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Article

Practical Steady-State Discrete-Time Kalman Filter Design for Uncertain LTI Systems

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Abstract

A practical methodology for designing a steady-state discrete-time Kalman filter for linear time-invariant (LTI) systems subject to parametric uncertainties is presented. Inspired by the Mean Value First-Order Second-Moment (MVFOSM) method, the approach models parameter deviations as zero-mean random variables with known second-order statistics, and extends De Koning's foundational framework to explicitly account for control input contributions to equivalent noise statistics. Closed-form approximations for the state- and input-dependent noise covariance matrices Q, R and M are derived as functions of parameter uncertainties and a representative nominal input magnitude. The steady-state filter gain is obtained by solving a generalized discrete algebraic Riccati equation (DARE), yielding a fixed-gain filter that remains robust to modelling errors without requiring on-line tuning. Filter robustness is parameterized by a single scalar choice — the nominal input level (e.g., worst-case or RMS value) — which provides a transparent physical trade-off between nominal performance and robustness to component tolerances. The methodology is particularly suited to embedded power converter control, where component parameter variations significantly affect system dynamics and real-time computational resources are constrained. An unbiased formulation via an augmented state that tracks second-order bias corrections is also presented alongside the simpler biased formulation that reduces to the canonical Kalman filter structure.

Keywords: Kalman filter; robust estimation; parametric uncertainty; power converter control; Mean Value First-Order Second-Moment (MVFOSM)

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1. Introduction

The Kalman filter, introduced by Rudolf E. Kálmán in 1960 [1], represents a fundamental tool in modern control theory and signal processing for optimal state estimation of linear dynamic systems in the presence of noise. The filter operates recursively on streams of noisy input data to produce statistically optimal estimates of the underlying system state, making it indispensable in applications ranging from navigation and guidance systems to power converter control [2].

1.1. State Estimation and the Filtering Problem

Consider a discrete-time linear time-invariant (LTI) system described by the state-space representation:

$$\begin{cases} x_{k+1} &= Ax_k + Bu_k + w_k \\ y_k &= Hx_k + v_k \end{cases} \tag{1}$$

where $x_k \in \mathbb{R}^n$ is the state vector, $u_k \in \mathbb{R}^q$ is the control input, $y_k \in \mathbb{R}^m$ is the measurement (output) vector, and $w_k \in \mathbb{R}^n$ and $v_k \in \mathbb{R}^m$ represent process and measurement noise, respectively.

The fundamental problem of state estimation is to reconstruct the internal state x_k from the available noisy measurements y_k and known inputs u_k. This problem is particularly challenging when:

- the system state is not directly measurable,
- measurements are corrupted by noise with unknown characteristics,

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- the system model itself contains uncertainties,
- real-time estimation is required for feedback control.

The Kalman filter addresses this problem by computing the *minimum variance* unbiased estimate of the state, i.e. the estimate that minimizes the mean squared error $\mathbb{E}\left[||x_k - \hat{x}_k||^2\right]$ ($\hat{x}_k$ being the state estimate) under the assumption of Gaussian noise statistics [3].

1.2. The Dual Role: Estimation and Filtering

The Kalman filter serves two complementary purposes in control and signal processing applications.

State Estimation

The filter provides optimal estimates of the system's internal state variables that are either unmeasurable or measured with excessive noise. By fusing information from the dynamic model (predictions) and measurements (corrections), the Kalman filter produces estimates that are more accurate than either source alone. The recursive structure, consisting of prediction and update steps, enables real-time implementation with minimal computational overhead:

$$\begin{cases} \hat{x}_{k+1}^- = A\hat{x}_k^+ + Bu_k & (a \textit{ priori} \text{ estimate}), \\ \hat{x}_k^+ = \hat{x}_k^- + K_k(y_k - H\hat{x}_k^-) & (a \textit{ posteriori} \text{ estimate}). \end{cases} \tag{2}$$

The Kalman gain K_k is computed to optimally weight the relative contributions of the prediction and measurement based on their respective noise characteristics.

Variance Minimization (Filtering)

Beyond point estimation, the Kalman filter explicitly minimizes the covariance of the estimation error. By propagating the error covariance matrix P_k through the prediction and update equations, the filter quantifies the uncertainty in its estimates and automatically adjusts the Kalman gain to minimize this uncertainty. This dual estimation of both the state mean and covariance distinguishes the Kalman filter from simpler estimation techniques and enables:

- optimal sensor fusion when multiple measurements with different noise levels are available,
- detection of degraded system performance through monitoring of innovation statistics,
- robust control design using the filtered state estimates with quantified uncertainty,
- adaptive adjustment of controller parameters based on estimation confidence.

The variance reduction property is particularly valuable in power converter applications, where measurement noise from analog-to-digital converters, electromagnetic interference, and quantization effects can significantly degrade control performance if not properly filtered.

1.3. Steady-State Kalman Filter

For time-invariant systems, the Kalman gain converges to a constant matrix, yielding the *steady-state Kalman filter*. This asymptotic solution, obtained by solving the discrete algebraic Riccati equation (DARE), offers several practical advantages:

- reduced computational burden (no recursive covariance propagation),
- simplified implementation and parameter tuning,
- guaranteed stability under standard detectability and stabilizability assumptions (subject to numerical regularization) [4],
- design-time computation of filter parameters,

The steady-state formulation is particularly attractive for embedded control systems where computational resources are limited and deterministic real-time performance is critical.

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1.4. Model Uncertainty and Robustness

Classical Kalman filter theory assumes perfect knowledge of the system matrices A, B, and H, as well as Gaussian white noise processes w and v with known covariances. In practice, these assumptions are rarely satisfied:

- ☐ component tolerances introduce uncertainty in physical parameters (resistance, inductance, capacitance),
- ☐ operating conditions vary (temperature, ageing, non-linearities),
- ☐ modelling approximations neglect high-frequency dynamics,
- ☐ external disturbances exhibit non-Gaussian characteristics.

Non-linearities are expressly addressed by Extended Kalman Filters (EKFs) and not investigated any further in this work; it is however worth mentioning that the standard EKF employs a similar linearization strategy — first-order Taylor expansion combined with second-order moment propagation — which is precisely the approach adopted here. When model uncertainties are significant, the nominal Kalman filter may provide far from optimal or even unstable estimates. This motivates the development of *robust* Kalman filtering techniques that explicitly account for parametric uncertainty. By incorporating the covariance of model parameters into the filter design, one can synthesize filters that maintain near-optimal performance (such as being the best linear estimator) despite modelling errors.

The literature on robust Kalman filtering has evolved along several distinct directions. The seminal work of De Koning [5] established the theoretical foundation by deriving equivalent noise covariances for systems with stochastic parameters variations modelled as sequences of i.i.d. (independent and identically distributed) zero-mean random variables, demonstrating that model uncertainties can be systematically incorporated into the classical Kalman framework and deriving augmented process and measurement noise statistics.

Building on these foundations, several researchers have developed alternative robust filtering methodologies. Xie et al. [6] proposed a min-max approach based on game theory, designing filters that minimize the worst-case estimation error over all admissible parameter uncertainties. Theodor and Shaked [7] extended this framework to derive robust minimum-variance filters with guaranteed performance bounds. Zhu et al. [8] provided a comprehensive design methodology combining linear matrix inequality (LMI) techniques with parameter-dependent Lyapunov functions. Luo and Bosch [9] analysed the performance robustness of Kalman filters, establishing relationships between parameter uncertainty levels and estimation error degradation.

More recent advances have addressed structured uncertainties as polytopic models: Yu et al. [10] developed robust Kalman filters for systems with convex polytopic uncertainties, where the uncertain system matrices lie within a convex hull of known vertices. Xie et al. [11] proposed improved H_2 and H_∞ filtering techniques that, although not strictly Kalman filters, offer relevant insights into handling model uncertainties.

While all the approaches mentioned above provide valuable theoretical insights and worst-case performance guarantees, they do not explicitly consider the control input u_k in the uncertainty propagation.

Rocha and Terra [12] presented the most comprehensive treatment to date (to the best of the author's knowledge), deriving robust filters that handle multiplicative noise, time-correlated uncertainties, and cross-coupled parameter variations through a unified recursive framework; it furthermore includes control input u_k. Such an approach is, however, considered difficult to apply operationally for practitioners that are not necessarily experts in robust control as it is based on *regularization* techniques that need non-trivial tuning.

The present work attempts to extend De Koning's approach [5] to explicitly account for control inputs; it however departs from it by modelling the parametric uncertainties more rigorously although only approximately. This extension is particularly important for feedback control applications where the system operates under varying inputs, and where the interaction between parametric uncertainty and control effort significantly affects the equivalent noise characteristics. By deriving closed-form expressions for the input-dependent covariance terms, it enables systematic tuning of the steady-state filter robustness through appropriate selection of the nominal operating point. The focus of the present work is mostly on the practicality of the proposed robust Kalman filter.

1.5. Scope and Contribution

This paper presents a steady-state discrete-time Kalman filter design methodology for LTI systems with parametric uncertainties similar to the MVFOSM approach [13,14]. The key contributions are:

- derivation of equivalent process and measurement noise covariances that incorporate both stochastic noise and deterministic model uncertainty, with explicit treatment of control input effects,
- closed-form expressions for the (approximated) state-dependant covariance matrices Q, R, and M as functions of parameter uncertainty and control input,
- extension to time-varying implementations for scenarios where reduced conservatism justifies increased computational cost.

The methodology enables systematic design of robust Kalman filters that explicitly trade off nominal performance and robustness to parameters' variations. By selecting an appropriate nominal input value (e.g., maximum expected input or RMS value for periodic signals), designers can tune the conservatism of the steady-state solution to match application requirements. This is particularly valuable for power converter control, where component tolerances can significantly affect dynamic behaviour, and where computational constraints favour steady-state over time-varying implementations.

The remainder of this paper is organized as follows: Section 2 develops the system model with parametric uncertainty and establishes the discretization procedure. Section 3 derives the steady-state Kalman filter equations together with detailed, although approximated, calculations of the state-dependent covariance matrices. Section 4 briefly outlines the extension to time-varying implementations. Conclusions and directions for future work are presented in Section 5.

2. Modeling and Discretization

The following LTI continuous-time model is considered:

$$\begin{cases} \dot{x} = (A_c + \delta A_c)x + (B_c + \delta B_c)(u + \delta u) + w^*, \\ y = (H + \delta H)x + v^*, \end{cases} \tag{3}$$

where:

$$\begin{cases} A_c &= A_c(\bar{\theta}), \\ B_c &= B_c(\bar{\theta}), \\ H &= H(\bar{\theta}), \end{cases}$$

are deterministic matrices and:

$$\begin{cases} \delta A_c &= \delta A_c(\delta\theta), \\ \delta B_c &= \delta B_c(\delta\theta), \\ \delta H &= \delta H(\delta\theta), \end{cases}$$

are random matrices due to the uncertainty of the parameters grouped in the vector θ, which can be expressed as:

$$\theta = \bar{\theta} + \delta\theta. \tag{4}$$

In eq. (4), $\bar{\theta}$ groups the nominal values of the parameters, and $\delta\theta$ represents their uncertainty such that $\mathbb{E}[\delta\theta] = 0$. Furthermore, it is assumed that the input u is affected by some noise δu (which could be quantization noise or coming from un-modelled dynamics), and the state is affected by noise w^*, while the output is corrupted by measurement noise v^*. It is also assumed that $\delta\theta$ is statistically independent from δu, w^*, and v^* and x_0, which is a natural assumption. In terms of dimensions: the state $x \in \mathbb{R}^n$, the output $y \in \mathbb{R}^m$, u and $\delta u \in \mathbb{R}^q$, A_c and $\delta A_c \in \mathbb{R}^{n\times n}$, B_c and $\delta B_c \in \mathbb{R}^{n\times q}$, H and $\delta H \in \mathbb{R}^{m\times n}$, $w^* \in \mathbb{R}^n$ and $v^* \in \mathbb{R}^m$.

For practical use, this model needs to be discretized (with sampling period T) as follows:

$$\begin{cases} x_{k+1} &= (A+\delta A)x_k + (B+\delta B)(u_k+\delta u_k) + w_k^*, \\ y_k &= (H+\delta H)x_k + v_k^*, \end{cases} \tag{5}$$

where:

$$\begin{cases} A = e^{A_c T}, \\ B = \int_0^T e^{A_c \tau} d\tau B_c = A_c^{-1}\left(e^{A_c T} - I_n\right) B_c = A_c^{-1}(A - I_n)\, B_c, \end{cases} \tag{6}$$

and where I_n is the $n \times n$ identity matrix and A_c is assumed invertible. It is convenient to express the last relationship in eq. (6) as follows:

$$B = A_c^{-1}(A - I_n)B_c = G\, B_c \tag{7}$$

It can be assumed (standard Kalman filters) that w_k^* and v_k^* are zero-mean uncorrelated white Gaussian noises. The LTI state equations can be therefore rewritten as:

$$\begin{cases} x_{k+1} &= Ax_k + Bu_k + (\delta A x_k + \delta B u_k + B\delta u_k + \delta B \delta u_k) + w_k^*, \\ y_k &= Hx_k + \left(\delta H x_k + v_k^*\right), \end{cases} \tag{8}$$

or equivalently as:

$$\begin{cases} x_{k+1} &= Ax_k + Bu_k + w_k, \\ y_k &= Hx_k + v_k, \end{cases} \tag{9}$$

where:

$$\begin{cases} w_k &= \delta A x_k + \delta B u_k + B\delta u_k + \delta B \delta u_k + w_k^*, \\ v_k &= \delta H x_k + v_k^*. \end{cases} \tag{10}$$

Continuous-time process noise $w^*(t)$ and measurement noise $v^*(t)$ are assumed zero-mean, white and Gaussian. In particular $\mathbb{Cov}(w^*(t)) = \mathbb{E}\left[w^*(t)w^{*T}(t-\tau)\right] = Q^c_{w^*}\delta(\tau)$; the discretized process noise w_k^* has a covariance [2]:

$$Q_{w^*} = \mathbb{Cov}(w_k^*) = \int_0^T e^{A_c(T-\tau)} Q^c_{w^*} e^{A_c^T(T-\tau)} d\tau. \tag{11}$$

Equation (11) can be used if a knowledge of the covariance of the continuous time process noise is available to numerically calculate the covariance of the discrete time process noise. In particular one might assume $Q^c_{w^*}$ to be diagonal; as shown by eq. (11) this is not the case in discrete time unless T is extremely small, in such a case it can be seen that $Q_{w^*} \approx Q^c_{w^*} T$ so diagonality is preserved.

Furthermore [2]:

$$R_{v^*} = \frac{R^c_{v^*}}{T}, \tag{12}$$

in this case, diagonality of the measurement noise covariance is preserved irrespective of the value of T.

Equation (10) defines equivalent noises similar to those presented in [5] (which, however, does not discuss the control input u_k) where the equivalent noises are state dependent.

However, for the case analysed here, the equivalent noises are not zero-mean:

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$$\begin{cases} \mathbb{E}[w_k] &= z_k, \\ \mathbb{E}[v_k] &= t_k. \end{cases} \tag{13}$$

For the second moments:

$$\begin{cases} \mathbb{E}\left[w_k w_k^T\right] &= P_{ww_k}, \\ \mathbb{E}\left[v_k v_k^T\right] &= P_{vv_k}, \\ \mathbb{E}\left[w_k v_k^T\right] &= P_{wv_k}, \end{cases} \tag{14}$$

and covariances:

$$\begin{cases} Q_k &= P_{ww_k} - z_k z_k^T, \\ R_k &= P_{vv_k} - t_k t_k^T, \\ M_k &= P_{wv_k} - z_k t_k^T. \end{cases} \tag{15}$$

The expected values of the equivalent noises (z_k and t_k respectively) will be considered as *additional states*. In terms of dimensions: $Q_k \in \mathbb{R}^{n\times n}$, $R_k \in \mathbb{R}^{m\times m}$, $M_k \in \mathbb{R}^{n\times m}$.

2.1. First-Order Modelling of the Uncertainty

At the first order, the continuous-time state-space uncertainties can be written as follows:

$$\begin{cases} \delta A_c \approx J_{A_c}\delta\theta, \\ \delta B_c \approx J_{B_c}\delta\theta, \\ \delta H \approx J_H\delta\theta, \end{cases} \tag{16}$$

where the $J_{(.)}$ are the *Jacobians* (used here as a shorthand for *Jacobian matrices*) of the state-space matrices with respect to the parameters θ. Actually, the Jacobian is defined for (column) vectors and needs to be generalized for matrices by stacking on top of each other the Jacobians of each column; this is also known as *vectorization*. In other words, the following should have been written as:

$$\begin{cases} \delta A_c \approx \operatorname{unvec}\left[\dfrac{\partial}{\partial\theta}\operatorname{vec}(A_c)\delta\theta\right] &= \operatorname{unvec}(J_{A_c}\delta\theta), \\ \delta H \approx \operatorname{unvec}\left[\dfrac{\partial}{\partial\theta}\operatorname{vec}(H)\delta\theta\right] &= \operatorname{unvec}(J_H\delta\theta), \end{cases} \tag{17}$$

however the simplified, but abused, notation in eq. (16) is preferred throughout this work. The (partial) derivatives $\partial\theta$ should be considered as evaluated at $\theta = \bar{\theta}$, but here a simplified notation has also been preferred. In terms of dimensions: $\theta \in \mathbb{R}^p$, $J_{A_c} \in \mathbb{R}^{n^2\times p}$, $J_{B_c} \in \mathbb{R}^{nq\times p}$, $J_H \in \mathbb{R}^{nm\times p}$.

These uncertainties need to be converted into discrete time, except for δH, which is identical to the continuous time counterpart.

This is done in eq. (18) where it is considered that $B = G(A_c)\,B_c$ is a function of both B_c and A_c, i.e. $B = g(B_c, A_c)$:

$$\begin{cases} \delta A \approx \dfrac{\partial\operatorname{vec}(A)}{\partial\operatorname{vec}(A_c)^T}\delta A_c = J_A\delta A_c \approx J_A J_{A_c}\delta\theta, \\ \delta B \approx \dfrac{\partial\operatorname{vec}(g)}{\partial\operatorname{vec}(B_c)^T}\delta B_c + \dfrac{\partial\operatorname{vec}(g)}{\partial\operatorname{vec}(A_c)^T}\delta A_c = (I_q\otimes G)\delta B_c + J_B\delta A_c \approx (I_q\otimes G)J_{B_c}\delta\theta + J_B J_{A_c}\delta\theta. \end{cases} \tag{18}$$

In eq. (18), the symbol $\otimes$ refers to the Kronecker (or tensor) product used in the *vectorization* that allowed the different contributions to be expressed as functions of the covariances of the state-space matrices. Furthermore in eqs. (17) and (18) the notation for the expression of the Jacobians strictly

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follows [15] although it is abused for the expression of the differentials which would require an unvec operation. This will be clarified only when a disambiguation is strictly needed.

In the following, the definition in eq. (19) is used:

$$V = (I_q \otimes G) J_{B_c} + J_B J_{A_c}, \tag{19}$$

hence it can finally be written, with abused notation (i.e. implicitly assuming *vectorization* and *de-vectorization*):

$$\delta B \approx V \delta\theta. \tag{20}$$

The other newly introduced Jacobians are the following:

$$\begin{cases} J_A = \dfrac{\partial A}{\partial A_c} = \dfrac{\partial \text{vec}(A)}{\partial \text{vec}(A_c)^T} = \displaystyle\int_0^T e^{A_c^T \tau} \otimes e^{A_c (T-\tau)} d\tau, \\ J_B = \dfrac{\partial B}{\partial A_c} = \dfrac{\partial \text{vec}(B)}{\partial \text{vec}(A_c)^T} = \left(B_c^T \otimes A_c^{-1}\right) J_A - \left(B^T \otimes A_c^{-1}\right). \end{cases} \tag{21}$$

The derivation of the expression of J_B is detailed in the appendix. In terms of dimensions: $V \in \mathbb{R}^{nq \times p}$, $J_A \in \mathbb{R}^{n^2 \times n^2}$, $J_B \in \mathbb{R}^{nq \times n^2}$.

2.1.1. Type B Uncertainty Characterization of Component Tolerances

The robust filter design methodology presented in this work requires knowledge of the parameter uncertainty covariance matrix $\Sigma_{\delta\theta}$. In practice, the filter designers have rarely access to statistical population data for individual components. Instead, parameter uncertainties are typically available only as manufacturer-specified tolerances. This section discusses how to construct $\Sigma_{\delta\theta}$ from tolerance specifications following the Guide to the Expression of Uncertainty in Measurement (GUM) [16] framework.

The GUM distinguishes between two fundamental approaches to uncertainty quantification:

- Type A evaluation: Based on statistical analysis of series of observations. Requires measuring a representative sample of components and computing sample statistics (mean, variance, correlations).
- Type B evaluation: Based on means other than statistical analysis, including manufacturer specifications, calibration certificates, handbooks, experience, or scientific judgment. This is the approach typically available to filter designers and is therefore adopted in this work.

Electrical component tolerances are usually specified by manufacturers as symmetric bounds around nominal values. Common examples include:

- Resistors: $R = 10\ \text{k}\Omega \pm 1\%$ (meaning $R \in [9.9, 10.1]\ \text{k}\Omega$)
- Capacitors: $C = 100\ \mu\text{F} \pm 10\%$ (meaning $C \in [90, 110]\ \mu\text{F}$)
- Inductors: $L = 1\ \text{mH} \pm 5\%$ (meaning $L \in [0.95, 1.05]\ \text{mH}$)

These specifications define bounded intervals $[\bar{\theta}_i - \Delta\theta_i, \bar{\theta}_i + \Delta\theta_i]$ but provide no information about the probability distribution within these bounds.

When only tolerance bounds are available, the GUM methodology recommends using a uniform distribution:

$$\delta\theta_i \sim \mathcal{U}(-\Delta\theta_i, \Delta\theta_i). \tag{22}$$

The corresponding variance is:

$$\sigma^2_{\delta\theta_i} = \frac{\Delta\theta_i^2}{3}. \tag{23}$$

For a system with p uncertain parameters $\theta = [\theta_1, \theta_2, \ldots, \theta_p]^T$, the covariance matrix is constructed as:

$$\Sigma_{\delta\theta} = \begin{bmatrix} \sigma^2_{\delta\theta_1} & \sigma_{\delta\theta_1,\delta\theta_2} & \cdots & \sigma_{\delta\theta_1,\delta\theta_p} \\ \sigma_{\delta\theta_2,\delta\theta_1} & \sigma^2_{\delta\theta_2} & \cdots & \sigma_{\delta\theta_2,\delta\theta_p} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{\delta\theta_p,\delta\theta_1} & \sigma_{\delta\theta_p,\delta\theta_2} & \cdots & \sigma^2_{\delta\theta_p} \end{bmatrix}. \tag{24}$$

Diagonal elements are computed from tolerance specifications using, as an example eq. (23); off-diagonal elements (covariances between different parameters) are typically set to zero in the absence of specific information; assuming that component parameters are statistically independent is a sound hypothesis most of the time (there could be exceptions for special arrangement such as current or voltage dividers where two or more components are known to track each other etc...). When assuming: (i) uniform distribution, (ii) diagonal covariance matrix and (iii) symmetry around the mean value, it is straightforward to calculate all higher order moments :

$$\mathbb{E}[\delta\theta_i^n] = \begin{cases} \frac{\Delta\theta_i^n}{n+1} & n \text{ even} \\ 0 & n \text{ odd} \end{cases} \tag{25}$$

What is presented in the rest of the paper does not specifically assume the uniform distribution or the diagonality of the covariance matrix, however the derivation of the proposed robust Kalman filter is driven exactly by the practical case where only tolerances of the components are known to the filter designer.

2.2. First-Order Statistical Modelling

The state can be expressed explicitly as:

$$\begin{cases} x_k &= (A+\delta A)^k x_0 + \sum_{j=0}^{k-1} (A+\delta A)^{k-1-j} \left[(B+\delta B)(u_j + \delta u_j) + w_j^* \right], \\ y_k &= (H+\delta H) x_k + v_k^*. \end{cases} \tag{26}$$

Even assuming statistical independence of x_0 from δA (which is a completely reasonable assumption), the problem as expressed in eq. (26) is very complex in general terms. The evolution of the state is governed by $(A+\delta A)^k$ which can be expressed as follows:

$$(A+\delta A)^k = A^k + \sum_{i=0}^{k-1} A^{k-1-i} \delta A A^i + \alpha_k(\delta A), \tag{27}$$

where $\alpha_k(\delta A)$ contains all terms of order $O(\delta A^2)$ (i.e. terms like $\sum_{0 \le i < j, i+j \le k-2} A^{k-2-i-j} \delta A A^i \delta A A^j$) and higher-order ones in the expansion of $(A+\delta A)^k$ (note that $\alpha_k(\delta A) = 0$ for $k < 0$).

From eqs. (26) and (27) the state can be expressed as:

$$\begin{aligned} x_k = &\left(A^k + \sum_{i=0}^{k-1} A^{k-1-i} \delta A A^i + \alpha_k(\delta A) \right) x_0 \\ &+ \sum_{j=0}^{k-1} \left[A^{k-1-j}(B+\delta B) + \left(\sum_{i=0}^{k-2-j} A^{k-2-j-i} \delta A A^i \right) (B+\delta B) + \alpha_{k-1-j}(\delta A)(B+\delta B) \right] (u_j + \delta u_j) \\ &+ \sum_{j=0}^{k-1} \left(A^{k-1-j} + \sum_{i=0}^{k-2-j} A^{k-2-j-i} \delta A A^i + \alpha_{k-1-j}(\delta A) \right) w_j^*, \end{aligned} \tag{28}$$

From eq. (28) one can decompose x_k as follows:

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$$
\begin{aligned}
x_k = x_k^{nom} + \delta x_k + \delta x_k^{ho} = A^k x_0 + \sum_{j=0}^{k-1} A^{k-1-j} B u_j + \widetilde{\delta A}_{k-1} x_0 + \sum_{j=0}^{k-1} A^{k-1-j} \delta B u_j \\
+ \sum_{j=0}^{k-1} \widetilde{\delta A}_{k-2-j} B u_j + \delta x_k^{ho},
\end{aligned} \tag{29}
$$

where:

$$
\begin{cases}
x_k^{nom} & = A^k x_0 + \sum_{j=0}^{k-1} A^{k-1-j} B u_j, \\
\delta x_k & = \widetilde{\delta A}_{k-1} x_0 + \sum_{j=0}^{k-1} A^{k-1-j} \delta B u_j + \sum_{j=0}^{k-1} \widetilde{\delta A}_{k-2-j} B u_j,
\end{cases} \tag{30}
$$

represent, respectively, the *nominal* state evolution (i.e. the state evolution assuming perfect knowledge of the state-space model) and the first order perturbation in δA and δB, or equivalently in $\delta\theta$. It is worth noting that $\mathbb{E}[\delta x_k] = 0$ because of the independence of x_0 from $\delta\theta$ and $\mathbb{E}[\delta A] = 0$ and $\mathbb{E}[\delta B] = 0$. The fundamental idea, in this work, is to design a robust Kalman filter neglecting the higher order contributions in δx_k^{ho}. This is similar to the MVFOSM approach (where only the zeroth order is considered for the mean and only contributions up to the second order are considered for the second moments such as the covariances) but slightly more accurate as it also allows to take into account the state and output biases induced by the model uncertainty, by augmenting the state as detailed in the next section.

2.3. Augmented Model

To build the augmented model it can be observed (thanks to the statistical independence assumptions) that:

$$
\begin{cases}
z_k = \mathbb{E}[w_k] = \mathbb{E}[\delta A x_k], \\
t_k = \mathbb{E}[v_k] = \mathbb{E}[\delta H x_k].
\end{cases} \tag{31}
$$

For the additional states, the dynamics is as follows:

$$
\begin{cases}
z_{k+1} & = \mathbb{E}[\delta A x_{k+1}] = \mathbb{E}[\delta A A x_k] + \mathbb{E}[\delta A \delta A x_k] + \mathbb{E}[\delta A \delta B u_k] \\
& \approx \mathbb{E}[\delta A A x_k] + \mathbb{E}[\delta A \delta A] \bar{x}_k^{nom} + \mathbb{E}[\delta A \delta B] u_k, \\
t_{k+1} & = \mathbb{E}[\delta H x_{k+1}] = \mathbb{E}[\delta H A x_k] + \mathbb{E}[\delta H \delta A x_k] + \mathbb{E}[\delta H \delta B u_k] \\
& \approx \mathbb{E}[\delta H A x_k] + \mathbb{E}[\delta H \delta A] \bar{x}_k^{nom} + \mathbb{E}[\delta H \delta B] u_k,
\end{cases} \tag{32}
$$

where in (32) all contributions that are higher than second order have been neglected.

In eq. (32) the difficulty comes from the fact that A does not commute with δA and δH so the terms $\mathbb{E}[\delta A A x_k]$ and $\mathbb{E}[\delta H A x_k]$ are not immediately expressible as functions of z_k and t_k respectively. However, it can be observed that:

$$
\begin{cases}
\mathbb{E}[\delta A A x_k] & = \mathrm{vec}(\mathbb{E}[\delta A A x_k]) = \mathbb{E}[(x_k^T \otimes \delta A)\mathrm{vec}(A)] = \mathbb{E}[x_k^T \otimes \delta A]\mathrm{vec}(A), \\
\mathbb{E}[\delta H A x_k] & = \mathrm{vec}(\mathbb{E}[\delta H A x_k]) = \mathbb{E}[(x_k^T \otimes \delta H)\mathrm{vec}(A)] = \mathbb{E}[x_k^T \otimes \delta H]\mathrm{vec}(A),
\end{cases} \tag{33}
$$

and that:

$$
\begin{cases}
z_{k+1} = & \mathbb{E}[\delta A x_{k+1}] = \mathrm{vec}(\mathbb{E}[\delta A I_n x_{k+1}]) = \mathbb{E}[x_{k+1}^T \otimes \delta A]\mathrm{vec}(I_n), \\
t_{k+1} = & \mathbb{E}[\delta H x_{k+1}] = \mathrm{vec}(\mathbb{E}[\delta H I_n x_{k+1}]) = \mathbb{E}[x_{k+1}^T \otimes \delta H]\mathrm{vec}(I_n).
\end{cases} \tag{34}
$$

Therefore, the recursion can be written as:

$$
\begin{cases}
z_{k+1} = & \mathbb{E}[x_{k+1}^T \otimes \delta A]\mathrm{vec}(I_n) \approx \mathbb{E}[x_k^T \otimes \delta A]\mathrm{vec}(A) + \mathbb{E}[\delta A \delta A] \bar{x}_k^{nom} + \mathbb{E}[\delta A \delta B] u_k, \\
t_{k+1} = & \mathbb{E}[x_{k+1}^T \otimes \delta H]\mathrm{vec}(I_n) \approx \mathbb{E}[x_k^T \otimes \delta H]\mathrm{vec}(A) + \mathbb{E}[\delta H \delta A] \bar{x}_k^{nom} + \mathbb{E}[\delta H \delta B] u_k.
\end{cases} \tag{35}
$$

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The recursion, as written in eq. (35), is still implicit; to be able to actually *run* it, z_k and t_k need to be expressed explicitly. This can be done as follows; first note, by means of the definition in eq. (34), that:

$$\begin{cases} z_k &= \mathbb{E}[x_k^T \otimes \delta A]\text{vec}(I_n), \\ t_k &= \mathbb{E}[x_k^T \otimes \delta H]\text{vec}(I_n), \end{cases} \tag{36}$$

then define:

$$\begin{cases} Z_k &= \mathbb{E}[x_k^T \otimes \delta A] \in \mathbb{R}^{n \times n^2}, \\ T_k &= \mathbb{E}[x_k^T \otimes \delta H] \in \mathbb{R}^{m \times n^2}, \end{cases} \tag{37}$$

the recursion for these newly introduced matrices is therefore the following:

$$\begin{cases} Z_{k+1} &\approx Z_k(A^T \otimes I_n) + (\bar{x}_k^{nom} \otimes I_n)\,\mathbb{E}[\delta A^T \otimes \delta A] + (u_k^T \otimes I_n)\,\mathbb{E}[\delta B^T \otimes \delta A] \\ T_{k+1} &\approx T_k(A^T \otimes I_n) + (\bar{x}_k^{nom} \otimes I_m)\,\mathbb{E}[\delta A^T \otimes \delta H] + (u_k^T \otimes I_m)\,\mathbb{E}[\delta B^T \otimes \delta H]. \end{cases} \tag{38}$$

The derivation of eq. (38) is detailed in the Appendix.
The following holds true for the expected values of state and output:

$$\begin{cases} \bar{x}_{k+1} &\approx A\bar{x}_k + Bu_k + z_k \\ Z_{k+1} &\approx Z_k(A^T \otimes I_n) + (\bar{x}_k^{nom} \otimes I_n)\,\mathbb{E}[\delta A^T \otimes \delta A] + (u_k^T \otimes I_n)\,\mathbb{E}[\delta B^T \otimes \delta A] \\ z_k &= Z_k\text{vec}(I_n) \\ T_{k+1} &\approx T_k(A^T \otimes I_n) + (\bar{x}_k^{nom} \otimes I_m)\,\mathbb{E}[\delta A^T \otimes \delta H] + (u_k^T \otimes I_m)\,\mathbb{E}[\delta B^T \otimes \delta H] \\ t_k &= T_k\text{vec}(I_n) \\ \bar{y}_k &\approx H\bar{x}_k + t_k \end{cases} \tag{39}$$

Equation (39) can be further simplified as follows by recognizing that $\bar{x}_k^{nom} \approx \bar{x}_k$ up to $O(\delta\theta^2)$:

$$\begin{cases} \bar{x}_{k+1} &\approx A\bar{x}_k + Bu_k + z_k \\ Z_{k+1} &\approx Z_k(A^T \otimes I_n) + (\bar{x}_k \otimes I_n)\,\mathbb{E}[\delta A^T \otimes \delta A] + (u_k^T \otimes I_n)\,\mathbb{E}[\delta B^T \otimes \delta A] \\ z_k &= Z_k\text{vec}(I_n) \\ T_{k+1} &\approx T_k(A^T \otimes I_n) + (\bar{x}_k \otimes I_m)\,\mathbb{E}[\delta A^T \otimes \delta H] + (u_k^T \otimes I_m)\,\mathbb{E}[\delta B^T \otimes \delta H] \\ t_k &= T_k\text{vec}(I_n) \\ \bar{y}_k &\approx H\bar{x}_k + t_k \end{cases} \tag{40}$$

this simplification is very important as it allows reducing the computational burden by avoiding the introduction of yet another *state* $d_k = \mathbb{E}[\delta x_k]$ to keep track of the difference between $\bar{x}_k$ and $\bar{x}_k^{nom}$ as detailed in the appendix.

3. Steady-State Kalman Filter

Equations (15) and (40) are sufficient to calculate the steady-state Kalman filter when assuming Q, R and M to be time invariant (which will be discussed in section 3.3) and that the system in (40) is stable (this is so because the homogeneous dynamics of Z_k and T_k are governed by the matrix $(A^T \otimes I_n) \in \mathbb{R}^{n^2 \times n^2}$ whose eigenvalues are those of A each with multiplicity n; and the $\bar{x}_k$ equation is driven by z_k which decays to a constant for constant input. Since all eigenvalues of A are strictly within the unit circle, all modes are stable.). Let the covariance matrix of the state estimation error, in steady-state, be denoted as P. It can be computed by solving the following generalized DARE (discrete algebraic Riccati equation) as reported in [2]:

$$P = APA^T - A\left(PH^T + M\right)\left(HPH^T + HM + M^TH^T + R\right)^{-1}\left(HP + M^T\right)A^T + Q \tag{41}$$

Once P is calculated, the steady-state Kalman gain K can be calculated as follows:

$$K = \left(PH^T + M\right)\left(HPH^T + HM + M^TH^T + R\right)^{-1} \tag{42}$$

It is useful to introduce the following Choleski decomposition: $LL^T = Q - MR^{-1}M^T$. A unique positive semi-definite solution P of eq. (41) exists under the following conditions [2]:

$$\begin{cases} Q \succeq 0, \\ R \succ 0, \\ (A, H) \text{ detectable}, \\ (A - MR^{-1}H, L) \text{ stabilizable}. \end{cases} \tag{43}$$

In such a case the solution leads to a stable steady-state Kalman filter i.e. all eigenvalues of $(I_n - KH)A$ are strictly inside the unit circle. The detectability condition is satisfied when all undetectable modes of (A, H) are stable. For a stable open-loop system (all eigenvalues of A strictly inside the unit circle), detectability is automatically satisfied. The stabilizability of $(A - MR^{-1}H, L)$ can be verified once Q, R, M are computed, typically by checking the controllability Gramian or by attempting to solve the DARE numerically. Their calculation is discussed in the remaining of this section.

3.1. Steady-State of the Augmented Model

It is not straightforward to calculate the asymptotic response (i.e. assuming u_k has been kept constant for long enough) from eq. (40) so a different way is presented here. Considering the first order perturbation δx_k introduced in eq. (28) the *additional states* z_k and t_k can be written explicitly as:

$$\begin{cases} z_k & = \mathbb{E}[\delta A x_k] \approx E[\delta A \delta x_k] = \mathbb{E}[\delta A \widetilde{\delta A}_{k-1}]\,\mathbb{E}[x_0] + \mathbb{E}[\delta A \sum_{j=0}^{k-1} A^{k-1-j}\delta B]u_j \\ & + \mathbb{E}[\delta A \sum_{j=0}^{k-1} \widetilde{\delta A}_{k-2-j}]Bu_j, \\ t_k & = \mathbb{E}[\delta H x_k] \approx E[\delta H \delta x_k] = \mathbb{E}[\delta H \widetilde{\delta A}_{k-1}]\,\mathbb{E}[x_0] + \mathbb{E}[\delta H \sum_{j=0}^{k-1} A^{k-1-j}\delta B]u_j \\ & + \mathbb{E}[\delta H \sum_{j=0}^{k-1} \widetilde{\delta A}_{k-2-j}]Bu_j. \end{cases} \tag{44}$$

It is worth introducing the following:

$$W = (I_n - A)^{-1} \tag{45}$$

The first term of both expressions in eq. (44), proportional to $\mathbb{E}[x_0]$, vanishes as $k \to \infty$ (the proof is reported in the appendix).

The steady-state values of the additional states read (with $u_j = u_k = u$):

$$\begin{cases} z_\infty & \approx \mathbb{E}[\delta A W \delta B]u_k + \mathbb{E}[\delta A W \delta A]WBu_k, \\ t_\infty & \approx \mathbb{E}[\delta H W \delta B]u_k + \mathbb{E}[\delta H W \delta A]WBu_k. \end{cases} \tag{46}$$

The proof of eq. (46) is also given in the appendix.

So the steady-state behaviour of the expected value is described by the following:

$$\begin{cases} \bar{x}_\infty & \approx A\bar{x}_\infty + Bu_k + z_\infty, \\ z_\infty & \approx \mathbb{E}[\delta A W \delta B]u_k + \mathbb{E}[\delta A W \delta A]WBu_k, \\ t_\infty & \approx \mathbb{E}[\delta H W \delta B]u_k + \mathbb{E}[\delta H W \delta A]WBu_k. \end{cases} \tag{47}$$

The solution of the algebraic system in eq. (47) can be written as follows:

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$$\begin{cases} \bar{x}_\infty & \approx F_x u_k \\ z_\infty & \approx F_z u_k \\ t_\infty & \approx F_t u_k \end{cases} \tag{48}$$

letting $F_{nom} = (I_n - A)^{-1}B = WB$ be the *nominal* DC gain, the augmented model gains are:

$$\begin{cases} F_x & = (I_n - A)^{-1}(B + F_z) = F_{nom} + WF_z, \\ F_z & = \mathbb{E}[\delta A W \delta B] + \mathbb{E}[\delta A W \delta A]F_{nom}, \\ F_t & = \mathbb{E}[\delta H W \delta B] + \mathbb{E}[\delta H W \delta A]F_{nom}. \end{cases} \tag{49}$$

For the actual computation of the above *gains*, the following relationships are useful:

$$\begin{cases} \mathbb{E}[\delta A W \delta B] & = \text{unvec}\big(\mathbb{E}[\delta B^T \otimes \delta A]\text{vec}(W)\big) \approx \text{unvec}\Big[\Sigma^*_{(BA)}\text{vec}(W)\Big], \\ \mathbb{E}[\delta A W \delta A] & = \text{unvec}\big(\mathbb{E}[\delta A^T \otimes \delta A]\text{vec}(W)\big) \approx \text{unvec}\Big[\Sigma^*_{(AA)}\text{vec}(W)\Big], \\ \mathbb{E}[\delta H W \delta B] & = \text{unvec}\big(\mathbb{E}[\delta B^T \otimes \delta H]\text{vec}(W)\big) \approx \text{unvec}\Big[\Sigma^*_{(BH)}\text{vec}(W)\Big], \\ \mathbb{E}[\delta H W \delta A] & = \text{unvec}\big(\mathbb{E}[\delta A^T \otimes \delta H]\text{vec}(W)\big) \approx \text{unvec}\Big[\Sigma^*_{(AH)}\text{vec}(W)\Big]. \end{cases} \tag{50}$$

Finally the *DC gains* of the augmented model are expressed by the following:

$$\begin{cases} F_x & = W(B + F_z) = F_{nom} + WF_z, \\ F_z & \approx \text{unvec}\Big[\Sigma^*_{(BA)}\text{vec}(W)\Big] + \text{unvec}\Big[\Sigma^*_{(AA)}\text{vec}(W)\Big]F_{nom}, \\ F_t & \approx \text{unvec}\Big[\Sigma^*_{(BH)}\text{vec}(W)\Big] + \text{unvec}\Big[\Sigma^*_{(AH)}\text{vec}(W)\Big]F_{nom}. \end{cases} \tag{51}$$

3.2. Calculation of Q, R and M

In accordance with the MVFOSM approach in the calculation of the Q, R and M matrices only contributions up to the second order in δA, δB and δH, or, in general, $O(\delta\theta^2)$ will be considered. However, the terms $z_k z_k^T$, $t_k t_k^T$ and $z_k t_k^T$ although being $O(\delta\theta^4)$ will be retained to ensure that Q_k, R_k and M_k exactly represent the covariances of the equivalent noises: $\mathbb{E}[(w_k - \bar{w}_k)(w_k - \bar{w}_k)^T]$, $\mathbb{E}[(v_k - \bar{v}_k)(v_k - \bar{v}_k)^T]$ and $\mathbb{E}[(w_k - \bar{w}_k)(v_k - \bar{v}_k)^T]$ as required by standard Kalman filter.

The following matrices are going to be useful in the upcoming calculations:

$$J^A_{A_c} = J_A J_{A_c} \in \mathbb{R}^{n^2 \times p}, \tag{52}$$

so

$$\Sigma_{\delta A} = \mathbb{E}\Big[\delta A \delta A^T\Big] = \mathbb{E}\Big[J_A J_{A_c} \delta\theta\delta\theta^T J^T_{A_c} J^T_A\Big] \approx J^A_{A_c}\,\mathbb{E}\Big[\delta\theta\delta\theta^T\Big] J^{A^T}_{A_c} = J^A_{A_c} \Sigma_{\delta\theta} J^{A^T}_{A_c}. \tag{53}$$

Furthermore:

$$\Sigma_{\delta u} = \mathbb{E}[\delta u_k \delta u_k^T], \tag{54}$$

where $\Sigma_{\delta u} \in \mathbb{R}^{q \times q}$.

It is also useful to define the following matrices:

$$
\begin{cases}
\Sigma_{(AA)} = \Sigma_{\delta A} = \mathbb{E}[\delta A \delta A^T] \approx J^A_{A_c} \Sigma_{\delta\theta} J^{A^T}_{A_c} \in \mathbb{R}^{n^2 \times n^2}, \\
\Sigma_{(BB)} = \Sigma_{\delta B} = \mathbb{E}[\delta B \delta B^T] \approx V \Sigma_{\delta\theta} V^T \in \mathbb{R}^{nq \times nq}, \\
\Sigma_{(HH)} = \Sigma_{\delta H} = \mathbb{E}[\delta H \delta H^T] \approx J_H \Sigma_{\delta\theta} J_H^T \in \mathbb{R}^{nm \times nm}, \\
\Sigma_{(AB)} = \mathbb{E}[\delta A \delta B^T] \approx J^A_{A_c} \Sigma_{\delta\theta} V^T \in \mathbb{R}^{n^2 \times nq}, \\
\Sigma_{(AH)} = \mathbb{E}[\delta A \delta H^T] \approx J^A_{A_c} \Sigma_{\delta\theta} J_H^T \in \mathbb{R}^{n^2 \times nm}, \\
\Sigma_{(BH)} = \mathbb{E}[\delta B \delta H^T] \approx V \Sigma_{\delta\theta} J_H^T \in \mathbb{R}^{nq \times nm}, \\
\Sigma^*_{(AA)} = \mathbb{E}[\delta A^T \otimes \delta A] \approx r(\Sigma_{(AA)}), \\
\Sigma^*_{(BA)} = \mathbb{E}[\delta B^T \otimes \delta A] \approx r(\Sigma_{(AB)}), \\
\Sigma^*_{(AH)} = \mathbb{E}[\delta A^T \otimes \delta H] \approx r(\Sigma_{(AH)}), \\
\Sigma^*_{(BH)} = \mathbb{E}[\delta B^T \otimes \delta H] \approx r(\Sigma_{(BH)}),
\end{cases}
\tag{55}
$$

where $r(.)$ is a reshaping function that *translates* the entries of the un-starred matrices in eq. (55) into the entries of the starred matrices that appear in the augmented state. Detailed expression of $r(.)$ is given in the appendix.

It is important to highlight that the equalities in eq. (55) have to be interpreted taking into account the abused notation that implicitly assumes *vectorization* and *de-vectorization* e.g. for $\Sigma_{(AB)} = \mathbb{E}[\delta A \delta B^T]$ the product $\delta A \delta B^T$ does not ordinarily exist as $\delta A \in \mathbb{R}^{n \times n}$ and $\delta B^T \in \mathbb{R}^{q \times n}$ so they cannot be multiplied; for the calculation of $\Sigma_{(AB)}$ one would more correctly need to write $\Sigma_{(AB)} = \mathbb{E}[\text{vec}(\delta A)\text{vec}(\delta B^T)] = \mathbb{E}[\text{vec}[\text{unvec}(J^A_{A_c}\delta\theta)]\text{vec}[\text{unvec}(\delta\theta^T V^T)]] = \mathbb{E}[J^A_{A_c}\delta\theta\delta\theta^T V^T] = J^A_{A_c}\Sigma_{\delta\theta}V^T$ and so on.

3.2.1. Calculation of Q

Neglecting the terms higher than quadratic it yields:

$$
\begin{aligned}
P_{ww_k} \approx \mathbb{E}\left[w_k w_k^T\right] &\approx \mathbb{E}\left[\delta A x_k x_k^T \delta A^T\right] + \mathbb{E}\left[\delta A x_k u_k^T \delta B^T\right] + \mathbb{E}\left[\delta B u_k x_k^T \delta A^T\right] + \\
&+ \mathbb{E}\left[\delta B u_k u_k^T \delta B^T\right] + \mathbb{E}\left[B \delta u_k \delta u_k^T B^T\right] + \mathbb{E}\left[w_k^* w_k^{*T}\right],
\end{aligned}
\tag{56}
$$

where the other terms are exactly zero because of statistical independence or approximately zero due to MVFOSM truncation.

The terms in eq. (56) can be expressed as follows:

$$
\begin{cases}
\mathbb{E}\left[\delta A x_k x_k^T \delta A^T\right] \approx \mathbb{E}\left[\delta A \bar{x}_k^{nom} \bar{x}_k^{nom^T} \delta A^T\right] \\
\qquad \approx (u_k^T \otimes I_n)\left(F_{nom}^T \otimes I_n\right)\Sigma_{(AA)}(F_{nom} \otimes I_n)(u_k \otimes I_n), \\
\mathbb{E}\left[\delta A x_k u_k^T \delta B^T\right] \approx \mathbb{E}\left[\delta A \bar{x}_k^{nom} u_k^T \delta B^T\right] \approx \mathbb{E}\left[\delta A F_{nom} u_k u_k^T \delta B^T\right] \\
\qquad \approx (u_k^T \otimes I_n)\left(F_{nom}^T \otimes I_n\right)\Sigma_{(AB)}(u_k \otimes I_n), \\
\mathbb{E}\left[\delta B u_k x_k^T \delta A^T\right] \approx \mathbb{E}\left[\delta B u_k \bar{x}_k^{nom^T} \delta A^T\right] \approx \mathbb{E}\left[\delta B u_k u_k^T F_{nom}^T \delta A^T\right] \\
\qquad \approx (u_k^T \otimes I_n)\Sigma_{(AB)}^T(F_{nom} \otimes I_n)(u_k \otimes I_n), \\
\mathbb{E}\left[\delta B u_k u_k^T \delta B^T\right] = (u_k^T \otimes I_n)\,\mathbb{E}\left[\delta B \delta B^T\right](u_k \otimes I_n) \approx (u_k^T \otimes I_n)\Sigma_{(BB)}(u_k \otimes I_n), \\
\mathbb{E}\left[B \delta u_k \delta u_k^T B^T\right] = B\,\mathbb{E}\left[\delta u_k \delta u_k^T\right]B^T = B\Sigma_{\delta u}B^T, \\
\mathbb{E}\left[w_k^* w_k^{*T}\right] = \Sigma_{w_k^*} = Q_{w^*}
\end{cases}
\tag{57}
$$

where it has been considered that $x_k = x_k^{nom} + \delta x_k^{full}$ and δx_k^{full} contains all the terms depending on δA and δB; when calculating the second moments only the term x_k^{nom} contributes up to the second order as δx_k^{full} would only contribute to third order or higher. As an example when computing $E[\delta A x_k x_k^T \delta A^T]$, expanding the products yields:

- ☐ $\mathbb{E}[\delta A x_k^{nom} x_k^{nom^T} \delta A^T]$: $O(\delta\theta^2)$ - retained,
- ☐ $\mathbb{E}[\delta A x_k^{nom} \delta x_k^{full^T} \delta A^T]$ and transpose: $O(\delta\theta^3)$ - neglected
- ☐ $\mathbb{E}[\delta A \delta x_k^{full} \delta x_k^{full^T} \delta A^T]$: $O(\delta\theta^4)$ -neglected.

It is important to note that x_k^{nom} depends only on x_0 (and the history of u_k that is however deterministic) which is independent of $\delta\theta$ and therefore of δA, δB and δH so $\mathbb{E}[\delta A x_k^{nom} x_k^{nom^T} \delta A^T] = \mathbb{E}[\delta A \bar{x}_k^{nom} \bar{x}_k^{nom^T} \delta A^T]$. Thus, retaining only second-order terms, $\bar{x}_k^{nom} = \mathbb{E}[x_k^{nom}]$ is used in place of x_k, which is consistent with MVFOSM-like truncation. Furthermore it has also been considered that the terms containing $A^k\,\mathbb{E}[x_0]$ vanishes in steady-state (as A has eigenvalues strictly inside the unit circle).

As already done for the augmented state, $\bar{x}_k^{nom}$ can be approximated by $\bar{x}_k$ leading to:

$$\begin{cases} \mathbb{E}\left[\delta A x_k x_k^T \delta A^T\right] \approx \mathbb{E}\left[\delta A \bar{x}_k \bar{x}_k^T \delta A^T\right] \approx (u_k^T \otimes I_n)\left(F_x^T \otimes I_n\right)\Sigma_{(AA)}(F_x \otimes I_n)(u_k \otimes I_n), \\ \mathbb{E}\left[\delta A x_k u_k^T \delta B^T\right] \approx \mathbb{E}\left[\delta A \bar{x}_k u_k^T \delta B^T\right] \approx \mathbb{E}\left[\delta A F_x u_k u_k^T \delta B^T\right] \\ \qquad \approx (u_k^T \otimes I_n)\left(F_x^T \otimes I_n\right)\Sigma_{(AB)}(u_k \otimes I_n), \\ \mathbb{E}\left[\delta B u_k x_k^T \delta A^T\right] \approx \mathbb{E}\left[\delta B u_k \bar{x}_k^T \delta A^T\right] \approx \mathbb{E}\left[\delta B u_k u_k^T F_x^T \delta A^T\right] \\ \qquad \approx (u_k^T \otimes I_n)\Sigma_{(AB)}^T(F_x \otimes I_n)(u_k \otimes I_n), \\ \mathbb{E}\left[\delta B u_k u_k^T \delta B^T\right] = (u_k^T \otimes I_n)\,\mathbb{E}\left[\delta B \delta B^T\right](u_k \otimes I_n) \approx (u_k^T \otimes I_n)\Sigma_{(BB)}(u_k \otimes I_n), \\ \mathbb{E}\left[B \delta u_k \delta u_k^T B^T\right] = B\,\mathbb{E}\left[\delta u_k \delta u_k^T\right]B^T = B\Sigma_{\delta u}B^T, \\ \mathbb{E}\left[w_k^* w_k^{*T}\right] = \Sigma_{w_k^*} = Q_{w^*}. \end{cases} \tag{58}$$

In the remainder of the paper this approximation is going to be adopted for the covariances computation by using the DC gain F_x instead of F_{nom}.

The following time invariant covariance matrix is useful for the upcoming computations:

$$Q_0 = Q_{w^*} + B\Sigma_{\delta u}B^T. \tag{59}$$

Thus, the full expression of P_{ww_k} can be written as:

$$\begin{aligned} P_{ww_k} &\approx Q_0 + (u_k^T \otimes I_n)P_u(u_k \otimes I_n) \\ P_u &= \left(F_x^T \otimes I_n\right)\Sigma_{(AA)}(F_x \otimes I_n) + (F_x^T \otimes I_n)\Sigma_{(AB)} + \Sigma_{(AB)}^T(F_x \otimes I_n) + \Sigma_{(BB)}. \end{aligned} \tag{60}$$

Finally:

$$Q_k^{th} \approx P_{ww_k} - z_k z_k^T = P_{ww_k} - F_z u_k u_k^T F_z^T. \tag{61}$$

3.2.2. Calculation of R

$$P_{vv_k} = \mathbb{E}\left[v_k v_k^T\right] = \mathbb{E}\left[\delta H x_k x_k^T \delta H^T\right] + \mathbb{E}\left[v_k^* v_k^{*T}\right] \approx \mathbb{E}\left[\delta H \bar{x}_k^{nom} \bar{x}_k^{nom^T} \delta H^T\right] + \mathbb{E}\left[v_k^* v_k^{*T}\right], \tag{62}$$

as the expectations $\mathbb{E}\left[\delta H x_k v_k^{*T}\right]$ and $\mathbb{E}\left[v_k^* x_k^T \delta H^T\right]$ are both (strictly) zero.

The individual contributions are (again using $\bar{x}_k$ instead of $\bar{x}_k^{nom}$):

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$$\begin{cases} \mathbb{E}\left[\delta H \bar{x}_k \bar{x}_k^T \delta H^T\right] \approx \mathbb{E}\left[\delta H F_x u_k u_k^T F_x^T \delta H^T\right] \\ \qquad \approx (u_k^T \otimes I_m)\left(F_x^T \otimes I_m\right)\Sigma_{(HH)}(F_x \otimes I_m)(u_k \otimes I_m), \\ \mathbb{E}\left[v_k^* v_k^{*T}\right] = \Sigma_{v^*} = R_{v^*}. \end{cases} \tag{63}$$

Finally:

$$P_{vv_k} \approx R_{v^*} + (u_k^T \otimes I_m)\left(F_x^T \otimes I_m\right)\Sigma_{(HH)}(F_x \otimes I_m)(u_k \otimes I_m), \tag{64}$$

and:

$$R_k^{th} \approx P_{vv_k} - t_k t_k^T = P_{vv_k} - F_t u_k u_k^T F_t^T. \tag{65}$$

3.2.3. Calculation of M

$$\begin{aligned} P_{wv_k} &= \mathbb{E}\left[w_k v_k^T\right] = \mathbb{E}\left[\delta A x_k x_k^T \delta H^T\right] + \mathbb{E}\left[\delta B u_k x_k^T \delta H^T\right] \\ &\approx \mathbb{E}\left[\delta A \bar{x}_k^{nom} \bar{x}_k^{nom^T} \delta H^T\right] + \mathbb{E}\left[\delta B u_k \bar{x}_k^T \delta H^T\right], \end{aligned} \tag{66}$$

as the expectations $\mathbb{E}\left[\delta A x_k v_k^{*T}\right]$, $\mathbb{E}\left[\delta B u_k v_k^{*T} \delta H^T\right]$, $\mathbb{E}\left[B \delta u_k x_k^T \delta H^T\right]$ and $\mathbb{E}\left[B \delta u_k v_k^{*T}\right]$ are all (strictly) zero.

The non zero contributions are:

$$\begin{cases} \mathbb{E}\left[\delta A \bar{x}_k \bar{x}_k^T \delta H^T\right] \approx \mathbb{E}\left[\delta A F_x u_k u_k^T F_x^T \delta H^T\right] \approx (u_k^T \otimes I_n)(F_x^T \otimes I_n)\Sigma_{(AH)}(F_x \otimes I_m)(u_k \otimes I_m), \\ \mathbb{E}\left[\delta B u_k \bar{x}_k^T \delta H^T\right] \approx \mathbb{E}\left[\delta B u_k u_k^T F_x^T \delta H^T\right] \approx (u_k^T \otimes I_n)\Sigma_{(BH)}(F_x \otimes I_m)(u_k \otimes I_m). \end{cases} \tag{67}$$

Finally:

$$P_{wv_k} \approx (u_k^T \otimes I_n)\left[(F_x^T \otimes I_n)\Sigma_{(AH)}(F_x \otimes I_m) + \Sigma_{(BH)}(F_x \otimes I_m)\right](u_k \otimes I_m), \tag{68}$$

and:

$$M_k \approx P_{wv_k} - z_k t_k^T = P_{wv_k} - F_z u_k u_k^T F_t^T. \tag{69}$$

3.3. Time Invariant Covariances

As derived in the previous sections Q, R and M have components proportional to u_k (or conversely proportional to U_{k-1}^*) as such they are time-varying matrices. A time-varying Kalman filter could be straightforwardly implemented if desired and computationally compatible with real time implementation as briefly discussed in Section 4. However, the purpose of this work is to design a steady-state Kalman filter; in order to do that, several choices are possible with different level of robustness.

The degree of robustness can, as an example, be tuned by properly choosing the value of the *nominal* input u.

For steady-state filter design, time-invariant covariance matrices Q, R and M are sought. Since these depend on u_k (eqs. (60), (64), (68)), a representative constant value must be selected. Two common approaches:

- ☐ Worst-case robustness: Set $u = u_{max}$ (componentwise) to bound uncertainties for all operating conditions,
- ☐ Average-case design: For periodic inputs, use $u = u_{rms}$ to match typical operating conditions.

The choice represents a trade-off: larger u increases Q, R, M, making the filter more conservative (higher estimation uncertainty) but more robust to parameter variations. The designer should select u based on the application's robustness requirements and typical operating regime.

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It can happen that, due to the different approximations introduced, Q_k^{th} in eq. (61) is not positive semi-definite and/or R_k^{th} in eq. (65) is not positive definite. If that is the case, one needs to *regularize* them to numerically solve the DARE equation as follows:

$$\begin{aligned} Q_k &= Q_k^{th} + \epsilon_Q I_n, \quad \text{where } \epsilon_Q = \max\big(0,\ |\lambda_-^{max}(Q_k^{th})|\big), \\ R_k &= R_k^{th} + \epsilon_R I_m, \quad \text{where } \epsilon_R = \max\big(\epsilon,\ |\lambda_-^{max}(R_k^{th})|\big). \end{aligned} \tag{70}$$

and ϵ is a sufficiently small positive number.

3.4. Filter Implementation

It is useful to recall the following property of the *a priori* and *a posteriori* estimations:

$$\begin{cases} \hat{x}_k^+ = \mathbb{E}[x_k | y_1 \dots y_k], \\ \hat{x}_k^- = \mathbb{E}[x_k | y_1 \dots y_{k-1}]. \end{cases} \tag{71}$$

It is important to note that for the "Predict Phase" x_k can be approximated by $\hat{x}_k^+$ since in this phase the measurement y_k is available; whereas for the "Update/Correction Phase" x_k can be approximated by $\hat{x}_k^-$ as only measurements up to y_{k-1} are available. Furthermore, $\hat{z}_k$ and $\hat{t}_k$ are the estimations of z_k and t_k respectively.

3.4.1. Predict Phase

$$\begin{cases} \hat{x}_{k+1}^- &= A\, \hat{x}_k^+ + B\, u_k + \hat{z}_k, \\ \hat{Z}_{k+1} &= \hat{Z}_k\big(A^T \otimes I_n\big) + \Big({\hat{x}_k^+}^T \otimes I_n\Big)\Sigma_{(AA)}^* + \big(u_k^T \otimes I_n\big)\Sigma_{(BA)}^*, \\ \hat{T}_{k+1} &= \hat{T}_k\big(A^T \otimes I_n\big) + \Big({\hat{x}_k^+}^T \otimes I_m\Big)\Sigma_{(AH)}^* + \big(u_k^T \otimes I_m\big)\Sigma_{(BH)}^*. \end{cases} \tag{72}$$

The initial values $\hat{Z}_0$ and $\hat{T}_0$ are both zero.

3.4.2. Update/Correction Phase

$$\begin{cases} \hat{z}_k &= Z_k \text{vec}(I_n), \\ \hat{t}_k &= T_k \text{vec}(I_n), \\ \hat{x}_k^+ &= \hat{x}_k^- + K\big[y_k - (H\hat{x}_k^- + \hat{t}_k)\big]. \end{cases} \tag{73}$$

It can be noticed that the correction does not depend only on the (steady-state) Kalman gain K but also on the estimation of the output *bias* $\hat{t}_k$.

3.5. Computational Considerations and Basic MVFOSM Implementation

As it can be seen in eqs. (72) and (73), even if there is no need to update the state estimation covariance and therefore the *Kalman gain*, to fully profit of the proposed robust filter one needs to update in real time the estimations $\hat{z}_k$ and $\hat{t}_k$ and therefore, during the "Predict Phase", one needs to compute an additional $n \times n^2$ matrix to keep track of $\hat{Z}_k$ and an additional $m \times n^2$ matrix to keep track of $\hat{T}_k$ with respect to the *canonical* steady-state Kalman filter (the additional computational burden during the "Update/Correction Phase" is negligible [1]).

A possible simplification can be realized by strictly following the MVFOSM framework i.e. performing:

- ☐ Zeroth-order approximation for means: $\mathbb{E}[x_k] = \bar{x}_k \approx \mathbb{E}[x_k^{nom}]$, therefore neglecting $O(\delta\theta^2)$ corrections

[1] As an example, the j-th component of $\hat{z}_k$ is $\hat{z}_k^{(j)} = \sum_{t=1}^{n} \hat{Z}_k^{(j,(t-1)n+t)}$, i.e. the trace of the j-th $n \times n$ block of $\hat{Z}_k$, which is computationally inexpensive.

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□ Second-order approximation for covariances: retaining all terms up to $O(\delta\theta^2)$ in Q, R and M

This would mean accepting a *biased* state estimation by considering $\hat{z}_k = 0$ and $\hat{t}_k = 0$, in this case the resulting robust steady-state Kalman filter would reduce to the canonical Kalman filter described in eq. (2) with $K_k = K$, where K is still the solution of DARE in eq. (42) where one used $Q = P_{ww_k}$, $R = P_{vv_k}$ and $M = P_{wv_k}$; indeed the terms $\hat{z}_k\hat{z}_k^T$, $\hat{t}_k\hat{t}_k^T$ and $\hat{z}_k\hat{t}_k^T$ would be neglected too as they are $O(\delta\theta^4)$.

$$\begin{cases} \hat{x}_{k+1}^- = A\hat{x}_k^+ + Bu_k & \textit{predict}\text{ phase} \\ \hat{x}_k^+ = \hat{x}_k^- + K(y_k - H\hat{x}_k^-) & \textit{correction/update}\text{ phase} \end{cases} \tag{74}$$

4. Time-Varying Kalman Filter

If computationally feasible a robust time-varying filter can be implemented quite straightforwardly, as briefly sketched in the following. Furthermore, if the original uncertain system is also time-varying but having uncertainties with zero first order moment and constant second order moment, a time-varying filter is the only available option (where one could substitute A_k for A, B_k for B and H_k for H).

4.1. Predict Phase

1. Calculate Q_k as follows:

$$\begin{aligned} Q_k \approx Q_0 &+ (\hat{x}_k^{+T} \otimes I_n)\Sigma_{(AB)}(u_k \otimes I_n) + (u_k^T \otimes I_n)\Sigma_{(AB)}^T(\hat{x}_k^+ \otimes I_n) \\ &+ (\hat{x}_k^{+T} \otimes I_n)\Sigma_{(AA)}(\hat{x}_k^+ \otimes I_n) + (u_k^T \otimes I_n)\Sigma_{(BB)}(u_k \otimes I_n) - \hat{z}_k\hat{z}_k^T. \end{aligned} \tag{75}$$

2. Perform the prediction step:

$$\begin{cases} \hat{x}_{k+1}^- &= A\,\hat{x}_k^+ + B\,u_k + \hat{z}_k, \\ \hat{Z}_{k+1} &= \hat{Z}_k(A^T \otimes I_n) + \left(\hat{x}_k^{+T} \otimes I_n\right)\Sigma_{(AA)}^* + (u_k^T \otimes I_n)\Sigma_{(BA)}^*, \\ \hat{T}_{k+1} &= \hat{T}_k(A^T \otimes I_n) + \left(\hat{x}_k^{+T} \otimes I_m\right)\Sigma_{(AH)}^* + (u_k^T \otimes I_m)\Sigma_{(BH)}^*, \\ P_{k+1}^- &= AP_k^+A^T + Q_k. \end{cases} \tag{76}$$

4.2. Update/Correction Phase

1. Calculate R_k and M_k as follows:

$$R_k \approx R_{v^*} + (\hat{x}_k^{-T} \otimes I_m)\Sigma_{(HH)}(\hat{x}_k^- \otimes I_m) - \hat{t}_k\hat{t}_k^T, \tag{77}$$

and

$$M_k \approx (\hat{x}_k^{-T} \otimes I_n)\Sigma_{(AH)}(\hat{x}_k^- \otimes I_m) + (u_k^T \otimes I_n)\Sigma_{(BH)}(\hat{x}_k^- \otimes I_m) - \hat{z}_k\hat{t}_k^T. \tag{78}$$

2. Perform the update/correction step (based on the most general formulation of Kalman filter in [2]):

$$\begin{cases} S_k &= HP_k^-H^T + HM_k + M_k^TH^T + R_k, \\ K_k &= (P_k^-H^T + M_k)S_k^{-1}, \\ \hat{z}_k &= Z_k\text{vec}(I_n), \\ \hat{t}_k &= T_k\text{vec}(I_n), \\ \hat{x}_k^+ &= \hat{x}_k^- + K_k\left[y_k - (H\hat{x}_k^- + \hat{t}_k)\right], \\ P_k^+ &= (I_n - K_kH)P_k^-(I_n - K_kH)^T + K_k(HM_k + M_k^TH^T + R_k)K_k^T - K_kM_k^T - M_kK_k^T. \end{cases} \tag{79}$$

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5. Conclusions

A practical and systematic methodology for robust steady-state Kalman filter design for uncertain LTI systems has been presented. Building on De Koning's classical framework, the key contribution is the explicit derivation, within an MVFOSM-like approximation, of input-dependent (and state-dependent) equivalent noise covariance matrices Q, R and M that systematically incorporate second-order statistics of parametric uncertainty alongside the usual process and measurement noise statistics. The resulting steady-state filter is obtained by solving a standard generalized DARE, requiring no iterative robust optimization.

A notable feature of the methodology is that filter robustness is governed by a single physically interpretable design parameter: the nominal input magnitude $\|u\|$ used to evaluate the input-dependent covariance contributions. Selecting $u = u_{\text{max}}$ yields a worst-case robust design; selecting $u = u_{\text{rms}}$ targets average operating conditions. This transparent parametrization distinguishes the proposed approach from LMI- or regularization-based methods that require less intuitive tuning.

Two levels of approximation have been detailed. The simpler formulation (standard MVFOSM truncation, $z_k = 0$ and $t_k = 0$) yields a biased estimator but retains the canonical Kalman filter prediction-correction structure, making it directly deployable with minimal implementation overhead. The refined formulation tracks second-order bias corrections via an state, improving mean estimate accuracy at the cost of increased memory and computation. An extension to the time-varying case — which replaces the fixed nominal input with the actual time-varying u_k and propagates the error covariance recursively — is also outlined, offering reduced conservatism where computational resources permit.

The methodology is especially relevant to embedded control in power electronics, where component tolerances are often significant, steady-state implementations are strongly preferred, and the physical interpretation of uncertainty parameters (component tolerance bounds) maps naturally onto the required second-order statistics via the GUM framework. Future work will focus on experimental validation in a power converter testbench.

Appendix A: Summary of Assumptions

All the assumptions mentioned in the paper are summarized here for the readers' convenience.

System Structure and Modelling

(1) Invertibility of A_c: The continuous-time system matrix A_c is assumed invertible, allowing the expression:

$$B = A_c^{-1}(A - I_n)B_c.$$

(This assumption simplifies the derivation of J_B; if A_c is singular, the integral form must be used and derivatives computed accordingly.)

(2) Parametric Uncertainty Structure: The matrices A_c, B_c, and H depend on a physical parameter vector $\theta \in \mathbb{R}^p$:

$$\theta = \bar{\theta} + \delta\theta,$$

where $\bar{\theta}$ is the nominal (mean) value and $\delta\theta$ is a zero-mean random vector with known covariance $\Sigma_{\delta\theta} = \mathbb{E}[\delta\theta\delta\theta^T]$.

(3) Small Parameter Variations: The parameter uncertainty is small in the sense that $\|\delta\theta\| \ll \|\bar{\theta}\|$, justifying first-order Taylor expansions (MVFOSM-like framework).

Noise Characteristics

(4) Zero-Mean Gaussian Process Noise: The continuous-time process noise $w^*(t)$ is zero-mean, white, and Gaussian with covariance:

$$\mathbb{E}[w^*(t)w^*(\tau)^T] = Q^c_{w^*}\delta(t - \tau), \quad Q^c_{w^*} \succeq 0.$$

Its discrete-time counterpart w_k^* is zero-mean, white, Gaussian with covariance Q_{w^*} given by:

$$Q_{w^*} = \int_0^T e^{A_c(T-\tau)} Q_{w^*}^c e^{A_c^T(T-\tau)} d\tau.$$

(5) Zero-Mean Gaussian Measurement Noise: The continuous-time measurement noise $v^*(t)$ is zero-mean, white, and Gaussian with covariance:

$$\mathbb{E}[v^*(t)v^*(\tau)^T] = R_{v^*}^c \delta(t-\tau), \quad R_{v^*}^c \succ 0.$$

Its discrete-time counterpart v_k^* is zero-mean, white, Gaussian with covariance $R_{v^*} = \frac{R_{v^*}^c}{T}$.

(6) Input Noise: The control input is corrupted by additive noise δu_k, zero-mean with covariance $\Sigma_{\delta u} = \mathbb{E}[\delta u_k \delta u_k^T]$, independent of $\delta\theta$, w_k^*, and v_k^*.

(7) Statistical Independence: The following independence conditions hold:
- ☐ $\delta\theta$ is independent of x_0, δu_k, w_k^*, and v_k^*.
- ☐ w_k^* and v_k^* are mutually independent and independent of x_0, $\delta\theta$, and δu_k.
- ☐ δu_k is independent of x_0, $\delta\theta$, w_k^*, and v_k^*.

(8) Time-Invariant Noise Statistics: The covariance matrices Q_{w^*}, R_{v^*}, and $\Sigma_{\delta u}$ are constant (stationary noise processes).

Mathematical Approximations and Methodological Assumptions

(9) First-Order Taylor Expansion of Matrices: Variations in A_c, B_c, and H are approximated to first order:

$$\delta A_c \approx J_{A_c}\delta\theta, \quad \delta B_c \approx J_{B_c}\delta\theta, \quad \delta H \approx J_H \delta\theta,$$

where Jacobians $J_{A_c} = \frac{\partial \mathrm{vec}(A_c)}{\partial \theta^T}$, etc., are evaluated at $\bar{\theta}$.

(10) Discretization of Uncertainties: The discrete-time variations are obtained via chain rule:

$$\delta A \approx J_A J_{A_c} \delta\theta, \quad \delta B \approx \left[(I_q \otimes G) J_{B_c} + J_B J_{A_c}\right]\delta\theta,$$

with $G = A_c^{-1}(A - I_n)$ and Jacobians $J_A = \frac{\partial \mathrm{vec}(A)}{\partial \mathrm{vec}(A_c)^T}$, $J_B = \frac{\partial \mathrm{vec}(B)}{\partial \mathrm{vec}(A_c)^T}$.

(11) MVFOSM-like Truncation and Extensions: The approximation framework employed in this work extends the standard Mean Value First-Order Second-Moment method:
Standard MVFOSM approximation:
- ☐ Zeroth-order means: $\mathbb{E}[x_k] \approx \mathbb{E}[x_k^{\mathrm{nom}}]$ (neglecting $O(\delta\theta^2)$ bias corrections).
- ☐ Second-order covariances: Retain all terms up to $O(\delta\theta^2)$ in $\mathbb{E}[x_k x_k^T]$ and so on.
- ☐ Neglect of higher-order terms: Terms $O(\delta\theta^3)$ and above are discarded.

This work's enhancement:
- ☐ First-order means: $\mathbb{E}[x_k] \approx \mathbb{E}[x_k^{\mathrm{nom}}] + \mathbb{E}[\delta x_k]$ where $\mathbb{E}[\delta x_k]$ is $O(\delta\theta^2)$.
- ☐ Explicitly track bias terms: $z_k = \mathbb{E}[\delta A x_k]$ and $t_k = \mathbb{E}[\delta H x_k]$.
- ☐ Second-order covariances: same as MVFOSM, i.e. retain $O(\delta\theta^2)$, but corrected for $z_k z_k^T$, $t_k t_k^T$ and $z_k t_k^T$.
- ☐ Neglect of higher-order terms: Same as MVFOSM, discard $O(\delta\theta^3)$ and above

The key distinction is that second-order bias corrections z_k and t_k are explicitly computed and tracked via the augmented state representation, rather than being neglected as in standard MVFOSM. This improves mean estimate accuracy at the computational cost of maintaining $Z_k \in \mathbb{R}^{n \times n^2}$ and $T_k \in \mathbb{R}^{m \times n^2}$. When computational constraints are severe, the standard MVFOSM approach can be recovered by setting $z_k = 0$ and $t_k = 0$ (Section 3.5).

(12) Steady-State Approximation for Mean State (Covariance Computation):

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Two distinct but related approximations are employed for the mean state in the computation of the covariance matrices Q, R, and M.
(a) Approximation $\bar{x}_k \approx \bar{x}_k^{\text{nom}}$ (MVFOSM consistency):
The true conditional mean $\bar{x}_k = \mathbb{E}[x_k]$ differs from the nominal trajectory $\bar{x}_k^{nom}$ by an $O(\delta\theta^2)$ correction:

$$\bar{x}_k = \bar{x}_k^{nom} + \underbrace{\mathbb{E}[\delta x_k]}_{O(\delta\theta^2)}.$$

Within the MVFOSM framework, only second-order terms in $\delta\theta$ are retained in the covariances. Substituting $\bar{x}_k^{\text{nom}}$ for $\bar{x}_k$ in expressions such as $\mathbb{E}[\delta A\, \bar{x}_k \bar{x}_k^T \delta A^T]$ introduces an error of order $O(\delta\theta^4)$, which is consistently neglected. This approximation is therefore exact at the order of truncation adopted throughout this work.
(b) Steady-state DC-gain substitution $\bar{x}_k^{\text{nom}} \approx F_x u_k$ (time-invariance of Q, R, M):
For stable A (spectral radius $\rho(A) < 1$) and a constant input $u_k \equiv u$, the nominal trajectory $\bar{x}_k^{\text{nom}} = A^k \mathbb{E}[x_0] + \sum_{j=0}^{k-1} A^{k-1-j} B u$ converges to the DC steady state $F_x u = (I_n - A)^{-1}(B + F_z)u$ as $k \to \infty$, since the transient $A^k \mathbb{E}[x_0] \to 0$ exponentially. Substituting $\bar{x}_k^{\text{nom}} \approx F_x u$ into eqs. (60)–(68) renders Q, R, and M time-invariant, which is a prerequisite for solving the DARE and obtaining a fixed Kalman gain. The nominal input u serves as the sole robustness tuning parameter (Section 3.3).

(13) Neglect of State-Dependent Higher-Order Terms: In covariance calculations (e.g., $\mathbb{E}[\delta A x_k x_k^T \delta A^T]$), only the nominal state x_k^{nom} is retained; contributions from δx_k^{full} are $O(\delta\theta^3)$ or higher and are neglected.

Stability and Existence Conditions

(14) Nominal Stability: The nominal discrete-time matrix A has all eigenvalues strictly inside the unit circle ($\rho(A) < 1$ where ρ is the spectral radius).
(15) Detectability: The pair (A, H) is detectable. Given A is stable, this condition is automatically satisfied.
(16) Stabilizability: The pair $(A - MR^{-1}H, L)$ is stabilizable, where L satisfies $LL^T = Q - MR^{-1}M^T$.
(17) Positive-Definiteness of R: The augmented measurement noise covariance R is positive definite.
(18) Positive Semi-Definiteness of Q: The matrix Q is positive semi-definite, ensuring a valid solution to the DARE.
(19) Regularization of Q and R: The theoretical covariances $Q_k^{th} = P_{ww_k} - z_k z_k^T$ and $R_k^{th} = P_{vv_k} - t_k t_k^T$ may not satisfy the required definiteness conditions ($Q \succeq 0$, $R \succ 0$) due to numerical errors from approximation truncation and finite precision arithmetic. When this occurs, regularization is applied as specified in eq. (70) :

$$Q_k = Q_k^{\text{th}} + \varepsilon_Q I_n, \quad \text{where } \varepsilon_Q = \max\big(0,\ |\lambda_-^{max}(Q_k^{\text{th}})|\big),$$
$$R_k = R_k^{\text{th}} + \varepsilon_R I_m, \quad \text{where } \varepsilon_R = \max\big(\varepsilon,\ |\lambda_-^{max}(R_k^{\text{th}})|\big).$$

Here $\lambda_-^{max}(\cdot)$ denotes the most negative eigenvalue, and $\varepsilon > 0$ is a small positive constant (typically 10^{-10}) ensuring strict positive definiteness of R, which is required for the innovation covariance to be invertible in the Kalman gain computation (eq. (42)). The regularization preserves the approximate nature of Q and R while ensuring numerical stability of the DARE solution.

Practical Implementation Assumptions

(20) Time-Invariant Covariances for Steady-State Filter: For steady-state filter design, Q, R, and M are approximated as constant by selecting a representative constant input u (e.g., worst-case $u_{\max}$ or RMS value u_{rms}).

(21) Jacobian Computability: The Jacobian matrices $J_{A_c}, J_{B_c}, J_H, J_A, J_B$ can be computed analytically or numerically.
(22) Known Statistical Moments: The covariance $\Sigma_{\delta\theta}$ is known (second moment). No specific distribution is assumed beyond zero mean and finite second moment although in practice, for parameters, tolerances can be assumed as drawn from uniform distributions.
(23) Discretization Accuracy: The discretization period T is sufficiently small to accurately capture continuous-time dynamics and noise properties.

Appendix B: Some Mathematical Derivations

B.1 Derivation of J_B Expression

The relationship given in eq. (21) is:

$$J_B = \frac{\partial \text{vec}(B)}{\partial \text{vec}(A_c)^T} = (B_c^T \otimes A_c^{-1})J_A - (B^T \otimes A_c^{-1}),$$

where $J_A = \frac{\partial \text{vec}(A)}{\partial \text{vec}(A_c)^T}$ and B is defined as

$$B = GB_c, \qquad G = A_c^{-1}(A - I_n).$$

Starting from the definition $B = A_c^{-1}(A - I_n)B_c$, its differential is :

$$dB = d\big(A_c^{-1}\big)(A - I_n)B_c + A_c^{-1} dA\, B_c.$$

Using the identity $d(A_c^{-1}) = -A_c^{-1}(dA_c)A_c^{-1}$ (valid for an invertible matrix), the first term becomes

$$-A_c^{-1}(dA_c)A_c^{-1}(A - I_n)B_c.$$

Now $A_c^{-1}(A - I_n) = G$ can be substituted to simplify:

$$-A_c^{-1}(dA_c)GB_c = -A_c^{-1}(dA_c)B.$$

Thus the differential is

$$dB = -A_c^{-1}(dA_c)B + A_c^{-1}(dA)B_c.$$

To convert this to a vectorized form, the vec operator is applied together with the identity in eq. (80):

$$d\text{vec}(B) = -\big(B^T \otimes A_c^{-1}\big)\, d\text{vec}(A_c) + \big(B_c^T \otimes A_c^{-1}\big)\, d\text{vec}(A).$$

Now observe that $d\text{vec}(A) = J_A\, d\text{vec}(A_c)$ by the definition of J_A. Substituting this gives

$$d\text{vec}(B) = -\big(B^T \otimes A_c^{-1}\big)\, d\text{vec}(A_c) + \big(B_c^T \otimes A_c^{-1}\big) J_A\, d\text{vec}(A_c).$$

Finally, factoring out $d\text{vec}(A_c)$ on the right yields the Jacobian matrix:

$$\frac{\partial \text{vec}(B)}{\partial \text{vec}(A_c)^T} = (B_c^T \otimes A_c^{-1})J_A - (B^T \otimes A_c^{-1}).$$

B.2 Second Moments

In the computation of the second moments such as the ones presented in eqs. (58), (63) and (67) the following identities have been used:

$$\text{vec}(XYZ) = (Z^T \otimes X)\text{vec}(Y), \tag{80}$$

$$(X \otimes Y)(W \otimes Z) = (XW) \otimes (YZ), \tag{81}$$

from which, assuming $Y = Z = I$, so $YZ = II = I$, it yields:

$$XW \otimes I = (X \otimes I)(W \otimes I). \tag{82}$$

As an example, the computation of $\mathbb{E}[\delta A \bar{x}_k \bar{x}_k^T \delta A^T]$ in eq. (58), with the identities $\bar{x}_k = F_x u_k$ and $\Sigma_{(AA)} = \Sigma_{\delta A} = \mathbb{E}[\text{vec}(\delta A)\text{vec}(\delta A)^T]$, goes as follows:

1.
$$\delta A \bar{x}_k = (x_k^T \otimes I_n)\text{vec}(\delta A)$$

2.
$$\begin{aligned}\delta A \bar{x}_k \bar{x}_k^T \delta A^T &= (\delta A \bar{x}_k)(\delta A \bar{x}_k)^T = \Big[(x_k^T \otimes I_n)\text{vec}(\delta A)\Big]\Big[(x_k^T \otimes I_n)\text{vec}(\delta A)\Big]^T \\ &= (\bar{x}_k^T \otimes I_n)\text{vec}(\delta A)\text{vec}(\delta A)^T(\bar{x}_k \otimes I_n)\end{aligned}$$

3.
$$\mathbb{E}[\delta A \bar{x}_k \bar{x}_k^T \delta A^T] = (\bar{x}_k^T \otimes I_n)\, \mathbb{E}[\text{vec}(\delta A)\text{vec}(\delta A)^T](\bar{x}_k \otimes I_n) = (\bar{x}_k^T \otimes I_n)\Sigma_{(AA)}(\bar{x}_k \otimes I_n)$$

4.
$$\begin{aligned}\bar{x}_k \otimes I_n &= (F_x u_k) \otimes I_n = (F_x \otimes I_n)(u_k \otimes I_n) \\ \bar{x}_k^T \otimes I_n &= (u_k^T F_x^T) \otimes I_n = (u_k^T \otimes I_n)(F_x \otimes I_n)\end{aligned}$$

5.
$$(\bar{x}_k^T \otimes I_n)\Sigma_{(AA)}(\bar{x}_k \otimes I_n) = (u_k^T \otimes I_n)(F_x \otimes I_n)\Sigma_{(AA)}(F_x \otimes I_n)(u_k \otimes I_n)$$

The other terms in the second moments can be calculated analogously.

B.3 Derivation of Eq. (38)

$$\begin{aligned}
\mathbb{E}[x_{k+1}^T \otimes \delta A] &= \mathbb{E}\Big[\Big(x_k^T (A+\delta A)^T + (u_k + \delta u_k)^T (B+\delta B)^T + w_k^{*T}\Big) \otimes \delta A\Big] \\
&= \mathbb{E}[x_k^T A^T \otimes \delta A] + \mathbb{E}[x_k^T \delta A^T \otimes \delta A] + \mathbb{E}[u_k^T \delta B^T \otimes \delta A] \\
&\approx \mathbb{E}[x_k^T A^T \otimes \delta A] + \mathbb{E}[\bar{x}_k^T \delta A^T \otimes \delta A] + \mathbb{E}[u_k^T \delta B^T \otimes \delta A] \\
&= \mathbb{E}[x_k^T \otimes \delta A]\Big(A^T \otimes I_n\Big) + \Big(\bar{x}_k^T \otimes I_n\Big)\, \mathbb{E}[\delta A^T \otimes \delta A] + \Big(u_k^T \otimes I_n\Big)\, \mathbb{E}[\delta B^T \otimes \delta A] \\
\mathbb{E}[x_{k+1}^T \otimes \delta H] &= \mathbb{E}\Big[\Big(x_k^T (A+\delta A)^T + (u_k + \delta u_k)^T (B+\delta B)^T + w_k^{*T}\Big) \otimes \delta H\Big] \\
&= \mathbb{E}[x_k^T A^T \otimes \delta H] + \mathbb{E}[x_k^T \delta A^T \otimes \delta H] + \mathbb{E}[u_k^T \delta B^T \otimes \delta H] \\
&\approx \mathbb{E}[x_k^T A^T \otimes \delta H] + \mathbb{E}[\bar{x}_k^T \delta A^T \otimes \delta H] + \mathbb{E}[u_k^T \delta B^T \otimes \delta H] \\
&= \mathbb{E}[x_k^T \otimes \delta H]\Big(A^T \otimes I_n\Big) + \Big(\bar{x}_k^T \otimes I_n\Big)\, \mathbb{E}[\delta A^T \otimes \delta H] + \Big(u_k^T \otimes I_n\Big)\, \mathbb{E}[\delta B^T \otimes \delta H]
\end{aligned} \tag{83}$$

The derivation of eq. (83) is a simple application of the property in eq. (81) once the terms higher than $O(\delta\theta^2)$ have been neglected; eq. (38) then comes from the definition of Z_k and T_k in eq. (37).

B.4 Asymptotic Behavior of $\mathbb{E}[\delta A \widetilde{\delta A}_{k-1}]$ and $\mathbb{E}[\delta H \widetilde{\delta A}_{k-1}]$

$$\lim_{k\to\infty} \widetilde{\delta A}_{k-1} = \lim_{k\to\infty} \sum_{j=1}^{k-1} A^{k-1-j} \delta A A^j \tag{84}$$

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Let A be Schur stable, i.e., its spectral radius satisfies $\rho(A) < 1$. Then there exist constants $C > 0$ and $\lambda \in (0,1)$ such that for all integers $p \geq 0$,

$$\|A^p\| \leq C\lambda^p,$$

where $\|\cdot\|$ denotes any submultiplicative matrix norm. For any $j = 1, \ldots, k-1$:

$$\|A^{k-1-j}\delta A A^j\| \leq \|A^{k-1-j}\| \, \|\delta A\| \, \|A^j\| \leq C^2\|\delta A\|\lambda^{k-1-j}\lambda^j = C^2\|\delta A\|\lambda^{k-1}.$$

Summing this bound over $j = 1$ to $k-1$ gives:

$$\|\widetilde{\delta A}_{k-1}\| = \left\|\sum_{j=1}^{k-1} A^{k-1-j}\delta A A^j\right\| \leq \sum_{j=1}^{k-1} \|A^{k-1-j}\delta A A^j\| \leq (k-1)C^2\|\delta A\|\lambda^{k-1}.$$

Since λ^{k-1} decays exponentially, the factor $(k-1)\lambda^{k-1}$ tends to zero as $k \to \infty$. Consequently,

$$\lim_{k\to\infty} \widetilde{\delta A}_{k-1} = 0.$$

B.5 Proof of Eq. (46)

Only the result for z_∞ is going to be proved; the derivation of t_∞ is entirely analogous (replace δA with δH everywhere).

Starting from eq. (44) and dropping the transient term (which vanishes as $k \to \infty$, see the proof of $\lim_{k\to\infty} \widetilde{\delta A}_{k-1} = 0$ in this appendix), for a *constant* input $u_k \equiv u$ one has:

$$z_\infty \;\approx\; \lim_{k\to\infty} \mathbb{E}\left[\delta A \sum_{j=0}^{k-1} A^{k-1-j}\delta B\right] u \;+\; \lim_{k\to\infty} \mathbb{E}\left[\delta A \sum_{j=0}^{k-1} \widetilde{\delta A}_{k-2-j}\right] Bu. \tag{85}$$

Step 1 - linearity of expectation over a finite sum:

For every finite k both sums contain finitely many terms; the expectation of a finite sum of random matrices equals the sum of expectations. Hence, for any finite k:

$$\mathbb{E}\left[\delta A \sum_{j=0}^{k-1} A^{k-1-j}\delta B\right] u = \sum_{j=0}^{k-1} \mathbb{E}\Big[\delta A \, A^{k-1-j}\delta B\Big] u = \sum_{j=0}^{k-1} \mathbb{E}[\delta A \, A^{k-1-j}\delta B] \, u, \tag{86}$$

where A^{k-1-j} is a deterministic matrix and has been factored out of the expectation (bilinearity). Substituting $\ell = k-1-j$ (so ℓ runs from 0 to $k-1$ as j runs from $k-1$ to 0) and using $\rho(A) < 1$ so that $\sum_{\ell=0}^{\infty} A^\ell = (I_n - A)^{-1}$ in the operator–norm sense:

$$\lim_{k\to\infty} \sum_{j=0}^{k-1} \mathbb{E}[\delta A \, A^{k-1-j}\delta B] \, u = \mathbb{E}\left[\delta A\left(\sum_{\ell=0}^{\infty} A^\ell\right)\delta B\right] u. \tag{87}$$

Justification of the limit–sum exchange : since δA and δB have finite second moments (assumption (3)) and $\rho(A) < 1$ (assumption (14)), there exist $C > 0$, $\lambda \in (0,1)$ such that $\|A^\ell\| \leq C\lambda^\ell$. Therefore:

$$\|\mathbb{E}[\delta A \, A^\ell \delta B]\| \;\leq\; \mathbb{E}[\|\delta A\| \, \|A^\ell\| \, \|\delta B\|] \;\leq\; C\lambda^\ell \; \mathbb{E}[\|\delta A\| \, \|\delta B\|].$$

Because $\sum_{\ell=0}^{\infty} C\lambda^\ell < \infty$, by the *dominated convergence* argument for matrix series the limit and expectation commute, and the partial sum converges in norm to $\mathbb{E}[\delta A \, (I_n - A)^{-1}\delta B]$, yielding the first term of eq. (46):

$$\lim_{k\to\infty} \mathbb{E}\left[\delta A \sum_{j=0}^{k-1} A^{k-1-j}\delta B\right] u = \mathbb{E}\Big[\delta A \, (I_n - A)^{-1}\delta B\Big] u = \mathbb{E}[\delta A \, W\delta B] \, u. \tag{88}$$

Step 2 - expansion of the double-sum term.
Expanding $\widetilde{\delta A}_{k-2-j}$ by its definition (eq. (27)):

$$\widetilde{\delta A}_{k-2-j} = \sum_{i=0}^{k-2-j} A^{k-2-j-i}\,\delta A\,A^i, \tag{89}$$

so the second term in eq. (85) becomes, again using linearity of expectation over a finite double sum:

$$\mathbb{E}\left[\delta A \sum_{j=0}^{k-1} \widetilde{\delta A}_{k-2-j}\right] Bu = \sum_{j=0}^{k-1}\sum_{i=0}^{k-2-j} \mathbb{E}\left[\delta A\,A^{k-2-j-i}\,\delta A\right] A^i\,Bu. \tag{90}$$

Justification of the double limit–sum exchange : setting $p = k-2-j-i$ and $q = i$, the general term has norm bounded by

$$\|\mathbb{E}[\delta A\,A^p\,\delta A]\|\,\|A^q\| \;\le\; C^2\lambda^{p+q}\,\mathbb{E}[\|\delta A\|^2].$$

The sum $\sum_{p=0}^{\infty}\sum_{q=0}^{\infty} C^2\lambda^{p+q}\,\mathbb{E}[\|\delta A\|^2] = C^2\,\mathbb{E}[\|\delta A\|^2]/(1-\lambda)^2 < \infty$, so the double series converges absolutely in norm. Therefore the limit $k \to \infty$ and both summation signs may be freely exchanged, giving:

$$\lim_{k\to\infty}\sum_{j=0}^{k-1}\sum_{i=0}^{k-2-j}\mathbb{E}[\delta A\,A^p\,\delta A]\,A^i Bu = \left(\sum_{p=0}^{\infty}\mathbb{E}[\delta A\,A^p\,\delta A]\right)\left(\sum_{i=0}^{\infty}A^i\right)Bu. \tag{91}$$

Note on the change of order : the two infinite sums decouple because the change of variables $(j,i) \mapsto (p,q)$ with $p = k-2-j-i$, $q = i$ maps the triangular region $\{0 \le i \le k-2-j,\ 0 \le j \le k-1\}$ (for fixed k) to the triangular region $\{p+q \le k-2,\ p,q \ge 0\}$, which exhausts all pairs $(p,q) \in \mathbb{N}_0^2$ as $k \to \infty$. Absolute convergence (established above) guarantees that the limiting double sum equals the product of the two individual geometric sums:

$$\sum_{p=0}^{\infty}\mathbb{E}[\delta A\,A^p\,\delta A] = \mathbb{E}\Big[\delta A\,(I_n - A)^{-1}\delta A\Big] = \mathbb{E}[\delta A\,W\,\delta A], \qquad \sum_{i=0}^{\infty}A^i = W, \tag{92}$$

where the same dominated-convergence argument as in Step 1 justifies moving $\mathbb{E}$ outside the sum $\sum_p$. Combining eqs. (91)–(92):

$$\lim_{k\to\infty}\mathbb{E}\left[\delta A\sum_{j=0}^{k-1}\widetilde{\delta A}_{k-2-j}\right]Bu = \mathbb{E}[\delta A\,W\,\delta A]\,W\,Bu. \tag{93}$$

Substituting eqs. (88) and (93) into eq. (85) gives:

$$z_\infty = \mathbb{E}[\delta A\,W\,\delta B]\,u + \mathbb{E}[\delta A\,W\,\delta A]\,W\,Bu = \mathbb{E}[\delta A\,W\,\delta B]\,u + \mathbb{E}[\delta A\,W\,\delta A]\,F_{nom}\,u. \tag{94}$$

The identical calculation with $\delta A \to \delta H$ yields:

$$t_\infty = \mathbb{E}[\delta H\,W\,\delta B]\,u + \mathbb{E}[\delta H\,W\,\delta A]\,W\,Bu = \mathbb{E}[\delta H\,W\,\delta B]\,u + \mathbb{E}[\delta H\,W\,\delta A]\,F_{nom}\,u, \tag{95}$$

which completes the proof of eq. (46).

B.6 Definition of the $r()$ Function

As discussed in section 3.2 the starred Σ matrices introduced in eq. (55) can be calculated by simply rearranging the elements of the unstarred Σ ones as detailed in the following:

$$\Big[r\Big(\Sigma_{(AA)}\Big)\Big]_{(p-1)n+q,(r-1)n+s} = \Big(\Sigma^*_{(AA)}\Big)_{(p-1)n+q,(r-1)n+s} = \Big(\Sigma_{(AA)}\Big)_{(p-1)n+r,(s-1)n+q} \qquad p,q,r,s = 1,\dots,n \tag{96}$$

$$\left[r\left(\Sigma_{(AB)}\right)\right]_{(c-1)n+i,(k-1)n+j} = \left(\Sigma^*_{(BA)}\right)_{(c-1)n+i,(k-1)n+j} = \left(\Sigma_{(AB)}\right)_{(j-1)n+i,(c-1)n+k} \quad i,j,k = 1,\dots,n \quad c = 1,\dots,q \tag{97}$$

$$\left[r\left(\Sigma_{(AH)}\right)\right]_{(i-1)m+r,(j-1)n+c} = \left(\Sigma^*_{(AH)}\right)_{(i-1)m+r,(j-1)n+c} = \left(\Sigma_{(AH)}\right)_{(i-1)n+j,(c-1)m+r} \quad i,j,c = 1,\dots,n \quad r = 1,\dots,m \tag{98}$$

$$\left[r\left(\Sigma_{(BH)}\right)\right]_{(j-1)m+r,(c-1)n+i} = \left(\Sigma^*_{(BH)}\right)_{(j-1)m+r,(c-1)n+i} = \left(\Sigma_{(BH)}\right)_{(j-1)n+c,(i-1)m+r} \quad i,c = 1,\dots,n \quad r = 1,\dots,m \quad j = 1,\dots,q \tag{99}$$

The above equations enable the computation of the starred matrices straightforwardly from the unstarred ones that have an explicit expression in terms of the Jacobians and the second moment $\Sigma_{\delta\theta}$.

B.7 Further Augmented State

Considering the augmented state described in eq. (39) and introducing an additional state $d_k = \bar{x} - \bar{x}^{nom}$ it is straightforward to write down the following system evolution:

$$\begin{cases} \bar{x}_{k+1} & \approx A\bar{x}_k + Bu_k + z_k \\ Z_{k+1} & \approx Z_k\left(A^T \otimes I_n\right) + \left((\bar{x}_k - d_k) \otimes I_n\right)\mathbb{E}[\delta A^T \otimes \delta A] + \left(u_k^T \otimes I_n\right)\mathbb{E}[\delta B^T \otimes \delta A] \\ z_k & = Z_k \mathrm{vec}(I_n) \\ T_{k+1} & \approx T_k\left(A^T \otimes I_n\right) + \left((\bar{x}_k - d_k) \otimes I_m\right)\mathbb{E}[\delta A^T \otimes \delta H] + \left(u_k^T \otimes I_m\right)\mathbb{E}[\delta B^T \otimes \delta H] \\ t_k & = T_k \mathrm{vec}(I_n) \\ d_{k+1} & = Ad_k + z_k \\ \bar{y}_k & \approx H\bar{x}_k + t_k \end{cases} \tag{100}$$

indeed $d_{k+1} = \bar{x}_{k+1} - \bar{x}^{nom}_{k+1} = A\bar{x}_k + Bu_k + z_k - \left(A\bar{x}^{nom}_k + Bu_k\right) = A\left(\bar{x}_k - \bar{x}^{nom}_k\right) + z_k$. As already mentioned the difference between $\bar{x}_k$ and $\bar{x}^{nom}_k$ is at least $O(\delta\theta^2)$ and can be neglected when the additional computational burden is not worth it. The evolution of the fully augmented system is reported in the following:

$$\begin{cases} \bar{x}_{k+1} & \approx A\bar{x}_k + Bu_k + z_k \\ Z_{k+1} & \approx Z_k\left(A^T \otimes I_n\right) + (\bar{x}_k \otimes I_n)\Sigma^*_{(AA)} - (d_k \otimes I_n)\Sigma^*_{(AA)} + \left(u_k^T \otimes I_n\right)\Sigma^*_{(BA)} \\ z_k & = Z_k \mathrm{vec}(I_n) \\ T_{k+1} & \approx T_k\left(A^T \otimes I_n\right) + (\bar{x}_k \otimes I_m)\Sigma^*_{(AH)} - (d_k \otimes I_m)\Sigma^*_{(AH)} + \left(u_k^T \otimes I_m\right)\Sigma^*_{(BH)} \\ t_k & = T_k \mathrm{vec}(I_n) \\ d_{k+1} & = Ad_k + z_k \\ \bar{y}_k & \approx H\bar{x}_k + t_k \end{cases} \tag{101}$$

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