K for a given y_G is affected by σ . At the position $y_G = y$, where λ_G and λ'_G are known, the new grounding-line conditions (of height and volume flux) yield

$$K = [(1 - \sigma) / \lambda_G]^{(n+2)/(n+1)}, \quad (\text{A15})$$

$$q = J^{k+\alpha/n} = \begin{cases} (K/\alpha) \left[(1 - \sigma) (-\lambda'_G \lambda_G^{n+1})^{1/n} + \beta \sigma y_G \right] & (2D) \\ (K^2 y_G / \alpha) \left[(1 - \sigma) (-\lambda'_G \lambda_G^{n+1})^{1/n} + \beta \sigma y_G \right] & (AXI), \end{cases} \quad (\text{A16})$$

where $k = 2$ for 2D and 3 for AXI. For given n, σ and y_G we obtain the values of J and K of the confined GC. For $\sigma = 0$ we recover the results of U25. The results of [8] and [7] are recovered for $n = 1$ and AXI geometry (but note that these papers use a different scaling of the variables).

Equation (A16) indicates that $K \sim J^{(2n+3)/(n+1)}$ in 2D case, and $K \sim J^{(3n+5)/(2(n+2))}$ in AXI case. These dependencies are in agreement with the box-model predictions (24) and (40). However, it is evident that the box model formulas are simpler and more insightful. (Recall that K in this appendix corresponds to $\tilde{K}$ in the main text.)

A nose meniscus can be implemented as follows. Assume that the height (dimensionless) is $\hat{\sigma}_N$. (Note that σ_N of Appendix A multiplies $h_N = 1 - \sigma$ of the box model. Hence $\hat{\sigma}_N = (1 - \sigma)\sigma_N$.) The relevant conditions at $y = 1$ are $h_N = \hat{\sigma}_N > 0$ and $u_N = d\tilde{\zeta}_N/dt = \beta K t^{\beta-1}$. The differential equation for $\lambda(y)$ and the flux condition at the grounding line are not affected (the derivation of (A9) and (A11) is slightly modified because $h_N > 0$). When $\hat{\sigma}_N > 0$, the condition for λ and λ' applied at $1 - \Delta$ according to (A14) must be changed to

$$h_N = K^{(n+1)/(n+2)} \lambda(1) = \hat{\sigma}_N, \quad (\text{A17})$$

$$\lambda'(1) = -\beta^n \lambda(1)^{-(n+1)}. \quad (\text{A18})$$

Equation (A18) expresses the condition $u_N = \beta K t^{\beta-1}$. The similarity solution calculation of $\lambda(y)$ is now implicit, because the unknown K appears at $y = 1$ and y_G , see (A17) and (A15). $\lambda(y)$ is no longer a universal profile. An iteration is needed, such as: for a given y_G , guess an initial K , use (A17)–(A18) to solve the $\lambda(y)$ equation from $y = 1$ to y_G , calculate the new K by (A15), repeat the solution with a corrected K , until convergence of K is achieved. Then calculate J by (A16). This iteration complicates the solution, but since the effect of σ_N is very small, the simpler solution without the nose meniscus is in general a sufficiently accurate prediction.

We used a Runge-Kutta method for the calculation of λ .

The present extension of the similarity solution of the basic lubrication-theory model (without surface tension effects) to GCs affected by surface-tension effects has been facilitated by the assumptions that the menisci lengths are fixed with respect to a geometric scale, H , present in the previous formulation. A more realistic modeling of the menisci may introduce a new lengthscale and require a more complex solution of the $h(\tilde{\zeta}, t)$ interface. This topic is left for future work.

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