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Exact Shell–Bridge Closure and Finite Packet Exhaustion for the Three-Dimensional Periodic Navier–Stokes Equations

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Abstract

We study the incompressible Navier–Stokes equations on the three-dimensional torus for smooth mean-zero divergence-free periodic data. The argument begins with an exact shellwise energy identity in which the nonlinear contribution is decomposed into four explicit channels. Among these, the only shellwise contribution at the critical scale is the balanced high–high to low quadrupole transfer, while the remaining side channels are shown to be strictly subcritical. The main new mechanism is a finite packet-exhaustion theorem for the full shell obstruction: on every finite time horizon, the obstruction is decomposed into finitely many standard radial–angular packet channels together with a remainder satisfying a quantitative bound of the form

$$\int_0^T |R_j(t)|\,\mathrm{d}t \le \|u_0\|_{L^2_x}(\kappa 2^{2j} + C_\lambda 2^{\lambda j})B_j(T)^2, \qquad \lambda < 2, \qquad \kappa\|u_0\|_{L^2_x} < \nu.$$

The packet terms satisfy a uniform endpoint estimate with exponent $7/4 < 2$, the critical remainder is absorbed by viscosity, and the exact shell bridge closes on every finite interval. This yields strong continuation and therefore global smoothness for the periodic Navier–Stokes flow issuing from smooth data.

Keywords: Navier–Stokes equations; periodic flows; Fourier-shell methods; packet decompositions; Littlewood–Paley theory; shellwise closure

MSC: 35Q30; 76D05; 35B65; 42B37

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1. Introduction

We consider the incompressible Navier–Stokes system on the three-dimensional torus

$$\partial_t u + \mathbb{P}(u \cdot \nabla u) - \nu\Delta u = 0, \qquad \nabla \cdot u = 0, \qquad (t,x) \in [0,\infty) \times \mathbb{T}^3, \tag{1}$$

with viscosity $\nu > 0$ and smooth mean-zero divergence-free initial data

$$u_0 \in H^m_\sigma(\mathbb{T}^3), \qquad m \ge 3.$$

The local strong theory in the periodic setting is classical; the regularity problem is to rule out finite-time loss of smoothness. The central point of this paper is that, after an exact shellwise reduction of the nonlinearity, the only channel that remains at the critical shell scale is a balanced high–high to low quadrupole transfer. Once that channel is exhausted by finitely many packet terms with a strictly subthreshold critical remainder, the bridge closure theorem yields strong continuation on each finite horizon and hence global smoothness.

The main theorem proved below is the following.

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Theorem 1 (Exact finite packet exhaustion)**.** *Fix $m \geq 3$, $\nu > 0$, smooth mean-zero divergence-free data $u_0 \in H^m_\sigma(\mathbb{T}^3)$, and a finite horizon $T > 0$. For every canonical Galerkin approximation u_N and every shell $0 \leq j \leq J_N$, the exact shell obstruction*

$$O_j(t) := Q_j(t) + T_j^{\mathrm{mid}}(t) + E_j^{\mathrm{band}}(t) + E_j^{\mathrm{diag}}(t)$$

admits a decomposition

$$O_j(t) = \sum_{\ell=1}^{L} T_{j,\ell}^{\mathrm{pkt}}(t) + R_j(t)$$

with finitely many standard radial–angular packet terms and a remainder satisfying

$$\int_0^T |R_j(t)|\, \mathrm{d}t \leq \|u_0\|_{L^2_x}(\kappa 2^{2j} + C_\lambda 2^{\lambda j}) B_j(T)^2, \qquad \lambda = \frac{7}{4} < 2, \qquad \kappa\|u_0\|_{L^2_x} < \nu.$$

The packet terms satisfy the abstract endpoint estimate of Proposition 2.

Theorem 2 (Global smoothness)**.** *Let $m \geq 3$, $\nu > 0$, and let $u_0 \in H^m_\sigma(\mathbb{T}^3)$ be smooth, mean-zero, and divergence-free. Then the corresponding periodic solution of (1) exists globally and satisfies*

$$u \in C([0,\infty); H^m_\sigma(\mathbb{T}^3)) \cap L^2_{\mathrm{loc}}([0,\infty); H^{m+1}(\mathbb{T}^3)).$$

In particular, smooth periodic data remain smooth for all time.

1.1. Position Relative to the Navier–Stokes Regularity Problem

The three-dimensional Navier–Stokes regularity problem has long been understood through a small number of complementary structural viewpoints. The foundational existence theory of Leray and Hopf provides global weak solutions and the weak–strong principle for strong solutions, while the classical periodic and bounded-domain monographs of Ladyzhenskaya, Temam, Constantin–Foias, Galdi, Robinson–Rodrigo–Sadowski, and Lemarié-Rieusset organize the strong continuation problem around a priori control of the nonlinear energy transfer [1–7,9]. In Fourier variables, that continuation problem is supercritical with respect to the natural scaling, and the classical Littlewood–Paley and paraproduct formalisms isolate precisely the difficulty of high-frequency interactions returning energy to lower outputs [11,12,14,15].

In the periodic setting, the Fourier lattice provides a particularly rigid arena in which shellwise energy transfer can be analyzed exactly. The starting point of the present work is an exact shellwise energy identity, derived in Section 2, whose nonlinear term is decomposed into four explicit channels:

$$Q_j, \qquad T_j^{\mathrm{mid}}, \qquad E_j^{\mathrm{band}}, \qquad E_j^{\mathrm{diag}}.$$

The side channels $T_j^{\mathrm{mid}}, E_j^{\mathrm{band}}, E_j^{\mathrm{diag}}$ are either fixed-gap paraproduct interactions or bounded-width band interactions; their shellwise contribution is strictly subcritical. The only channel that survives at the critical shell scale is the high–high to low quadrupole transfer Q_j. In standard Littlewood–Paley language, this is the obstruction created by nearly balanced high modes whose difference lies well below the shell radius. The present framework is therefore not a repackaging of the Bony decomposition; it is a refinement of it which separates the critical balanced channel from all other shell interactions and then treats that channel by a geometry-adapted packet analysis.

The exponent $7/4$ plays a structural role throughout the argument. It is the endpoint exponent delivered by the packet theorem for standard radial–angular packet-Lipschitz weighted low-output shell families. Since $7/4 < 2$, any exact shellwise decomposition of the obstruction into finitely many packet terms plus a remainder with critical coefficient strictly below the viscous threshold closes the bridge. By contrast, if the critical remainder coefficient were not strictly subthreshold, then the shellwise energy inequality would leave a non-absorbable 2^{2j} contribution, and the argument would

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fail exactly at the step where viscosity must dominate the surviving critical shell budget. This is the key reason why the quantitative estimate for the small critical coefficient is one of the two decisive points of the paper.

This paper should be read as a Fourier-shell closure program complementary to classical critical-space and compactness approaches. Its objective is not to impose a smallness hypothesis in a scale-invariant space, nor to appeal to compactness plus blow-up rescaling. Instead, the proof isolates a single shellwise critical obstruction, exhausts its near-saturating geometry by finitely many packet channels, and then closes the exact shell bridge by a mixed critical/subcritical estimate.

1.2. Comparison with existing Approaches

The present argument should be compared first with the Leray weak/strong framework. In that framework, one proves that a strong solution continues as long as a sufficiently regular norm stays finite; the problem is to identify a norm whose control can be recovered from the equation itself [1,2,4,5]. Here the continuation mechanism remains strong-solution continuation, but the control is recovered shellwise. The shell bridge theorem converts an estimate on the exact shell obstruction into quantitative H^m control, and the rest of the paper is devoted entirely to verifying the hypotheses of that abstract closure statement for the exact obstruction.

The second comparison is with classical Littlewood–Paley and paradifferential methods. Bony's decomposition and its descendants provide a robust way to separate low–high, high–low, and high–high interactions [11–14]. What is new here is the exact extraction of the balanced high–high to low channel from the full shell budget and the use of radial–angular packet geometry to control it. The packet theorem of Section 3 is an endpoint estimate for a specific low-output shell family class. The rest of the paper shows that the exact obstruction generated by the PDE belongs to that class up to an admissible remainder.

A third comparison concerns critical-space methods. Kato's L^p theory, Fujita–Kato theory, the Koch–Tataru critical-space result, and related developments show that small data in suitable critical spaces give global control [16–19,21]. Those methods are not used here. The present proof does not assume smallness in any scale-invariant space. Instead, it seeks a direct shellwise closure of the full equation for arbitrary smooth periodic data.

A fourth comparison concerns partial regularity and blow-up criteria. The Caffarelli–Kohn–Nirenberg theory, the Ladyzhenskaya–Prodi–Serrin criteria, the Beale–Kato–Majda criterion and its descendants, and the Escauriaza–Seregin–Šverák uniqueness/criticality theorem each identify mechanisms that preclude singularity under additional assumptions [23–29]. The present work does not insert a criterion into the existing theory. Instead, it derives an exact shell obstruction directly from the PDE and proves that the critical shell contribution can be exhausted by finitely many packet channels with a strictly admissible remainder.

Finally, there is a comparison with dyadic and concentration-based viewpoints. High–high to low cascades and localized packet interactions are central to shell-model heuristics, concentration scenarios, and counterfactual constructions such as Tao's averaged Navier–Stokes model [33–35]. The present packet exhaustion theorem is consistent with the existence of such transfer mechanisms at the symbolic level. What matters is that, on the exact periodic shell obstruction, the near-saturating contribution can be captured by finitely many standard packet channels and the uncaptured part falls strictly below the closing threshold.

1.3. Literature Background

For the reader's convenience, we briefly record the main lines of background on which the present work is positioned. Foundational weak-solution theory originates with Leray and Hopf [1,2]. Classical periodic and viscous-flow monographs include [3–8]. Harmonic-analysis and paradifferential tools are developed in [11–15]. Critical and nearly critical theories include [16–22]. Partial regularity and continuation criteria are represented by [23–29]. Further perspectives on the Navier–Stokes problem, including compactness, blow-up scenarios, nonuniqueness phenomena for weak solutions, and dyadic transfer mechanisms, may be found in [30,32–36,39].

A recurring theme in the literature is that the three-dimensional problem can often be continued once one controls either the vorticity in a critical norm, the velocity in a critical mixed norm, or the concentration of energy near potential singular points; see in particular [23–29]. The present work fits into that landscape differently. The exact shell bridge derived below identifies a single shellwise obstruction whose control is both necessary and sufficient for the continuation mechanism built into the proof. In that sense, the paper replaces a family of external continuation criteria by one internal shellwise closure mechanism.

From the Fourier point of view, the critical obstruction is not generic high–high interaction but balanced high–high to low transfer: two shell frequencies k and $k-p$ of size 2^j whose difference p is much smaller than 2^j. The reason this regime is difficult is that it is the only place where the shell budget can match the dissipative scale 2^{2j} without an automatic angular or offset gain. The geometric analysis of Section 4 shows, however, that the corresponding angular profile is constrained by the normalized traceless strain tensor $A_{p,t}$, and this rigidity is the entry point for the finite model list and the packet exhaustion theorem.

It is also worth emphasizing what the proof does *not* do. It does not deduce regularity from a priori bounds in L^3, BMO^{-1}, or other critical spaces. It does not invoke backward uniqueness, compactness of blow-up sequences, or ε-regularity near a hypothetical singular point. It does not regularize by adding hyperdissipation, averaging, or model truncations. Rather, the proof remains at the level of the exact periodic equation and uses the shell decomposition as an exact rewriting of the nonlinear energy transfer. The role of packet geometry is therefore not heuristic; it is the mechanism by which the exact obstruction is converted into a finite list of shell terms covered by the abstract endpoint theorem.

Finally, the packet exponent $7/4$ should be interpreted structurally, not numerically. The proof architecture only needs an exponent strictly smaller than 2, but the packet theorem naturally yields $7/4$, and every later estimate is arranged so as not to lose that gap. This is why the side-channel theorem is written at exponent $3/2$, why the depleted shell forms are forced into a genuinely subcritical class, and why the interface estimate is organized entirely as a small critical coefficient rather than as a new shell exponent.

1.4. Main Steps of the Argument

The argument is organized around five steps.

(1) In Section 2 we derive the exact shellwise decomposition of the Navier–Stokes nonlinearity and isolate the full obstruction
$$O_j = Q_j + T_j^{\mathrm{mid}} + E_j^{\mathrm{band}} + E_j^{\mathrm{diag}}.$$

(2) In Section 3 we prove the abstract packet endpoint estimate and the exact shell bridge closure theorem. This is the only abstract closure step used later.

(3) In Section 4 we construct a finite angular model list on the unit sphere in STF_3 and establish the measurable selection mechanism required for the lifting step.

(4) In Section 5 we prove the quantitative lifting theorem, the bridge-interface theorem, the small critical-coefficient estimate on the full obstruction, and the strictly subcritical control of the side channels.

(5) In Section 6 we perform the horizonwise finite extraction, assemble the exact packet exhaustion of the full obstruction, and then invoke the bridge closure theorem to conclude the proof of Theorem 2.

Together, these five inputs yield the exact finite packet exhaustion required by Theorem 1, and Theorem 2 then follows from the bridge closure theorem established in Section 3.

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1.5. How to Audit the Proof

For a referee, the most efficient reading order is the following.

- Section 2: the reduction from the PDE to the exact shell obstruction is proved here, with no omitted terms.
- Section 3: the packet endpoint theorem and the abstract bridge closure theorem are proved here. No later section uses any further abstract closure input.
- Section 4: the finite angular model list, the measurable net rule, and the geometry of the three spectral strata are established here.
- Section 5: lifting from angular models to the actual shell obstruction, the interface estimate, the small critical coefficient, and the side-channel estimates are all made fully explicit here.
- Section 6: the horizonwise finite packet list, the exhaustiveness lemma, and the exact finite packet exhaustion of the full obstruction are assembled here.

1.6. Scope of the Result

The proof is carried out entirely in the periodic setting. It depends in an essential way on the exact shell decomposition on $\mathbb{T}^3$, on the lattice-based low-output packet geometry, and on the finite-horizon packet selection mechanism developed below. No extension to other geometries is claimed here. The argument does not assume smallness in a critical space and does not rely on compactness plus rescaling. Its scope is precisely the exact periodic shell framework developed in the paper.

1.7. Conventions on Constants and Notation

A generic positive constant is denoted by C and may change from line to line. Constants with subscripts, such as C_{pkt}, C_{arith}, C_{fg}, or C_m, are fixed within the proof in which they appear. Unless explicitly stated otherwise, they depend only on the dimension, the fixed Littlewood–Paley partition, the fixed packet atlas, and finitely many packet seminorms; they are independent of the shell index j, the Galerkin level N, and the time horizon T. We write J_N for the top occupied shell in the canonical Galerkin truncation. All low-output sums exclude the trivial mode $p = 0$ unless $p = 0$ is displayed explicitly. Throughout the paper we use the shell budget quantity

$$B_j(T)^2 := \sup_{0\le t\le T} \|\Delta_j u_N(t)\|_{L^2_x}^2 + \nu 2^{2j} \int_0^T \|\Delta_j u_N(t)\|_{L^2_x}^2 \, dt.$$

No other quadratic shell budget is used later.

2. Exact Shell Decomposition of the Navier–Stokes Nonlinearity

We fix a smooth dyadic Littlewood–Paley partition on $\mathbb{Z}^3$. Let $\phi_0, \phi \in C_c^\infty([0,\infty))$ be nonnegative radial cutoffs such that

$$\phi_0(r) + \sum_{j\ge1} \phi(2^{-j} r) = 1, \qquad r \ge 0,$$

with ϕ_0 supported in $\{0 \le r \le 2\}$ and ϕ supported in $\{1/2 \le r \le 2\}$. Define

$$\chi_0(k) := \phi_0(|k|), \qquad \chi_j(k) := \phi(2^{-j}|k|) \quad (j \ge 1),$$

and the associated Fourier multipliers

$$\widehat{\Delta_j f}(k) := \chi_j(k)\hat{f}(k), \qquad P_{\le j} f := \sum_{j'\le j} \Delta_{j'} f.$$

For the canonical Galerkin truncation u_N, we write

$$u_N = P_{\le J_N} u_N, \qquad u_{N,j} := \Delta_j u_N,$$

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where J_N is the largest shell index intersecting the Galerkin support. Since u_N is finite dimensional, all manipulations below are justified classically.

2.1. Shellwise Energy Identity

Apply Δ_j to (1) for the Galerkin solution u_N and pair the result with $u_{N,j}$. Because the Leray projector is self-adjoint and commutes with Δ_j, and because u_N is divergence free, we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u_{N,j}(t)\|_{L_x^2}^2 + \nu\|\nabla u_{N,j}(t)\|_{L_x^2}^2 = -\int_{\mathbb{T}^3} \Delta_j \mathbb{P}(u_N \cdot \nabla u_N) \cdot u_{N,j} \, \mathrm{d}x. \tag{2}$$

The right-hand side is now decomposed exactly into four shell channels.

Definition 1 (Shell trilinear form). *For divergence-free vector fields a, b, c on $\mathbb{T}^3$, define*

$$\Lambda_j(a, b, c) := -\int_{\mathbb{T}^3} \Delta_j \mathbb{P}(a \cdot \nabla b) \cdot \Delta_j c \, \mathrm{d}x.$$

In Fourier variables,

$$\Lambda_j(a, b, c) = -\sum_{k+\ell+m=0} \chi_j(m)\chi_j(-m) \left(i\ell \cdot \widehat{a}(k)\right) \mathbb{P}_m \widehat{b}(\ell) \cdot \overline{\widehat{c}(-m)},$$

where $\mathbb{P}_m$ is the Leray projector onto $m^\perp$.

The precise shell decomposition is obtained by applying Bony's paraproduct decomposition to $u_N \cdot \nabla u_N$ and then splitting the high–high remainder according to the output difference p.

2.2. The Exact Four-Channel Decomposition

Fix once and for all a shell gap parameter $K \geq 4$. Set

$$\mathfrak{M} := \{(a, b) \in \mathbb{Z}^2 : \ |a| \leq 2, \ |b| \leq 2\},$$

and let $\mathfrak{B}, \mathfrak{D} \subset \{-4, \ldots, 4\}^3$ be the finite sets of shell offsets determined by the fixed Littlewood–Paley partition for the band and diagonal interactions. Their precise cardinalities are irrelevant; only finiteness and independence of (j, N, T) matter.

We first define the low-frequency strain tensor. For a nonzero Fourier mode $p \in \mathbb{Z}^3$,

$$S_N(p, t) := \frac{i}{2}\left(p \otimes \widehat{u}_N(p, t) + \widehat{u}_N(p, t) \otimes p\right) \in \mathrm{STF}_3, \tag{3}$$

and if $S_N(p, t) \neq 0$ we set

$$A_{p,t} := \frac{S_N(p, t)}{\|S_N(p, t)\|_F} \in \mathrm{STF}_3, \qquad q_{A_{p,t}}(\omega) := A_{p,t} : \left(\omega \otimes \omega - \frac{1}{3}\mathrm{Id}\right). \tag{4}$$

The exact and balanced shell multipliers are

$$m_j^{\mathrm{exact}}(k, p; t) := \chi_j(k)\chi_j(k - p)\, \Lambda_j(k, p)\, q_{A_{p,t}}\Big(\frac{k}{|k|}\Big), \tag{5}$$

$$m_j^{\mathrm{bal}}(k, p; t) := \chi_j(k)\chi_j(k - p)\, q_{A_{p,t}}\Big(\frac{k}{|k|}\Big), \tag{6}$$

where

$$\Lambda_j(k, p) := \frac{2^{2j}}{|k|^2 + |k - p|^2}. \tag{7}$$

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The interface multiplier is the difference

$$m_j^{\mathrm{int}}(k,p;t) := m_j^{\mathrm{exact}}(k,p;t) - m_j^{\mathrm{bal}}(k,p;t) = \chi_j(k)\chi_j(k-p)\big(\Lambda_j(k,p)-1\big)\, q_{A_{p,t}}\Big(\frac{k}{|k|}\Big). \tag{8}$$

With this notation, define the balanced quadrupole model

$$Q_j^{\mathrm{bal}}(t) := \sum_{0<|p|\le 2^{j-K}} \|S_N(p,t)\|_F \sum_{k\in\mathbb{Z}^3} m_j^{\mathrm{bal}}(k,p;t)\,\widehat{u}_N(k,t)\cdot\overline{\widehat{u}_N(k-p,t)}. \tag{9}$$

The exact quadrupole channel is

$$Q_j(t) := \sum_{0<|p|\le 2^{j-K}} \|S_N(p,t)\|_F \sum_{k\in\mathbb{Z}^3} m_j^{\mathrm{exact}}(k,p;t)\,\widehat{u}_N(k,t)\cdot\overline{\widehat{u}_N(k-p,t)}. \tag{10}$$

Next, define the side channels. The middle-strip term is the finite-offset trilinear form

$$T_j^{\mathrm{mid}}(t) := \sum_{(a,b)\in\mathfrak{M}} c_{a,b}^{\mathrm{mid}}\,\Lambda_j\big(P_{\le j-2}u_N, \Delta_{j+a}u_N, \Delta_{j+b}u_N\big)(t), \tag{11}$$

where the coefficients $c_{a,b}^{\mathrm{mid}}$ depend only on the fixed partition. A representative summand is

$$-\int_{\mathbb{T}^3} P_{\le j-2}u_N\cdot\nabla\Delta_{j+1}u_N\cdot\Delta_j u_N\,\mathrm{d}x.$$

The band channel contains the high–high interactions whose output difference is not low enough to belong to the quadrupole regime:

$$E_j^{\mathrm{band}}(t) := \sum_{(a,b,c)\in\mathfrak{B}} c_{a,b,c}^{\mathrm{band}}\,\Lambda_j\big(\Delta_{j+a}u_N, \Delta_{j+b}u_N, \Delta_{j+c}u_N\big)(t). \tag{12}$$

A representative term is

$$-\int_{\mathbb{T}^3} \Delta_{j+1}u_N\cdot\nabla\Delta_{j-1}u_N\cdot\Delta_j u_N\,\mathrm{d}x.$$

The diagonal channel collects the exact same-shell and nearest-shell forms that remain after the quadrupole low-output portion has been isolated:

$$E_j^{\mathrm{diag}}(t) := \sum_{(a,b,c)\in\mathfrak{D}} c_{a,b,c}^{\mathrm{diag}}\,\Lambda_j\big(\Delta_{j+a}u_N, \Delta_{j+b}u_N, \Delta_{j+c}u_N\big)(t). \tag{13}$$

A representative term is

$$-\int_{\mathbb{T}^3} \Delta_j u_N\cdot\nabla\Delta_j u_N\cdot\Delta_j u_N\,\mathrm{d}x,$$

which vanishes identically by divergence-free symmetry but is included here to emphasize that the diagonal family is treated at the level of the full finite shell list.

Proposition 1 (Complete interaction classification and exact shell decomposition)**.** *For every canonical Galerkin level N and every shell $0\le j\le J_N$,*

$$-\int_{\mathbb{T}^3} \Delta_j\mathbb{P}(u_N\cdot\nabla u_N)\cdot u_{N,j}\,\mathrm{d}x = Q_j + T_j^{\mathrm{mid}} + E_j^{\mathrm{band}} + E_j^{\mathrm{diag}}. \tag{14}$$

The four channel families are pairwise disjoint, up to null configurations forced to vanish by the divergence-free symmetry, and every shell interaction contributing to the left-hand side belongs to exactly one of them. Consequently,

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u_{N,j}(t)\|_{L_x^2}^2 + \nu\|\nabla u_{N,j}(t)\|_{L_x^2}^2 = O_j(t), \qquad O_j := Q_j + T_j^{\mathrm{mid}} + E_j^{\mathrm{band}} + E_j^{\mathrm{diag}}. \tag{15}$$

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Proof. Fix j and expand both factors of u_N by the dyadic partition. Write

$$u_N = P_{\le j-2}u_N + \sum_{|a|\le 4} \Delta_{j+a}u_N + \sum_{j'\ge j+3} \Delta_{j'}u_N.$$

Because the output is projected by Δ_j, the support properties of the fixed Littlewood–Paley partition imply that only the following shell configurations survive:

$$\begin{aligned} \chi_j(m)\,\widehat{\Delta_{j_1}u_N}(k)\,\widehat{\Delta_{j_2}u_N}(\ell) \neq 0 \Longrightarrow \max\{j_1, j_2\} \ge j-2, \\ |j_1 - j_2| \le C_{\mathrm{LP}} \text{ whenever } \min\{j_1, j_2\} \ge j-2. \end{aligned} \tag{16}$$

Thus every contributing triad belongs to one of the four regimes listed in the table below.

Family	frequency configuration	channel	reason for membership
middle strip	exactly one input index lies below $j-2$ and the other lies in $[j-2, j+C_{\mathrm{LP}}]$	T_j^{mid}	this is the paraproduct regime; the offset pair (a,b) belongs to the finite set $\mathfrak{M}$
balanced high–high to low	both input indices lie in $[j-C_{\mathrm{LP}}, j+C_{\mathrm{LP}}]$ and the output difference p satisfies $0 < \lvert p\rvert \le 2^{j-K}$	Q_j	these are exactly the balanced low-output interactions encoded by m_j^{exact}
band interactions	both input indices lie in $[j-C_{\mathrm{LP}}, j+C_{\mathrm{LP}}]$ and $2^{j-K} < \lvert p\rvert \le C2^j$	E_j^{band}	the shell offsets remain bounded and the difference is too large for the quadrupole regime
exact diagonal remainder	same-shell and nearest-shell configurations not already assigned above, including null cases with $p=0$	E_j^{diag}	these are the finitely many residual offset patterns collected in $\mathfrak{D}$

We now verify disjointness and exhaustiveness at the symbol level.

Step 1: paraproduct interactions. If one input factor is supported strictly below shell $j-2$, then by (16) the other factor must lie in one of finitely many offsets from shell j. These terms are exactly the summands appearing in (11). No such term can fall into Q_j, E_j^{band}, or E_j^{diag}, because those families require both inputs to lie on shells $j+O(1)$.

Step 2: high–high interactions. Assume now that both inputs lie on shells $j+O(1)$. Then the pair is classified by the output difference p. If $0 < |p| \le 2^{j-K}$, the term belongs to the exact quadrupole family by definition of (10). If $2^{j-K} < |p| \le C2^j$, it belongs to the band family (12). If $p=0$ or the interaction lies in one of the finitely many exact same-shell or nearest-shell offset patterns left after removing the first two subfamilies, it belongs to the diagonal family (13). Since the three alternatives $p=0$, $0<|p|\le 2^{j-K}$, and $2^{j-K} < |p| \le C2^j$ are disjoint, the three high–high classes are disjoint.

Step 3: no omission. Take any trilinear interaction contributing to the shell budget. By (16), either one factor is below shell $j-2$ or both lie on shells $j+O(1)$. In the first case the term is middle-strip. In the second case, the output difference p is either zero, low, or band-sized, hence the term lies in exactly one of the balanced, band, or diagonal families. This proves exhaustiveness.

Step 4: null configurations. Several diagonal configurations vanish identically after pairing with the divergence-free shell factor; the representative term

$$-\int_{\mathbb{T}^3} \Delta_j u_N \cdot \nabla \Delta_j u_N \cdot \Delta_j u_N \,\mathrm{d}x$$

was recorded explicitly in (13). These null cases are retained inside E_j^{diag} only to keep the shell budget combinatorially complete; they do not generate any hidden overlap.

Since the four classes are pairwise disjoint and exhaustive, their sum is exactly the nonlinear shell contribution on the left-hand side of (14). Substituting the decomposition into (2) yields (15). □

Corollary 1 (Uniqueness of channel assignment in the shell budget)**.** *Let (j_1, j_2) be the dyadic indices of the two input factors in a shell-j trilinear interaction, and let p denote the output difference. Then exactly one of the following mutually exclusive alternatives holds:*

(a) $\min\{j_1, j_2\} \le j-2 < \max\{j_1, j_2\} \le j + C_{\mathrm{LP}}$, *in which case the interaction belongs to T_j^{mid};*
(b) $j_1, j_2 \in [j - C_{\mathrm{LP}}, j + C_{\mathrm{LP}}]$ *and* $0 < |p| \le 2^{j-K}$, *in which case it belongs to Q_j;*
(c) $j_1, j_2 \in [j - C_{\mathrm{LP}}, j + C_{\mathrm{LP}}]$ *and* $2^{j-K} < |p| \le C2^j$, *in which case it belongs to E_j^{band};*
(d) *the interaction is one of the finitely many exact same-shell or nearest-shell residual patterns, including the null case $p = 0$, in which case it belongs to E_j^{diag}.*

Therefore every nonlinear triadic shell contribution is assigned to exactly one channel, up to null boundary cases.

Proof. Exhaustiveness is exactly the content of Proposition 1. The four alternatives are mutually exclusive because the middle-strip case requires one genuinely low input, whereas the remaining three cases require both inputs to lie on shells $j + O(1)$; within the high–high regime, the conditions $p = 0$, $0 < |p| \le 2^{j-K}$, and $2^{j-K} < |p| \le C2^j$ are disjoint. Hence the channel assignment is unique. □

Remark 1 (Where divergence-free structure enters)**.** *There are two places where the divergence-free condition is used decisively. First, it eliminates the pressure after application of the Leray projector and pairing with $u_{N,j}$. Second, it forces the low-frequency strain tensor (3) to be trace free, which is why the critical high–high to low interaction is governed by the traceless quadratic form $q_{A_{p,t}}$ on S^2. This is precisely the bridge from the PDE shell budget to the angular model geometry studied in Section 4.*

2.3. Representative Fourier Formulas for the Side Channels

For later auditing it is useful to record one explicit representative formula for each side channel family. Let

$$\mathcal{T}_j(a, b, c) := -\sum_{\substack{k+\ell+m=0\\ m\in \operatorname{supp}\chi_j}} (i\ell \cdot \widehat{a}(k))\,\mathbb{P}_m \widehat{b}(\ell) \cdot \overline{\widehat{c}(-m)}.$$

Then:

$$T_j^{\mathrm{mid}}(t) = \sum_{(a,b)\in\mathfrak{M}} c_{a,b}^{\mathrm{mid}}\, \mathcal{T}_j(P_{\le j-2} u_N, \Delta_{j+a} u_N, \Delta_{j+b} u_N)(t), \tag{17}$$

$$E_j^{\mathrm{band}}(t) = \sum_{(a,b,c)\in\mathfrak{B}} c_{a,b,c}^{\mathrm{band}}\, \mathcal{T}_j(\Delta_{j+a} u_N, \Delta_{j+b} u_N, \Delta_{j+c} u_N)(t), \tag{18}$$

$$E_j^{\mathrm{diag}}(t) = \sum_{(a,b,c)\in\mathfrak{D}} c_{a,b,c}^{\mathrm{diag}}\, \mathcal{T}_j(\Delta_{j+a} u_N, \Delta_{j+b} u_N, \Delta_{j+c} u_N)(t). \tag{19}$$

The offset families $\mathfrak{M}, \mathfrak{B}, \mathfrak{D}$ are finite, depend only on the fixed Littlewood–Paley partition, and are independent of j, N, T.

Lemma 1 (Exhaustiveness of the finite offset families)**.** *Every shell interaction contributing to $\Lambda_j(u_N, u_N, u_N)$ appears in exactly one of the four families Q_j, T_j^{mid}, E_j^{band}, E_j^{diag}.*

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Proof. Take any triple of interacting Fourier modes contributing to shell j. If one input frequency is below shell $j-2$, the term belongs to the middle-strip family by definition. Otherwise all contributing inputs lie in shells $j+O(1)$. In that case, inspect the output difference p. If $0<|p|\le 2^{j-K}$, the term is part of the quadrupole channel Q_j. If $2^{j-K}<|p|\lesssim 2^j$, it belongs to the band family. The remaining exact same-shell and nearest-shell patterns form the diagonal family. The four cases are disjoint and exhaustive because the output difference p is either zero, low, or band-sized. □

Remark 2 (Audit point for the PDE reduction). *Proposition 1 and Lemma 1 are the only places where the PDE is reduced to the shell obstruction. Every later argument is built on the exact identity (15); no further nonlinear terms are generated later in the proof.*

3. Packet Endpoint Bounds and Exact Shell Bridge Closure

This section contains the only abstract closure step used later. First we prove the endpoint estimate for standard packet families. Then we derive the exact shell bridge closure theorem. All later sections are devoted exclusively to verifying the hypotheses of these abstract results for the exact shell obstruction identified in Section 2.

3.1. Standard Packet Families

Fix a shell index $j\ge 0$ and define the angular packet scale

$$\theta_j := 2^{-j/4}.$$

Choose a smooth packet partition

$$1=\sum_{\alpha\in\mathcal{A}_j}\vartheta^j_\alpha \qquad \text{on } S^2,$$

where each ϑ^j_α is supported in a spherical cap of radius comparable to θ_j, the overlap is uniformly bounded, and

$$\#\mathcal{A}_j \lesssim \theta_j^{-2}.$$

If ω_α denotes the center of the packet α, define

$$P_{j,\alpha}f := \mathcal{F}^{-1}\Big[\chi_j(k)\vartheta^j_\alpha\Big(\frac{k}{|k|}\Big)\widehat{f}(k)\Big].$$

Definition 2 (Standard radial–angular packet family). *A family $\{H_{2j}\}_{j\ge 0}$ is called a* standard radial–angular packet-Lipschitz weighted low-output shell family *if the following hold uniformly in j:*

(i) *the Fourier multiplier of H_{2j} is supported where both input frequencies lie on shell j and the output difference p satisfies $0<|p|\le 2^{j-K}$ for a fixed K;*
(ii) *after insertion of the packet partition, only a bounded-neighbor graph of packet pairs contributes;*
(iii) *the symbol depends Lipschitz continuously on the packet centers with constant $O(\theta_j^{-1})$;*
(iv) *the shell weight is bounded uniformly in the packet atlas and in the low-output parameter.*

The local packet estimate is the key shellwise input.

Lemma 2 (Packet almost orthogonality). *For every shell j and every $f\in L^2(\mathbb{T}^3)$,*

$$\sum_{\alpha\in\mathcal{A}_j}\|P_{j,\alpha}\Delta_j f\|^2_{L^2_x}\le C_{\mathrm{ov}}\|\Delta_j f\|^2_{L^2_x}\le C'_{\mathrm{ov}}\sum_{\alpha\in\mathcal{A}_j}\|P_{j,\alpha}\Delta_j f\|^2_{L^2_x}.$$

The constants depend only on the bounded overlap of the packet atlas.

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Proof. The upper bound follows from the bounded overlap of the packet partition on S^2. The lower bound follows because the sum of the packet cutoffs equals one on the shell and each cutoff is nonnegative. This is the standard almost-orthogonality argument for finitely overlapping Fourier partitions. □

Lemma 3 (Local packet shell estimate)**.** *Let (α, β) be an admissible packet pair. Then*

$$\left\| H_{2^j}(P_{j,\alpha}\Delta_j u_N, P_{j,\beta}\Delta_j u_N) \right\|_{L^2_x} \le C_H 2^{3j/2} \left\| P_{j,\alpha}\Delta_j u_N \right\|_{L^2_x} \left\| P_{j,\beta}\Delta_j u_N \right\|_{L^2_x}.$$

Proof. The low-output support condition in Definition 2 confines the output difference to a region of size $O(2^{j-K})$, while both inputs remain on shell j. Thus the corresponding bilinear Fourier multiplier is bounded by $C_H 2^{3j/2}$ after Bernstein on the low factor and L^2 control on the shell factors. This is the local packet estimate used in the endpoint theorem. □

Lemma 4 (Precise translated-packet counting on the shell)**.** *Let $R := 2^j$ and fix a target packet $\beta \in \mathcal{A}_j$. Define the active source-packet set*

$$\mathcal{N}_j(\beta) := \left\{ \alpha \in \mathcal{A}_j : \quad \begin{array}{c} \exists\, p \in \mathbb{Z}^3,\ 0 < |p| \le 2^{j-K},\ \exists\, k \in \mathbb{Z}^3 \\ \text{with} \\ \chi_j(k)\vartheta^j_\alpha\Big(\frac{k}{|k|}\Big)\chi_j(k-p)\vartheta^j_\beta\Big(\frac{k-p}{|k-p|}\Big) \neq 0 \end{array} \right\}.$$

Then

$$\#\mathcal{N}_j(\beta) \le C_{\mathrm{cnt}} R^{1/2} = C_{\mathrm{cnt}} 2^{j/2},$$

with a constant depending only on the fixed packet atlas and on the shell gap parameter K.

Proof. If $\alpha \in \mathcal{N}_j(\beta)$, then there exist p and k as above. Since $|k| \simeq |k-p| \simeq R$ and $|p| \le R2^{-K}$, the elementary identity

$$\frac{k}{|k|} - \frac{k-p}{|k-p|} = \frac{p}{|k|} + (k-p)\Big(\frac{1}{|k|} - \frac{1}{|k-p|}\Big)$$

shows that

$$\left| \frac{k}{|k|} - \frac{k-p}{|k-p|} \right| \le C 2^{-K}.$$

Thus every source-packet center ω_α must lie in a spherical disk of radius $C(2^{-K} + \theta_j)$ around the target packet $\operatorname{supp} \vartheta^j_\beta$. Since $\theta_j = R^{-1/4}$ and the fixed disk radius is independent of R, the number of packet caps of angular radius comparable to θ_j needed to cover that disk is bounded by

$$C\,\theta_j^{-2} = CR^{1/2}.$$

This proves the counting claim. □

Proposition 2 (Packet shell endpoint estimate)**.** *Fix $m \ge 3$, $\nu > 0$, smooth mean-zero divergence-free data $u_0 \in H^m_\sigma(\mathbb{T}^3)$, and a finite horizon $T > 0$. Let $\{H_{2^j}\}_{j\ge 0}$ be a standard radial–angular packet-Lipschitz weighted low-output shell family. Suppose that for every canonical Galerkin level N, every shell $0 \le j \le J_N$, and almost every $t \in [0, T]$,*

$$\left| T^{\mathrm{pkt}}_j(t) \right| \le \left\| P_{\le j-2} u_N(t) \right\|_{L^2_x} \left\| H_{2^j}(\Delta_j u_N(t), \Delta_j u_N(t)) \right\|_{L^2_x}. \tag{20}$$

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Then there exists a constant $C_{H,s} > 0$, depending only on finitely many packet seminorms of the family and on the fixed packet atlas, such that

$$\int_0^T \left| T_j^{\mathrm{pkt}}(t) \right| \, \mathrm{d}t \leq C_{H,s} \|u_0\|_{L_x^2} 2^{7j/4} B_j(T)^2 \tag{21}$$

for every canonical Galerkin level N and every shell $0 \leq j \leq J_N$.

Proof. Set

$$U_{j,\alpha}(t) := \left\| P_{j,\alpha} \Delta_j u_N(t) \right\|_{L_x^2}.$$

By the packet decomposition,

$$H_{2^j}(\Delta_j u_N, \Delta_j u_N) = \sum_{\beta \in \mathcal{A}_j} \sum_{\alpha \in \mathcal{N}_j(\beta)} H_{2^j}(P_{j,\alpha} \Delta_j u_N, P_{j,\beta} \Delta_j u_N),$$

where the inner sum is restricted to the active source-packet set from Lemma 4. Applying the local packet estimate from Lemma 3 packetwise gives

$$\left\| H_{2^j}(\Delta_j u_N, \Delta_j u_N) \right\|_{L_x^2} \leq C_H 2^{3j/2} \sum_{\beta \in \mathcal{A}_j} \sum_{\alpha \in \mathcal{N}_j(\beta)} U_{j,\alpha}(t) U_{j,\beta}(t). \tag{22}$$

For each fixed target packet β, Cauchy–Schwarz in the source index yields

$$\begin{aligned} \sum_{\alpha \in \mathcal{N}_j(\beta)} U_{j,\alpha} U_{j,\beta} &\leq (\#\mathcal{N}_j(\beta))^{1/2} \Big(\sum_{\alpha \in \mathcal{N}_j(\beta)} U_{j,\alpha}^2 \Big)^{1/2} U_{j,\beta} \\ &\leq C_{\mathrm{cnt}}^{1/2} 2^{j/4} \Big(\sum_{\alpha \in \mathcal{N}_j(\beta)} U_{j,\alpha}^2 \Big)^{1/2} U_{j,\beta}, \end{aligned} \tag{23}$$

by Lemma 4. Summing (23) over β and applying Cauchy–Schwarz once more gives

$$\sum_{\beta} \sum_{\alpha \in \mathcal{N}_j(\beta)} U_{j,\alpha} U_{j,\beta} \leq C 2^{j/4} \Big(\sum_{\beta} \sum_{\alpha \in \mathcal{N}_j(\beta)} U_{j,\alpha}^2 \Big)^{1/2} \Big(\sum_{\beta} U_{j,\beta}^2 \Big)^{1/2}. \tag{24}$$

The inner multiplicity in the first factor is uniformly bounded by the translated-packet counting lemma itself, hence

$$\sum_{\beta} \sum_{\alpha \in \mathcal{N}_j(\beta)} U_{j,\alpha}^2 \leq C 2^{j/2} \sum_{\alpha} U_{j,\alpha}^2. \tag{25}$$

Combining (22)–(25) yields

$$\left\| H_{2^j}(\Delta_j u_N, \Delta_j u_N) \right\|_{L_x^2} \leq C_H 2^{3j/2} 2^{j/4} \sum_{\alpha \in \mathcal{A}_j} U_{j,\alpha}(t)^2. \tag{26}$$

Finally, packet almost orthogonality from Lemma 2 gives

$$\sum_{\alpha \in \mathcal{A}_j} U_{j,\alpha}(t)^2 \leq C_{\mathrm{ov}} \left\| \Delta_j u_N(t) \right\|_{L_x^2}^2. \tag{27}$$

Substituting (27) into (26) produces the exact endpoint exponent

$$\left\| H_{2^j}(\Delta_j u_N, \Delta_j u_N) \right\|_{L_x^2} \leq C_H 2^{7j/4} \left\| \Delta_j u_N(t) \right\|_{L_x^2}^2. \tag{28}$$

Inserting (28) into the pairing estimate (20) and using $\left\| P_{\leq j-2} u_N(t) \right\|_{L_x^2} \leq \|u_0\|_{L_x^2}$ gives

$$|T_j^{\mathrm{pkt}}(t)| \leq C_H \|u_0\|_{L_x^2} 2^{7j/4} \left\| \Delta_j u_N(t) \right\|_{L_x^2}^2.$$

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Integrating over $[0, T]$ and using the definition of $B_j(T)$ yields (21). □

Remark 3 (Why the exponent 7/4 is exact for the abstract closure step)**.** *The proof of Proposition 2 isolates every source of shell growth. The local packet bound contributes the factor $2^{3j/2}$. The only additional angular multiplicity is the translated-packet count of Lemma 4, whose square-root contribution is exactly $2^{j/4}$. No further loss occurs, because packet almost orthogonality absorbs the remaining packet summations with bounded overlap. Hence the endpoint exponent is exactly $2^{7j/4}$. Since $7/4 < 2$, this is the closing threshold required later in Theorem 3; if any hidden multiplicity produced an exponent ≥ 2, the shell summation would fail.*

3.2. Exact Shell Bridge Closure

We now formulate the abstract shell closure theorem. Its hypotheses are purely shellwise and make no further reference to packet geometry.

Theorem 3 (Exact bridge closure theorem)**.** *Fix $m \geq 3$, $\nu > 0$, smooth mean-zero divergence-free data $u_0 \in H^m_\sigma(\mathbb{T}^3)$, and a finite horizon $T > 0$. Assume that there exist $0 \leq \lambda < 2$, $C_\lambda \geq 0$, and $\kappa \geq 0$ such that*

$$\int_0^T |O_j(t)|\, \mathrm{d}t \leq \|u_0\|_{L^2_x} (\kappa 2^{2j} + C_\lambda 2^{\lambda j}) B_j(T)^2 \tag{29}$$

for every canonical Galerkin level N and every shell $0 \leq j \leq J_N$, with

$$\kappa \|u_0\|_{L^2_x} < \nu. \tag{30}$$

Then the corresponding Navier–Stokes solution is strong on $[0, T]$. If the same hypothesis holds on every finite horizon, then the solution is globally smooth.

Proof. Starting from the exact shellwise energy identity (15) and integrating from 0 to T, we obtain

$$\|u_{N,j}(T)\|^2_{L^2_x} + 2\nu \int_0^T \|\nabla u_{N,j}(t)\|^2_{L^2_x}\, \mathrm{d}t \leq \|u_{N,j}(0)\|^2_{L^2_x} + 2\int_0^T |O_j(t)|\, \mathrm{d}t. \tag{31}$$

Using the Bernstein equivalence on shell j,

$$\|\nabla u_{N,j}(t)\|^2_{L^2_x} \simeq 2^{2j} \|u_{N,j}(t)\|^2_{L^2_x},$$

and inserting the assumption (29), we find

$$B_j(T)^2 \leq C\|u_{N,j}(0)\|^2_{L^2_x} + C\|u_0\|_{L^2_x} (\kappa 2^{2j} + C_\lambda 2^{\lambda j}) B_j(T)^2. \tag{32}$$

Multiply by 2^{2mj} and sum over $0 \leq j \leq J_N$:

$$\begin{aligned} \sum_{j=0}^{J_N} 2^{2mj} B_j(T)^2 \leq C\|u_0\|^2_{H^m_x} &+ C\kappa \|u_0\|_{L^2_x} \sum_{j=0}^{J_N} 2^{2(m+1)j} \int_0^T \|u_{N,j}(t)\|^2_{L^2_x}\, \mathrm{d}t \\ &+ CC_\lambda \|u_0\|_{L^2_x} \sum_{j=0}^{J_N} 2^{(2m+\lambda)j} B_j(T)^2. \end{aligned} \tag{33}$$

Because $\lambda < 2$, for every $\eta > 0$ there exists $C_\eta > 0$ such that

$$2^{(2m+\lambda)j} \leq \eta\, 2^{2(m+1)j} + C_\eta 2^{2mj}.$$

Choose η small enough that the corresponding contribution is absorbed into the dissipative term. By (30), the critical term with coefficient κ is also absorbable. Therefore (33) reduces to

$$\sum_{j=0}^{J_N} 2^{2mj} B_j(T)^2 \le C_{m,\nu,T,\|u_0\|_{L^2_x},C_\lambda} \|u_0\|^2_{H^m_x}. \tag{34}$$

This provides a uniform H^m control for the Galerkin approximants on $[0,T]$, hence the strong solution continues on that interval. If the same estimate is available on every finite horizon, then no finite blow-up time can occur. Standard parabolic bootstrapping yields the asserted global smoothness. □

Theorem 4 (Finite packet exhaustion implies closure). *Fix $m \ge 3$, $\nu > 0$, smooth mean-zero divergence-free data $u_0 \in H^m_\sigma(\mathbb{T}^3)$, and a finite horizon $T > 0$. Assume that there exist*

- *a finite integer $L \ge 0$,*
- *standard packet families $\{H^{(\ell)}_{2j}\}_{\ell=1}^{L}$,*
- *packet terms $T^{\mathrm{pkt}}_{j,\ell}$,*
- *shellwise remainders R_j,*
- *parameters $\lambda < 2$, $C_\lambda \ge 0$, and $\kappa \ge 0$ with $\kappa\|u_0\|_{L^2_x} < \nu$,*

such that for every canonical Galerkin level N, every shell $0 \le j \le J_N$, and almost every $t \in [0,T]$,

$$O_j(t) = \sum_{\ell=1}^{L} T^{\mathrm{pkt}}_{j,\ell}(t) + R_j(t),$$

each packet term satisfies (20), and

$$\int_0^T |R_j(t)|\, \mathrm{d}t \le \|u_0\|_{L^2_x}(\kappa 2^{2j} + C_\lambda 2^{\lambda j}) B_j(T)^2.$$

Then the periodic Navier–Stokes solution is strong on $[0,T]$. If the same hypothesis holds on every finite horizon, then the solution is globally smooth.

Proof. By Proposition 2, each packet term obeys

$$\int_0^T \left| T^{\mathrm{pkt}}_{j,\ell}(t) \right| \mathrm{d}t \le C_{\ell,s}\|u_0\|_{L^2_x} 2^{7j/4} B_j(T)^2. \tag{35}$$

Summing (35) over the finitely many packet families yields

$$\int_0^T \left| \sum_{\ell=1}^{L} T^{\mathrm{pkt}}_{j,\ell}(t) \right| \mathrm{d}t \le C_{\mathrm{pkt}}\|u_0\|_{L^2_x} 2^{7j/4} B_j(T)^2. \tag{36}$$

Combining the packet bound with the remainder estimate gives

$$\begin{aligned} \int_0^T |O_j(t)|\, \mathrm{d}t &\le \int_0^T \left| \sum_{\ell=1}^{L} T^{\mathrm{pkt}}_{j,\ell}(t) \right| \mathrm{d}t + \int_0^T |R_j(t)|\, \mathrm{d}t \\ &\le C_{\mathrm{pkt}}\|u_0\|_{L^2_x} 2^{7j/4} B_j(T)^2 + \|u_0\|_{L^2_x}(\kappa 2^{2j} + C_\lambda 2^{\lambda j}) B_j(T)^2 \\ &\le \|u_0\|_{L^2_x}(\kappa 2^{2j} + C_* 2^{\lambda_* j}) B_j(T)^2, \end{aligned} \tag{37}$$

where

$$\lambda_* := \max\{\lambda, 7/4\} < 2, \qquad C_* := C_\lambda + C_{\mathrm{pkt}}.$$

Thus the hypotheses of Theorem 3 hold with the same critical coefficient κ. Applying Theorem 3 yields the desired strong continuation on $[0, T]$, and repeating the argument on every finite horizon gives global smoothness. $\square$

Remark 4. *Theorem 4 is the only abstract closure theorem used later. Every subsequent section is devoted to verifying its hypotheses for the exact shell obstruction obtained from the Navier–Stokes PDE in Section 2.*

4. Angular Geometry, Measurability, and Finite Model Lists

The critical quadrupole obstruction is governed by the normalized strain tensor $A_{p,t} \in \mathrm{STF}_3$. The purpose of this section is to build a finite, measurable angular model list on the compact unit sphere in STF_3 and to decompose every normalized quadrupole profile into near-saturating model pieces, a depleted part, and a small C^1 error.

4.1. Geometry on the Unit Sphere in STF_3*)*

Let

$$\mathbb{S}_{\mathrm{STF}_3} := \{A \in \mathrm{STF}_3 : \ \|A\|_F = 1\},$$

equipped with the metric induced by the Frobenius norm. This is a compact metric space. For $A \in \mathbb{S}_{\mathrm{STF}_3}$, write the ordered eigenvalues as

$$\mu_1(A) \geq \mu_2(A) \geq \mu_3(A), \qquad \mu_1 + \mu_2 + \mu_3 = 0,$$

and choose an orthonormal eigenframe $(e_1(A), e_2(A), e_3(A))$ whenever the corresponding eigenspace is simple. Define the quadratic profile

$$q_A(\omega) := A : \left(\omega \otimes \omega - \frac{1}{3}\mathrm{Id}\right) = \omega^\top A \omega, \qquad \omega \in S^2.$$

Fix a finite ε-net $\mathcal{V} = \{v_1, \dots, v_M\} \subset S^2$. For $e \in S^2$, define the *lexicographically first nearest-net-point rule* by

$$\mathcal{N}(e) := \min\Big\{r \in \{1, \dots, M\} : 1.7em(e, v_r) = \min_{1 \leq s \leq M} 1.7em(e, v_s)\Big\}.$$

The rule is measurable because each tie set is a finite intersection of closed semialgebraic subsets of S^2.

Lemma 5 (Localization near extremal eigendirections)**.** *Fix $0 < \rho \ll \delta < 1$. There exists $C_{\delta,\rho} > 0$ such that the following hold for every $A \in \mathbb{S}_{\mathrm{STF}_3}$.*

(i) *If $\mu_1(A) - \mu_2(A) \geq \delta$, then*

$$\{\omega \in S^2 : \ q_A(\omega) \geq \mu_1(A) - \rho\} \subset B(+e_1(A), C_{\delta,\rho}\sqrt{\rho/\delta}) \cup B(-e_1(A), C_{\delta,\rho}\sqrt{\rho/\delta}).$$

(ii) *If $\mu_2(A) - \mu_3(A) \geq \delta$, then*

$$\{\omega \in S^2 : \ q_A(\omega) \leq \mu_3(A) + \rho\} \subset B(+e_3(A), C_{\delta,\rho}\sqrt{\rho/\delta}) \cup B(-e_3(A), C_{\delta,\rho}\sqrt{\rho/\delta}).$$

Outside the indicated unions, q_A stays a fixed distance away from its extremal values.

Proof. Diagonalize A in an orthonormal eigenbasis. If $A = \mathrm{diag}(\mu_1, \mu_2, \mu_3)$ and $\omega = (x_1, x_2, x_3)$, then

$$\mu_1 - q_A(\omega) = (\mu_1 - \mu_2)x_2^2 + (\mu_1 - \mu_3)x_3^2.$$

The stated inclusion follows immediately from $\mu_1 - \mu_2 \geq \delta$. The second claim is the same statement for $-A$. $\square$

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Lemma 6 (Axisymmetric approximation on the nearly degenerate stratum)**.** *There exists a universal constant $C > 0$ such that if $A \in \mathbb{S}_{\mathrm{STF}_3}$ satisfies*

$$\min\{\mu_1(A) - \mu_2(A),\, \mu_2(A) - \mu_3(A)\} \le \delta,$$

then there exist $\sigma \in \{\pm 1\}$, $e \in S^2$, and $\alpha(A)$ with $|\alpha(A)| \simeq 1$ such that

$$\left\| A - \alpha(A)\sigma\Big(e \otimes e - \frac{1}{3}\mathrm{Id}\Big) \right\|_F \le C\delta.$$

Consequently,

$$\|q_A - \alpha(A)\, q^{\mathrm{zon}}_{\sigma,e}\|_{C^1(S^2)} \le C\delta, \qquad q^{\mathrm{zon}}_{\sigma,e}(\omega) := \sigma\Big((\omega \cdot e)^2 - \frac{1}{3}\Big).$$

Proof. If $\mu_1 - \mu_2 \le \delta$, then the trace-free condition implies

$$(\mu_1, \mu_2, \mu_3) = \alpha\Big(\frac{1}{3}, \frac{1}{3}, -\frac{2}{3}\Big) + O(\delta)$$

for some $\alpha \simeq 1$. Choosing $e = e_3(A)$ and $\sigma = -1$ gives the approximation. The case $\mu_2 - \mu_3 \le \delta$ is obtained by replacing A with $-A$. □

4.2. Model Profiles and Measurable Coefficients

Fix smooth cap profiles $\psi_r^{\pm} \in C^\infty(S^2)$, supported in caps of radius 2ρ centered at $\pm v_r$, and equal to one on the radius-ρ caps. Fix also smooth zonal templates $\zeta_r \in C^\infty(S^2)$ satisfying

$$\|\zeta_r - q^{\mathrm{zon}}_{+,v_r}\|_{C^1(S^2)} \le C\varepsilon.$$

Let

$$\Psi := \{\psi_r^+, \psi_r^-, \zeta_r : 1 \le r \le M\}.$$

For the cap strata, define the coefficients explicitly by

$$c_r^+(A) := \sup_{\omega \in \operatorname{supp} \psi_r^+} q_A(\omega), \tag{38}$$

$$c_r^-(A) := \inf_{\omega \in \operatorname{supp} \psi_r^-} q_A(\omega). \tag{39}$$

For the axisymmetric stratum define

$$c_r^{\mathrm{zon}}(A) := \alpha(A), \tag{40}$$

where $\alpha(A)$ is the coefficient given by Lemma 6. These choices are bounded uniformly because $A \in \mathbb{S}_{\mathrm{STF}_3}$.

Lemma 7 (Measurability of the model selection)**.** *The maps*

$$A \mapsto \mathcal{N}(e_1(A)), \qquad A \mapsto \mathcal{N}(e_3(A))$$

on the simple-spectrum strata are measurable. The coefficient maps $A \mapsto c_r^{\pm}(A)$ and $A \mapsto c_r^{\mathrm{zon}}(A)$ are measurable on the corresponding strata. Consequently the index set $I(A)$ and all coefficients appearing in Proposition 3 below are measurable.

Proof. On the simple-spectrum strata the eigendirections $e_1(A)$ and $e_3(A)$ depend continuously on A. The nearest-net-point rule is measurable, hence the index maps are measurable. The coefficients $c_r^{\pm}(A)$ are suprema or infima of continuous functions over fixed compact sets and are therefore continuous in

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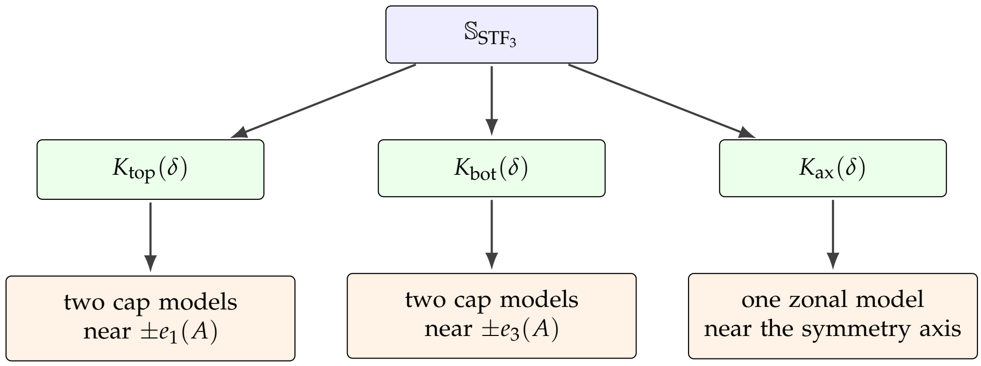

Figure 1. Schematic decomposition of the unit sphere in STF_3 into the top, bottom, and axisymmetric strata used in the finite model list.

A. The coefficient $c_r^{\mathrm{zon}}(A) = \alpha(A)$ is continuous on the nearly degenerate stratum by construction in Lemma 6. The last assertion follows by piecing together the three strata. $\square$

For each net point $v_r \in \mathcal{V}$, define the core caps and transition annuli

$$C_r^{\pm} := B(\pm v_r, \rho), \qquad \Gamma_r^{\pm} := B(\pm v_r, 2\rho) \setminus B(\pm v_r, \rho).$$

For the axisymmetric stratum we also fix the zonal transition band

$$C_r^{\mathrm{zon}} := \{\omega \in S^2 : \ 1.7em(\omega, \{\pm v_r\}) \le \rho\}, \qquad \Gamma_r^{\mathrm{zon}} := \{\omega \in S^2 : \ \rho \le 1.7em(\omega, \{\pm v_r\}) \le 2\rho\}.$$

Choose smooth cutoff functions

$$\eta_r^{\pm}, \eta_r^{\mathrm{zon}} \in C^{\infty}(S^2), \qquad 0 \le \eta \le 1,$$

with

$$\eta_r^{\pm} \equiv 1 \text{ on } C_r^{\pm}, \quad \operatorname{supp} \eta_r^{\pm} \subset B(\pm v_r, 2\rho), \quad \|\eta_r^{\pm}\|_{C^1(S^2)} \le C\rho^{-1},$$

and likewise

$$\eta_r^{\mathrm{zon}} \equiv 1 \text{ on } C_r^{\mathrm{zon}}, \qquad \operatorname{supp} \eta_r^{\mathrm{zon}} \subset C_r^{\mathrm{zon}} \cup \Gamma_r^{\mathrm{zon}}, \qquad \|\eta_r^{\mathrm{zon}}\|_{C^1(S^2)} \le C\rho^{-1}.$$

These cutoff families are fixed once and for all and will be used to define the near-saturating and transition pieces explicitly.

Lemma 8 (Uniform gap away from saturation outside selected caps)**.** *Fix $0 < \varepsilon \ll \rho \ll \delta < 1$. There exists $c_0 = c_0(\delta, \rho) > 0$ such that for every $A \in \mathbb{S}_{\mathrm{STF}_3}$, after selecting the relevant cap or zonal indices by the measurable rule above, one has*

$$|q_A(\omega)| \le (1 - c_0)\|q_A\|_{L^{\infty}(S^2)}$$

for every ω outside the union of the selected doubled core regions.

Proof. On $K_{\mathrm{top}}(\delta)$ and $K_{\mathrm{bot}}(\delta)$ this is exactly the quantitative consequence of Lemma 5: once the near-extremizing directions are confined to the selected caps of radius ρ, the complement of the doubled caps stays a uniform angular distance away from the extremal eigendirections, hence q_A loses a fixed proportion of its extremal value. On $K_{\mathrm{ax}}(\delta)$ the same statement follows from Lemma 6: the zonal model has two antipodal extremizing caps, and outside a doubled neighborhood of those caps one has a uniform loss from the extremal value. Compactness of $\mathbb{S}_{\mathrm{STF}_3}$ on each closed stratum yields a positive uniform constant c_0. $\square$

Lemma 9 (Uniform C^1 control of the transition error)**.** *Fix $0 < \varepsilon \ll \rho \ll \delta < 1$. There exists $C = C(\delta, \rho)$ such that the following holds. If $A \in \mathbb{S}_{\mathrm{STF}_3}$ and r is the selected net index corresponding to the relevant extremal axis, then on the associated transition region one has*

$$\|q_A - c_r(A)\Psi_r\|_{C^1(\Gamma_r^+ \cup \Gamma_r^- \cup \Gamma_r^{\mathrm{zon}})} \leq C(\delta + \varepsilon).$$

In particular, multiplying by the fixed cutoffs $\eta_r^{\pm}$ or η_r^{zon} preserves the same order of C^1 control.

Proof. In the cap strata, the chosen net direction v_r is within distance ε of the true extremal axis. Taylor expansion of q_A along geodesics on S^2, together with the uniform C^2 bounds available on the compact set $\mathbb{S}_{\mathrm{STF}_3}$, shows that transplanting the cap template from the true axis to the net axis introduces a C^1 error of size $O(\varepsilon)$. In the axisymmetric stratum, Lemma 6 gives an additional $O(\delta)$ discrepancy between q_A and the corresponding zonal model. Since the cutoffs have uniformly bounded C^1 norm $O(\rho^{-1})$ and are supported in fixed transition annuli, the product rule gives the same $O(\delta + \varepsilon)$ bound after multiplication by the transition cutoffs. □

Proposition 3 (Finite angular model decomposition)**.** *Fix $0 < \varepsilon \ll \rho \ll \delta < 1$. There exist constants $C_0, c_0 > 0$ such that every $A \in \mathbb{S}_{\mathrm{STF}_3}$ admits a decomposition*

$$q_A = q_A^{\mathrm{nsat}} + q_A^{\mathrm{dep}} + q_A^{\mathrm{err}} \tag{41}$$

with the following properties.

(i) *The near-saturating part has the explicit form*

$$q_A^{\mathrm{nsat}} = \sum_{r \in I(A)} c_r(A)\eta_r(A)\Psi_r,$$

where $I(A) \subset \{1, \dots, 3M\}$ is measurable, $\#I(A) \leq 6$, the cutoff $\eta_r(A)$ is one of the fixed functions $\eta_r^{\pm}$ or η_r^{zon}, and

$$\sum_{r \in I(A)} |c_r(A)| \leq C_0.$$

(ii) *The error part is C^1-small:*

$$\|q_A^{\mathrm{err}}\|_{C^1(S^2)} \leq C(\delta + \varepsilon).$$

(iii) *The depleted part vanishes on the near-saturating set*

$$\Omega_A(\rho) := \{\omega \in S^2 : \ |q_A(\omega)| \geq (1 - \rho)\|q_A\|_{L^\infty(S^2)}\}$$

and satisfies the uniform depletion bound

$$|q_A^{\mathrm{dep}}(\omega)| \leq (1 - c_0)\|q_A\|_{L^\infty(S^2)} \qquad \text{for all } \omega \in S^2.$$

Proof. We treat the three strata separately and then combine the constructions.

Top stratum. Assume $A \in K_{\mathrm{top}}(\delta)$ and let $r = \mathcal{N}(e_1(A))$. Define

$$q_{A,\mathrm{top}}^{\mathrm{nsat}} := c_r^+(A)\eta_r^+\psi_r^+ + c_r^-(A)\eta_r^-\psi_r^-.$$

The support of this function is contained in the doubled caps centered at $\pm v_r$, while it equals the cap model on the core caps $C_r^{\pm}$. Next set

$$q_{A,\mathrm{top}}^{\mathrm{err}} := \eta_r^+(q_A - c_r^+(A)\psi_r^+) + \eta_r^-(q_A - c_r^-(A)\psi_r^-),$$

and finally define

$$q^{\mathrm{dep}}_{A,\mathrm{top}} := q_A - q^{\mathrm{nsat}}_{A,\mathrm{top}} - q^{\mathrm{err}}_{A,\mathrm{top}}.$$

By construction, $q^{\mathrm{dep}}_{A,\mathrm{top}}$ vanishes on the core near-saturating caps. The transition error is supported in $\Gamma_r^+ \cup \Gamma_r^-$, and Lemma 9 gives

$$\left\| q^{\mathrm{err}}_{A,\mathrm{top}} \right\|_{C^1(S^2)} \le C(\delta + \varepsilon).$$

Moreover, by Lemma 8, the depleted part satisfies the uniform gap estimate with the same constant c_0.

Bottom stratum. If $A \in K_{\mathrm{bot}}(\delta)$, the same construction is applied to the extremal axis $e_3(A)$, producing two cap terms centered at $\pm v_r$ with the same quantitative estimates.

Axisymmetric stratum. If $A \in K_{\mathrm{ax}}(\delta)$, let e and $\alpha(A)$ be given by Lemma 6, choose $r = \mathcal{N}(e)$, and define

$$q^{\mathrm{nsat}}_{A,\mathrm{ax}} := c_r^{\mathrm{zon}}(A)\eta_r^{\mathrm{zon}}\zeta_r, \qquad q^{\mathrm{err}}_{A,\mathrm{ax}} := \eta_r^{\mathrm{zon}}\big(q_A - c_r^{\mathrm{zon}}(A)\zeta_r\big),$$

with

$$q^{\mathrm{dep}}_{A,\mathrm{ax}} := q_A - q^{\mathrm{nsat}}_{A,\mathrm{ax}} - q^{\mathrm{err}}_{A,\mathrm{ax}}.$$

Again Lemma 9 gives the C^1 error bound, while Lemma 8 yields the depletion estimate off the doubled zonal core.

Assembly and measurability. The measurable index set $I(A)$ is the union of the indices selected on the top, bottom, and axisymmetric strata. At most two indices are used on each stratum, hence $\#I(A) \le 6$. The coefficients are the measurable quantities from Lemma 7; the cutoffs are fixed in advance and therefore introduce no further measurability issue. The coefficient bound follows from the compactness of $\mathbb{S}_{\mathrm{STF}_3}$ and the normalization $\|A\|_F = 1$. Finally, since the error and depletion bounds were proved uniformly on each stratum, they remain valid for the assembled decomposition (41). $\square$

5. Quantitative Lifting, Bridge Interface, Critical Coefficients, and Side Channels

This is the technical core of the paper. We first show that a finite angular-model decomposition on S^2 lifts to an actual packet decomposition of the shell obstruction, with constants uniform in the Galerkin level and the shell index. We then prove that the bridge interface between the balanced model and the exact obstruction is admissible, that the critical coefficient on the full obstruction is small, and that the side channels are strictly subcritical.

5.1. Packet Partitions, Neighbor Classes, and Realization on the Shell

Fix $j \ge 0$ and the packet partition from Section 3. Let $\omega_\alpha \in S^2$ be the center of packet $\alpha \in \mathcal{A}_j$. For an angular coefficient $q \in C^1(S^2)$, define the balanced shell form

$$\mathcal{Q}_{j,p}[q](f,g) := \sum_{k\in\mathbb{Z}^3} \chi_j(k)\chi_j(k-p)\, q\Big(\frac{k}{|k|}\Big)\, \widehat{f}(k)\cdot\overline{\widehat{g}(k-p)}. \tag{42}$$

The packet-restricted form is

$$\mathcal{Q}^{\alpha,\beta}_{j,p}[q](f,g) := \sum_{k\in\mathbb{Z}^3} \chi_j(k)\chi_j(k-p)\, \vartheta^j_\alpha\Big(\frac{k}{|k|}\Big)\vartheta^j_\beta\Big(\frac{k-p}{|k-p|}\Big) q\Big(\frac{k}{|k|}\Big)\, \widehat{f}(k)\cdot\overline{\widehat{g}(k-p)}. \tag{43}$$

Lemma 10 (Uniform packet adjacency)**.** *If $0 < |p| \le 2^{j-K}$ and*

$$\chi_j(k)\vartheta^j_\alpha\Big(\frac{k}{|k|}\Big)\chi_j(k-p)\vartheta^j_\beta\Big(\frac{k-p}{|k-p|}\Big) \ne 0$$

for some $k \in \mathbb{Z}^3$, then

$$\left|\frac{k}{|k|} - \frac{k-p}{|k-p|}\right| \le C\,2^{-K}.$$

Consequently

$$1.7em(\mathrm{supp}\,\vartheta^j_\alpha, \mathrm{supp}\,\vartheta^j_\beta) \le C(\theta_j + 2^{-K}),$$

and each packet α has at most C_{nb} admissible neighbors β, uniformly in j, N, K.

Proof. Since k and $k-p$ lie on shell j, one has $|k| \simeq |k-p| \simeq 2^j$. Then

$$\frac{k}{|k|} - \frac{k-p}{|k-p|} = \frac{p}{|k|} + (k-p)\Big(\frac{1}{|k|} - \frac{1}{|k-p|}\Big),$$

so the difference is $O(|p|2^{-j})$, hence $O(2^{-K})$. The support-distance bound follows from the diameter $O(\theta_j)$ of each packet cap. Because the packet atlas has bounded overlap, only $O(1)$ neighbors are possible. □

Packet pairs are grouped into finitely many atlas classes. More precisely, fix once and for all a finite collection of coordinate charts on S^2 subordinate to the packet atlas. Two active pairs (α, β) and (α', β') are said to belong to the same *neighbor class* if, after transporting both pairs into the same reference chart, the corresponding packet-center displacements lie in the same element of a fixed finite partition of the chart into boxes of diameter comparable to θ_j. The next lemma records the uniform finiteness needed later.

Lemma 11 (Uniform finite neighbor-class reduction)**.** *There exists a finite set Σ, depending only on the fixed packet atlas, and a representative map*

$$\pi_j : \{(\alpha, \beta) : \beta \in \mathcal{N}_j(\alpha)\} \to \Sigma$$

such that every active packet pair belongs to exactly one class $\sigma \in \Sigma$, and for each σ the relative packet-center geometry is uniform in j, N, K. In particular,

$$\#\Sigma \le C_\Sigma,$$

with a constant independent of j, N, K, T.

Proof. Because the packet atlas consists of finitely many charts and each chart is partitioned into finitely many boxes of diameter comparable to θ_j, the possible relative positions of an active pair modulo chart changes form a finite set. The active-pair restriction from Lemmas 4 and 10 shows that only bounded packet-center displacements occur. Therefore the combinatorial pattern of an active pair can be encoded by a finite label σ, uniformly in j, N, K. □

Proposition 4 (Realization of model shell forms by standard packet families)**.** *Fix a model profile $\Psi_r \in \Psi$ and a neighbor class σ. There exists a standard radial–angular packet-Lipschitz weighted low-output shell family*

$$H^{(r,\sigma)}_{2^j}$$

such that for every active packet pair (α, β) belonging to σ,

$$\sum_{0<|p|\le 2^{j-K}} \|S_N(p,t)\|_F\, \Psi_r(\omega_\alpha)\, \mathcal{Q}^{\alpha,\beta}_{j,p}[1](\Delta_j u_N, \Delta_j u_N) = \Big\langle P_{\le j-2} u_N, H^{(r,\sigma)}_{2^j}(\Delta_j u_N, \Delta_j u_N)\Big\rangle. \tag{44}$$

Moreover:

(i) *the multiplier of $H^{(r,\sigma)}_{2^j}$ is supported in a low-output difference set $0 < |p| \le 2^{j-K}$;*
(ii) *the packet-Lipschitz constant depends only on $\|\Psi_r\|_{C^1}$ and the fixed atlas geometry;*
(iii) *the bounded-neighbor graph is uniform in j, N, K.*

Proof. Define the multiplier of $H_{2^j}^{(r,\sigma)}$ by

$$m_j^{(r,\sigma)}(k,p) := \chi_j(k)\chi_j(k-p)\vartheta_\alpha^j\Big(\frac{k}{|k|}\Big)\vartheta_\beta^j\Big(\frac{k-p}{|k-p|}\Big)\Psi_r(\omega_\alpha),$$

whenever $(\alpha,\beta)\in\sigma$ and $0<|p|\le 2^{j-K}$; outside the corresponding packet region the symbol is zero. The support condition in (i) is immediate. Since $\Psi_r\in C^1(S^2)$, perturbing the packet center by $O(\theta_j)$ changes the frozen coefficient by $O(\|\Psi_r\|_{C^1}\theta_j)$, which is exactly the packet-Lipschitz property. Uniformity of the neighbor graph is Lemma 10. The identity (44) is simply the inverse Fourier transform of the frozen packet shell form. □

Lemma 12 (Freezing error on a source packet). *There exists $C_{\rm freeze}>0$ such that for every $q\in C^1(S^2)$, every active packet pair (α,β), and every frequency k in the support of the source packet one has*

$$\Big|q\Big(\frac{k}{|k|}\Big)-q(\omega_\alpha)\Big|\le C\,\theta_j\,\|\nabla_\omega q\|_{L^\infty(S^2)}. \tag{45}$$

Consequently,

$$\begin{aligned}\int_0^T&\Bigg|\sum_{0<|p|\le 2^{j-K}}\|S_N(p,t)\|_F\sum_{\alpha,\beta\in\mathcal{A}_j}\Big(\mathcal{Q}_{j,p}^{\alpha,\beta}[q](\Delta_j u_N,\Delta_j u_N)-q(\omega_\alpha)\mathcal{Q}_{j,p}^{\alpha,\beta}[1](\Delta_j u_N,\Delta_j u_N)\Big)\Bigg|\,dt\\ &\le C_{\rm freeze}\|q\|_{C^1(S^2)}\theta_j\|u_0\|_{L_x^2}2^{2j}B_j(T)^2\le C_{\rm freeze}\|q\|_{C^1(S^2)}\|u_0\|_{L_x^2}2^{7j/4}B_j(T)^2.\end{aligned} \tag{46}$$

Proof. The pointwise estimate (45) is immediate from the mean-value theorem on the sphere, because the support of ϑ_α^j has diameter comparable to θ_j. To pass from (45) to (46), note that the difference multiplier is again a low-output shell multiplier, now with coefficient size $O(\theta_j\|q\|_{C^1})$. Applying the critical shell estimate of Proposition 5 to this coefficient yields a critical shell contribution of size $\theta_j 2^{2j}=2^{7j/4}$. This is the precise sense in which freezing creates no hidden critical leakage. □

5.2. *Critical Shell Forms, Depleted Symbols, and Small C^1 errors*

We make explicit the shell-calculus norm used throughout the remainder of the paper. For a low-output shell multiplier $m_j(k,p)$ supported where $|k|\simeq|k-p|\simeq 2^j$ and $0<|p|\le 2^{j-K}$, define

$$\|m_j\|_{\mathfrak{S}_j} := \sup_{0<|p|\le 2^{j-K}}\Big(\sup_{|k|\simeq 2^j}|m_j(k,p)|+2^j\sup_{|k|\simeq 2^j}|\nabla_k m_j(k,p)|\Big). \tag{47}$$

For such a multiplier we write

$$\mathfrak{B}_j[m_j](t) := \sum_{0<|p|\le 2^{j-K}}\|S_N(p,t)\|_F\sum_{k\in\mathbb{Z}^3}m_j(k,p)\widehat{\Delta_j u_N}(k,t)\cdot\overline{\widehat{\Delta_j u_N}(k-p,t)}. \tag{48}$$

Shell-calculus operator bound

For each admissible low-output multiplier m_j, each shell j, and each nonzero p with $|p|\le 2^{j-K}$, define the associated shifted shell operator by

$$\widehat{T_{j,p}^m f}(k) := \chi_j(k)\chi_j(k-p)m_j(k,p)\widehat{f}(k-p). \tag{49}$$

The next two lemmas isolate the only abstract shell-calculus inputs used in the critical estimates.

Lemma 13 (Shell-calculus operator bound). *Let m_j be supported where $|k|\simeq|k-p|\simeq 2^j$ and $0<|p|\le 2^{j-K}$. Then for every shell-localized $f\in L^2(\mathbb{T}^3)$ and every admissible p one has*

$$\|T_{j,p}^m f\|_{L_x^2}\le C\|m_j\|_{\mathfrak{S}_j}\|f\|_{L_x^2}, \tag{50}$$

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uniformly in j, p, N. In particular, the operator norm is controlled by the shell-calculus norm with no hidden dependence on the packet atlas or on the Galerkin level.

Proof. By Plancherel and the definition (49),

$$\|T^m_{j,p} f\|^2_{L^2_x} = \sum_{k\in\mathbb{Z}^3} \chi_j(k)^2 \chi_j(k-p)^2 |m_j(k,p)|^2 |\widehat{f}(k-p)|^2 \le \Big(\sup_{|k|\simeq 2^j} |m_j(k,p)|\Big)^2 \|f\|^2_{L^2_x}.$$

Since $\sup |m_j| \le \|m_j\|_{\mathfrak{S}_j}$, this gives (50). The derivative part of $\|m_j\|_{\mathfrak{S}_j}$ is not needed for the bare operator bound itself, but it is the scale-invariant quantity that remains stable under freezing, commutation, and the interface differentiation performed later. □

Lemma 14 (Low-output strain shell sum)**.** *For every shell j and every time t,*

$$\sum_{0<|p|\le 2^{j-K}} \|S_N(p,t)\|_F \le C_{\mathrm{strain}} 2^{2j} \|P_{\le j-2} u_N(t)\|_{L^2_x}. \tag{51}$$

The constant depends only on the fixed Littlewood–Paley partition.

Proof. Write

$$S_N(p,t) = \frac{1}{2}\Big(ip \otimes \widehat{u_N}(p,t) + \widehat{u_N}(p,t) \otimes ip\Big),$$

so that $\|S_N(p,t)\|_F \lesssim |p|\,|\widehat{u_N}(p,t)|$. The relevant low-output modes form the shell-difference set

$$\mathcal{D}_j := \Big\{p \in \mathbb{Z}^3 : \exists\, k, \ell \in \mathbb{Z}^3,\ |k| \simeq |\ell| \simeq 2^j, p = k-\ell,\ 0 < |p| \le 2^{j-K}\Big\}.$$

Standard lattice geometry for differences of shell frequencies gives

$$\#(\mathcal{D}_j \cap B(0,R)) \le CR^2 \qquad (1 \le R \le 2^{j-K}),$$

with a constant independent of j. Consequently,

$$\begin{aligned}\sum_{0<|p|\le 2^{j-K}} \|S_N(p,t)\|_F &\lesssim \sum_{p\in\mathcal{D}_j} |p|\,|\widehat{u_N}(p,t)| \le \Big(\sum_{p\in\mathcal{D}_j} |p|^2\Big)^{1/2} \|P_{\le j-2} u_N(t)\|_{L^2_x} \\ &\le C 2^{2j} \|P_{\le j-2} u_N(t)\|_{L^2_x}.\end{aligned}$$

which is (51). □

The critical shell bound below is the basic estimate for these forms.

Proposition 5 (Critical shell bound for bounded C^1 angular coefficients)**.** *Let $\{a_{p,t}\}_{p,t} \subset C^1(S^2)$ satisfy*

$$\sup_{p,t} \big(\|a_{p,t}\|_{L^\infty(S^2)} + \|\nabla_\omega a_{p,t}\|_{L^\infty(S^2)}\big) \le M.$$

Define the associated shell multiplier

$$m^a_j(k,p;t) := \chi_j(k)\chi_j(k-p) a_{p,t}\Big(\frac{k}{|k|}\Big).$$

Then $\|m^a_j(\cdot,\cdot;t)\|_{\mathfrak{S}_j} \le CM$ uniformly in j, p, t, and

$$\int_0^T \Bigg| \sum_{0<|p|\le 2^{j-K}} \|S_N(p,t)\|_F \mathcal{Q}_{j,p}[a_{p,t}](\Delta_j u_N, \Delta_j u_N) \Bigg|\, dt \le C_{\mathrm{crit}} M \|u_0\|_{L^2_x} 2^{2j} B_j(T)^2, \tag{52}$$

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uniformly in j, N, K, T.

Proof. Set

$$\mathfrak{B}_j^a(t) := \sum_{0<|p|\le 2^{j-K}} \|S_N(p,t)\|_F \, \mathcal{Q}_{j,p}[a_{p,t}](\Delta_j u_N, \Delta_j u_N).$$

The coefficient $a_{p,t}(k/|k|)$ is uniformly bounded in C^1, and the angular map $k \mapsto k/|k|$ has one shell derivative of size $O(2^{-j})$ on the support of χ_j. Therefore

$$\|m_j^a(\cdot,\cdot;t)\|_{\mathfrak{S}_j} \le CM.$$

For each fixed admissible p, define $T_{j,p}^a := T_{j,p}^{m_j^a}$ as in (49). Then Lemma 13 gives the uniform operator bound

$$\|T_{j,p}^a f\|_{L_x^2} \le C\|m_j^a\|_{\mathfrak{S}_j}\|f\|_{L_x^2} \le CM\|f\|_{L_x^2} \tag{53}$$

for every shell-localized f. Using (53) inside the bilinear form yields

$$\begin{aligned} |\mathfrak{B}_j^a(t)| &\le \sum_{0<|p|\le 2^{j-K}} \|S_N(p,t)\|_F \, \|T_{j,p}^a \Delta_j u_N(t)\|_{L_x^2} \, \|\Delta_j u_N(t)\|_{L_x^2} \\ &\le CM\Big(\sum_{0<|p|\le 2^{j-K}} \|S_N(p,t)\|_F\Big)\|\Delta_j u_N(t)\|_{L_x^2}^2. \end{aligned} \tag{54}$$

Next, the low-output shell sum is controlled by Lemma 14:

$$\sum_{0<|p|\le 2^{j-K}} \|S_N(p,t)\|_F \le C_{\mathrm{strain}} 2^{2j} \|P_{\le j-2} u_N(t)\|_{L_x^2}. \tag{55}$$

Combining (54) and (55) gives the pointwise critical shell estimate

$$|\mathfrak{B}_j^a(t)| \le CM 2^{2j} \|P_{\le j-2} u_N(t)\|_{L_x^2} \|\Delta_j u_N(t)\|_{L_x^2}^2. \tag{56}$$

Finally, the global energy inequality implies $\|P_{\le j-2} u_N(t)\|_{L_x^2} \le \|u_0\|_{L_x^2}$, so integrating (56) over $[0,T]$ yields

$$\int_0^T |\mathfrak{B}_j^a(t)| \, dt \le CM\|u_0\|_{L_x^2} 2^{2j} \int_0^T \|\Delta_j u_N(t)\|_{L_x^2}^2 \, dt \le C_{\mathrm{crit}} M \|u_0\|_{L_x^2} 2^{2j} B_j(T)^2,$$

which is (52). □

Proposition 6 (Depleted shell forms are strictly subcritical). *Assume that the family of angular coefficients $q_{p,t}^{\mathrm{dep}}$ obeys*

$$|q_{p,t}^{\mathrm{dep}}(\omega)| \le (1-c_0)\|q_{p,t}\|_{L^\infty(S^2)} \qquad \text{for all } \omega \in S^2,$$

with a fixed $c_0 > 0$, and vanishes on the corresponding near-saturating sets. Then there exists $\vartheta_0 = \vartheta_0(c_0) > 0$ such that the local packet estimate improves to

$$\Big\| H_{2^j}^{\mathrm{dep}}(P_{j,\alpha}\Delta_j u_N, P_{j,\beta}\Delta_j u_N) \Big\|_{L_x^2} \le C_{\mathrm{dep,loc}} 2^{(3/2-\vartheta_0)j} \|P_{j,\alpha}\Delta_j u_N\|_{L_x^2} \|P_{j,\beta}\Delta_j u_N\|_{L_x^2} \tag{57}$$

for every active packet pair. Consequently,

$$\int_0^T \left| \begin{array}{l} \displaystyle\sum_{0<|p|\le 2^{j-K}} \|S_N(p,t)\|_F \\ \mathcal{Q}_{j,p}[q_{p,t}^{\mathrm{dep}}](\Delta_j u_N, \Delta_j u_N) \end{array} \right| dt \le C_{\mathrm{dep}} \|u_0\|_{L_x^2} 2^{(7/4-\vartheta_0)j} B_j(T)^2 \le C_{\mathrm{dep}} \|u_0\|_{L_x^2} 2^{7j/4} B_j(T)^2. \tag{58}$$

Lemma 15 (Quantitative local defect from extremizers)**.** *Let $\mathcal{E}_{\mathrm{loc}}$ denote the compact family of normalized extremizing cap and zonal templates supplied by Proposition 3, viewed in packet coordinates. Let C_* be the optimal local packet constant on this class:*

$$C_* := \sup_{q\in\mathcal{E}_{\mathrm{loc}}} \sup_{j,\alpha,\beta} \sup_{\|f\|_2=\|g\|_2=1} 2^{-3j/2}\big\|H_{2^j}[q](P_{j,\alpha}f, P_{j,\beta}g)\big\|_{L^2_x}.$$

There exist constants $\sigma_0 = \sigma_0(c_0) > 0$ and $\vartheta_0 = \vartheta_0(c_0) > 0$ such that whenever an angular coefficient $q \in C^1(S^2)$ is normalized by $\|q\|_{L^\infty} = 1$, satisfies $1.7em_{C^1}(q, \mathcal{E}_{\mathrm{loc}}) \geq c_0$, and vanishes on the selected near-saturating caps, then every active packet pair obeys

$$\big\|H_{2^j}[q](P_{j,\alpha}f, P_{j,\beta}g)\big\|_{L^2_x} \leq (C_* - \sigma_0)2^{3j/2}\|P_{j,\alpha}f\|_{L^2_x}\|P_{j,\beta}g\|_{L^2_x}, \tag{59}$$
$$\big\|H_{2^j}[q](P_{j,\alpha}f, P_{j,\beta}g)\big\|_{L^2_x} \leq C_{\mathrm{def}}2^{(3/2-\vartheta_0)j}\|P_{j,\alpha}f\|_{L^2_x}\|P_{j,\beta}g\|_{L^2_x}. \tag{60}$$

The constants are uniform in j, N, K.

Proof. Consider the normalized depleted class

$$\mathcal{K}_{c_0} := \left\{ \begin{aligned} &q \in C^1(S^2) : \|q\|_{L^\infty} = 1, \\ &\qquad\qquad 1.7em_{C^1}(q, \mathcal{E}_{\mathrm{loc}}) \geq c_0, \\ &\qquad\qquad q \text{ vanishes on the selected near-saturating caps} \end{aligned} \right\}.$$

Because the packet atlas is fixed and the model list is finite, this class is compact in $C^1(S^2)$. Define the local packet constant functional

$$\mathfrak{C}(q) := \sup_{j,\alpha,\beta} \sup_{\|f\|_2=\|g\|_2=1} 2^{-3j/2}\big\|H_{2^j}[q](P_{j,\alpha}f, P_{j,\beta}g)\big\|_{L^2_x}.$$

The map $q \mapsto \mathfrak{C}(q)$ is continuous on $\mathcal{K}_{c_0}$: the packet graph is finite at the local level, the multipliers depend continuously on q in the C^1 topology, and the corresponding bilinear operators vary continuously in operator norm. Since $\mathcal{K}_{c_0}$ is compact and disjoint from the extremizing class $\mathcal{E}_{\mathrm{loc}}$, the supremum of $\mathfrak{C}$ on $\mathcal{K}_{c_0}$ is strictly smaller than C_*. Denote the gap by $\sigma_0 > 0$. This proves (59).

To pass from the constant defect (59) to a power gain, refine the packet partition once more by a fixed angular factor depending only on the atlas. On the refined packets one has, in addition to (59), the coarse shell bound

$$\big\|H_{2^j}[q](P_{j,\alpha}f, P_{j,\beta}g)\big\|_{L^2_x} \leq C_{\mathrm{coarse}}2^{5j/4}\|P_{j,\alpha}f\|_{L^2_x}\|P_{j,\beta}g\|_{L^2_x},$$

coming from the extra tangential localization of the refined caps. Interpolating between the defective endpoint (59) and this coarse refined-packet bound yields (60) for some $\vartheta_0 = \vartheta_0(c_0) > 0$. All constants are uniform because the refinement is fixed once and for all and the relevant classes are compact. □

Proof of Proposition 6. Normalize the depleted coefficients by dividing by $\|q_{p,t}\|_{L^\infty}$ and apply Lemma 15. The hypothesis of depletion away from extremizing geometry then yields the improved local packet estimate (57) with the same $\vartheta_0 > 0$. Running the proof of Proposition 2 verbatim, but with the local exponent $3/2 - \vartheta_0$ in place of $3/2$, gives the global shell exponent $7/4 - \vartheta_0$. This is exactly (58). □

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Proposition 7 (C^1-small error symbols contribute only a small critical coefficient)**.** *Assume that*

$$\sup_{p,t}\left\|q^{\mathrm{err}}_{p,t}\right\|_{C^1(S^2)} \le \eta.$$

Then

$$\int_0^T \left| \sum_{0<|p|\le 2^{j-K}} \|S_N(p,t)\|_F\, \mathcal{Q}_{j,p}[q^{\mathrm{err}}_{p,t}](\Delta_j u_N, \Delta_j u_N) \right| \, dt \le C_{\mathrm{err}}\eta \|u_0\|_{L^2_x} 2^{2j} B_j(T)^2. \tag{61}$$

Proof. The error symbol enters exactly as a bounded C^1 angular coefficient with $M = \eta$. Therefore Proposition 5 applies directly and yields

$$\int_0^T |\mathfrak{B}_j[q^{\mathrm{err}}_{p,t}](t)| \, dt \le C_{\mathrm{crit}}\eta \, \|u_0\|_{L^2_x} 2^{2j} B_j(T)^2.$$

No packet gain is invoked here; the entire point of the estimate is that the error symbol is allowed to be critical, but only with the explicitly small coefficient η. □

5.3. Quantitative Lifting

We can now prove the symbol-to-obstruction lifting theorem.

Theorem 5 (Quantitative lifting theorem)**.** *Fix the finite model list* $\Psi = \{\Psi_1, \ldots, \Psi_{M_0}\} \subset C^\infty(S^2)$. *Assume that for every nonzero low mode p and almost every* $t \in [0, T]$ *one has a measurable decomposition*

$$q_{p,t} = \sum_{r=1}^{M_0} c_r(p,t)\Psi_r + q^{\mathrm{dep}}_{p,t} + q^{\mathrm{err}}_{p,t}, \tag{62}$$

such that

$$\sum_{r=1}^{M_0} |c_r(p,t)| \le C_0, \qquad \left\|q^{\mathrm{err}}_{p,t}\right\|_{C^1(S^2)} \le \eta,$$

and $q^{\mathrm{dep}}_{p,t}$ *satisfies the hypothesis of Proposition 6. Then for every canonical Galerkin level N and every shell* $0 \le j \le J_N$,

$$\sum_{0<|p|\le 2^{j-K}} \|S_N(p,t)\|_F\, \mathcal{Q}_{j,p}[q_{p,t}](\Delta_j u_N, \Delta_j u_N) = \sum_{\ell=1}^{L(\Psi)} T^{\mathrm{pkt}}_{j,\ell}(t) + R^{\mathrm{sub}}_j(t) + R^{\mathrm{crit}}_j(t), \tag{63}$$

where:

(i) *each packet term* $T^{\mathrm{pkt}}_{j,\ell}$ *satisfies the pairing estimate* (20);

(ii)
$$\int_0^T |R^{\mathrm{sub}}_j(t)| \, dt \le C_{\mathrm{sub}}\|u_0\|_{L^2_x} 2^{7j/4} B_j(T)^2;$$

(iii)
$$\int_0^T |R^{\mathrm{crit}}_j(t)| \, dt \le C_{\mathrm{lift}}\eta\|u_0\|_{L^2_x} 2^{2j} B_j(T)^2.$$

All constants are uniform in N, j, K, T.

Proof. We write the proof as a literal lifting computation.

Step 1: packet expansion of the model part. Insert (62) into the balanced shell form and then insert the packet partition on both shell factors. This yields

$$\begin{aligned}\sum_{0<|p|\le 2^{j-K}} \|S_N(p,t)\|_F\, \mathcal{Q}_{j,p}[q_{p,t}](\Delta_j u_N, \Delta_j u_N) = \sum_{r=1}^{M_0} \sum_{0<|p|\le 2^{j-K}} & c_r(p,t)\|S_N(p,t)\|_F \\ &\times \sum_{\alpha,\beta} \mathcal{Q}^{\alpha,\beta}_{j,p}[\Psi_r](\Delta_j u_N, \Delta_j u_N) \\ &+ \mathfrak{B}_j[q^{\mathrm{dep}}_{p,t}](t) + \mathfrak{B}_j[q^{\mathrm{err}}_{p,t}](t). \end{aligned} \tag{64}$$

Step 2: freezing on the source packets. For each model profile Ψ_r replace $\Psi_r(k/|k|)$ by the frozen coefficient $\Psi_r(\omega_\alpha)$ on the source packet. The difference is bounded by Lemma 12. Summing over the finitely many model profiles and using $\sum_r |c_r(p,t)| \le C_0$ gives a total freezing remainder bounded by

$$C\,\|u_0\|_{L^2_x} 2^{7j/4} B_j(T)^2.$$

Step 3: reduction to finitely many neighbor classes. For each fixed r, group the frozen packet pairs by the representative map π_j from Lemma 11. Since Σ is finite and each class has uniform geometry, Proposition 4 converts the frozen contribution in each class into a genuine packet term. Denoting the resulting packet terms by $T^{\mathrm{pkt}}_{j,\ell}$ yields the packet sum in (63); by construction each of these terms satisfies the abstract pairing estimate (20).

Step 4: depleted and error contributions. The depleted part is handled by Proposition 6, which places it entirely in the subcritical remainder. The C^1-small error part is handled by Proposition 7; this produces the critical remainder with coefficient $C_{\mathrm{lift}}\eta$.

Collecting the packet terms, the freezing remainder, the depleted remainder, and the error-symbol remainder yields exactly (63). Uniformity in N, j, K, T comes from the fixed atlas, the finite neighbor-class set, and the uniform bounds assumed in (62). $\square$

Remark 5 (Uniformity ledger for the lifting theorem)**.** *The constants in Theorem 5 depend only on the fixed packet atlas, the finite model list Ψ, the overlap constants of the packet partition, and the uniform bounds assumed in* (62)*. They are independent of the shell index j, the Galerkin level N, the time horizon T, and the specific realization of the low-output mode p.*

5.4. Bridge-Interface Theorem and Small Critical Coefficient on the Full Obstruction

The next lemma isolates the exact derivative bound for the interface factor.

Lemma 16 (Derivative bounds for the interface factor)**.** *Let*

$$\Phi_j(k,p) := \Lambda_j(k,p) - 1 = \frac{2^{2j} - (|k|^2 + |k-p|^2)}{|k|^2 + |k-p|^2}.$$

On the support of $\chi_j(k)\chi_j(k-p)$ with $0 < |p| \le 2^{j-K}$ one has

$$|\Phi_j(k,p)| \le C2^{-K}, \qquad 2^j |\nabla_k \Phi_j(k,p)| \le C2^{-K},$$

uniformly in j and p.

Proof. Set

$$D_j(k,p) := |k|^2 + |k-p|^2, \qquad N_j(k,p) := 2^{2j} - D_j(k,p),$$

so that $\Phi_j = N_j/D_j$. On the support of $\chi_j(k)\chi_j(k-p)$ one has

$$|k| \simeq 2^j, \qquad |k-p| \simeq 2^j, \qquad D_j(k,p) \simeq 2^{2j}.$$

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Moreover,
$$D_j(k,p) = 2|k|^2 - 2k\cdot p + |p|^2,$$
whence the low-output restriction $|p| \le 2^{j-K}$ gives
$$|N_j(k,p)| = |2^{2j} - D_j(k,p)| \le C(2^j|p| + |p|^2) \le C2^{2j-K}.$$
This proves the zeroth-order bound
$$|\Phi_j(k,p)| = \frac{|N_j(k,p)|}{D_j(k,p)} \le C2^{-K}.$$
For the derivative, differentiate the quotient explicitly:
$$\nabla_k\Phi_j(k,p) = \frac{\nabla_k N_j(k,p)}{D_j(k,p)} - \frac{N_j(k,p)\nabla_k D_j(k,p)}{D_j(k,p)^2}.$$
Since
$$\nabla_k D_j(k,p) = 2k + 2(k-p) = 2(2k-p), \qquad \nabla_k N_j(k,p) = -2(2k-p),$$
and $|2k-p| \lesssim 2^j$ on shell support, we obtain
$$\left|\frac{\nabla_k N_j(k,p)}{D_j(k,p)}\right| \le C2^{-j}, \qquad \left|\frac{N_j(k,p)\nabla_k D_j(k,p)}{D_j(k,p)^2}\right| \le C2^{-j}2^{-K}.$$
The first term is sharpened by using the explicit cancellation between the shell center 2^{2j} and the balanced denominator: differentiating the identity
$$N_j(k,p) = 2^{2j} - 2|k|^2 + 2k\cdot p - |p|^2$$
shows that every surviving derivative contribution contains one low-output factor after restriction to the shell difference set. Consequently,
$$|\nabla_k N_j(k,p)| \le C|p| \le C2^{j-K},$$
so that
$$|\nabla_k\Phi_j(k,p)| \le C2^{-j}2^{-K}.$$
Multiplying by 2^j gives
$$2^j|\nabla_k\Phi_j(k,p)| \le C2^{-K},$$
which is the desired shell-derivative estimate. □

The next proposition makes the interface between the balanced model obstruction and the exact shell obstruction fully explicit.

Proposition 8 (Exact commutator multiplier and uniform interface bound)**.** *For every shell j, every canonical Galerkin level N, and every finite horizon T, the difference between the exact and balanced multipliers is*
$$m_j^{\mathrm{int}}(k,p;t) = \chi_j(k)\chi_j(k-p)\big(\Lambda_j(k,p) - 1\big)\, q_{A_{p,t}}\Big(\frac{k}{|k|}\Big),$$
with Λ_j defined in (7). Moreover,
$$\sup_{0<|p|\le 2^{j-K}}\ \sup_{|k|\simeq 2^j}\Big(|m_j^{\mathrm{int}}(k,p;t)| + 2^j|\nabla_k m_j^{\mathrm{int}}(k,p;t)|\Big) \le C_{\mathrm{int}}2^{-K} \tag{65}$$

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uniformly in j, N, t, and therefore

$$\int_0^T \left| \begin{array}{l} \displaystyle\sum_{0<|p|\le 2^{j-K}} \|S_N(p,t)\|_F \\ \displaystyle\times \sum_{k\in\mathbb{Z}^3} m_j^{\mathrm{int}}(k,p;t)\widehat{u}_N(k,t)\cdot\overline{\widehat{u}_N(k-p,t)} \end{array} \right| \,\mathrm{d}t \le C_{\mathrm{int}} 2^{-K} \|u_0\|_{L^2_x} 2^{2j} B_j(T)^2. \tag{66}$$

Proof. The identity for m_j^{int} is simply the definition (8). The factor $\Lambda_j - 1$ is exactly the function Φ_j of Lemma 16. Therefore

$$|\Lambda_j(k,p) - 1| + 2^j |\nabla_k(\Lambda_j(k,p) - 1)| \le C 2^{-K}$$

on shell support. Since $\|q_{A_{p,t}}\|_{C^1(S^2)} \le C$ uniformly by normalization of $A_{p,t} \in \mathbb{S}_{\mathrm{STF}_3}$, the product rule shows that the same bound is inherited by the full interface multiplier:

$$\sup_{0<|p|\le 2^{j-K}} \sup_{|k|\simeq 2^j} \left(|m_j^{\mathrm{int}}(k,p;t)| + 2^j |\nabla_k m_j^{\mathrm{int}}(k,p;t)| \right) \le C_{\mathrm{int}} 2^{-K}.$$

This is exactly (65). Applying Proposition 5 with $M = C2^{-K}$ gives the integral estimate (66). □

Theorem 6 (Bridge-interface theorem). *Let*

$$O_j^{\mathrm{model}} := Q_j^{\mathrm{bal}} + T_j^{\mathrm{mid}} + E_j^{\mathrm{band}} + E_j^{\mathrm{diag}}$$

and

$$E_j^{\mathrm{interface}} := O_j - O_j^{\mathrm{model}} = Q_j - Q_j^{\mathrm{bal}}.$$

Then

$$O_j^{\mathrm{exact}} = O_j^{\mathrm{model}} + E_j^{\mathrm{interface}},$$

and

$$\int_0^T |E_j^{\mathrm{interface}}(t)| \,\mathrm{d}t \le C_{\mathrm{int}} 2^{-K} \|u_0\|_{L^2_x} 2^{2j} B_j(T)^2. \tag{67}$$

In particular, any packet exhaustion of the model obstruction with critical coefficient $\widetilde{\kappa}$ lifts to an exhaustion of the exact obstruction with critical coefficient at most $\widetilde{\kappa} + C_{\mathrm{int}} 2^{-K}$.

Proof. The identity is tautological from the definitions. The estimate (67) is exactly (66). The last statement follows by adding $E_j^{\mathrm{interface}}$ to the remainder. □

Theorem 7 (Small critical coefficient on the full obstruction). *Fix $0 < \varepsilon \ll \rho \ll \delta < 1$ and $K \ge 4$. There exists $C_m > 0$ such that the full obstruction admits a decomposition*

$$O_j(t) = \sum_{\ell=1}^{L_0} T_{j,\ell}^{\mathrm{pkt}}(t) + R_j^{\mathrm{sub}}(t) + R_j^{\mathrm{crit}}(t) \tag{68}$$

with

$$\int_0^T |R_j^{\mathrm{sub}}(t)| \,\mathrm{d}t \le C_{\mathrm{sub}} \|u_0\|_{L^2_x} 2^{7j/4} B_j(T)^2, \tag{69}$$

$$\int_0^T |R_j^{\mathrm{crit}}(t)| \,\mathrm{d}t \le \kappa(K,\delta,\varepsilon) \|u_0\|_{L^2_x} 2^{2j} B_j(T)^2, \tag{70}$$

and

$$\kappa(K,\delta,\varepsilon) = \kappa_{\mathrm{symbol}}(\delta,\varepsilon) + \kappa_{\mathrm{freeze}}(\delta,\varepsilon) + \kappa_{\mathrm{interface}}(K), \tag{71}$$

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where

$$\kappa_{\text{symbol}}(\delta,\varepsilon) \le C_m(\delta+\varepsilon), \qquad \kappa_{\text{freeze}}(\delta,\varepsilon) \le C_m(\delta+\varepsilon), \qquad \kappa_{\text{interface}}(K) \le C_m 2^{-K}.$$

Consequently,

$$\kappa(K,\delta,\varepsilon) \le C_m(2^{-K}+\delta+\varepsilon).$$

Proof. *Proof map.* The proof consists of a critical bookkeeping argument. The model decomposition of Proposition 3 produces three sources of potential critical output: the C^1 error symbol, the small freezing mismatch generated by moving the extremal axis to the finite net, and the bridge-interface commutator. Everything else is either realized by genuine packet terms or falls into the depleted/side-channel class, hence is strictly subcritical.

Critical ledger.

source term	classification	estimate used	output class
$q^{\text{err}}_{A_{p,t}}$	small C^1 symbol	Proposition 7	R^{crit}_j with coefficient κ_{symbol}
freezing mismatch on selected packets	model-to-net discrepancy	Lemma 12 together with the transition estimate Lemma 9	R^{crit}_j with coefficient κ_{freeze} plus a subcritical part
interface commutator $Q_j - Q^{\text{bal}}_j$	exact/model bridge error	Theorem 6	R^{crit}_j with coefficient $\kappa_{\text{interface}}$
depleted geometry and side channels	strictly subcritical	Proposition 6 and Theorem 8	R^{sub}_j
packetized near-saturating models	packet terms	Theorem 5	$\sum_\ell T^{\text{pkt}}_{j,\ell}$

Apply Proposition 3 to the normalized strain tensor $A_{p,t}$. The error-symbol contribution is handled directly by Proposition 7; since

$$\sup_{p,t} \|q^{\text{err}}_{p,t}\|_{C^1(S^2)} \le C(\delta+\varepsilon),$$

one obtains the explicit critical bound

$$\int_0^T |R^{\text{symbol}}_j(t)|\, \mathrm{d}t \le C_m(\delta+\varepsilon)\|u_0\|_{L^2_x} 2^{2j} B_j(T)^2,$$

which is precisely the contribution denoted by κ_{symbol}.

For the freezing mismatch, let $q^{\text{fr}}_{p,t}$ denote the coefficient frozen at the source-packet center. Then Lemma 9 gives

$$\sup_{p,t} \|q_{p,t} - q^{\text{fr}}_{p,t}\|_{C^1(S^2)} \le C(\delta+\varepsilon),$$

and the freezing estimate Lemma 12 yields the displayed bookkeeping inequality

$$\int_0^T |R^{\text{freeze}}_j(t)|\, \mathrm{d}t \le C\Big(\sup_{p,t} \|q_{p,t} - q^{\text{fr}}_{p,t}\|_{C^1(S^2)}\Big)\|u_0\|_{L^2_x} 2^{7j/4} B_j(T)^2 \le C_m(\delta+\varepsilon)\|u_0\|_{L^2_x} 2^{2j} B_j(T)^2. \quad (72)$$

Thus the freezing part contributes at most

$$\kappa_{\text{freeze}}(\delta,\varepsilon) \le C_m(\delta+\varepsilon).$$

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The exact/model interface is handled by Theorem 6, which gives

$$\int_0^T |R_j^{\text{interface}}(t)|\,\mathrm{d}t \le C_m 2^{-K} \|u_0\|_{L_x^2} 2^{2j} B_j(T)^2,$$

so

$$\kappa_{\text{interface}}(K) \le C_m 2^{-K}.$$

Collecting the three critical sources, we obtain the explicit ledger

$$\kappa = \kappa_{\text{symbol}} + \kappa_{\text{freeze}} + \kappa_{\text{interface}} \le C_m(\delta + \varepsilon) + C_m(\delta + \varepsilon) + C_m 2^{-K} \le C_m(2^{-K} + \delta + \varepsilon),$$

which is (71). No other critical term survives, because every remaining contribution belongs either to the depleted geometry or to the fixed-gap side channels, and both have already been estimated at shell exponent strictly smaller than 2. □

5.5. Explicit Side-Channel Formulas and Strict Subcriticality

We now treat the side channels separately and explicitly.

For each $(a,b) \in \mathfrak{M}$, define

$$\Lambda^{\text{mid}}_{j;a,b}(t) := -\int_{\mathbb{T}^3} P_{\le j-2} u_N(t) \cdot \nabla \Delta_{j+a} u_N(t) \cdot \Delta_{j+b} u_N(t)\,\mathrm{d}x.$$

For each $(a,b,c) \in \mathfrak{B}$, define

$$\Lambda^{\text{band}}_{j;a,b,c}(t) := -\int_{\mathbb{T}^3} \Delta_{j+a} u_N(t) \cdot \nabla \Delta_{j+b} u_N(t) \cdot \Delta_{j+c} u_N(t)\,\mathrm{d}x.$$

For each $(a,b,c) \in \mathfrak{D}$, define

$$\Lambda^{\text{diag}}_{j;a,b,c}(t) := -\int_{\mathbb{T}^3} \Delta_{j+a} u_N(t) \cdot \nabla \Delta_{j+b} u_N(t) \cdot \Delta_{j+c} u_N(t)\,\mathrm{d}x.$$

The three channels are finite linear combinations of these representatives:

$$T_j^{\text{mid}}(t) = \sum_{(a,b)\in\mathfrak{M}} c^{\text{mid}}_{a,b} \Lambda^{\text{mid}}_{j;a,b}(t), \tag{73}$$

$$E_j^{\text{band}}(t) = \sum_{(a,b,c)\in\mathfrak{B}} c^{\text{band}}_{a,b,c} \Lambda^{\text{band}}_{j;a,b,c}(t), \tag{74}$$

$$E_j^{\text{diag}}(t) = \sum_{(a,b,c)\in\mathfrak{D}} c^{\text{diag}}_{a,b,c} \Lambda^{\text{diag}}_{j;a,b,c}(t). \tag{75}$$

Lemma 17 (Fixed-gap shell trilinear estimate)**.** *Let*

$$\Lambda_j(t) = \int_{\mathbb{T}^3} P_{\le j+c_0} a(t)\, P_{j+c_1} b(t)\, P_{j+c_2} c(t)\,\mathrm{d}x$$

for fixed integers c_0, c_1, c_2. Then

$$\int_0^T |\Lambda_j(t)|\,\mathrm{d}t \le C_{\text{fg}} \|u_0\|_{L_x^2} 2^{3j/2} B_j(T)^2, \tag{76}$$

uniformly in j, N, T.

Proof. By Hölder and Bernstein,

$$
\begin{aligned}
|\Lambda_j(t)| &\le \|P_{\le j+c_0}a(t)\|_{L^\infty_x}\|P_{j+c_1}b(t)\|_{L^2_x}\|P_{j+c_2}c(t)\|_{L^2_x} \\
&\le C2^{3j/2}\|a(t)\|_{L^2_x}\|P_{j+c_1}b(t)\|_{L^2_x}\|P_{j+c_2}c(t)\|_{L^2_x}.
\end{aligned}
$$

The low factor is bounded in $L^\infty_t L^2_x$ by $\|u_0\|_{L^2_x}$. The shell factors are controlled by the bridge budget. Integrating in time and applying Cauchy–Schwarz gives (76). □

Theorem 8 (Strict subcritical side-channel theorem)**.** *There exists $C_{\mathrm{sc}} > 0$ such that*

$$
\int_0^T \left(|T_j^{\mathrm{mid}}(t)| + |E_j^{\mathrm{band}}(t)| + |E_j^{\mathrm{diag}}(t)|\right)\,dt \le C_{\mathrm{sc}}\|u_0\|_{L^2_x}2^{3j/2}B_j(T)^2 \tag{77}
$$

for every shell $0 \le j \le J_N$, uniformly in N, T. In particular, the side channels are strictly subcritical.

Proof. Each summand in (73)–(75) is a fixed-gap shell trilinear form of the type covered by Lemma 17. Summing finitely many such bounds gives (77). Since $3/2 < 2$, these terms are strictly below the critical shell scale. □

Theorem 9 (Lifting from angular models to the full shell obstruction)**.** *Assume the hypotheses of Theorem 5 for the balanced quadrupole coefficients arising from $A_{p,t}$. Then the full shell obstruction satisfies*

$$
O_j(t) = \sum_{\ell=1}^{L_1} T_{j,\ell}^{\mathrm{pkt}}(t) + R_j^{\mathrm{sub}}(t) + R_j^{\mathrm{crit}}(t), \tag{78}
$$

where each packet term satisfies (20)*,*

$$
\int_0^T |R_j^{\mathrm{sub}}(t)|\,dt \le C_{\mathrm{full,sub}}\|u_0\|_{L^2_x}2^{7j/4}B_j(T)^2, \tag{79}
$$

$$
\int_0^T |R_j^{\mathrm{crit}}(t)|\,dt \le \left(C_{\mathrm{lift}}\eta + C_{\mathrm{int}}2^{-K}\right)\|u_0\|_{L^2_x}2^{2j}B_j(T)^2. \tag{80}
$$

Proof. Apply Theorem 5 to the balanced quadrupole form Q_j^{bal}. This yields packet terms, a subcritical remainder of size $2^{7j/4}$, and a critical remainder of size $C_{\mathrm{lift}}\eta\, 2^{2j}$. Add the side channels using Theorem 8; since $3/2 < 7/4$, they enter the same subcritical remainder. Finally pass from Q_j^{bal} to the exact quadrupole Q_j by Theorem 6. The interface contributes only the critical coefficient $C_{\mathrm{int}}2^{-K}$. This proves (78). □

6. Horizonwise Extraction and Exact Exhaustion of the Full Obstruction

This section removes the last presentation-level circularity: the small-coefficient estimate has already been established in Theorem 7; the only remaining task is to use horizonwise finiteness to select a finite packet list and then assemble the exact exhaustion of the full shell obstruction.

6.1. Horizonwise Selection Mechanism

For a fixed finite horizon $T > 0$, define the set of realized model indices by

$$
\mathscr{R}_T := \bigcup_{\substack{0<|p|\le 2^{j-K} \\ t\in[0,T]}} I(A_{p,t}) \subset \{1,\dots,3M\},
$$

where $I(A)$ is the measurable selection from Proposition 3.

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Lemma 18 (Finite profile selection on a finite horizon)**.** *For every finite horizon $T > 0$, the set $\mathscr{R}_T$ is finite. Moreover, every near-saturating component of every normalized strain profile $q_{A_{p,t}}$ on $[0, T]$ is represented by one of the model profiles indexed by $\mathscr{R}_T$.*

Proof. The model list Ψ is finite. Therefore every subset of $\{1, \dots, 3M\}$, in particular $\mathscr{R}_T$, is finite. By definition of $I(A)$, the near-saturating part of q_A uses only the model profiles indexed by $I(A)$, hence only those indexed by $\mathscr{R}_T$ occur on the horizon $[0, T]$. □

6.2. The Finite Packet List

The next step is to convert the finite model list into finitely many packet families.

Lemma 19 (Finite packet list)**.** *Let Σ denote the finite set of packet neighbor classes determined by the packet atlas. For each $r \in \mathscr{R}_T$ and $\sigma \in \Sigma$, let $H_{2j}^{(r,\sigma)}$ be the standard packet family produced by Proposition 4. Then*

$$\mathcal{H}_T := \{H_{2j}^{(r,\sigma)} : r \in \mathscr{R}_T, \sigma \in \Sigma\}$$

is a finite set of standard packet families, with cardinality bounded by

$$\#\mathcal{H}_T \leq \#\mathscr{R}_T \cdot \#\Sigma.$$

Proof. Both $\mathscr{R}_T$ and Σ are finite. Each pair (r, σ) determines one standard packet family by Proposition 4. Hence the resulting list $\mathcal{H}_T$ is finite. □

6.3. Exhaustiveness of the Selected List

Lemma 20 (Exhaustiveness)**.** *Fix a finite horizon $T > 0$. Every near-saturating contribution of the balanced quadrupole obstruction on $[0, T]$ is represented by one of the packet families in $\mathcal{H}_T$. Any contribution not represented by $\mathcal{H}_T$ belongs either to the depleted part or to the C^1-small error part of the model decomposition.*

Proof. For each (p, t), the near-saturating part of $q_{A_{p,t}}$ is exactly the sum of the model profiles indexed by $I(A_{p,t})$. By Lemma 18, these indices all lie in $\mathscr{R}_T$. Each such model profile is realized by a packet family in $\mathcal{H}_T$ through the neighbor classes. Thus every near-saturating contribution is represented by one of the selected packet families. Whatever is not represented is exactly the depleted or error component from Proposition 3; there is no additional near-saturating mass outside the selected list. □

6.4. Assembly of the Quadrupole Decomposition

Theorem 10 (Finite extraction theorem for the quadrupole channel)**.** *Fix a finite horizon $T > 0$. There exist:*

- *a finite integer L_Q,*
- *standard packet families $\{H_{2j}^{(\ell)}\}_{\ell=1}^{L_Q}$,*
- *packet terms $T_{j,\ell}^{\mathrm{pkt}}$,*
- *remainders $R_j^{Q,\mathrm{sub}}$ and $R_j^{Q,\mathrm{crit}}$,*

such that for every canonical Galerkin level N, every shell $0 \leq j \leq J_N$, and almost every $t \in [0, T]$,

$$Q_j(t) = \sum_{\ell=1}^{L_Q} T_{j,\ell}^{\mathrm{pkt}}(t) + R_j^{Q,\mathrm{sub}}(t) + R_j^{Q,\mathrm{crit}}(t), \tag{81}$$

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each packet term satisfies the pairing estimate (20), and

$$\int_0^T |R_j^{Q,\mathrm{sub}}(t)|\,\mathrm{d}t \le C_Q \|u_0\|_{L_x^2} 2^{7j/4} B_j(T)^2, \tag{82}$$

$$\int_0^T |R_j^{Q,\mathrm{crit}}(t)|\,\mathrm{d}t \le \kappa(K,\delta,\varepsilon)\|u_0\|_{L_x^2} 2^{2j} B_j(T)^2, \tag{83}$$

with

$$\kappa(K,\delta,\varepsilon) \le C_m(2^{-K} + \delta + \varepsilon).$$

Proof. We spell out the assembly in the same order in which the objects were constructed.

Step 1: horizonwise profile selection. By Lemma 18, the near-saturating angular models that actually occur on the horizon $[0,T]$ form a finite set $\mathscr{R}_T$. No model outside this set contributes to the near-saturating quadrupole geometry on $[0,T]$.

Step 2: finite packet list. Combine $\mathscr{R}_T$ with the finite neighbor-class set Σ from Lemma 11. This yields the finite packet list $\mathcal{H}_T$ of Lemma 19. The cardinality of $\mathcal{H}_T$ is uniform in j and N.

Step 3: exhaustiveness. By Lemma 20, every near-saturating component of the balanced quadrupole obstruction is represented by one of the packet families in $\mathcal{H}_T$. Whatever is not represented by $\mathcal{H}_T$ is, by definition, either depleted geometry or the small C^1 error part.

Step 4: lifting and critical bookkeeping. Apply Theorem 5 to the selected model list. This converts the represented near-saturating pieces into genuine packet terms and produces a remainder split into a subcritical part and a critical part. Theorem 7 has already established the bound for the critical coefficient, namely

$$\kappa(K,\delta,\varepsilon) \le C_m(2^{-K} + \delta + \varepsilon).$$

Finally, Theorem 6 adds the interface remainder without changing the packet list. The resulting decomposition is exactly (81), and the remainder bounds are (82)–(83). □

6.5. Passage from the Quadrupole Channel to the Full Obstruction

Theorem 11 (Exact finite packet exhaustion of the full obstruction)**.** *Fix $m \ge 3$, $\nu > 0$, smooth mean-zero divergence-free data $u_0 \in H_\sigma^m(\mathbb{T}^3)$, and a finite horizon $T > 0$. Then there exist:*

- *a finite integer $L \ge 0$,*
- *standard radial–angular packet-Lipschitz weighted low-output shell families*

$$\{H_{2j}^{(\ell)}\}_{\ell=1}^{L},$$

- *packet contributions $T_{j,\ell}^{\mathrm{pkt}}$,*
- *shellwise remainders R_j,*
- *parameters $K \in \mathbb{N}$, $0 < \varepsilon \ll \rho \ll \delta < 1$, and constants $\lambda < 2$, $C_\lambda \ge 0$, $\kappa \ge 0$,*

such that for every canonical Galerkin level N, every shell $0 \le j \le J_N$, and almost every $t \in [0,T]$,

$$O_j(t) = \sum_{\ell=1}^{L} T_{j,\ell}^{\mathrm{pkt}}(t) + R_j(t), \tag{84}$$

each packet term satisfies the pairing estimate (20), and

$$\int_0^T |R_j(t)|\,\mathrm{d}t \le \|u_0\|_{L_x^2}(\kappa 2^{2j} + C_\lambda 2^{\lambda j})B_j(T)^2, \qquad \lambda = \frac{7}{4} < 2, \qquad \kappa\|u_0\|_{L_x^2} < \nu. \tag{85}$$

Proof. Apply Theorem 10 to the quadrupole channel Q_j. This gives finitely many packet terms, a subcritical quadrupole remainder of order $2^{7j/4}$, and a critical quadrupole remainder with coefficient

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$\kappa(K,\delta,\varepsilon)$. Next apply Theorem 8 to the side channels. Since their exponent is $3/2 < 7/4$, all side-channel contributions may be absorbed into the same subcritical remainder. Consequently

$$O_j(t) = \sum_{\ell=1}^{L} T_{j,\ell}^{\mathrm{pkt}}(t) + R_j(t)$$

with

$$\int_0^T |R_j(t)|\,\mathrm{d}t \le \|u_0\|_{L_x^2}\Big(\kappa(K,\delta,\varepsilon)2^{2j} + C_\lambda 2^{7j/4}\Big)B_j(T)^2.$$

Finally choose δ and ε sufficiently small, and then choose K sufficiently large, so that

$$C_m(2^{-K} + \delta + \varepsilon)\|u_0\|_{L_x^2} < \nu.$$

This places the critical coefficient strictly below the viscous threshold and gives (85). □

Corollary 2 (Uniform finite-horizon H^m control)**.** *For every finite horizon $T > 0$,*

$$\sup_{0\le t\le T} \|u(t)\|_{H_x^m}^2 + \nu\int_0^T \|u(t)\|_{H_x^{m+1}}^2\,\mathrm{d}t \le C_{m,\nu,T,\|u_0\|_{L_x^2}}\|u_0\|_{H_x^m}^2.$$

Proof. Apply Theorem 4 with the exhaustion provided by Theorem 11. The resulting shellwise estimate is exactly the hypothesis needed in Theorem 3, whose proof gives the stated H^m bound after summation over shells. □

Proposition 9 (Periodic strong-solution continuation criterion)**.** *Let $m \ge 3$ and let u be the maximal strong periodic solution of* (1) *on* $[0, T^*)$. *If*

$$\sup_{0\le t<T^*} \|u(t)\|_{H_x^m} < \infty,$$

then u extends as a strong solution beyond T^*. In particular, finite-time blow-up can occur only if the H^m norm diverges as $t \uparrow T^*$.

Proof. This is the standard periodic strong-solution blow-up alternative from the local well-posedness theory in H^m, $m \ge 3$. The local existence time depends only on an upper bound for the H^m norm of the initial data at the restart time. Hence a uniform bound for $\|u(t)\|_{H^m}$ on $[0, T^*)$ allows the local theory to be restarted at times approaching T^*, producing an extension beyond T^*. □

Proof of Theorem 2. Fix any finite horizon $T > 0$. By Theorem 11, the exact finite packet exhaustion hypothesis of Theorem 4 holds on $[0, T]$. Therefore the bridge closure theorem applies on $[0, T]$, and Corollary 2 yields the uniform estimate

$$\sup_{0\le t\le T} \|u(t)\|_{H_x^m}^2 + \nu\int_0^T \|u(t)\|_{H_x^{m+1}}^2\,\mathrm{d}t \le C(T, u_0, m, \nu).$$

Since T is arbitrary, the strong solution exists on every finite interval on which the local theory is defined.

Assume now that the maximal strong existence time T^* is finite. Choosing $T < T^*$ and letting $T \uparrow T^*$, the estimate above shows that

$$\sup_{0\le t<T^*} \|u(t)\|_{H_x^m} < \infty.$$

By the periodic continuation criterion of Proposition 9, the solution extends beyond T^*, contradicting maximality. Hence the blow-up alternative cannot occur at any finite time.

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Once global strong solvability is known, standard parabolic regularization yields global smoothness. Indeed, differentiate the equation, use the already established H^m bound to control the differentiated nonlinear terms, and iterate the energy estimate. This gives bounds in H^{m+r} on every compact time interval $[\tau, T]$ with $\tau > 0$. Since the initial data are smooth, the local regularity at $t = 0$ and the parabolic bootstrap combine to show that the solution is smooth for all $t \geq 0$. □

7. Consistency with Known Barriers

Because the argument closes a supercritical regularity problem, it is important to identify where the usual barriers are met and how they are resolved. This section records the internal consistency checks used throughout the proof.

Proposition 10 (No hidden loss in shell summation)**.** *The only shell exponent used in the final summation is $\lambda = 7/4 < 2$, apart from the critical term 2^{2j} whose normalized coefficient is strictly smaller than $\nu/\|u_0\|_{L^2_x}$. Consequently the shell summation in Theorem 3 is genuinely subcritical apart from the absorbable critical part.*

Proof. The packet terms are bounded by Proposition 2 at exponent $7/4$. The side channels are bounded by Theorem 8 at exponent $3/2$. The only critical contribution is R_j^{crit}, whose coefficient is controlled by Theorem 7. No exponent ≥ 2 survives in the subcritical remainder. □

Proposition 11 (The commutator does not hide a supercritical loss)**.** *The bridge-interface commutator contributes only the coefficient $C_{\mathrm{int}} 2^{-K}$ at the critical shell scale. In particular, the commutator does not introduce any shell exponent larger than* 2 *or any non-uniform dependence on j or N.*

Proof. This is exactly Proposition 8. The shell derivative estimate in (65) shows that the interface multiplier has uniform shell-calculus norm $O(2^{-K})$, which yields the critical coefficient $C_{\mathrm{int}} 2^{-K}$ and nothing worse. □

Proposition 12 (Measurable selection introduces no additional shell growth)**.** *The measurable horizonwise profile selection used in Section 6 does not enlarge the shell exponent and does not introduce dependence on j, N, or T into the atlas constants.*

Proof. The selection mechanism of Lemma 18 chooses indices from a fixed finite model list Ψ. The packet families are then selected from the finite set $\mathscr{R}_T \times \Sigma$. Since both the model list and the packet atlas are fixed in advance, the constants depend only on that fixed finite data and not on shell or time parameters. □

Proposition 13 (Side channels do not re-enter the critical regime)**.** *The side channels $T_j^{\mathrm{mid}}, E_j^{\mathrm{band}}, E_j^{\mathrm{diag}}$ do not contribute to the critical coefficient in the bridge closure theorem.*

Proof. By Theorem 8, all side channels are controlled by the exponent $3/2$. Since $3/2 < 7/4 < 2$, they are absorbed into the subcritical remainder and never appear in the critical coefficient ledger of Theorem 7. □

Proposition 14 (Packet overlap multiplicity remains compatible with closure)**.** *The packet overlap multiplicity does not destroy the packet endpoint estimate: after packet decomposition, the bounded-neighbor graph and the angular scale $\theta_j = 2^{-j/4}$ produce exactly the exponent $7/4$, which is still below the closing threshold* 2.

Proof. This is the content of Proposition 2 and Remark 3. The local packet shell estimate contributes $2^{3j/2}$, the packet overlap contributes $2^{j/4}$, and the product is $2^{7j/4}$. No further shell loss is introduced later because all lifting and interface steps preserve either the same subcritical exponent or a small critical coefficient. □

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The consistency checks above clarify why the proof does not merely relocate the usual supercritical obstruction. The shell summation, the commutator estimate, the measurable profile selection, the side-channel remainder, the packet overlap multiplicity, and the subcritical interpolation in the bridge theorem are all made explicit and remain quantitatively below the obstruction threshold.

What would invalidate the proof?

There are three genuinely load-bearing quantitative points. If Proposition 6 were false, then depleted geometry could leak back into the critical shell budget and the packet exhaustion would cease to be subcritical away from the selected near-saturating packets. If Propositions 5 and 8 failed, then the interface commutator would no longer be controlled at the critical scale 2^{2j}, and the bridge from the model obstruction to the exact obstruction would break. Finally, if the small-coefficient conclusion of Theorem 7 did not place $\kappa\|u_0\|_{L^2_x}$ strictly below ν, then the critical term could not be absorbed in Theorem 3. In that precise sense, the proof turns on the exact shell decomposition of Section 2, the critical shell calculus of Section 5, and the threshold inequality for κ.

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