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Not peer-reviewed version

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Posted Date: 3 March 2026

doi: [10.20944/preprints202603.0254.v1](https://doi.org/10.20944/preprints202603.0254.v1)

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Article

Generic Singularities in Elliptic Refractive Systems

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Abstract

In this work, we present a generic classification of a class of n -dimensional piecewise smooth vector fields known as refractive systems. Our purpose is to characterize the local structural stability of refractive fold-folds of elliptic type. To this end, we reduce the study of such piecewise smooth vector fields to the analysis of the first return maps and their structural stability. Normal forms for such systems are also provided.

Keywords: piecewise smooth vector fields; structural stability; singularities

MSC: 34A36; 34D30; 37C10; 37G05

1. Introduction

The theory of dynamical systems is a powerful tool for the study of systems whose state changes over time. Today, this theory is well developed for smooth systems. However, several problems are modeled by systems that exhibit discontinuous behavior, such as systems that involve friction or impact. Examples of such systems arise in control theory [1], biology [2] and various other areas.

In this text, we work with local properties and therefore we use the notion of germs of mappings and vector fields. Let p be a point and let A and B be two neighborhoods of p (in some global topological space). We say that $f : A \rightarrow X$ and $g : B \rightarrow X$ are *germ-equivalent* at p if there exists a neighborhood $C \subset A \cap B$ of p such that f and g coincide on C . As the name suggests, germ-equivalence is an equivalence relation. A *germ* of a function at p is its equivalence class in the germ-equivalence. The germ of f at p is denoted by f, p or, in the extensive notation, $f : A, p \rightarrow X$. If we require that $f(p) = q$, we denote $f : A, p \rightarrow X, q$. A *representative* of a germ is a (global) function that belongs to this equivalence class.

Throughout this text, unless stated otherwise, we do not make distinction between functions and germs of functions. Germs of functions and germs of vector fields will be referred simply by functions and vector fields, respectively, if there is no risk for ambiguity.

Let $\alpha : \mathbb{R}^{n+1}, 0 \rightarrow \mathbb{R}, 0$ be the function defined by $\alpha(x_1, \dots, x_{n+1}) = x_{n+1}$. We define the set

$$\Sigma = \{p \in \mathbb{R}^{n+1} ; \alpha(p) = 0\}$$

which separates the space $\mathbb{R}^{n+1}$ in the two open regions

$$\Sigma^+ = \{p \in \mathbb{R}^{n+1} ; \alpha(p) > 0\} \quad \text{and} \quad \Sigma^- = \{p \in \mathbb{R}^{n+1} ; \alpha(p) < 0\}.$$

Let X and Y be two smooth (class C^r , r large enough) vector fields around the origin. We define a *piecewise smooth vector field* $Z = (X, Y)$ by

$$Z(p) = \begin{cases} X(p) & \text{if } p \in \Sigma^+, \\ Y(p) & \text{if } p \in \Sigma^-. \end{cases}$$

The set Σ is called *switching boundary* or simply *boundary*.

Let $U \subset \mathbb{R}^{n+1}$ be an open set with compact closure containing the origin. Denote by $\mathfrak{X}(U)$, or simply $\mathfrak{X}$, the set of smooth vector fields in U . Endow $\mathfrak{X}$ with the C^r topology. We consider the space of piecewise smooth vector fields defined in U being $\mathfrak{X}(U) \times \mathfrak{X}(U)$ with the product topology.

The points $p \in \Sigma$ where $X\alpha(p) = 0$ are called *tangency points* (or Σ -singularities) of X . Here $X\alpha(p) = X(p) \cdot \nabla\alpha(p)$ denotes the Lie derivative of α relative to X at p .

A point $p \in \Sigma$ that satisfies $X\alpha(p) = 0$ and $X^2\alpha(p) \neq 0$ is called a *fold point*. If $X^2\alpha(p) > 0$, we say that p is a *visible fold* while if $X^2\alpha(p) < 0$, p is said to be an *invisible fold* of X . If $X\alpha(p) = X^2\alpha(p) = 0$ and $X^3\alpha(p) \neq 0$, we say that p is a *cusp point* of X .

In an analogous way, we define fold and cusp points of Y . However, the inequalities of visibility are reversed: $Y^2\alpha(p) < 0$ for visible folds and $Y^2\alpha(p) > 0$ for invisible folds.

A point $p \in \Sigma$ is called a *fold-fold point* if it is a fold of both vector fields X and Y . A fold-fold point p is called *hyperbolic* if it is a visible fold of both fields, *elliptic* if it is an invisible fold of both fields, and *parabolic* if it is a visible fold of one field and an invisible fold of the other (Figure 1).

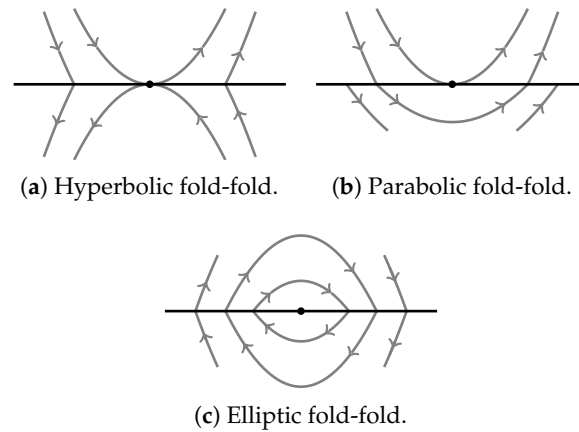


Figure 1. Examples of fold-folds.

Definition 1. A refractive system is a piecewise smooth vector field $Z = (X, Y)$ satisfying $X\alpha(p) = Y\alpha(p)$ for all $p \in \Sigma$.

In a refractive system, there are no sliding or escaping regions, only crossing regions and tangency points. Also, if a point in the boundary is a tangency point of one of the fields, then it must be a tangency point of the other.

A local trajectory of $X \in \mathfrak{X}$ through a point $p \in U$ is a curve $\gamma : (-\varepsilon, \varepsilon) \rightarrow U$ that satisfies $\gamma(0) = p$ and $\gamma'(t) = X(\gamma(t))$. A local orbit of a point is the image of a local trajectory of this point. The orientation of an orbit is the orientation induced by the trajectory.

For refractive systems, we define local trajectories as follows.

Definition 2. Let $Z = (X, Y)$ be a refractive vector field defined in $U \subset \mathbb{R}^{n+1}$. A local trajectory of Z through a point $p \in U$ is a curve $\gamma : (-\varepsilon, \varepsilon) \rightarrow U$ satisfying

- (i) If $p \in \Sigma^+$ (respect. Σ^-) then $\gamma(t)$ is the usual trajectory of X (respect. Y) through the point p in Σ^+ (respect. Σ^-).
- (ii) If $p \in \Sigma$ is a crossing point¹ then, in the case $X\alpha(p) = Y\alpha(p) > 0$, we define $\gamma(t) = \varphi_Y(t)$ for $t < 0$ and $\gamma(t) = \varphi_X(t)$ for $t > 0$, where φ_X and φ_Y are the usual trajectories of the fields X and Y through p . An analogous definition can be done when $X\alpha(p) = Y\alpha(p) < 0$.

¹ A point $p \in \Sigma$ is said a crossing point of $Z = (X, Y)$ if $X\alpha(p)Y\alpha(p) > 0$. For refractive systems, this is equivalent to $X\alpha(p) \neq 0$.

(iii) For tangency points, if it is possible to extend the definitions of the trajectories at points of the boundary near p , then the local trajectory is defined as being this extension. If it is not possible to extend the definition, we define the local trajectory of p as being the constant curve $\gamma(t) = p, \forall t$.

Let Z_1 and Z_2 be two refractive vector fields defined in the open sets U_1 and U_2 , respectively, both sets containing the origin. We say that Z_1 and Z_2 are Σ -equivalent if there exists a homeomorphism $h : U_1 \rightarrow U_2$ that preserves the boundaries (i.e., $h(U_1 \cap \Sigma) = U_2 \cap \Sigma$) and carries orbits of Z_1 to orbits of Z_2 preserving the orientation of the orbits.

We say that a refractive vector field Z is (globally) *structurally stable* if there exists a neighborhood $B \subset \mathfrak{X} \times \mathfrak{X}$ of Z such that every refractive system in B is Σ -equivalent to Z .

The definition of local Σ -equivalence is analogous to the definition above, just considering h as a germ of homeomorphism.

A germ of a refractive vector field is said to be *structurally stable* if it admits a structurally stable representative.

It is worth to say that the definition of structural stability above is relative to the space of refractive vector fields. Some fold-folds are structurally stable in the space of refractive systems (as we will see later) whereas fold-folds are never structurally stable in the context of general vector fields.

For definitions on general theory of piecewise smooth dynamical systems (Filippov systems) we refer the reader to the texts [3,4], for example, and references therein.

The main motivation for the study of refractive systems comes from the work of Ivar Ekeland [5] about Variational Calculus published in 1977. In this work, the author establishes results on existence of solutions for a one-dimensional variational problem and presents a local classification of what are now called refractive systems in the plane. In 1984, Fopke Klok [6] studied the same problem for two-dimensional systems, adding a homogeneity hypothesis to the system.

Another example of an application of refractive systems arises when we consider a discontinuous ODE of order $k > 1$

$$x^{(k)} = f^\pm(x)$$

where $x = (x_1, \dots, x_n)$ and

$$f^\pm(x) = \begin{cases} f^+(x) & \text{if } x_n > 0 \\ f^-(x) & \text{if } x_n < 0 \end{cases}$$

for f^+ and f^- smooth functions.

When we introduce the variables $x_{i,j} = x_i^{(j)}$ for $1 \leq i \leq n$ and $0 \leq j \leq k-1$, the resulting system is a first-order refractive ODE. This type of system appears, for instance, when we work with forces ($k=2$) including friction in the system. Another example where this type of construction appears is in the modeling of blocks over surfaces in motion. This model is used in the study of earthquakes. See [7–9] for more details.

A recent study that uses the construction above was done by Jacquemard, Pereira and Teixeira [10] in the so-called *relay systems*. These systems can be described by the equation

$$y^{(n)} = \beta(x) + \operatorname{sgn}(y)\alpha(x)$$

where $\alpha, \beta : U \subset \mathbb{R}^n \rightarrow \mathbb{R}$ are smooth functions and $x = (x_1, \dots, x_n) = (y, \dot{y}, \dots, y^{(n-1)})$.

This paper is organized as follows. In Section 2, we describe the main results and the methodology used in this paper. In Section 3, we prove that the structural stability of the fields can be reduced to the structural stability of the first return maps. In Section 4, we find the series expansion of the first return maps up to the second-order terms. In Sections 5, 6, 7 and 8, we prove the main results of this text.

This work presents the results obtained in Siller’s doctoral thesis [11], where the proofs of the main results and the methodology of this paper are developed.

2. Main Results

Throughout the next sections, $[x]_j$ denotes the j -th coordinate of x , that is, if $x = (x_1, \dots, x_n)$ then $[x]_j = x_j$. $\mathbb{R}_+^n$ and $\mathbb{R}_-^n$ denote the sets $\{x \in \mathbb{R}^n; x_n > 0\}$ and $\{x \in \mathbb{R}^n; x_n < 0\}$ respectively.

In this text, we are concerned with local structural stability of n -dimensional refractive fold-fold systems around the origin.

Denote by $\mathcal{R}_f$ the space of germs of fold-fold refractive vector fields at $0 \in \mathbb{R}^{n+1}$. Every hyperbolic refractive fold-fold has the same topological behavior around the origin. It is not difficult to find a local Σ -equivalence between any two hyperbolic refractive fold-folds. The same occurs with the parabolic ones. Therefore, the hyperbolic and parabolic refractive fold-folds are structurally stable on $\mathcal{R}_f$.

On the other hand, the behavior of the elliptic refractive fold-folds is more complicated and their classification is not trivial. The main purpose of this work is to characterize the structural stability of these systems. To this end, it is natural to reduce the study of the systems to the analysis of their first return maps.

Let $\mathcal{R}_{ell} \subset \mathcal{R}_f$ be the space of germs of elliptic refractive fold-fold systems at $0 \in \mathbb{R}^{n+1}$. When we take a system $Z = (X, Y) \in \mathcal{R}_{ell}$ we can assume, without loss of generality, that it satisfies:

- (P1) Z is a germ of a system around the origin;
- (P2) $\alpha(x) = x_{n+1}$ and therefore the boundary is $\Sigma = \mathbb{R}^n \times 0$;
- (P3) The set of tangency points of Z is $\mathbb{R}^{n-1} \times 0 \subset \Sigma$;
- (P4) The field Y is $(0, \dots, 0, 1, x_n)$;
- (P5) The last coordinate of the field X is $X_{n+1}(x) = x_n$.

Below we briefly discuss these properties.

The first two properties are already considered since the beginning of this text. For the property (P3), the set of tangency points of Z is a codimension-one submanifold of Σ and, by a smooth change of variables, it can be assumed to be $\mathbb{R}^{n-1} \times 0$. The property (P4) is known as *Vishik Normal Form* of Y . See [12,13] for more details.

The Property (P5) is non-trivial. Since Z is a refractive system, we have that

$$X_{n+1}(p) = Y_{n+1}(p) = x_n, \quad \forall p \in \Sigma.$$

However, it may fail to hold for points outside the boundary. As the origin is a fold point of X , the $(n+1)$ -th coordinate of X satisfies $X_{n+1}(0) = 0$ and

$$X^2 \alpha(x) = \sum_i^{n+1} X_i(x) \cdot \frac{\partial X_{n+1}}{\partial x_i}(x).$$

Moreover, from (P3), $\partial X_{n+1} / \partial x_i(0) = 0$ for $i < n$. So

$$X_n(0) \cdot \partial X_{n+1} / \partial x_n(0) = X^2 \alpha(0) \neq 0$$

and therefore $\partial X_{n+1} / \partial x_n(0) \neq 0$. By the Implicit Function Theorem, there exists a smooth function $f : \mathbb{R}^n, 0 \rightarrow \mathbb{R}, 0$ such that

$$X_{n+1}(x_1, \dots, x_{n-1}, f(x_1, \dots, x_{n-1}, x_{n+1}), x_{n+1}) = 0.$$

So, the set of points x near the origin where $X_{n+1}(x) = 0$ is the codimension-one submanifold

$$M = \{(x_1, \dots, x_{n-1}, f(x_1, \dots, x_{n-1}, x_{n+1}), x_{n+1})\}.$$

After the change of variables

$$x \mapsto (x_1, \dots, x_{n-1}, x_n - f(x_1, \dots, x_{n-1}, x_{n+1}), x_{n+1})$$

for the points where $x_{n+1} > 0$, the manifold M becomes

$$M = \{(x_1, \dots, x_{n-1}, 0, x_{n+1})\}.$$

Then the $(n+1)$ -th coordinate of X can be written as $X_{n+1}(x) = x_n \cdot g(x)$ for some smooth and positive function $g : \mathbb{R}^{n+1}, 0 \rightarrow \mathbb{R}$.

Define the positive scalar function $s : \mathbb{R}^{n+1}, 0 \rightarrow \mathbb{R}$ by

$$s(x) := \frac{1}{g(x)}$$

for $x_{n+1} \geq 0$ and $s(x) = 1$ otherwise. Since $g(x) = 1$ in the boundary and it is piecewise smooth, the function s is clearly continuous and positive near origin.

Multiplying the field X by s , we have that $X_{n+1}(x)$ becomes x_n for all x near the origin and therefore we have Property (P4).

Since X is a fold at the origin, the trajectory of each point $x \in \Sigma$ (unless the tangency points) hits the boundary at another point $\varphi_X(x)$ (Figure 2). For the tangency points, we define $\varphi_X(x) = x$. The function $\varphi_X : \Sigma, 0 \rightarrow \Sigma, 0$ is a map associated to X that satisfies

- φ_X is an involution, that is, $\varphi_X \circ \varphi_X = Id$;
- φ_X is smooth. It can be seen applying the change of coordinates given by Vishik in [13], noticing that in these coordinates the involution φ_X becomes the smooth function

$$(x_1, \dots, x_n) \mapsto (x_1, \dots, x_{n-1}, -x_n)$$

and then returning to the original coordinates.

- $\text{Fix}(\varphi_X) = \mathbb{R}^{n-1} \times 0 \subset \Sigma$, where $\text{Fix}(\varphi_X)$ denotes the set of fixed points of φ_X .

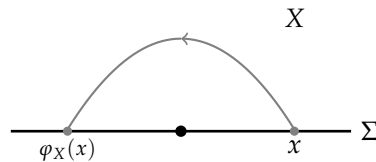


Figure 2. Involution φ_X associated to X . The trajectory of x hits Σ again in the point $\varphi_X(x)$.

In an analogous way, we can define the involution φ_Y associated to Y . From (P4), φ_Y is the involution

$$\varphi_Y(x_1, \dots, x_n) = (x_1, \dots, x_{n-1}, -x_n).$$

Denote $S := \text{Fix}(\varphi_X) = \mathbb{R}^{n-1} \times 0 \subset \Sigma$ the set of fixed points of φ_X . Let

$$S^+ := \{x \in \Sigma; x_n \geq 0\} \quad \text{and} \quad S^- := \{x \in \Sigma; x_n \leq 0\}.$$

The first return map of Z is the application $\varphi : S^+, 0 \rightarrow S^+, 0$ given by

$$\varphi := \varphi_Y \circ \varphi_X.$$

Remark 1. For the region S^- , the first return map is defined as $\varphi_X \circ \varphi_Y$. However, such map is conjugate to $\varphi_Y \circ \varphi_X$ defined on S^+ and therefore they are topologically equivalent. Hence we can restrict the domain of the first return map to the set S^+ .

Since $\text{Fix}(\varphi) \supset S$, the map φ can be written

$$\varphi(x) = x + x_n \cdot F(x)$$

where $F : S^+ \rightarrow \mathbb{R}^n$ is a C^{r-1} function. In Section 4 we study the series expansion of φ .

So, each $Z = (X, Y) \in \mathcal{R}_{ell}$ can be associated to a smooth function $F : S^+ \rightarrow \mathbb{R}^n$. The main result is described in terms of such mappings.

Theorem 1. *The field $Z = (X, Y) \in \mathcal{R}_{ell}$ is structurally stable on $\mathcal{R}_{ell}$ if, and only if, either*

- (a) *(Regular case) $F(0) \neq 0$, or*
- (b) *(Singular case) $F(0) = 0$ and F is hyperbolic at 0.*

Here, the hyperbolicity of F is understood in the sense of vector fields, that is, all the eigenvalues of its derivative have nonzero real parts. For this reason, from now on, F is regarded as a vector field on S^+ .

The proof of this theorem is divided in the following three lemmas.

Lemma 1 (Regular case). *If $F(0) \neq 0$ then Z is structurally stable on $\mathcal{R}_{ell}$.*

Lemma 2 (Singular case). *If $F(0) = 0$ and F hyperbolic at 0, then Z is structurally stable on $\mathcal{R}_{ell}$.*

Lemma 3 (Converse). *If $F(0) = 0$ and F is non-hyperbolic at 0, then Z is not structurally stable on $\mathcal{R}_{ell}$.*

The normal forms are expressed in the following result.

Theorem 2. [Normal Forms] *A field $Z \in \mathcal{R}_{ell}$ is structurally stable if, and only if, it is Σ -equivalent to one of the following normal forms*

- (a) *Regular case: $Z_0 = (X_0, Y_0) = ((a, 0, \dots, 0, -1, x_n)^T, Y_0)$ with $a \neq 0$;*
- (b) *Singular case: $Z_0 = (X_0, Y_0) = ((0, \dots, 0, -1, x_n)^T + Bx, Y_0)$ for some*

$$B = \begin{pmatrix} B_0 & & \\ & d & \\ & & 0 \end{pmatrix}$$

with B_0 hyperbolic and $d \neq 0$.

Where $Y_0 = (0, \dots, 0, 1, x_n)^T$.

The particular case of Theorems 1 and 2 for dimension 3 was already proved by Buzzi, Medrado and Teixeira in [14]. The authors reduce the study of elliptic refractive fold-folds to the analysis of the associated first return maps. For the regular case, the authors have proved that the first return map is conjugate to its linear part and therefore it is structurally stable. The main tool used in the proof is a result in [15] which states that every planar diffeomorphism with linear part $(x + y, y)$ is conjugate to this linear part. However, such result actually does not hold in general. A counterexample is given by the map $f(x, y) = (x + y, y + xy + y^2/2)$ which has the curve $(t, t^2/2)$ as an invariant set. The orbits of the points on this curve converge (in positive or negative time) to the origin, whereas the orbits of its linear part are parallel to the x -axis.

Here we use the same methodology as in [14]. For the regular case, we prove that the first return map is conjugate to its linear part. We introduce an additional hypothesis that allows us to construct a conjugacy between the first return map and its linear part. For the singular case, we prove that the first return maps are conjugate to their perturbations. The proofs of Lemmas 1 and 2 are given in Sections 5 and 6, respectively.

For Lemma 3 and Theorem 2, we explicitly compute the first terms of the series expansion of the first return maps in function of the series expansion of the fields. The proof of Lemma 3 is given in Section 7 and the proof of Theorem 2 is given in Section 8.

3. First Return Maps

In this section, we establish the equivalence between the structural stability of an elliptic refractive system and of its first return map.

For each $Z = (X, Y) \in \mathcal{R}_{ell}$, denote by $j(Z) = \varphi_Y \circ \varphi_X$ its first return map. Since the trajectories depend continuously on the vector fields, the application j is continuous.

Let $\mathcal{I}$ be the space of germs of involutions φ on $\mathbb{R}^n, 0$ such that $\text{Fix}(\varphi) = S = \mathbb{R}^{n-1} \times 0$. Denote by $\mathcal{J}$ the space of compositions

$$\mathcal{J} = \{f \circ g; f, g \in \mathcal{I}\}.$$

As observed in the previous section, we assume that the functions in $\mathcal{J}$ are restricted to the region S^+ .

To prove the main result of this section, we use the lemma below. The proof of this lemma is similar to that given in [14], adapted to arbitrary dimension.

Lemma 4. *Let $Z_1, Z_2 \in \mathcal{R}_{ell}$. The systems Z_1 and Z_2 are Σ -equivalent at the origin if and only if $j(Z_1)$ and $j(Z_2)$ are conjugate.*

Proof. Suppose $Z_1 = (X_1, Y_1)$ and $Z_2 = (X_2, Y_2)$ are Σ -equivalent and let h be a Σ -equivalence between these fields. The boundary Σ is invariant by h . We claim that h restricted to Σ conjugate $j(Z_1)$ and $j(Z_2)$.

Indeed, let $p \in S^+$. The orbit of X_1 leaving p intersects Σ again at the point $\varphi_{X_1}(p)$. As h is an equivalence between X_1 and X_2 , the orbit of X_2 through $h(p)$ intersects Σ at $h(\varphi_{X_1}(p))$. On the other hand, by the definition of φ_{X_2} , the orbit of X_2 through $h(p)$ intersects Σ at $\varphi_{X_2}(h(p))$. It follows that $h \circ \varphi_{X_1} = \varphi_{X_2} \circ h$. The same argument applied to Y_1 and Y_2 shows that $h \circ \varphi_{Y_1} = \varphi_{Y_2} \circ h$. Therefore

$$h \circ j(Z_1) = h \circ \varphi_{Y_1} \circ \varphi_{X_1} = \varphi_{Y_2} \circ h \circ \varphi_{X_1} = \varphi_{Y_2} \circ \varphi_{X_2} \circ h = j(Z_2) \circ h$$

and $j(Z_1)$ and $j(Z_2)$ are conjugate.

Conversely, assume that some homeomorphism $\tilde{h} : S^+ \rightarrow S^+$ conjugates $j(Z_1)$ and $j(Z_2)$. Denote by $\Phi_i(t, x)$ the flow of X_i , for $i = 1, 2$. For each point $x \in \overline{\mathbb{R}_+^{n+1}}$ near the origin, define $\pi(x) \in S^+$ and $\tau(x) \geq 0$ such that

$$\Phi_1(\tau(x), \pi(x)) = x.$$

Let $\tau_1, \tau_2 : S^+ \rightarrow \mathbb{R}$ be the functions satisfying

$$\Phi_i(\tau_i(p), p) = \varphi_{X_i}(p), \quad \text{for } i = 1, 2,$$

that is, $\tau_i(p)$ is the time that the trajectory of $p \in S^+$ takes to reach again Σ by the field X_i . Define

$$H(x) = \Phi_2\left(\frac{\tau(x)}{\tau_1(\pi(x))} \cdot \tau_2(\tilde{h}(\pi(x))), \tilde{h}(\pi(x))\right).$$

For $x \in S$, set $H(x) = \tilde{h}(x)$. This function is clearly continuous.

Now define analogous functions $\tilde{\pi} : \overline{\mathbb{R}_+^{n+1}} \rightarrow S^+$ and $\tilde{\tau} : \overline{\mathbb{R}_+^{n+1}} \rightarrow \mathbb{R}$ for the field X_2 , that is, satisfying

$$\Phi_2(\tilde{\tau}(x), \tilde{\pi}(x)) = x.$$

The function $\tilde{H}$ defined by

$$\tilde{H}(x) = \Phi_1\left(\frac{\tilde{\tau}(x)}{\tau_2(\tilde{\pi}(x))} \cdot \tau_1(\tilde{h}^{-1}(\tilde{\pi}(x))), \tilde{h}^{-1}(\tilde{\pi}(x))\right).$$

is continuous and it is easy to show that $\tilde{H}$ is the inverse of the function H around the origin. So H is a local homeomorphism over $\overline{\mathbb{R}_+^{n+1}}$ and is a Σ -equivalence between the fields X_1 and X_2 .

In the same way, we define $H : \overline{\mathbb{R}_-^{n+1}} \rightarrow \overline{\mathbb{R}_-^{n+1}}$ obtaining a function H defined in a neighborhood of the origin. The function H coincides with $\tilde{h}$ when restricted to S^+ and with $\varphi_{X_2} \circ \tilde{h} \circ \varphi_{X_1} = \varphi_{Y_2} \circ \tilde{h} \circ \varphi_{Y_1}$ when restricted to S^- . Hence H is continuous and, applying the same argument to $\tilde{H} = H^{-1}$, we conclude that H is a local homeomorphism around the origin. Therefore H is the desired Σ -equivalence and it follows that Z_1 and Z_2 are Σ -equivalent. $\square$

Let $\iota \in \mathcal{I}$ be the involution in $\mathbb{R}^n$ given by

$$\iota(x_1, \dots, x_n) = (x_1, \dots, x_{n-1}, -x_n). \quad (1)$$

By Lemma 4 and Property (P4), for each map in $\mathcal{J}$ there exists a change of coordinates so that it can be written as $\iota \circ f$ for some involution $f \in \mathcal{I}$.

The main result of this section follows.

Theorem 3. *Let $Z \in \mathcal{R}_{ell}$. If $j(Z)$ is structurally stable on $j(\mathcal{R}_{ell})$, then Z is structurally stable on $\mathcal{R}_{ell}$.*

Proof. Suppose that $j(Z)$ is structurally stable on $j(\mathcal{R}_{ell})$. Then there exists an open set $V \subset j(\mathcal{R}_{ell})$ containing $j(Z)$ such that all $j(Z_0) \in V$ are conjugate to $j(Z)$. By the continuity of j , there exists an open $U \subset \mathcal{R}_{ell}$ containing Z such that $j(U) \subset V$. It follows, by Lemma 4, that every $Z_0 \in U$ is Σ -equivalent to Z and therefore Z is structurally stable on $\mathcal{R}_{ell}$. $\square$

The converse of Theorem 3 also holds, however this assertion is not applied here and we do not prove it. A proof for the two-dimensional case can be found in [14].

4. Series Expansion of First Return Maps

Recall that the first return map associated with an elliptic refractive field $Z \in \mathcal{R}_{ell}$ is a map of the form $\varphi = \iota \circ \varphi_X$, where φ_X is an involution with $\text{Fix}(\varphi_X) = S$. It follows that φ_X can be written as

$$\varphi_X(x) = \begin{pmatrix} 1 & 0 & \cdots & 0 & a_1 \\ 0 & 1 & \cdots & 0 & a_2 \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & \cdots & 1 & a_{n-1} \\ 0 & 0 & \cdots & 0 & -1 \end{pmatrix} x + x_n \mathcal{O}(|x|). \quad (2)$$

We can write φ_X in the following way

$$\varphi_X(x) = \begin{pmatrix} x_1 + x_n \varphi_1(x) \\ \vdots \\ x_{n-1} + x_n \varphi_{n-1}(x) \\ -x_n + x_n f(x_1, \dots, x_{n-1}) + x_n^2 \varphi_n(x) \end{pmatrix}$$

where we decompose the last coordinate in terms that contain exactly one factor x_n and terms that contain at least two factors x_n in the series expansion of φ_X . We also require that $f(0) = 0$.

Computing the last coordinate of the expansion of $\varphi_X \circ \varphi_X$, we have

$$\begin{aligned} [\varphi_X \circ \varphi_X]_n(x) = & -(-x_n + x_n f(x) + x_n^2 \varphi_n(x)) + (-x_n + x_n f(x) + x_n^2 \varphi_n(x)) \cdot f(\varphi_X) + \\ & + (-x_n + x_n f(x) + x_n^2 \varphi_n(x))^2 \cdot \varphi_n(\varphi_X). \end{aligned}$$

As φ_X is an involution, $\varphi_X \circ \varphi_X$ is the identity function and therefore

$$x_n = -(-x_n + x_n f(x) + x_n^2 \varphi_n(x)) + (-x_n + x_n f(x) + x_n^2 \varphi_n(x)) f(\varphi_X) + (-x_n + x_n f(x) + x_n^2 \varphi_n(x))^2 \varphi_n(\varphi_X).$$

So

$$0 = x_n(-f(x) - x_n \varphi_n(x) + (-1 + f(x) + x_n \varphi_n(x)) f(\varphi_X) + x_n(-1 + f(x) + x_n \varphi_n(x))^2 \varphi_n(\varphi_X))$$

for all x near the origin. When $x_n \neq 0$, we have

$$0 = -f(x) - x_n \varphi_n(x) + (-1 + f(x) + x_n \varphi_n(x)) f(\varphi_X) + x_n(-1 + f(x) + x_n \varphi_n(x))^2 \varphi_n(\varphi_X).$$

By the continuity of the right-hand side expression, this equality also holds when $x_n = 0$. Replacing $x_n = 0$, it follows that

$$0 = -f(x) + (-1 + f(x)) f(x) = (f(x) - 2) f(x).$$

As f is continuous and $f(0) = 0$, we conclude that f is identically zero. Therefore the involution φ_X has the following format

$$\varphi_X(x) = \begin{pmatrix} x_1 + x_n \varphi_1(x) \\ \vdots \\ x_{n-1} + x_n \varphi_{n-1}(x) \\ -x_n + x_n^2 \varphi_n(x) \end{pmatrix}. \quad (3)$$

In the case where all a_i are zero in (2), the map φ_X can be written as

$$\varphi_X(x) = \iota(x) + x_n \cdot Ax + x_n \mathcal{O}(|x|^2)$$

where A is an $n \times n$ matrix. Rewrite φ_X in the following form

$$\varphi_X(x) = \begin{pmatrix} x_1 + \sum_i a_{1i} x_i x_n + \sum_{i \leq j} b_{1ij} x_i x_j x_n \\ x_2 + \sum_i a_{2i} x_i x_n + \sum_{i \leq j} b_{2ij} x_i x_j x_n \\ \vdots \\ -x_n + a_{nn} x_n^2 + \sum_i b_{nin} x_i x_n^2 \end{pmatrix} + x_n \mathcal{O}(|x|^3).$$

The first coordinate of $\varphi_X \circ \varphi_X$ is

$$\begin{aligned} [\varphi_X \circ \varphi_X]_1 &= x_1 + 2a_{1n} x_n^2 + \sum_{i < n} \left(2b_{1in} - \sum_{k < n} a_{1k} a_{ki} + a_{1i} a_{nn} \right) x_i x_n^2 + \\ &\quad + \left(- \sum_{i < n} a_{1i} a_{in} - 2a_{1n} a_{nn} \right) x_n^3 + x_n \mathcal{O}(|x|^3). \end{aligned}$$

But as φ_X is an involution, $[\varphi_X \circ \varphi_X]_1 = x_1$ and therefore

$$a_{1n} = 0 \quad (4)$$

and

$$b_{1in} = \frac{1}{2} \left(\sum_{k < n} a_{1k} a_{ki} - a_{1i} a_{nn} \right).$$

What interests us is expression (4). Clearly, the same arguments hold for all coordinates $k \leq n-1$ and therefore we have

$$a_{kn} = 0, \quad \forall k \leq n-1. \quad (5)$$

The last coordinate of $\varphi_X \circ \varphi_X$ is given by

$$\begin{aligned} [\varphi_X \circ \varphi_X]_n &= x_n - \sum_{i \leq n} b_{nin} x_i x_n^2 - 2a_{nn}^2 x_n^3 + \sum_{i < n} b_{nin} x_i x_n^2 - b_{nnn} x_n^3 + x_n \mathcal{O}(|x|^3) = \\ &= x_n + (-2b_{nnn} - 2a_{nn}^2) x_n^3 + x_n \mathcal{O}(|x|^3) \end{aligned}$$

and therefore follows

$$b_{nnn} = -a_{nn}^2.$$

The next sections are devoted to prove Lemmas 1, 2 and 3 and Theorem 2.

5. Lemma 1

In the following sections, we are restricting the diffeomorphisms to the region $S^+ = \overline{\mathbb{R}_+^n}$. From (3), the first return maps can be written as

$$\varphi(x) = x + x_n \begin{pmatrix} \varphi_1(x) \\ \vdots \\ \varphi_{n-1}(x) \\ x_n \varphi_n(x) \end{pmatrix}. \quad (6)$$

For $Z = (X, Y)$ satisfying the conditions of Lemma 1, we have

$$F(0) = (\varphi_1(0), \dots, \varphi_{n-1}(0), 0)^T \neq 0.$$

To prove Lemma 1, we use the following auxiliary result.

Lemma 5. Consider a diffeomorphism φ with the format (6). The linear part of φ is given by

$$D\varphi(0) = \begin{pmatrix} 1 & 0 & \cdots & 0 & A_1 \\ 0 & 1 & \cdots & 0 & A_2 \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & \cdots & 1 & A_{n-1} \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix} \quad (7)$$

where $A_i = \varphi_i(0)$. If $(A_1, \dots, A_{n-1}) \neq 0$, then φ is locally conjugate to its linear part.

Proof of Lemma 1. Let $Z = (X, Y)$ be as in Lemma 1. The linear part of $j(Z)$ is the transformation $x \mapsto x + x_n F(0)$ with $F(0) \neq 0$. By Lemma 5, $j(Z)$ is conjugate to its linear part.

There exists $\varepsilon > 0$ such that every map $j(Z_0)$ in $j(\mathcal{R}_{ell})$ with $d(j(Z), j(Z_0)) < \varepsilon$ has its linear part having the form $x \mapsto x + x_n F_0(0)$ with $F_0(0) \neq 0$. Again by Lemma 5, $j(Z_0)$ is conjugate to its linear part.

Therefore, $j(Z_0)$ is conjugate to $j(Z)$ for all Z_0 with $d(j(Z), j(Z_0)) < \varepsilon$. It follows that $j(Z)$ is structurally stable on $j(\mathcal{R}_{ell})$ and, by Theorem 3, Z is structurally stable on $\mathcal{R}_{ell}$. $\square$

The remainder of this section is devoted to the proof of Lemma 5.

Consider a diffeomorphism φ as in (6). Its linear part is given by (7) and suppose that $(A_1, \dots, A_{n-1}) \neq 0$. The following lemma reduces the form of (7) to a simpler one.

Lemma 6. The linear transformation $D\varphi(0)$ is conjugate to

$$x \mapsto \begin{pmatrix} 1 & 0 & \cdots & 0 & 1 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix} x. \quad (8)$$

Proof. Take a change of variables that preserves the x_n -axis and sends the vector $(A_1, \dots, A_{n-1}, 0)$ to $(1, 0, \dots, 0)$. $\square$

Therefore, from now on, we will suppose that the derivative $D\varphi(0)$ has the form given in (8). For each $t \in [0, 1]$, define the following function

$$\varphi^t(x) = (1 - t)x + t\varphi(x).$$

We extend φ^t for all $t \in \mathbb{R}$ in the following way

$$\varphi^t = \varphi^{t - \lfloor t \rfloor} \circ \varphi^{\lfloor t \rfloor}$$

where $\varphi^{\lfloor t \rfloor}$ is the $\lfloor t \rfloor$ -th iterate of φ . For each x , the curve $t \mapsto \varphi^t(x)$ is the union of straight line segments connecting $\varphi^i(x)$ to $\varphi^{i+1}(x)$ for $i \in \mathbb{Z}$. The function $\varphi^t(x)$ is continuous in x and t and smooth when $t \notin \mathbb{Z}$. The curves $\varphi^t(x)$ are right-differentiable in t for all x .

The notation φ^t does not create ambiguity with the notation of the iterates φ^n , since both definitions coincide when $t \in \mathbb{Z}$. It is worth noting that, despite the notation, the curves $\varphi^t(x)$ are not flows of vector fields, since we cannot guarantee the relation $\varphi^a \circ \varphi^b = \varphi^{a+b}$ when a and b are not integers.

Lemma 7. For each $t \in \mathbb{R}$, the transformation φ^t is a diffeomorphism around the origin.

Proof. For $t \in [0, 1]$, we have

$$\varphi^t(x) = \begin{pmatrix} x_1 + tx_n \\ x_2 \\ \vdots \\ x_n \end{pmatrix} + \mathcal{O}(|x|^2).$$

Then

$$D\varphi^t(0) = \begin{pmatrix} 1 & 0 & \cdots & 0 & t \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix}.$$

So $\det[D\varphi^t(0)] = 1$ and the result follows by the Inverse Function Theorem. The result for $t \in \mathbb{R}$ follows immediately. $\square$

Define the transversal $V^0 = \{x \in S^+ ; x_1 = 0\} \subset \mathbb{R}^n$ to the map φ (that is, the vectors $\varphi(x) - x$ are transverse to V^0 in a neighborhood of the origin). Let

$$V^+ = \{x \in S^+ ; x_1 > 0\} \quad \text{and} \quad V^- = \{x \in S^+ ; x_1 < 0\}$$

(Figure 3). The following lemma states that the orbits of points near the origin always cross this transversal.

Lemma 8. *There exists a neighborhood V of the origin such that for all $x \in V \cap V^-$ there exists an integer $n_0 > 0$ with $\varphi^{n_0}(x) \in V \cap V^+$ (see Figure 3).*

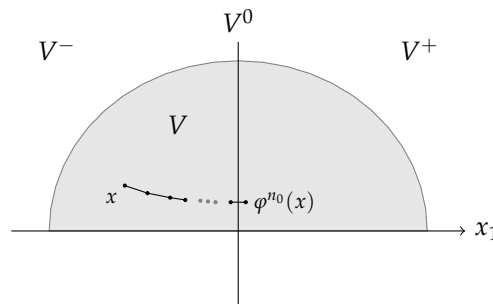


Figure 3. The construction of Lemma 8.

The proof of the lemma above depends on the following algebraic result.

Lemma 9. *Let (x_i) and (y_i) be two sequences of real numbers satisfying*

- (i) $y_i > 0$ for all $i \in \mathbb{N}$.
- (ii) $y_{i+1} = y_i - By_i^2$ for some $B > 0$.
- (iii) $x_{i+1} = x_i + Cy_i$ for some $C > 0$.

Then $\lim_{i \rightarrow \infty} x_i = \infty$.

Proof. Consider the sequence $z_i = B \cdot i \cdot y_i$. This sequence satisfies

$$z_{i+1} = z_i \left(1 - \frac{z_i}{i} + \frac{1}{i} - \frac{z_i}{i^2} \right).$$

Let

$$f_i(z) = z \left(1 - \frac{z}{i} + \frac{1}{i} - \frac{z}{i^2} \right) = z \left(1 + \frac{1}{i} \right) (1 - z).$$

We have that $z_{i+1} = f_i(z_i)$.

Notice that the fixed points of f_i are 0 and $1 - 1/i$ and that $f_i(z) > z$ between these two values. We must have $z_i \in (0, 1)$ for all i . Indeed, otherwise we would have $z_{i+1} = f_i(z_i) \leq 0$ which would imply $y_{i+1} \leq 0$, contradicting the condition (i).

If $z_i < 1 - 1/i$ then $z_{i+1} = f_i(z_i) > z_i$. If $z_i \geq 1 - 1/i$ then $z_{i+1} = f_i(z_i) \leq z_i$ and therefore exists $j > i$ such that $z_j < 1 - 1/j$. So (z_i) is increasing for i large enough and it is bounded by 1. Therefore the sequence (z_i) is convergent. It is easy to see that $z_i \rightarrow 1$.

So, for $\varepsilon > 0$ small enough, there exists $n_0 > 0$ such that $z_i > 1 - \varepsilon$ for all $i \geq n_0$. Then

$$y_i > \frac{1 - \varepsilon}{Bi}$$

for all $i \geq n_0$. Therefore

$$x_i = x_{n_0} + C \sum_{j=n_0}^{i-1} y_j > x_{n_0} + C \frac{1 - \varepsilon}{B} \sum_{j=n_0}^{i-1} \frac{1}{j} \rightarrow \infty$$

when $i \rightarrow \infty$ as we want to demonstrate. $\square$

Proof of Lemma 8. Let V be a bounded neighborhood of the origin such that $\varphi_1(x) > \frac{1}{2}$ for all $x \in V$. As the closure of V is compact, there exists some constant $C > 0$ such that $|\varphi_n(x)| < C$ for all $x \in V$.

Let $x = (x_1, \dots, x_n) \in V \cap V^-$. Define the sequences (x^i) and (y^i) as being the first and the n -th coordinate of the terms of the orbit $(\varphi^i(x))$, that is, $x^i = [\varphi^i(x)]_1$ and $y^i = [\varphi^i(x)]_n$. Define also the

sequences a_i and b_i such that $a_0 = x_1$, $b_0 = x_n$, $b_{i+1} = b_i - Cb_i^2$ and $a_{i+1} = a_i + \frac{1}{2}b_i$ (we can assume that V is small enough such that b_i is always positive).

We have

$$x^{i+1} = [\varphi^{i+1}(x)]_1 = x^i + y^i \varphi_1(\varphi^i(x)) > x^i + \frac{1}{2}y^i$$

and

$$y^{i+1} = [\varphi^{i+1}(x)]_n = y^i + (y^i)^2 \varphi_n(\varphi^i(x)) > y^i - C \cdot (y^i)^2.$$

Taking V smaller if necessary, we can assume that $y^i < 1/(2C)$ for all i . We know that $y^0 = x_n = b_0$ and, supposing that $y^j \geq b_j$ for some j , we have

$$y^{j+1} > y^j - C \cdot (y^j)^2 \geq b_j - C \cdot (b_j)^2 = b_{j+1},$$

since the function $\xi \mapsto \xi - C\xi^2$ is increasing in the interval $(0, 1/(2C))$. It follows, by induction, that $y^i \geq b_i$ and then $x^i \geq a_i$ for all i with $\varphi^i(x) \in V$. Notice that the sequences (a_i) and (b_i) satisfy the hypothesis of Lemma 9 and therefore $a_i \rightarrow \infty$. It implies that there exists $n_0 > 0$ such that $x^{n_0} > 0$ and therefore $\varphi^{n_0}(x) \in V^+$, as we want to demonstrate. $\square$

By the same arguments as above, we can prove

Lemma 10. *For all $x \in V \cap V^+$ there exists an integer $n_0 < 0$ such that $\varphi^{n_0}(x) \in V \cap V^-$.*

From Lemmas 8 and 10, it follows that the orbit of each point in V crosses (in positive or negative time) the transversal V^0 .

Thus, it is possible to define two functions $\pi : V \cap \mathbb{R}_+^n \rightarrow V^0$ and $\tau : V \cap \mathbb{R}_+^n \rightarrow \mathbb{R}$ such that $\varphi^{\tau(p)} = \pi(p)$. In other words, for each point $p \in V$, the curve $\varphi^t(p)$ intersects the transversal V^0 at the point $\pi(p)$ and at time $\tau(p)$. These functions are clearly continuous.

Define

$$h(x) = \pi(x) - [\pi(x)]_n \tau(x) e_1$$

where $e_1 = (1, 0, \dots, 0) \in \mathbb{R}^n$. This function is continuous and well-defined in a neighborhood V of the origin intersected with $\mathbb{R}_+^n$.

We prove that the function h can be extended continuously to the region $V \cap \overline{\mathbb{R}_+^n}$ (taking V smaller, if necessary), and this extension conjugates the map φ with its linear part $D\varphi(0)$. To do it, we need a sequence of lemmas.

Lemma 11. *The function π is locally Lipschitz at the origin².*

Proof. Let $\varphi(x) = x + x_n F(x)$ and write $F = (F_1, \dots, F_n)$.

Consider a bounded neighborhood V of the origin such that the orbit of every point in $V \cap \mathbb{R}_+^n$ crosses the transversal V^0 , that $F_1(x) > 1/2$ and the angle between $F(x)$ and $F(0) = (1, 0, \dots, 0)$ is less than $\pi/3$ for all $x \in V \cap \mathbb{R}_+^n$.

Since the closure $\overline{V}$ of this neighborhood is compact and F is continuously differentiable, there exists a constant $D > 0$ such that $\|DF(x)\| \leq D$ for all $x \in V$, where $\|\cdot\|$ denotes the usual norm in the space of matrices. This implies that

$$\|F(p_2) - F(p_1)\| \leq D\|p_2 - p_1\| \quad \text{for all } p_1, p_2 \in V \cap \mathbb{R}_+^n.$$

Take two points $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_n)$ in $V \cap \mathbb{R}_+^n$ and let $d = 1.7em(x, y)$. Consider the reparametrizations $\gamma_1(t)$ and $\gamma_2(t)$ of the curves $\varphi^t(x)$ and $\varphi^t(y)$, respectively, such that its derivatives in the first coordinate are equal to 1, that is, $[\gamma_1(t)]_1 = x_1 + t$ and $[\gamma_2(t)]_1 = y_1 + t$.

² A function f is said *locally Lipschitz* at p if there exists some neighborhood V of p such that f restricted to V is Lipschitz.

Consider the function $v(t) = \gamma_2(t) - \gamma_1(t)$. For each t , there exists $\varepsilon > 0$ such that the curve γ_1 is a line segment for all time between t and $t + \varepsilon$. This segment connects points of the form $\varphi^{t_1}(x)$ and $\varphi^{t_1+1}(x)$ for some integer t_1 . Let w_1 be the vector in the direction of this segment such that its first coordinate is 1 (that is, w_1 is the right derivative of γ_1 in t). In the same way, we define the vector w_2 for the curve γ_2 . For $\varepsilon > 0$ sufficiently small, we have

$$v(t + \varepsilon) - v(t) = \varepsilon \cdot (w_2 - w_1).$$

However, for each $i = 1, 2$, the vector w_i has the form $F(p_i)/F_1(p_i)$ for some $p_i = \gamma(t_i)$. It follows that

$$\|w_2 - w_1\| = \left\| \frac{F(p_2)}{F_1(p_2)} - \frac{F(p_1)}{F_1(p_1)} \right\| \leq 2\|F(p_2) - F(p_1)\| \leq 2D \cdot \|p_2 - p_1\|.$$

So

$$\|v(t + \varepsilon) - v(t)\| \leq \varepsilon \cdot 2D \cdot \|p_2 - p_1\|$$

If $\|p_2 - p_1\| > \|v(t)\|$, then $\|v(\cdot)\|$ is a decreasing function in the interval $[t, t + \varepsilon]$.

In the other hand, if $\|p_2 - p_1\| \leq \|v(t)\|$, we have

$$\|v(t + \varepsilon) - v(t)\| \leq \varepsilon \cdot 2D \cdot \|v(t)\|.$$

Then

$$\|v(t)\| \leq \exp(2D \cdot t) \|v(0)\|$$

for $t \geq 0$. By the same argument for $t \leq 0$, we have

$$\|v(t)\| \leq \exp(2D \cdot |t|) \|v(0)\|$$

Since the neighborhood V is bounded, the function $\exp(2D \cdot |t|)$ is bounded and therefore there exists $C_0 > 0$ such that

$$\|v(t)\| \leq C_0 \|v(0)\| = C_0 \cdot d.$$

Finally, consider the points $A = \pi(x) = \gamma_1(\tau(x))$, $B = \pi(y)$ and $C = \gamma_2(\tau(x))$. They form a triangle ABC with $\pi/2 - \pi/3 < \hat{A}BC < \pi/2 + \pi/3$.

We have $AC \leq C_0 \cdot d$, $BC < |y_1 - x_1| / \cos(\pi/3) \leq d / \cos(\pi/3) = 2d$. Then

$$1.7em(\pi(x), \pi(y)) = AB \leq AC + BC < (C_0 + 2)d(x, y).$$

This estimation holds for any two initial points $x, y \in V$ and so the proof is reached. $\square$

Lemma 12. *There exists a neighborhood of the origin in which the function π satisfies*

$$\frac{x_n}{2} < [\pi(x)]_n < 2x_n$$

for all $x = (x_1, \dots, x_n)$ in this neighborhood.

Proof. Consider a bounded neighborhood V of the origin such that $\|F(x)\| > 1/2$ for all $x \in V$ (remember we are assuming $F(0) = (1, 0, \dots, 0)$ by (8)).

As F is continuously differentiable and V has compact closure, there exist constants $M, D > 0$ such that $\|x\| < M$ and $\|DF(x)\| < D$ for all $x \in V$. So we have

$$\|F(x) - F(x_1, \dots, x_{n-1}, 0)\| < D \cdot x_n$$

for $x = (x_1, \dots, x_n) \in V$. We can assume, without loss of generality, that $x_1 < 0$.

If θ is the angle between $F(x_1, \dots, x_{n-1}, 0)$ and $F(x)$, we have that

$$\tan \theta < \frac{Dx_n}{\|F(x)\|} < 2Dx_n \quad (9)$$

and this inequality holds for all $x \in V$. Remember that $[F(x_1, \dots, x_{n-1}, 0)]_n = 0$ by (6). It means that the angle between $F(x)$ and the horizontal hyperplane containing x is less than θ .

Therefore, the point $\pi(x)$ must satisfy

$$x_n - [\pi(x)]_n < |x_1| \cdot \tan \theta < 2Dx_n|x_1|.$$

By intersecting V with the region where $|x_1| < 1/(4D)$, we have the left-hand inequality.

On the other hand, let $\gamma(t)$ be the reparametrization of $\varphi^t(x)$ by the first coordinate, that is, the $\gamma(t)$ and $\varphi^t(x)$ have the same image and $[\gamma(t)]_1 = t$. From (9) we have that

$$[\gamma'(t)]_n < 2D[\gamma(t)]_n$$

where γ' denotes the right-derivative of γ . Hence

$$[\gamma(t)]_n < [\gamma(x_1)]_n \exp(2D(t - x_1))$$

and therefore

$$[\pi(x)]_n < x_n \exp(2D|x_1|).$$

By intersecting the neighborhood with the region where $\exp(2D|x_1|) < 2$, we obtain the right-hand inequality. The case $x_1 > 0$ is analogous. $\square$

Lemma 13. *The function $x \mapsto x_n \cdot \tau(x)$ is locally Lipschitz at the origin*

Proof. Let V be a bounded neighborhood of the origin such that $|F_1(x) - F_1(0)| < 1/2$ for all $x \in V$. We also assume that V satisfies the conditions of Lemma 12. There exist constants $M, D > 0$ such that $\|x\| < M$ and $\|DF(x)\| < D$ in V .

Let $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_n)$ in V . We assume, without loss of generality, that $x_1, y_1 < 0$.

Define the curves

$$\gamma_1(t) = \varphi^{t/x_n}(x) \quad \text{and} \quad \gamma_2(t) = \varphi^{t/y_n}(y),$$

the function difference $v(t) = \gamma_1(t) - \gamma_2(t)$ and $d(t) = |v(t)|$.

The curves γ_1, γ_2 and v are right-differentiable and we denote by γ'_1, γ'_2, v' its right-derivatives, respectively. The notation φ' denotes the right-derivative of φ with respect to time t .

We want to maximize the value of $d(t)$ for each t . From Lemma 12, for $0 < t < \tau(x)$, we have

$$\begin{aligned} d'(t) &< |F_n(\gamma_1(t))F(\gamma_1(t))/x_n - F_n(\gamma_2(t))F(\gamma_2(t))/y_n| < \\ &< 2|F(\gamma_1(t)) - F(\gamma_2(t))| < 2D|\gamma_1(t) - \gamma_2(t)| = 2Dd(t). \end{aligned}$$

Therefore, for $t > 0$,

$$|v(t)| = d(t) < d(0) \exp(2Dt) = 1.7em(x, y) \exp(2Dt).$$

The curve γ_i hits the transversal V^0 in time T_i for $i = 1, 2$. Then $T_1 = x_n \tau(x)$ and $T_2 = y_n \tau(y)$. Since $[\gamma'_1(t)]_1 > [\gamma_1(t)]_n F_1(\gamma_1(t))/x_n > 1/4$, then

$$T_1 < 4|x_1| < 4M.$$

We also have

$$[\gamma'_2(t)]_1 > \frac{1}{4}.$$

From

$$1.7em(\gamma_1(T_1), \gamma_2(T_1)) = |v(T_1)| < 1.7em(x, y) \exp(2DT_1),$$

it follows that

$$|T_2 - T_1| < 4 \cdot 1.7em(\gamma_1(T_1), \gamma_2(T_1)) < 4 \cdot 1.7em(x, y) \exp(2DT_1) < 4 \cdot 1.7em(x, y) \exp(8DM).$$

Since $T_1 = x_n \tau(x)$ and $T_2 = y_n \tau(y)$, it follows that the function $x \mapsto x_n \tau(x)$ is Lipschitz in V . $\square$

From the Lemmas 11, 12 and 13, it follows:

Corollary 1. *The function $x \mapsto [\pi(x)]_n \cdot \tau(x)$ is locally Lipschitz at the origin.*

We now combine the previous Lemmas to show that h is well behaved.

Lemma 14. *The function h is locally Lipschitz at the origin.*

Proof. This follows straightforwardly from Corollary 1. $\square$

Finally, we prove the main lemma of this section.

Proof of Lemma 5. From Lemma 14, we can extend the function h continuously to the region $V \cap S^+$. This extension is clearly a local homeomorphism around the origin. We claim that h conjugates the diffeomorphisms φ and $D\varphi(0)$.

First, notice that $D\varphi(0)(x) = x + x_n F(0) = x + x_n e_1$ (Lemma 6). If $x_n = 0$, then $\varphi(x) = DF(0)x = x$ and therefore $h \circ \varphi(x) = D\varphi(0) \circ h(x)$. If $x_n > 0$, we have

$$\begin{aligned} h \circ \varphi(x) &= \pi(\varphi(x)) - [\pi(\varphi(x))]_n \tau(\varphi(x)) e_1 = \pi(x) - [\pi(x)]_n (\tau(x) - 1) e_1 = \\ &= \pi(x) - [\pi(x)]_n \tau(x) e_1 + [\pi(x)]_n e_1 = h(x) + [\pi(x)]_n e_1 \end{aligned}$$

and

$$D\varphi(0) \circ h(x) = h(x) + [h(x)]_n e_1 = h(x) + [\pi(x)]_n e_1,$$

which finishes the proof. $\square$

The proof of Lemma 5 concludes the proof of Lemma 1.

6. Lemma 2

The approach for Lemma 2 is somehow similar to that done for Lemma 1. We define the curves φ^t and φ_0^t associated with the first return map and one of its perturbations, and find a neighborhood of the origin whose boundary is transverse to these curves. By using this curves we find a local conjugacy between the maps defined when $x_n > 0$. Our final step is to extend this conjugacy continuously to the region $x_n = 0$. Since the maps are the identity when restricted to the region $x_n = 0$, this extension is a local conjugacy of the maps in the space $x_n \geq 0$.

Consider $\varphi = j(Z)$ satisfying the conditions of Lemma 2. Write $\varphi(x) = x + x_n F(x)$. From (3) and (5), we have that $DF(0)$ has the form

$$DF(0) = \begin{pmatrix} A_0 & 0 \\ 0 & c \end{pmatrix}$$

where A is an $(n-1) \times (n-1)$ hyperbolic matrix and $c \neq 0$. Assume, without loss of generality, that $c < 0$ and

$$DF(0) = \begin{pmatrix} A_+ & 0 & 0 \\ 0 & A_- & 0 \\ 0 & 0 & c \end{pmatrix}$$

where all eigenvalues of A_+ have positive real parts and all eigenvalues of A_- have negative real parts. Let $m = \dim A_+$.

Consider a region $V \subset S^+$ around the origin homeomorphic to $D^m \times D^{n-m}$ intersected with S^+ such that its boundary is written as $\partial V = V_0 \cup T_1 \cup T_2$, where $V_0 = V \cap S$, T_1 is homeomorphic to $(D^m \times S^{n-m-1}) \cap S^+$, T_2 is homeomorphic to $(S^{m-1} \times D^{n-m}) \cap S^+$ (S^k and D^k are the sphere and the disk of dimension k , respectively; see Figure 4) and such that the orbits of φ in T_1 (respect. T_2) point inward V (respect. outward V), that is, the vectors $\varphi(x) - x$ are transverse and point inward (respect. outward) of V when $x \in T_1$ (respect. $x \in T_2$).

To show that it is possible to choose such a region, notice that we want to find a region whose boundary is transverse to the vectors $\varphi(x) - x$. As we have the constraint $x_n > 0$, this is equivalent to finding a region whose boundary is transverse to the vectors $(\varphi(x) - x)/x_n = F(x)$. As F is a hyperbolic vector field, there exists a change of variables that transform the field F to the linear field $DF(0)$ (Hartman-Grobman Theorem). Since $DF(0)$ has the format (6), the region D^m satisfies those conditions for the first m coordinates of this system and the region D^{n-m} for the last $n-m$ coordinates. In this way, our desired region is $D^m \times D^{n-m}$. Finally return to the original coordinates to obtain the desired region.

We assume that the region V is small enough such that all arguments from now on hold.

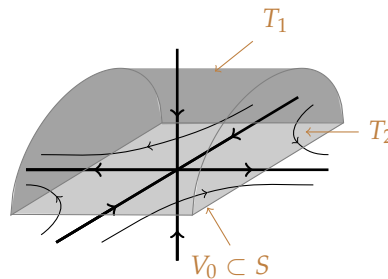


Figure 4. Regions V , V_0 , T_1 and T_2 .

Now, we consider a small perturbation $\varphi_0 = j(Z_0)$ of φ written as $\varphi_0(x) = x + x_n F_0(x)$ with

$$DF_0(0) = \begin{pmatrix} B_+ & 0 & 0 \\ 0 & B_- & 0 \\ 0 & 0 & d \end{pmatrix},$$

where B_+ is an $m \times m$ matrix with eigenvalues all having positive real parts, B_- is an $(n-m-1) \times (n-m-1)$ matrix with eigenvalues all having negative real parts and $d < 0$ is a real number.

We may assume that F_0 is topologically equivalent to F and the orbits of φ_0 are transverse to the boundary of V , always pointing inward V in T_1 and outward in T_2 .

Let φ^t and φ_0^t be the maps associated to φ and φ_0 as done in Section 5. For each $x \in V \setminus V_0$, there exists a unique point $\pi(x) \in T_1$ and a non negative real number $\tau(x)$ such that

$$\varphi^{\tau(x)}(\pi(x)) = x.$$

Observe that the functions π and τ are continuous.

Define the function $h : V \setminus V_0 \rightarrow \mathbb{R}_+^n$ by

$$h(x) = \varphi_0^{\tau(x)}(\pi(x)).$$

The function h is a local homeomorphism around the origin and conjugates φ and φ_0 in the region $V \setminus V_0$. We need to extend the function h continuously to V_0 .

To do this, consider $x_\varepsilon = (x_1, \dots, x_{n-1}, \varepsilon) \in V$ with $\varepsilon > 0$ sufficiently small. Consider the curve $\varphi^t(x_\varepsilon)$, take its orthogonal projection to V_0 and reparametrize the time by $t \mapsto \varepsilon t$. The resulting curve $\gamma(t)$ satisfies

$$\gamma(\varepsilon) = x_0 + \varepsilon F(x_\varepsilon)$$

where $x_0 = (x_1, \dots, x_{n-1}, 0)$. So, the curve γ can be seen as an approximation of the solution of the ODE

$$\begin{cases} \dot{x} = F(x) \\ x(0) = x_0 \end{cases} \quad (10)$$

on V_0 by the Euler method.

Notice that, by (3), the last coordinate of F has the form

$$[F(x)]_n = x_n \cdot \varphi_n(x)$$

and therefore V_0 is invariant under the field F .

Restricting F to V_0 , we have that F is a continuously differentiable vector field defined in the compact region V_0 . So the curves given by the Euler method converge uniformly to the solutions of (10) (see [16], for instance) and therefore the curve γ converge uniformly to this solution when $\varepsilon \rightarrow 0$.

Let F^t and F_0^t be the flows of F and F_0 , respectively. Define $\tau(x)$ as the time that the flow $F^t(x)$ reaches the boundary T_1 in negative time, that is, $F^{-\tau(x)}(x) \in T_1$. The extension of h to the region V_0 is given by

$$h(x) = F_0^{\tau(x)} \circ F^{-\tau(x)}(x). \quad (11)$$

However, this function is not well defined on the entire region V_0 . The function $\tau(x)$ is not defined when $x \in W^u(0)$, the unstable manifold of the origin relative to the field F . Thus, our next step is to extend h continuously to the whole region V_0 .

To this end, notice that the vector field F restricted to V_0 is hyperbolic and hence structurally stable. Then F and F_0 , when restricted to V_0 , are topologically equivalent (since they are sufficiently close) and therefore there exists a change of coordinates $\tilde{H} : V_0 \rightarrow V_0$ such that $\tilde{H}_* \circ F_0 \circ \tilde{H}^{-1} = F$ on V_0 .

Lift $\tilde{H}$ to dimension n by defining

$$H(x_1, \dots, x_n) = (\tilde{H}(x_1, \dots, x_{n-1}), x_n).$$

Applying the change of coordinates H to φ_0 , we obtain the new map $\tilde{\varphi} = H \circ \varphi_0 \circ H^{-1}$.

Writing $\tilde{\varphi}(x) = x + x_n \tilde{F}(x)$, we have $\tilde{F} = F$ on V_0 . Now, we apply the same arguments as above to find the function h which conjugates the maps F and $\tilde{F}$. In this case, the change of coordinates given in (11) reduces to the identity. We can easily extend h to $W^u(0)$ defining $h = id_{V_0}$. Returning to the original coordinates via H^{-1} , we obtain the desired conjugacy between the original maps.

It proves that φ is conjugate to all of its perturbations in $\mathcal{R}_{ell}$ and therefore it is structurally stable. It concludes the proof of Lemma 2.

7. Lemma 3

Consider a piecewise smooth vector field $Z = (X, Y)$ satisfying the hypotheses of Lemma 3. To prove Lemma 3, we show that it is possible to perturb the system Z in such a way that the first return

map is not topologically equivalent to $j(Z)$. It follows, from Lemma 4, that this perturbation is not Σ -equivalent to Z and therefore Z is not structurally stable on $\mathcal{R}_{ell}$.

We know that the first return map $j(Z)$ has the form

$$j(Z)(x) = x + x_n \cdot Ax + x_n \mathcal{O}(x) \quad (12)$$

where

$$A = \begin{pmatrix} A_0 & 0 \\ 0 & c \end{pmatrix}$$

is a non-hyperbolic $n \times n$ matrix. Let φ_X be the involution associated to the field X . Define the function $\tau : S^+ \rightarrow \mathbb{R}$ as the "first return time", that is,

$$X^{\tau(p)}(p) \in S^-$$

for all $p \in S^+$, where $X^t(p)$ denotes the flow of X . The series expansion of τ is given by the following lemma.

Lemma 15. $\tau(x) = -\frac{2}{X_n(0)}x_n + x_n \mathcal{O}(|x|)$, where X_n is the n -th coordinate of X .

Proof. When $x_n = 0$, the function τ is equal zero, then $\tau(x) = ax_n + x_n \mathcal{O}(|x|)$ for some $a \in \mathbb{R}$.

Consider an $r > 0$ small and let V be a neighborhood of $0 \in \mathbb{R}^{n+1}$ such that $|X_n(x) - X_n(0)| < r$ for all $x \in V$. Take $p = (p_1, \dots, p_n, 0) \in S^+ = \overline{\mathbb{R}}_+^n \times 0$ and let $\gamma(t) = (\gamma_1(t), \dots, \gamma_{n+1}(t))$ be the trajectory of p by the field X . We can assume that $\gamma(t) \in V$ for all $t \in [0, \tau(p)]$. We have

$$\gamma'_n(t) = X_n(\gamma(t)) \in (X_n(0) - r, X_n(0) + r)$$

for all $t \in [0, \tau(p)]$. It follows that

$$\tau(p)(X_n(0) - r) \leq \gamma_n(\tau(p)) - p_n \leq \tau(p)(X_n(0) + r).$$

Then

$$\frac{\tau(p)}{p_n}(X_n(0) - r) \leq \frac{\gamma_n(\tau(p)) - p_n}{p_n} \leq \frac{\tau(p)}{p_n}(X_n(0) + r).$$

Taking $r \rightarrow 0$,

$$\lim_{p \rightarrow 0} \frac{\tau(p)}{p_n} \cdot X_n(0) = \lim_{p \rightarrow 0} \frac{\gamma_n(\tau(p)) - p_n}{p_n}.$$

As

$$\frac{\gamma_n(\tau(p)) - p_n}{p_n} = \frac{[\varphi_X(p)]_n}{p_n} - 1 = -2 + p_n \mathcal{O}(p_n),$$

taking the limit, we have

$$\frac{\partial \tau}{\partial x_n}(0) \cdot X_n(0) = -2.$$

Since $\partial \tau / \partial x_n(0) = a$, it follows that

$$a = -\frac{2}{X_n(0)}.$$

□

The following result relates the series expansion of the field with the expansion of its first return map.

Lemma 16. Write $Z = (X, Y)$ with

$$X = (0, \dots, 0, a, x_n)^T + \begin{pmatrix} B_0 & & \\ & d & \\ & & 0 \end{pmatrix} x + \mathcal{O}(|x|^2)$$

where $a = X_n(0) < 0$ and let $j(Z)(x) = x + x_n \cdot Ax + x_n \mathcal{O}(|x|^2)$ with

$$A = \begin{pmatrix} A_0 & \\ & c \end{pmatrix},$$

A_0 and B_0 being $(n-1) \times (n-1)$ matrices. Then $A_0 = -\frac{2}{a}B_0$ and $c = -\frac{2d}{3a}$.

Proof. Let $A_0 = (a_{ij})$ and $B_0 = (b_{ij})$, $1 \leq i, j \leq n-1$ and denote

$$A_j = \begin{pmatrix} a_{1j} \\ \vdots \\ a_{n-1,j} \end{pmatrix} \quad \text{and} \quad B_j = \begin{pmatrix} b_{1j} \\ \vdots \\ b_{n-1,j} \end{pmatrix}$$

the j -th columns of A_0 and B_0 , respectively. We want to show that $A_j = -\frac{2}{a}B_j$ for every $1 \leq j \leq n-1$.

Take a neighborhood V of the origin such that $\|X(x) - X(y)\| \leq D\|x - y\|$ for all $x, y \in V$ for some $D > 0$ (it is possible to do this since X is smooth). Fix $j \leq n-1$ and consider the points

$$p = (0, \dots, 0, \varepsilon_j, 0, \dots, 0, \varepsilon_n) \in \mathbb{R}_+^n$$

and

$$q = (0, \dots, 0, \varepsilon_j, 0, \dots, 0, 0) \in \mathbb{R}^n$$

(ε_j in the j -th entry and ε_n in the n -th entry) with $|\varepsilon_j|$ and ε_n positives and near enough to zero such that the orbit of p by the field X stays inside V until it touches again the boundary Σ (Figure 5).

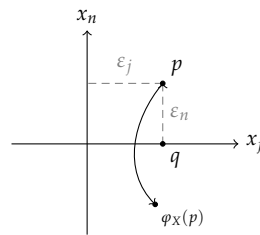


Figure 5. Construction in the proof of Lemma 16.

Let $\gamma(t)$ be the trajectory of X through p :

$$\gamma(0) = p \quad \text{and} \quad \gamma'(t) = X(\gamma(t)).$$

Then

$$\|\gamma'(t) - X(q)\| = \|X(\gamma(t)) - X(q)\| \leq D\|\gamma(t) - q\|.$$

We have

$$\left\| \int_0^{\tau(p)} \gamma'(t) - X(q) dt \right\| \leq \int_0^{\tau(p)} \|\gamma'(t) - X(q)\| dt \leq D \int_0^{\tau(p)} \|\gamma(t) - q\| dt.$$

Since

$$\begin{aligned} \int_0^{\tau(p)} \gamma'(t) - X(q) dt &= \int_0^{\tau(p)} \gamma'(t) dt - \tau(p)X(q) = \gamma(\tau(p)) - \gamma(0) - \tau(p)X(q) = \\ &= \varphi_X(p) - p - \tau(p)X(q). \end{aligned}$$

It follows that

$$\|\varphi_X(p) - p - \tau(p)X(q)\| \leq D \int_0^{\tau(p)} \|\gamma(t) - q\| dt \leq D \cdot \tau(p) \cdot \max_{t \in [0, \tau(p)]} \|\gamma(t) - q\|.$$

Dividing both sides by ε_n , we have

$$\left\| \frac{\varphi_X(p) - p}{\varepsilon_n} - \frac{\tau(p)X(q)}{\varepsilon_n} \right\| \leq D \cdot \frac{\tau(p)}{\varepsilon_n} \cdot \max_{t \in [0, \tau(p)]} \|\gamma(t) - q\|.$$

Letting $\varepsilon_n \rightarrow 0$ and noticing that

$$\frac{\tau(p)}{\varepsilon_n} \rightarrow \frac{\partial \tau}{\partial x_n}(q) = -\frac{2}{X_n(q)}$$

(the last equality is Lemma 15 applied to the tangency point q instead of the origin) and that $\max \|\gamma(t) - q\| \rightarrow 0$, we have that

$$\left\| \frac{\varphi_X(p) - p}{\varepsilon_n} - \frac{\tau(p)X(q)}{\varepsilon_n} \right\| \rightarrow 0$$

and therefore

$$\lim_{\varepsilon_n \rightarrow 0} \frac{\varphi_X(p) - p}{\varepsilon_n} = \lim_{\varepsilon_n \rightarrow 0} \frac{\tau(p)X(p)}{\varepsilon_n} = \frac{\partial \tau}{\partial x_n}(q) \cdot X(q).$$

On one hand, we have

$$\lim_{\varepsilon_n \rightarrow 0} \frac{\varphi_X(p) - p}{\varepsilon_n} = \begin{pmatrix} \varepsilon_j A_j \\ -2 \end{pmatrix} + \mathcal{O}(\varepsilon_j^2).$$

On the other hand,

$$X(q) = \begin{pmatrix} \varepsilon_j B_j \\ a \\ 0 \end{pmatrix} + \mathcal{O}(\varepsilon_j^2).$$

Then

$$\begin{pmatrix} \varepsilon_j A_j \\ -2 \end{pmatrix} + \mathcal{O}(\varepsilon_j^2) = \frac{\partial \tau}{\partial x_n}(q) X(q) = -\frac{2}{X_n(q)} X(q) = \begin{pmatrix} -\frac{2\varepsilon_j}{a} B_j \\ -2 \end{pmatrix} + \mathcal{O}(\varepsilon_j^2)$$

and therefore $A_j = -\frac{2}{a} B_j$, proving the first equality.

Consider now a point $p = (p_1, \dots, p_n, 0) \in \mathbb{R}^{n+1}$ near the origin and let $\gamma(t)$ be the trajectory of X through p . Write $\gamma(t) = (\gamma_1(t), \dots, \gamma_{n+1}(t))$. Since γ is a trajectory of X , the last two coordinates of γ satisfy:

$$\begin{cases} \gamma'_n(t) &= X_n(\gamma(t)) &= a + d\gamma_n(t) + \mathcal{O}(t^2), \\ \gamma'_{n+1}(t) &= X_{n+1}(\gamma(t)) &= \gamma_n(t). \end{cases}$$

Therefore, the series expansion of the solution is given by

$$\begin{cases} \gamma_n(t) &= p_n + (a + dp_n)t + \frac{d}{2}(a + dp_n)t^2 + \mathcal{O}(t^3), \\ \gamma_{n+1}(t) &= p_n t + \frac{1}{2}(a + dp_n)t^2 + \frac{d}{6}(a + dp_n)t^3 + \mathcal{O}(t^4). \end{cases}$$

The trajectory γ touches the boundary Σ when $\gamma_{n+1}(t) = 0$. It occurs at the time

$$t = -\frac{2}{a}p_n + \frac{2d}{3a^2}p_n^2 + \mathcal{O}(p_n^3).$$

So that, γ touches Σ at the point

$$\gamma_n(t) = -p_n + \frac{2d}{3a}p_n^2 + \mathcal{O}(p_n^3).$$

Thus

$$[j(Z)(x)]_n = [\varphi(x)]_n = x_n - \frac{2d}{3a}x_n^2 + \mathcal{O}(x^3)$$

and therefore

$$c = -\frac{2d}{3a}.$$

This completes the proof. $\square$

Finally, we prove Lemma 3.

Proof of Lemma 3. Consider $Z = (X, Y) \in \mathcal{R}_{ell}$ as in conditions of Lemma 3. We may write

$$X = (0, \dots, 0, a, x_n)^T + \begin{pmatrix} B_0 & & \\ & d & \\ & & 0 \end{pmatrix} x + \mathcal{O}(|x|^2)$$

with $a < 0$ and $Y = (0, \dots, 0, 1, x_n)^T$. Define $Z^\varepsilon = (X^\varepsilon, Y)$ as

$$X^\varepsilon = X + \begin{pmatrix} \varepsilon I_n & \\ & 0 \end{pmatrix} x,$$

where I_n is the $n \times n$ identity matrix. Then

$$X^\varepsilon = (0, \dots, 0, a, x_n)^T + \begin{pmatrix} B_0 + \varepsilon I_n & & \\ & d + \varepsilon & \\ & & 0 \end{pmatrix} x + \mathcal{O}(|x|^2).$$

Define $\varphi = j(Z)$ and $\varphi^\varepsilon = j(Z^\varepsilon)$. By Lemma 16, we have $\varphi^\varepsilon = x + x_n \cdot A^\varepsilon x + x_n \mathcal{O}(x)$ with

$$A^\varepsilon = \begin{pmatrix} -\frac{2}{a}(B_0 + \varepsilon I_n) & \\ & -\frac{2}{3a}(d + \varepsilon) \end{pmatrix}.$$

(Remember that $a < 0$). There exists an $\varepsilon_0 > 0$ such that, for all $\varepsilon \in (-\varepsilon_0, \varepsilon_0) \setminus \{0\}$, the matrix A^ε is hyperbolic. When $-\varepsilon_0 < \varepsilon < 0$, the eigenvalues of A^0 whose real part is equal to zero, now have negative real part in A^ε and when $0 < \varepsilon < \varepsilon_0$, they have positive real part in A^ε . Consequently, the maps $j(Z^\varepsilon)$ cannot be topologically equivalent for $\varepsilon > 0$ and $\varepsilon < 0$ and therefore, by Lemma 4, the perturbations Z^ε for $\varepsilon > 0$ cannot be Σ -equivalents to those with $\varepsilon < 0$. Then it follows that Z is not structurally stable, as we want to demonstrate. $\square$

The proof of this lemma concludes the proof of Theorem 1.

8. Normal Forms

We now prove Theorem 2.

Proof of Theorem 2. The fields Z_0 given in Theorem 2 are structurally stable by Lemma 16 and Lemmas 1 and 2.

If $Z \in \mathcal{R}_{ell}$ is structurally stable then, by Theorem 1, Z must belong either to the regular or to the singular case. In the regular case, $j(Z)$ and $j(Z_0)$ are conjugate to (8), therefore Z and Z_0 are Σ -equivalent and we are done. In the singular case, Let $Z = (X, Y_0)$ be written in the following form

$$X = (0, \dots, 0, a, x_n)^T + \begin{pmatrix} B_0 & & \\ & d & \\ & & 0 \end{pmatrix} x + \mathcal{O}(|x|^2)$$

with B_0 hyperbolic, $d \neq 0$ and $Y_0 = (0, \dots, 0, 1, x_n)^T$. Let $Z_0 = (X_0, Y_0)$ be the field

$$X_0 = (0, \dots, 0, a, x_n)^T + \begin{pmatrix} B_0 & & \\ & d & \\ & & 0 \end{pmatrix} x.$$

Define the family of systems

$$Z_t = tZ + (1-t)Z_0, \quad \text{for } t \in [0, 1].$$

All systems Z_t are refractive and $j(Z_t)(x) = x + x_n \cdot Ax + x_n \mathcal{O}(|x|^2)$ for all $t \in [0, 1]$ with A given by Lemma 16 not depending on t .

If $j(Z_0)$ were not conjugate to $j(Z)$, there would exist a $t_0 \in [0, 1]$ such that $j(Z_{t_0})$ is not structurally stable on $\mathcal{R}_{ell}$. This contradicts Theorem 1 and therefore $j(Z)$ is conjugate to $j(Z_0)$ and, by Lemma 4, Z is Σ -equivalent to Z_0 . $\square$

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