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Relativistic Plastino-Plastino Equation

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Abstract

The Plastino-Plastino Equation (PPE) is essential in non-extensive statistics in the study of systems that exhibit anomalous diffusion and do not fit conventional statistics, thus being a nonlinear extension of the Fokker-Planck Equation (FPE). This equation has been applied in various fields of physics (Cosmology, astrophysics and hadrons, specifically in Quark-Gluon Plasma) and other disciplines. In this work, a relativistic approach will be carried out on a system of particles for which the relativistic Boltzmann equation is obtained. Here, grazing collisions are considered to obtain the FPE integrated with special relativity. Subsequently, through fractal derivations, a modification of the FPE is made, resulting in the PPE in a relativistic context.

Keywords: fractal; Plastino-Plastino; Fokker-Planck; relativistic

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1. Introduction

The Boltzmann equation is a cornerstone of nonequilibrium statistical mechanics, allowing for the dynamical evolution of a distribution function $f(\mathbf{r}, \mathbf{p}, t)$, which describes how a system of particles evolves through collisions by quantifying the particles' positions and velocities [1–3]. This equation has found applicability in many systems, including those at relativistic energies, an extension that was made possible by incorporating relativity. Thus, the Boltzmann equation is now applied in astrophysics, in studies of the early universe, and in investigations of the Quark-Gluon Plasma (QGP) [4–6]. Another very common application of the relativistic Boltzmann equation is in the area of electromagnetic plasmas [7].

The Fokker-Planck Equation (FPE) can be derived from the Boltzmann equation through a second-order expansion of the probability density, resulting in a second-order differential equation that describes the statistical behaviour of the system through transport coefficients: the drag (or drift) coefficient, associated with external forces, and the diffusion coefficient, associated with stochastic collisions among the many particles in the system [8,9]. This equation must be covariant under Lorentz transformations to be applied to relativistic systems [10,11]. Among its applications is plasma physics, allowing the analysis of the dynamics of charged particles, as well as astrophysics and cosmology, where the evolution of nebulae is studied. All these systems are subject to the combined effects of drag and relativistic diffusion [12,13]. With such an extension, the transport coefficients must be reinterpreted in terms of the exchange of four-momentum during particle interactions.

The FPE, however, is restricted to a particular class of systems, namely those for which the collision term of the Boltzmann equation has a correlation functional that is bilinear in the interacting particle distributions. This assumption is useful and valid for interactions that are local, uncorrelated, and follow a Markovian sequence. However, there is increasing interest in systems that fall outside this class and are instead described by nonlinear Fokker-Planck equations. Within this family of nonlinear equations, a particular class is of broad applicability, namely the Plastino-Plastino Equation (PPE) [14], which was proposed in connection with systems following Tsallis statistics [14–17].

Tsallis statistics has been applied in different fields of physics and beyond, yielding numerous studies, especially in high-energy physics. The emergence of Tsallis statistics in quantum field theory can be traced to the renormalisation properties of these theories and to self-energy interactions. Together, these properties provide the necessary conditions for the formation of thermofractals [18], and their mathematical tools have uncovered a deep relationship between the Tsallis index q and field-theoretical parameters, which in the case of QCD are the number of colours and flavours. This connection has shown that q is a structural parameter within quantum field theory, rather than merely a fitting parameter.

The description of the dynamics of heavy quarks in the QGP within this statistical framework, particularly through the PPE [14], is of great importance. This nonlinear dynamics, within nonextensive statistics, provides a means to describe the evolution of complex and random systems and has proven relevant in research across different areas of physics [17]. Recently, the PPE was used to investigate the dynamical origin of the nuclear modification factor in high-energy nuclear collisions [19] by employing a nonrelativistic version of the equation. Although relativistic effects may be limited in some cases, it is important to extend the PPE so that it behaves covariantly under Lorentz transformations.

In this work, a derivation of the PPE is carried out by incorporating special relativity, leading to the Relativistic Plastino-Plastino Equation (RPPE) within the framework of Tsallis q-statistics. The paper is structured as follows: Section 2 derives the relativistic Boltzmann equation for a particle collision system. In addition, the relativistic Fokker-Planck equation is deduced from the relativistic Boltzmann equation, and in Section 4, the RPPE is derived from the RFPE using fractal derivatives, ultimately yielding a relativistic equation expressed in terms of the velocity variable. Final remarks are presented in Section 5.

2. Relativistic Boltzmann and Fokker-Planck Equations

A particle of rest mass m is described by the space-time coordinates $(x^\alpha) = (ct, \mathbf{x})$ and by the four-momentum $(p^\alpha) = (p^0, \mathbf{p})$, where p^0 is given by $p^0 = \sqrt{|\mathbf{p}|^2 + m^2c^2}$. The one-particle distribution function, defined in terms of space-time and momentum coordinates as $f(x^\alpha, p^\alpha) = f(\mathbf{x}, \mathbf{p}, t)$, is such that

$$f(\mathbf{x}, \mathbf{p}, t) d^3x\, d^3p = f(\mathbf{x}, \mathbf{p}, t) dx^1 dx^2 dx^3 dp^1 dp^2 dp^3, \tag{1}$$

where t is the instant at which the number of particles in a volume element d^3x around $\mathbf{x}$ and with momentum within d^3p around $\mathbf{p}$ is measured [1]. The number of particles in the volume is a scalar invariant, since all observers count the same number of particles. Therefore, the distribution function $f(\mathbf{x}, \mathbf{p}, t)$ is itself a scalar invariant [1].

Defining the phase-space volume at time t as $d\mu(t) = d^3x\, d^3p$, the number of particles in this volume is

$$N(t) = f(\mathbf{x}, \mathbf{p}, t) d\mu(t). \tag{2}$$

At time $t + \Delta t$, the number of particles becomes

$$N(t + \Delta t) = f(\mathbf{x} + \Delta\mathbf{x}, \mathbf{p} + \Delta\mathbf{p}, t + \Delta t) d\mu(t + \Delta t). \tag{3}$$

Due to particle collisions, $N(t) \neq N(t + \Delta t)$, and the variation is

$$\Delta N = f(\mathbf{x} + \Delta\mathbf{x}, \mathbf{p} + \Delta\mathbf{p}, t + \Delta t) d\mu(t + \Delta t) - f(\mathbf{x}, \mathbf{p}, t) d\mu(t). \tag{4}$$

where the increments in the position and in the momentum are given as,

$$\Delta\mathbf{x} = \mathbf{v}\Delta t, \quad \Delta\mathbf{p} = \mathbf{F}\Delta t. \tag{5}$$

The relationship between $d\mu(t + \Delta t)$ and $d\mu(t)$ is given by, $d\mu(t + \Delta t) = |J| d\mu(t)$ with J denoting the Jacobian of the transformation,

$$J = \frac{\partial(x^1(t + \Delta t), x^2(t + \Delta t), \ldots, p^3(t + \Delta t))}{\partial(x^1(t), x^2(t), \ldots, p^3(t))}. \tag{6}$$

The increments in position and momentum are $\Delta\mathbf{x} = \mathbf{v}\Delta t$ and $\Delta\mathbf{p} = \mathbf{F}\Delta t$, where $\mathbf{F}(\mathbf{x}, \mathbf{p}, t)$ is the external force and $\mathbf{v} = c\mathbf{p}/p^0$ is the particle velocity [1]. The Jacobian relating $d\mu(t + \Delta t)$ and $d\mu(t)$ is

$$J = 1 + \frac{\partial F^i}{\partial p^i}\Delta t + \mathcal{O}\Big[(\Delta t)^2\Big], \tag{7}$$

with $i = 1, 2, 3$.

Expanding $f(\mathbf{x} + \Delta\mathbf{x}, \mathbf{p} + \Delta\mathbf{p}, t + \Delta t)$ to first order in Δt yields

$$f \approx f + \frac{\partial f}{\partial t}\Delta t + \frac{\partial f}{\partial x^i}\Delta x^i + \frac{\partial f}{\partial p^i}\Delta p^i. \tag{8}$$

Combining Eqs. (7) and (8), and retaining linear terms, one finds

$$\frac{\Delta N}{\Delta t} = \left[\frac{\partial f}{\partial t} + v^i\frac{\partial f}{\partial x^i} + \frac{\partial}{\partial p^i}\left(fF^i\right)\right]d\mu(t). \tag{9}$$

Since ΔN and the proper time $\Delta\tau = \Delta t/\gamma$ are scalar invariants,

$$\frac{\Delta N}{\Delta \tau} = \gamma\left[\frac{\partial f}{\partial t} + v^i\frac{\partial f}{\partial x^i} + \frac{\partial}{\partial p^i}\left(fF^i\right)\right]d\mu(t). \tag{10}$$

ΔN is a scalar invariant as well as the proper time $\Delta\tau = \Delta t/\gamma$, hence

$$\gamma\frac{\Delta N}{\Delta t} = \frac{\Delta N}{\Delta \tau} = \gamma\left[\frac{\partial f}{\partial t} + v^i\frac{\partial f}{\partial x^i} + \frac{\partial fF^i}{\partial p^i}\right]d\mu(t), \tag{11}$$

is a scalar invariant [1]. Where $d\mu = d^3x d^3p$ is a scalar invariant, and as a consequence the expression multiplying, must have the same property. We first consider the term

$$\gamma\left[\frac{\partial f}{\partial t} + v^i\frac{\partial f}{\partial x^i}\right] = \gamma\left[\frac{\partial f}{\partial t} + \frac{cp^i}{p^0}\frac{\partial f}{\partial x^i}\right], \tag{12}$$

in which $\mathbf{v} = c\mathbf{p}/p^0$ and multiplying and dividing the first term by cp_0, we have that

$$\gamma\left[\frac{\partial f}{\partial t} + v^i\frac{\partial f}{\partial x^i}\right] = \frac{1}{m}p^\alpha\frac{\partial f}{\partial x^\alpha}. \tag{13}$$

Since f is a scalar invariant, $\partial f/\partial x^\alpha$ is a 4-vector, and the scalar product $p^\alpha \partial f/\partial x^\alpha$ is a scalar invariant. We consider the Minkowski force k^α defined by

$$k^\alpha = \frac{\partial p^\alpha}{\partial \tau}, \tag{14}$$

that satisfies, $k^\alpha p_\alpha = k^0 p_0 - \mathbf{k}\cdot\mathbf{p} = 0$ and the relationship,

$$\mathbf{F} = \frac{\mathbf{k}}{\gamma} = \frac{mc\mathbf{k}}{p^0}. \tag{15}$$

If we consider p^0 as an independent variable and make use of the chain rule:

$$\frac{\partial}{\partial p^i} \to \frac{\partial p^0}{\partial \mathbf{p}}\frac{\partial}{\partial p^0} + \frac{\partial}{\partial \mathbf{p}} = \frac{\mathbf{p}}{p^0}\frac{\partial}{\partial p^0} + \frac{\partial}{\partial \mathbf{p}}. \tag{16}$$

We can write $\gamma\partial fF^i/\partial p^i$ by introducing Eqs. (16) and (15) as follows,

$$\gamma\frac{\partial fF^i}{\partial p^i} = \frac{\partial fk^\alpha}{\partial p^\alpha}, \tag{17}$$

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which is a scalar invariant [1].

Substituting Eqs. (13) and (17) into Eq. (11) yields

$$\frac{\Delta N}{\Delta t} = \frac{c}{p^0}\left[p^\alpha \frac{\partial f}{\partial x^\alpha} + m\frac{\partial}{\partial p^\alpha}(fK^\alpha)\right] d\mu(t). \tag{18}$$

To determine $\Delta N/\Delta t$, we decompose it in two terms

$$\frac{\Delta N}{\Delta t} = \frac{(\Delta N)^+}{\Delta t} - \frac{(\Delta N)^-}{\Delta t}, \tag{19}$$

where $(\Delta N)^-/\Delta t$ corresponds to the particles that leave the volume $d^3x d^3p$, whereas $(\Delta N)^+/\Delta t$ corresponds to those particles that enter in the same volume. Further, we assume the following:

a) Only collisions between pairs of particles are taken into account, i. e. only binary collisions are considered (See Figure 1) [1].
b) If $\mathbf{p}$ and $\mathbf{p}_*$ denote the momenta of two particles before collision they are not correlated. This will be applied to the momenta $\mathbf{p}$ of the particle that we are following, and $\mathbf{p}_*$ of its collision partner, as well as to two momenta $\mathbf{p}'$ and $\mathbf{p}'_*$, possessed by two particles before a collision that will transform them into particles with momenta $\mathbf{p}$ and $\mathbf{p}_*$ after collision [1].
c) The one-particle distribution function $f(\mathbf{x}, \mathbf{p}, t)$ does not vary very much over a time interval which is larger than the duration of a collision but smaller the time between collisions. The same applies to the change of $f(\mathbf{x}, \mathbf{p}, t)$ over a distance of the order of the interaction range [1].

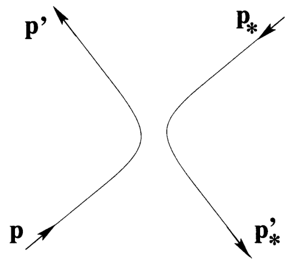

Figure 1. Schematic representation of a binary collision.

We consider a collision between two beams of particles with velocities $\mathbf{v} = c\mathbf{p}/p_0$ and $\mathbf{v}_* = c\mathbf{p}_*/p^0_*$. The particle number densities of these two beams in their own frames are denoted by dn and dn_*. The d in front of n and n_* indicates that these number densities are infinitesimal because they refer to volume elements d^3p and d^3p_* of momentum space ($dn = f(\mathbf{x}, \mathbf{p}, t)d^3p$ and $dn_* = f(\mathbf{x}, \mathbf{p}_*, t)d^3p_*$) [1].

The total number of particles around $\mathbf{x}$ is dnd^3x. The total number of particles that collide in the volume dV_* will be $dn_*dV_* = dn_*dV/\sqrt{1 - v^2_{rel}/c^2}$, where v_{rel} is the relative velocity and $dV/\sqrt{1 - v^2_{rel}/c^2}$ is a proper volume [1].

The particles with density dn_* in the volume dV are differently scattered by their partners in the collision through different angles. Each collision will occur in a plane with some scattering angle Θ; another angle is needed to single out the plane and two infinitesimal neighborhoods of the two angles together single out a solid angle element $d\Omega$ [1]. The volume element dV can be written in terms of the so-called collision cylinder of base $\sigma d\Omega$ and height $v_{rel}\Delta t$. Δt is identified with the differential of the proper time, because of the choice of the reference frame. The factor σ has clearly the dimensions of an area and is called the differential cross-section of the scattering process corresponding to the relative speed v_{rel} and the scattering angle Θ. In another reference system where $\mathbf{v} \neq \mathbf{0}$, $d^3x\Delta t$, σ, $d\Omega$ and v_{rel} are scalar invariants [1].

The total number of collisions will be given then by the product of the particle numbers corresponding to the velocities $\mathbf{v}$ and $\mathbf{v}_*$ (See Figure 2):

$$dnd^3x\frac{dn_*}{\sqrt{1-v_{rel}^2/c^2}}dV = dnd^3x\frac{dn_*}{\sqrt{1-v_{rel}^2/c^2}}(\sigma d\Omega v_{rel}\Delta t), \tag{20}$$

where we have rewritten the volume element dV in terms of the collision cylinder. Let us consider the product of the particle number densities in a system where $\mathbf{v} \neq \mathbf{0}$:

$$\frac{dndn_*}{\sqrt{1-v_{rel}^2/c^2}} = f(\mathbf{x},\mathbf{p},t)d^3p\, f(\mathbf{x},\mathbf{p}_*,t)d^3p_*\frac{p_\alpha p_*^\alpha}{p^0p_*^0}. \tag{21}$$

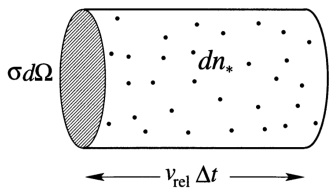

Figure 2. Schematic representation of the collision cylinder.

We have that the total number of collisions given by Eq. (20) is

$$\frac{dnd^3xdn_*}{\sqrt{1-v_{rel}^2/c^2}}dV = f(\mathbf{x},\mathbf{p},t)d^3pf(\mathbf{x},\mathbf{p}_*,t)d^3p_*\frac{p_\alpha p_*^\alpha}{p^0p_*^0}d^3x(\sigma d\Omega v_{rel}\Delta t). \tag{22}$$

In which another form of relative speed is used, known as Mϕller relative velocity,

$$g_\phi = \sqrt{(\mathbf{v}-\mathbf{v}_*)^2 - \frac{1}{c^2}(\mathbf{v}\times\mathbf{v}_*)^2} = v_{rel}\frac{p_\alpha p_*^\alpha}{p^0p_*^0}, \tag{23}$$

then,

$$\begin{aligned} f(\mathbf{x},\mathbf{p},t)d^3pf(\mathbf{x},\mathbf{p}_*,t)d^3p_*v_{rel}\frac{p_\alpha p_*^\alpha}{p^0p_*^0}\sigma d\Omega d^3x\Delta t = \\ f(\mathbf{x},\mathbf{p},t)d^3pf(\mathbf{x},\mathbf{p}_*,t)d^3p_*g_\phi\sigma d\Omega d^3x\Delta t, \end{aligned} \tag{24}$$

where we have introduced Mϕller´s relative velocity g_ϕ [1]. Now the total number of particles that leave the volume d^3xd^3p is obtained by integrating it over all momenta $\mathbf{p}_*$ and over all solid angle $d\Omega$, yielding

$$(\Delta N)^- = \int_\Omega\int_{\mathbf{p}_*} f(\mathbf{x},\mathbf{p},t)f(\mathbf{x},\mathbf{p}_*,t)g_\phi\sigma d\Omega d^3p_*d^3xd^3p\Delta t. \tag{25}$$

This is frequently called the loss term because it describes the loss of particles in the volume d^3xd^3p in phase space, due to collisions. We consider a collision between two beams of particles with velocities $\mathbf{v}'_* = c\mathbf{p}'_*/p'^0_*$ and $\mathbf{v}' = c\mathbf{p}'/p'^0$ and we write the total number of particles that leave the volume element $d^3x'd^3p'$ as,

$$(\Delta N)^+ = \int_{\Omega'}\int_{\mathbf{p}'_*} f(\mathbf{x},\mathbf{p}',t)f(\mathbf{x},\mathbf{p}'_*,t)g'_\phi\sigma' d\Omega' d^3p'_*d^3x'd^3p'\Delta t', \tag{26}$$

which is called the gain term since it describes the gain of particles in the volume element d^3xd^3p. For relativistic particles we have that $g_\phi \neq g'_\phi$ [1]. Therefore, through Liouville's theorem, it is obtained that the volume in phase space does not change over time. Here, we have that

$$g_\phi \Delta t \sigma d\Omega d^3p_* d^3x d^3p = g'_\phi \Delta t' \sigma' d\Omega' d^3p'_* d^3x' d^3p', \tag{27}$$

since $d^3x\Delta t = d^3x'\Delta t'$ is an invariant, we have that

$$\int_\Omega g_\phi \sigma d\Omega d^3p_* d^3p = \int_{\Omega'} g'_\phi \sigma' d\Omega' d^3p'_* d^3p'. \tag{28}$$

From Eq. (19), and introducing in Eq. (18), we have that

$$\begin{aligned} &\frac{c}{p^0}\left[p^\alpha \frac{\partial f}{\partial x^\alpha} + m\frac{\partial f k^\alpha}{\partial p^\alpha}\right] d^3x d^3p \\ &= \int_\Omega \int_{\mathbf{p}_*} f(\mathbf{x},\mathbf{p}',t) f(\mathbf{x},\mathbf{p}'_*,t) g_\phi \sigma d\Omega d^3p_* d^3x d^3p \\ &\quad - \int_\Omega \int_{\mathbf{p}_*} f(\mathbf{x},\mathbf{p},t) f(\mathbf{x},\mathbf{p}_*,t) g_\phi \sigma d\Omega d^3p_* d^3x d^3p, \end{aligned} \tag{29}$$

where

$$f'_* \equiv f(\mathbf{x},\mathbf{p}'_*,t); \quad f' \equiv f(\mathbf{x},\mathbf{p}',t); \quad f_* \equiv f(\mathbf{x},\mathbf{p}_*,t); \quad f \equiv f(\mathbf{x},\mathbf{p},t), \tag{30}$$

so,

$$\frac{c}{p^0}\left[p^\alpha \frac{\partial f}{\partial x^\alpha} + m\frac{\partial f k^\alpha}{\partial p^\alpha}\right] d^3x d^3p = \int (f'f'_* - ff_*) g_\phi \sigma d\Omega d^3p_* d^3x d^3p, \tag{31}$$

we have denoted by only one symbol the integrals over Ω and $\mathbf{p}_*$. If we denote by F the invariant flux

$$F = \frac{p^0 p^0_*}{c} g_\phi = \sqrt{(p^\alpha_* p_\alpha)^2 - m^4c^4}, \tag{32}$$

then,

$$p^\alpha \frac{\partial f}{\partial x^\alpha} + m\frac{\partial f k^\alpha}{\partial p^\alpha} = \int (f'f'_* - ff_*) F \sigma d\Omega \frac{d^3p_*}{p^0_*}, \tag{33}$$

which is the final form of the relativistic Boltzmann equation for a single non-degenerate relativistic gas [1].

3. Relativistic Fokker-Planck Equation

Under the assumption of grazing collisions that could take place in long-range interactions, only small changes in the momentum of the particles occur due to small deflections in the scattering angle. Then the collision term of the Boltzmann equation, denoted by $\mathcal{Q}(f, f_*)$ [7]. The total P^α and the relative Q^α four-momentum, defined by

$$P^\alpha = p^\alpha + p^\alpha_* = P'^\alpha, \quad Q^\alpha = p^\alpha - p^\alpha_*, \quad Q'^\alpha = p'^\alpha - p'^\alpha_*. \tag{34}$$

For these quantities the following relationships hold are,

$$P^\alpha P_\alpha = P^2, \quad P^\alpha Q_\alpha = 0 \quad \text{and} \quad P^2 = Q^2 + 4m^2c^2. \tag{35}$$

The differences between the post- and pre-collision four-momentum,

$$\begin{aligned} \Delta p^\alpha &= p'^\alpha - p^\alpha = \frac{1}{2}(Q'^\alpha - Q^\alpha) = \frac{1}{2}\Delta Q^\alpha, \\ \Delta p^\alpha_* &= p'^\alpha_* - p^\alpha_* = -\frac{1}{2}(Q'^\alpha - Q^\alpha) = -\frac{1}{2}\Delta Q^\alpha. \end{aligned} \tag{36}$$

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Hence for small changes of the momentum of the particles at collision one can expand the one-particle distribution function in Taylor series, which up to the second-order terms,

$$f\left(p'^i\right) \approx f\left(p^i\right) + \frac{1}{2}\Delta Q^i \frac{\partial f}{\partial p^i} + \frac{1}{8}\Delta Q^i \Delta Q^j \frac{\partial^2 f}{\partial p^i \partial p^j}, \tag{37}$$

with a similar expression for $f(p'^i_*)$. Now it is possible to approximate the collision term of the Boltzmann equation as [7],

$$\mathcal{Q}(f, f_*) = \int (f'_* f' - f_* f) F\sigma d\Omega \frac{d^3 p_*}{p^0_*}, \tag{38}$$

then, introducing Eq. (37), with the help of the relationship,

$$\frac{\partial}{\partial Q^i} = \frac{1}{2}\left(\frac{\partial}{\partial p^i_*} - \frac{\partial}{\partial p^i}\right), \tag{39}$$

then,

$$\mathcal{Q}(f, f_*) = \int \left[\Delta Q^i \frac{\partial (f f_*)}{\partial Q^i} + \frac{1}{2}\Delta Q^i \Delta Q^j \frac{\partial}{\partial Q^i}\frac{\partial}{\partial Q^j}(f f_*)\right] F\sigma d\Omega \frac{d^3 p_*}{p^0_*}. \tag{40}$$

In order to transform the integral, the center-of-mass system is chosen where the spatial components of the total four-momentum vanish, i.e., $(p^\alpha) = \left(p^0, \mathbf{0}\right)$ and $(Q^\alpha) = (0, \mathbf{Q})$. Now the element of solid angle can be written as $d\Omega = \sin\Theta d\Theta d\Phi$, where Θ and Φ are polar angles of Q'^α with respect to Q^α and such that Θ represents the scattering angle [7]. Further, without loss of generality, Q^α is chosen in the direction of the three axis, so that one can write Q^α and Q'^α as

$$(Q^\alpha) = Q\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \quad (Q'^\alpha) = Q\begin{pmatrix} 0 \\ \sin\Theta\cos\Phi \\ \sin\Theta\sin\Phi \\ \cos\Theta \end{pmatrix}, \tag{41}$$

by using the above representations, the integrals in the variable $0 \le \Phi \le 2\pi$, yielding

$$\int \Delta Q^i \frac{F}{p^0_*}\sigma d\Omega = \int \Delta Q^i \frac{F}{p^0_*}\sigma \sin\Theta d\Theta d\Phi = 2\pi \int \Delta Q^i \frac{F}{p^0_*}\sigma \sin\Theta d\Theta, \tag{42}$$

where $\Delta Q^i = -(1 - \cos\Theta) Q^i$ and $Q = F/p^0_*$. Note that the differential cross-section is a function of $\sigma = (Q, \Theta)$. Then,

$$\int \Delta Q^i \frac{F}{p^0_*}\sigma d\Omega = -2\pi Q^i Q \int (1 - \cos\Theta)\sigma \sin\Theta d\Theta, \tag{43}$$

where

$$\Sigma = 2\pi \int (1 - \cos\Theta)\sigma \sin\Theta d\Theta, \tag{44}$$

then,

$$\int \Delta Q^i \Delta Q^j \frac{F}{p^0_*}\sigma d\Omega = -\left(Q^2 \eta^{ij} + Q^i Q^j\right) Q\Sigma, \tag{45}$$

and η^{ij} are the spatial components of the metric tensor $(\eta_{\alpha\beta}) = \mathrm{diag}(1, -1, -1, -1)$ [7]. By differentiating with respect to Q^i, we have

$$\frac{\partial}{\partial Q^j}\int \Delta Q^i \Delta Q^j \frac{F}{p^0_*}\sigma \sin\Theta d\Theta d\Phi = -2Q^i Q\Sigma, \tag{46}$$

here we have used Eq. (45), therefore,

$$\frac{\partial}{\partial Q^j}\int \Delta Q^i \Delta Q^j Q\sigma d\Omega = 2\int \Delta Q^i Q\sigma d\Omega, \tag{47}$$

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therefore,

$$\mathcal{Q}(f,f_*) = \int \left[\Delta Q^i \frac{\partial}{\partial Q^i}(ff_*) + \frac{1}{2}\Delta Q^i \Delta Q^j \frac{\partial}{\partial Q^i}\frac{\partial}{\partial Q^j}(ff_*) \right] Q\sigma d\Omega d^3 p_*, \tag{48}$$

where $F/p^0_* = Q$ and using Eq. (47), we have that

$$\mathcal{Q}(f,f_*) = \frac{1}{4}\int \left[\frac{\partial}{\partial p^i_*}\left(\int \frac{\partial(ff_*)}{\partial Q^j}\Delta Q^i \Delta Q^j Q\sigma d\Omega \right) - \frac{\partial}{\partial p^i}\left(\int \frac{\partial(ff_*)}{\partial Q^j}\Delta Q^i \Delta Q^j Q\sigma d\Omega \right) \right] d^3 p_*. \tag{49}$$

The first term on the right-hand side of the above equation vanishes, since the hypothesis of grazing collisions is used and it is possible to convert; thanks to the divergence theorem; the volume integral in the momentum space into an integral at an infinitely far surface where the distribution functions tend to zero [7]. By invoking the divergence theorem again,

$$\mathcal{Q}(f,f_*) = -\frac{\partial}{\partial p^i}\left(fA^i - \frac{\partial f D^{ij}}{\partial p^j} \right), \tag{50}$$

where the spatial components of the coefficient of dynamic friction A^i and the diffusion coefficient D^{ij} are given by

$$A^i = \int f_* \Delta p^i_* F\sigma d\Omega \frac{d^3 p_*}{p^0_*}, \quad D^{ij} = \frac{1}{2}\int f_* \Delta p^i_* \Delta p^j_* F\sigma d\Omega \frac{d^3 p_*}{p^0_*}. \tag{51}$$

In this system $\Delta p^0_* = 0$, and one can include the zero components

$$\begin{aligned} A^0 &= \int f_* \Delta p^0_* F\sigma d\Omega \frac{d^3 p_*}{p^0_*}, \\ D^{i0} = D^{0i} &= \frac{1}{2}\int f_* \Delta p^i_* \Delta p^0_* F\sigma d\Omega \frac{d^3 p_*}{p^0_*}, \\ D^{00} &= \frac{1}{2}\int f_* \Delta p^0_* \Delta p^0_* F\sigma d\Omega \frac{d^3 p_*}{p^0_*}. \end{aligned} \tag{52}$$

Hence the Relativistic Boltzmann Equation reduces to the Relativistic Fokker-Planck Equation, namely [7]

$$p^\alpha \frac{\partial f}{\partial x^\alpha} + m\frac{\partial f k^\alpha}{\partial p^\alpha} = -\frac{\partial}{\partial p^\alpha}\left(fA^\alpha - \frac{\partial f D^{\alpha\beta}}{\partial p^\beta} \right). \tag{53}$$

4. Relativistic Plastino-Plastino Equation

The Relativistic Plastino-Plastino Equation is derived from the Relativistic Fokker-Planck equation given in Eq. (53). We consider the spatially homogeneous case without external forces, which reduces the relativistic Fokker-Planck equation to

$$\frac{p^0}{c}\frac{\partial f}{\partial t} = \frac{\partial}{\partial p^i}\left(-fA^i + \frac{\partial}{\partial p^j}\left(fD^{ij} \right) \right). \tag{54}$$

Here, fractal derivatives are employed. More specifically, ordinary derivatives are replaced by fractal derivatives, transforming the above equation into

$$\left[D^\zeta_{F',t} \right]\left(\frac{p_0}{c} f(\mathbf{p},t) \right) = \left[D^\zeta_{F,p_{0,i}} \right]\left(-A_i f(\mathbf{p},t) + \left[D^\zeta_{F,p_{0,j}} \right]\left(D_{ij} f(\mathbf{p},t) \right) \right), \tag{55}$$

where $\left[D^\zeta_{F',t} \right]$ denotes the fractal derivative and $\zeta = 2 - q$, with q being the Tsallis entropic index, thus establishing the connection with nonextensive statistics.

Starting from

$$\left[D^\alpha_{\mathbb{F},\varphi} \right]^{-1} f(x) = \frac{A(\alpha)}{\alpha} = A(\alpha) f^{\alpha-1}(x)\frac{df(x)}{dx} \propto \frac{\partial f^{2-q}}{\partial x}, \tag{56}$$

where $A(\alpha) := 2\pi^{\alpha/2}/\Gamma(\alpha/2)$ and $\alpha \in \mathbb{R}$, which relates ordinary differentiation to fractal differentiation, Eq. (55) is modified, yielding

$$\frac{p^0}{c}\frac{\partial f}{\partial t} = \frac{\partial}{\partial p_i}\left[-A_i(\mathbf{p})f + \frac{\partial}{\partial p_j}\left(B_{ij}(\mathbf{p})f^{2-q}\right)\right], \tag{57}$$

which is the Relativistic Plastino-Plastino Equation.

4.1. Rapidity Space

A useful modification of the Relativistic Plastino-Plastino Equation involves the rapidity variable, placing Eq. (57) fully within a relativistic framework. Using natural units, $c = \hbar = 1$, we have

$$p_0\frac{\partial f}{\partial t} = \frac{\partial}{\partial p_i}\left[-A_i(\mathbf{p})f + \frac{\partial}{\partial p_j}\left(B_{ij}(\mathbf{p})f^{2-q}\right)\right]. \tag{58}$$

Introducing the change of variables $p_0 = m_T \cosh y$ and $p_z = m_T \sinh y$ [20], the above equation becomes

$$p_0\frac{\partial f}{\partial t} = p_0\frac{\partial}{\partial y}\left[-A_i(y)f + \frac{\partial}{\partial y}\left(B_{ij}(y)f^{2-q}\right)\right], \tag{59}$$

and therefore

$$\frac{\partial f}{\partial t} = \frac{\partial}{\partial y}\left[-A_i(y)f + B_{ij}\frac{\partial}{\partial y}\left(f^{2-q}\right)\right], \tag{60}$$

which is the Plastino-Plastino equation expressed in terms of the rapidity variable.

5. Conclusions

This work provides the first demonstration of the Relativistic Plastino-Plastino Equation derived from the Relativistic Boltzmann Equation. Initially, the relativistic version of the Fokker-Planck equation is obtained, and it is then shown that this equation is limited to a specific form of correlators in the collision term. As a consequence, it cannot be applied to systems that do not follow a Markovian sequence of collisions, such as systems exhibiting memory effects or nonlocal correlations.

Motivated by recent results, this work identifies the Plastino-Plastino Equation as the appropriate framework for describing systems with such characteristics, including quark-gluon plasma, hadronic systems, and electromagnetic plasmas. Following the methodology of previous studies, the Relativistic Plastino-Plastino Equation is derived from the relativistic Fokker-Planck equation by exploiting the known connections between the fractal Fokker-Planck equation and the Plastino-Plastino Equation. These connections rely on the recently established relationship between fractal derivatives and q-deformed derivatives.

The relativistic version of the Plastino-Plastino Equation opens new possibilities for investigating the dynamical evolution of systems such as solar plasmas, Tokamak plasmas, quark-gluon plasma, and neutron star cores. The use of the relativistic formulation enables more precise analyses and a more accurate determination of the key physical parameters governing these systems.

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Abbreviations

The following abbreviations are used in this manuscript:

FPE	Fokker-Planck Equation
RFPE	Relativistic Fokker-Planck Equation
PPE	Plastino-Plastino Equation
RPPE	Relativistic Plastino-Plastino Equation
QGP	Quark-Gluon Plasma

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