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Article

Opinion Dynamics with Higher-order Bounded Confidence

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Abstract: The higher-order interactions in complex systems are gaining attention. Extending the classic bounded confidence model where an agent's opinion update is the average opinion of its peers, this paper proposes a higher-order version of the bounded confidence model. Each agent organizes a group opinion discussion among its peers. Then, the discussion's result influences all participants' opinions. Since an agent is also the peer of its peers, the agent actually participates in multiple group discussions. We assume the agent's opinion update is the average over multiple group discussions. The opinion dynamics rules can be arbitrary in each discussion. In this work, we experiment with two discussion rules: centralized and decentralized. We show that the centralized rule is equivalent to the classic bounded confidence model. The decentralized rule, however, can promote opinion consensus. In need of modeling specific real-life scenarios, the higher-order bounded confidence is convenient to combine with other higher-order dynamics, from the contagion process to evolutionary dynamics.

Keywords: Ppinion dynamics; bounded confidence; higher-order interaction; HK model

1. Introduction

Opinion dynamics, being one of the essential branches of sociophysics, studies the statistical physics of collective opinion evolution driven by microscopic rules of individuals [1]. Opinion dynamics models can be broadly classified into two categories concerning the opinion space [2]: the discrete opinion space [3–11], and the continuous opinion space [12–16]. The models based on discrete opinion space usually assume two opposing opinions in the system (e.g., +1, -1, or A, B, *etc.*). The classic discrete opinion dynamics models include the voter model [3–5], the Sznajd model [6–8], and the Galam model [9–11]. Another class of models is based on continuous opinion space, where an individual's opinion is measured by a real number between 0 and 1, inclusive. One of the most classic models with continuous opinion space is the DeGrootian model [2,12–14]. Then, it was not until researchers introduced the bounded confidence into the continuous opinion dynamics, that the well-known Deffuant–Weisbuch (DW) model [15] and Hegselmann–Krause (HK) model [16] were born. We consider the HK model a mean-field approximation to the DW model, and refer to both as the classic bounded confidence model.

The bounded confidence model assumes that an agent (i.e., an individual) only accepts opinions that do not differ from its own by more than a critical value. This critical value is labeled as the bounded confidence. This work denotes the bounded confidence by r. Suppose there are N agents in the system. The opinion of agent i at time step t is denoted by $x_i(t)$. In the classic HK model, an agent's opinion update is the average of all acceptable opinions:

$$x_i(t+1) = \frac{1}{\mathbb{N}_i(t)} \sum_{j \in \mathbb{N}_i(t)} x_j(t) \quad (1)$$

where, $\mathbb{N}_i(t) = \{j \big| |x_i(t) - x_j(t)| \leq r,\ j = 1, 2, \ldots, N\}$. The opinion updates of all agents are synchronous. Letting the system evolve according to Eq. (1), we can obtain a stability opinion profile. The opinion profile switches from consensus to polarization and

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fragmentation as the bounded confidence r decreases, intuitively elucidating the so-called "information cocoon."

A variety of mathematical tools have been used to investigate the properties of the bounded confidence model, by which the convergence [17,18], the pattern formation [19], the entropy [20], and the control theory [21] in the bounded confidence model have been studied. Other works focus on innovations in the model itself. Some of them introduced various new factors [22–32], such as the opinion leader [22], the memory [23], the expression and private opinion [24], the fuzzy inference [25], the stubbornness [26–28], and the noise [22,29–32], to the classic bounded confidence model. Others consider different possibilities of evolutionary mechanisms of the system [26,33–38], such as the heterogeneous bounded confidence [26,33,34], the heterogeneous pressure [35,36], and the circular opinion space [37,38]. One of the most important topics in the bounded confidence model is how to promote the opinion consensus. In this regard, some works have investigated the conditions of consensus formation [39,40]. Other works introduced new factors or mechanisms, such as the external activation [41], and the combination of pairwise and group interactions [42], with the aim of promoting opinion consensus.

As we mentioned previously, in the classic bounded confidence model, an agent's opinion update is directly the average opinion of its peers. In other words, in the framework of the classic bounded confidence model, it is not straightforward to consider higher-order dynamics. This is an important entry point, since higher-order dynamics beyond pairwise interactions can model real-life scenarios in a more intuitively way and have been revealed for non-trivial phenomena that do not exist in pairwise interactions [43–45]. With these attractive advantages, higher-order interactions have been introduced into a wide range of complex systems, from contagion process [46–48] to evolutionary games [49–51], by means of hypergraphs or simplicial complexes. In particular, opinion dynamics with higher-order interactions have sprouted [53–56]. Neuh′auser *et al.* [53] studied opinion consensus dynamics by multibody interactions and found the resulting dynamics can cause shifts away from the average system state. Sahasrabuddhe *et al.* [54] further explored consensus dynamics on hypergraphs based on sociological theories and investigated relevant dynamics on real-world structures. Hickok *et al.* [55] studied Deffuant–Weisbuch bounded confidence model on hypergraphs and found agents can jump from one opinion cluster to another in a single time step, which is impossible in bounded confidence models with pairwise interactions. In addition, Horstmeyer and Kuehn [56] investigated a coevolutionary voter model on simplicial complexes.

The work mentioned above on opinion dynamics was carried out on higher-order networks in a strict way, but did not relate the concept of higher-order interactions to the bounded confidence directly. The theoretical concept of "higher-order bounded confidence" has corresponding realistic scenarios; for example, when opinion discussions can happen among a group of people instead of two-by-two, a person may want to join in a discussion because her opinion is close to the discussion's organizer. As a result, she is involved in the group opinion discussion even if the opinions of some participants are not close to her.

Considering both the theoretical and practical importance, this work tries to provide the introduction of higher-order dynamics with bounded confidence. Similar algorithms can be found in many previous multidisciplinary fields, but let us employ a simple one to analog, the multiplayer evolutionary games (e.g., the public goods game [52]). In multiplayer games, each focal agent organizes a game among its neighbors and itself. Meanwhile, its neighbors also perform the same action. As a result, each agent actually participates in multiple games organized by its neighbors and itself (see Fig. 1, left). In this regard, the common algorithm is to average the results obtained by these multiple games. In this work, we analog this algorithm to the bounded confidence model. While the multiplayer games are based on constant networks, the peers that an agent interacts with in the bounded confidence model are determined by the opinion distance, which varies at each time step. Here, the homogeneity of bounded confidence ensures that the "peer network"

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is always undirected (i.e., interactions are always mutual, see Fig. 1, right). Therefore, we can perform the following analogous migration of the higher-order interaction algorithm. First, each agent organizes a group opinion discussion among its peers. Second, since the peers perform the same action, each agent participates in multiple opinion discussions organized by its peers. Finally, the opinion update of an agent is the average over the results obtained from these multiple discussions.

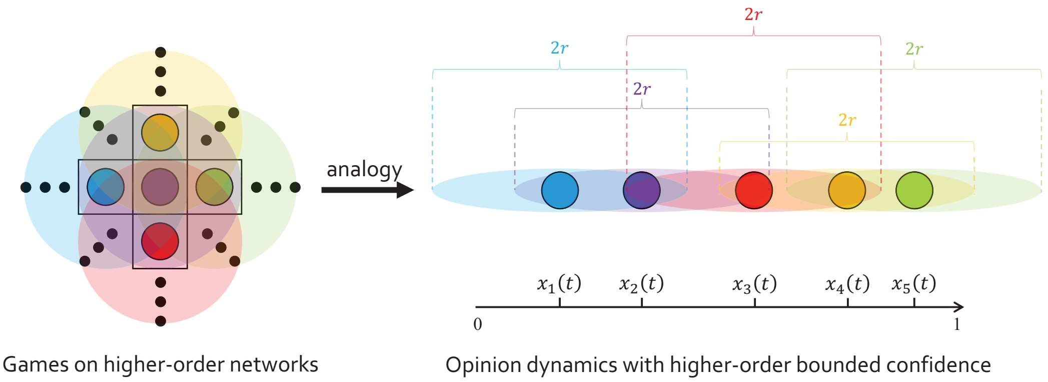

Figure 1. Schematic of the analogy, from games on higher-order networks (left), to opinion dynamics with higher-order bounded confidence (right). Left: five agents on a regular square lattice. The purple agent organizes a multiplayer game among the five agents (its nearest neighbors), while also participates in the games organized by the other four agents. Right: five agents on a continuous one-dimensional opinion space. The purple agent organizes a group opinion discussion among the blue, purple, and red agents within its bounded confidence, while also participates in the discussions organized by the blue and red agents.

The structure of this paper is described below. While the rules followed by a single group opinion discussion could be arbitrary, Sec. 2 gives two basic rules: centralized and decentralized. The former is equivalent to the classic HK model, while the latter leads to "higher-order" interactions. In Sec. 3, we explore the role of decentralized discussion in promoting the opinion consensus, compared to the classic HK model. In Sec. 4, we review the higher-order bounded confidence framework and discuss potential future development.

2. Model

Consider a system of N agents. At time step t, each agent $i = 1, 2, \dots, N$ holds an opinion $x_i(t)$. Suppose the opinion is represented by a continuous real number between 0 and 1: $0 \le x_i(t) \le 1$. For each agent, we denote a peer set $\mathbb{N}_i(t) = \{j \,|\, |x_i(t) - x_j(t)| \le r,\ j = 1, 2, \dots, N\}$, where r represents the bounded confidence. We assume an agent i only interacts with its peer agents in $\mathbb{N}_i(t)$, whose opinions are not more than r away from agent i.

The interactions are higher-order. At time step t, we go through the N agents. Each focal agent i organizes a group opinion discussion among its peers $j \in \mathbb{N}_i(t)$. The opinions of all participants $x_j(t)$ can influence the discussion's outcome. We denote the discussion's outcome by $o_i(t) = f(x_j(t) | j \in \mathbb{N}_i(t))$. The discussion dynamics rule, denoted by f, can be arbitrary. The N agents organize their discussions synchronously.

Note that an agent is also the peer of its peers. In this way, an agent i should participate in $|\mathbb{N}_i(t)|$ discussions at each time step, where $|\mathbb{N}_i(t)|$ denotes the number of elements in the set $\mathbb{N}_i(t)$. We assume each discussion works in the opinion updates of all participants, and the opinion update of each agent is the average over all discussions it participates in. That is, for an agent i, the opinion update is

$$x_i(t+1) = \frac{1}{\mathbb{N}_i(t)} \sum_{j \in \mathbb{N}_i(t)} o_j(t) \tag{2}$$

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The N agents update their opinions synchronously.

In this work, we further give $o_j(t)$ a concrete form. In general, $o_j(t)$ could be any function as the opinions of j's peers k, $o_j(t) = f(x_k(t)|k \in \mathbb{N}_j(t))$. For example, Eq. (2) degenerates to the classic HK model, if we give $o_j(t) = x_j(t)$. In this case, the discussion organized by agent j is "centralized," because the organizer j directly adopts its own opinion as the discussion's outcome. Other than the "centralized" one, let us propose another rule—the "decentralized." Literally, if the discussion is decentralized, the discussion's outcome is the average opinion over all participants. To sum up,

$$o_j(t) = \begin{cases} x_j(t), & \text{if agent } j \text{ is centralized,} \\ \dfrac{1}{\mathbb{N}_j(t)} \sum\limits_{k \in \mathbb{N}_j(t)} x_k(t), & \text{if agent } j \text{ is decentralized .} \end{cases} \tag{3}$$

We classify agent types by centralized and decentralized, who only organize centralized & decentralized discussions, respectively. We denote the fraction of decentralized agents in the population by α, while $1 - \alpha$ is the fraction of centralized agents. The type of an agent does not change with time.

3. Results and discussion

In the simulation, we fix $N = 1000$. At $t = 0$, we randomly set each agent's initial opinion $x_i(0)$ between 0 and 1, inclusive. Among the N agents, the decentralized agents totaling αN are randomly designated, and the remaining $(1 - \alpha)N$ are centralized. Then, we simulate the system according to the rules established in Sec. 2.

Figure. 2 shows each agent's opinion $x_i(t)$ as a function of time step t at $\alpha = 1$ (all agents are decentralized). Within finite time steps, the opinions in the system converge to clusters and no longer change with t; that is, the system achieves stability. When the system achieves stability, the opinion profile is fragmentation, polarization, and consensus at $r = 0.05$, $r = 0.15$, and $r = 0.25$, respectively, similar to the classic HK model [16].

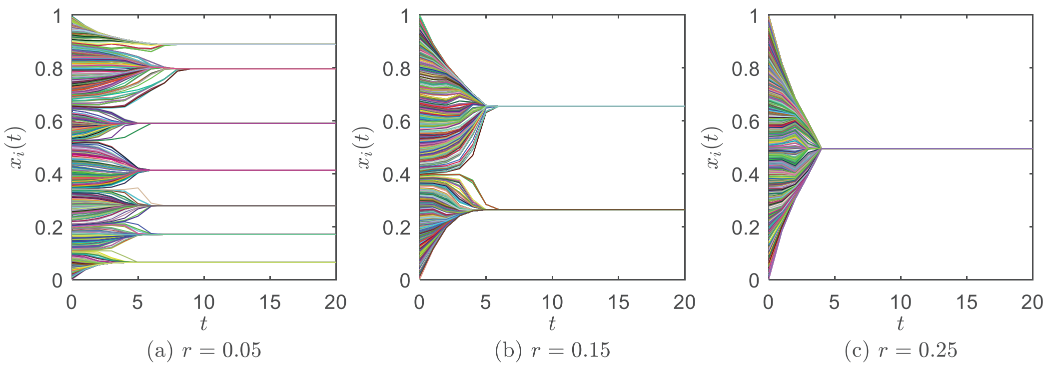

Figure 2. Each agent's opinion $x_i(t)$, $i = 1, 2, \ldots, N$, as a function of time step t at $\alpha = 1$ and different r. (a) $r = 0.05$. (b) $r = 0.15$. (c) $r = 0.25$.

Below, we define the system achieves stability if $|x_i(t) - x_i(t-1)| < 0.0001$, $\forall i = 1, 2, \ldots, N$. We set the following statistical quantities to measure the system's property at stability:

- P_C, the frequency of consensus in multiple runs. In a run, if there is only one opinion cluster left in the system (e.g., Fig. 2(c)), we say the system achieves consensus.
- r_1, the lower bounded confidence above which the system may consistently achieve consensus (i.e., $P_C < 1$, $\forall r < r_1$, and $P_C = 1$, $\exists r \geq r_1$). Similarly, r_0, the upper bounded confidence below which the system cannot achieve consensus (i.e., $P_C = 0$, $\forall r < r_0$, and $P_C > 1$, $\exists r \geq r_0$).

- N_C, the number of opinion clusters. For example, in Fig. 2(a), (b), and (c), we have $N_C = 7$, $N_C = 2$ and $N_C = 1$, respectively. $N_C = 1$ means the system achieves consensus.
- $C_{\max}$, the relative size of the largest opinion cluster. We find the opinion cluster with the highest number of agents and divide it by N. Obviously, this quantity yields $1/N \le C_{\max} \le 1$.
- $\rho[x_i(T^*)]$, the distribution of stability opinions. We divide the range between 0 and 1 into 100 equal parts, and denote $\Delta x = 1/100 = 0.01$. If $n\Delta x \le x_i(T^*) < (n+1)\Delta x$, we add 1 to the distribution function at the nth part ($n = 1, 2, \ldots, 100$). After going through $i = 1, 2, \ldots, N$, we divide the result in each part by N, and acquire the normalized opinion distribution.
- T^*, the convergence time. If $|x_i(t) - x_i(t-1)| < 0.0001$, $i = 1, 2, \ldots, N$, then, we denote $T^* = t$.

All the statistical quantities are the average over 10^5 independent runs.

In Fig. 3, we study the frequency of consensus P_C. Figure 3(a) shows P_C as a function of the bounded confidence r at different α. When $\alpha = 0$, the results are the same as the classic HK model. It is seen that as r increases, P_C gradually increases from 0 to 1 in the interval $0.15 \lesssim r \lesssim 0.25$. The curves at different α show the same pattern. However, the larger the α, the larger the P_C value of the corresponding curve at each point. To validate this, Fig. 3(b) shows P_C as a function of α at different r selected from the interval $0.15 \lesssim r \lesssim 0.25$. It can be seen that P_C always increases with an increase in α, which means the more decentralized agents in the system, the greater the frequency of complete consensus is.

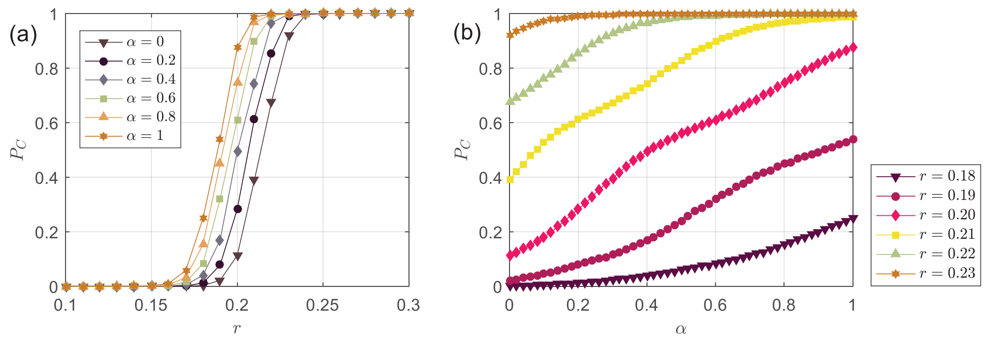

Figure 3. (a) The frequency of consensus P_C as a function of the bounded confidence r at different α. (b) The frequency of complete consensus P_C as a function of the fraction of decentralized agents α at different r.

A further approach to Fig. 3 is studying the critical point where the opinion consensus emerges. Figure 4 shows the lower bounded confidence r_1 (above which the system may consistently achieve consensus) and the upper bounded confidence r_0 (below which the system cannot achieve consensus) as a function of α. Since the data points are scattered, a linear fit is performed to reveal the trend of the data. It is revealed that either r_0 or r_1 decreases with an increase in α. This illustrates that, the larger the α, on the one hand, the earlier the P_C starts to increase from 0 to 1, and on the other hand, the earlier the P_C ends the change from 0 to 1, finally reaching 1. Decentralized agents can advance the critical point of opinion consensus emergence.

More generally, we can study the number of opinion clusters N_C when stability. Figure 5(a) demonstrates N_C as a function of r at different α. Similar to the results of P_C, the function N_C at different α share the same pattern. As r increases, N_C decreases, and the trend always presents a "steplike" behavior at different α. The breakpoints are distributed in $0.15 \lesssim r \lesssim 0.2$, where "sharp steps" appear. The position of breakpoints is consistent with r_0 (see the panel inside Fig. 5(a)), foretelling that opinion consensus will emerge as r continues to increase. In addition, we notice that the larger the α, the smaller the N_C

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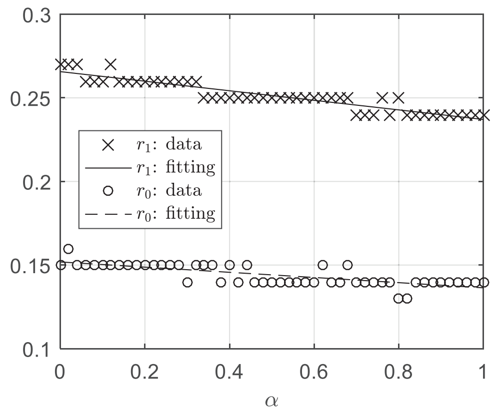

Figure 4. The lower bounded confidence r_1, above which the system may consistently achieve consensus (i.e., $P_C < 1, \forall r < r_1$, and $P_C = 1, \exists r \geq r_1$), as a function of α. The upper bounded confidence r_0, below which the system cannot achieve consensus (i.e., $P_C = 0, \forall r < r_0$, and $P_C > 1, \exists r \geq r_0$), as a function of α. The "data" derives from simulation, while the "fitting" derives from fitting a linear function to "data" using the least squares method.

value of the corresponding curve at each point. Figure 5(b) further shows N_C as a function of α at different r, which tells N_C always decreases with an increase in α; that is, more decentralized agents lead to fewer opinion clusters in the system.

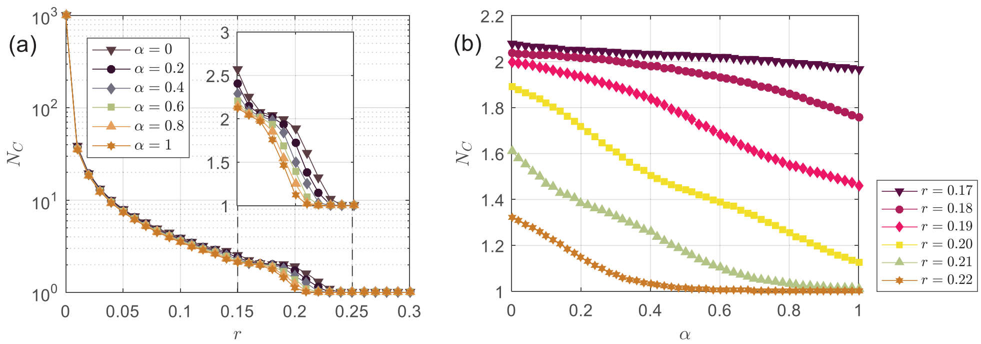

Figure 5. (a) The number of opinion clusters N_C as a function of the bounded confidence r at different α. (b) The number of opinion clusters N_C as a function of the fraction of decentralized agents α at different r.

Let us dig into more details. We show the relative size of the largest opinion cluster $C_{\max}$ as a function of r in Fig. 6(a). With an increase in r, the largest opinion cluster's relative size $C_{\max}$ increases, indicating greater consensus in the system, because more agents gather in the largest opinion cluster. At a larger α, the $C_{\max}$ value of the corresponding curve is greater; that is, decentralized agents facilitate the agents in the system to gather in the largest opinion cluster, forming opinion consensus. The "steplike" behavior can also be observed in the function $C_{\max}$, and the sharp steps appear in $0.15 \lesssim r \lesssim 0.2$. The position of breakpoints is also consistent with those in Fig. 5, where opinion consensus starts to emerge, implying that there is indeed a correlation between the relative size of the largest opinion cluster and the degree of opinion consensus. It is also worth noting that in the "step-like" stage, α has non-monotonous effects on $C_{\max}$, as seen in Fig. 6(b), which is different from most situations observed in Fig. 6(a). Such non-trivial marginal phenomena may be worth exploring in the future.

Furthermore, Fig. 7 presents the distribution of stability opinions $\rho[x_i(T^*)]$ at $r = 0.2$, which provides more details than a relative size of the largest opinion cluster. In Fig. 7(a), $\alpha = 0$. From Fig. 3 and Fig. 5, we have $P_C \approx 0.11$ and $N_C \approx 1.89$. The distribution of stability opinions is mainly polarized, as shown on the two sides in Fig. 7(a). The consensus brings about the less central distribution reached cases. In Fig. 7(b), $\alpha = 1$. We

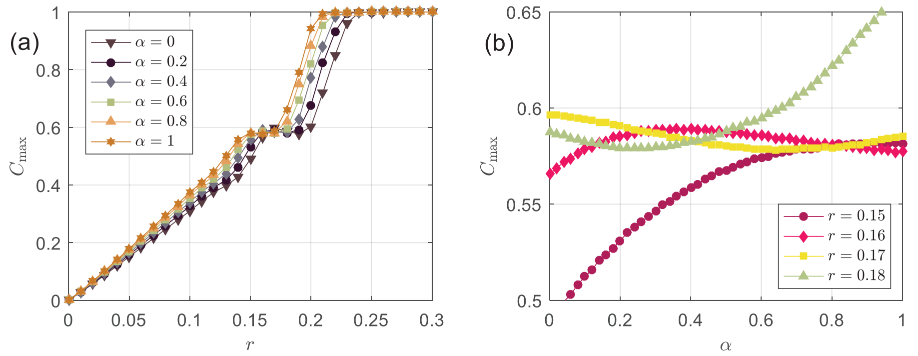

Figure 6. (a) The relative size of the largest opinion cluster $C_{\max}$ as a function of the bounded confidence r at different α. (b) The relative size of the largest opinion cluster $C_{\max}$ as a function of the fraction of decentralized agents α at different r.

have $P_C \approx 0.88$ and $N_C \approx 1.12$ from Fig. 3 and Fig. 5, respectively; opinion consensus takes the big lead. It can be seen from Fig. 7(b) that the distribution on both sides is already sparse, and the opinions are mainly concentrated in the central area, $x_i(T^*) \sim 0.5$. Comparing Fig. 7(a) and (b), we say that more decentralized agents guide the stability opinions toward the central area in opinion space, promoting the opinion consensus.

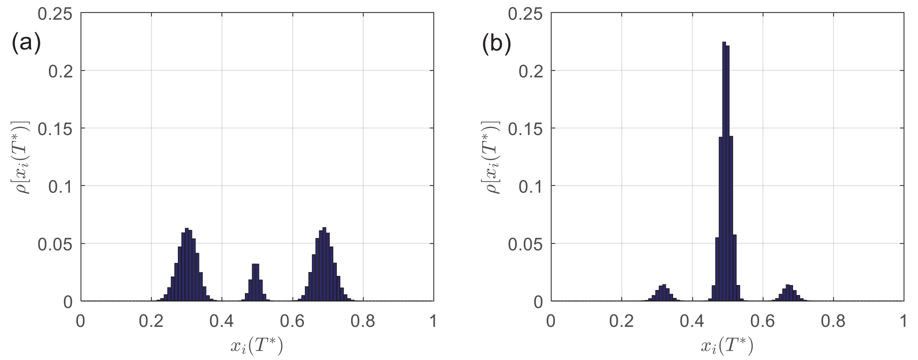

Figure 7. The distribution of stability opinions $\rho[x_i(T^*)]$ at $r = 0.2$ and different α. (a) $\alpha = 0$. (b) $\alpha = 1$. The results are the average of 10^5 independent runs.

Figure 8 shows the distribution of stability opinions $\rho[x_i(T^*)]$ as a function of parameters; that is, the transverse profile in Fig. 8 corresponding to a given vertical coordinate can be drawn in the form of Fig. 7. Figure 8(a) shows $\rho[x_i(T^*)]$ as a function of r at $\alpha = 1$. As r increases, the system tends to consensus, and the stability opinions gradually concentrate towards the center area $x_i(T^*) \sim 0.5$ rather than an evenly distribution $0 < x_i(T^*) < 1$. At a qualitative level, though all agents are decentralized, the pattern in Fig. 8(a) is the same as the classic HK model [16]. Figure 8(b) shows $\rho[x_i(T^*)]$ as a function of α at $r = 0.2$, in which we can observe the process of decentralized agents promoting consensus. Consistent with Fig. 7(a) and (b), with an increase in α, the opinion distribution on the two sides gradually whitens, and the one in the central area fades blue. The stability opinion profile transforms from polarization to consensus.

Finally, we study the convergence time T^* as a binary function of r and α in Fig. 9. The convergence time can also be used as a side measure of the role of decentralized agents on opinion consensus. It can be seen that the relatively time-consuming areas are two banded areas up and down. Looking at it vertically with r, the upper narrower band area corresponds to the region where P_C increases from 0 to 1 in Fig. 3. Looking horizontally at its variation with α, the narrower banded area gradually shifts downward as α increases, and its edges correspond qualitatively to r_0 and r_1 in Fig. 4. This likewise indicates that the convergence time becomes larger in the process of consensus emergence (i.e., $0 < P_C < 1$).

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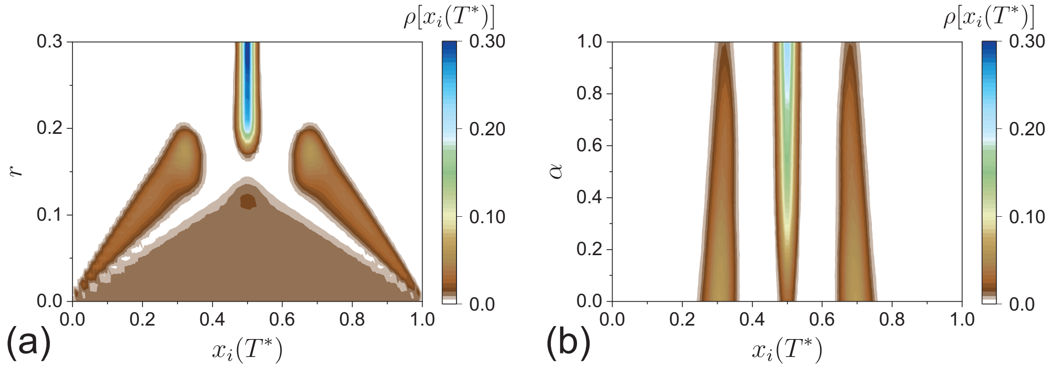

Figure 8. (a) The distribution of stability opinions $\rho[x_i(T^*)]$ as a function of the bounded confidence r at $\alpha = 1$. (b) The distribution of stability opinions $\rho[x_i(T^*)]$ as a function of the fraction of decentralized agents α at $r = 0.2$.

It is concluded from Fig. 9 that, first, decentralized agents accelerate the convergence of opinions. Second, the variation pattern of T^* with r does not change qualitatively with α.

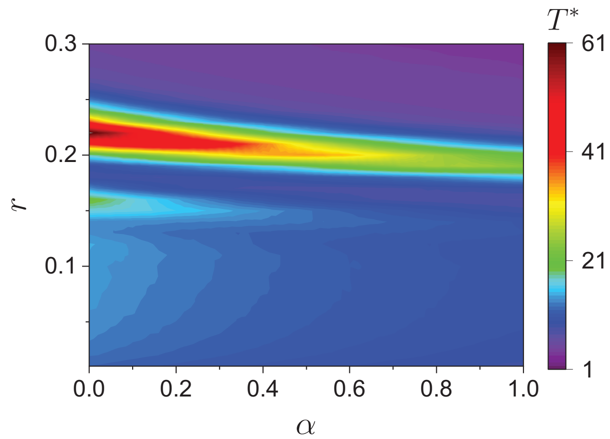

Figure 9. The convergence time T^* as a binary function of the bounded confidence r and the fraction of decentralized agents α.

4. Conclusion

As an extension to the classic bounded confidence model where agents are influenced by peers through pairwise interactions, this paper introduced a possible framework of higher-order bounded confidence. The opinions of agents are influenced by group opinion discussions instead of by peers directly. The microscopic rule in each group discussion can be arbitrary, and we experimented with two underlying rules: centralized and decentralized. The former is equivalent to the classic HK model. From a series of statistic quantities, we showed that the decentralized rule, which represents a higher-order interaction compared with the centralized one, can promote opinion consensus and accelerate opinion convergence. Not surprisingly, the decentralized rule allows the interaction with opinions outside an agent's original bounded confidence, which is somewhat equivalent to enlarging the bounded confidence.

However, the idea borne in the model is more important than simply numerical results. In this work, the focal object for interactions is not agents, but rather groups. The group-based perspective to the classic bounded confidence model may bring the convenience of introducing other group-based dynamics into the bounded confidence model, such as the majority rule and other interdisciplinary dynamics. Since the function $o_i(t) = f(x_j(t) | j \in N_i(t))$ determining the outcome of a single discussion is open-end, the possible microscopic rules to be introduced are extensive.

To sum up, the "higher-order" interaction in this paper has two levels of inspiration. The first level is extending the first-order peers in opinion updating to the second-order, (i.e., the "decentralized" rule). The second level is to reconstruct the classic bounded-confidence model from the group-based perspective.

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Data Availability Statement

The theoretical data used to support the findings of this study are already included in the article.

Declaration of competing interest

None.

1. F. Baumann, P. Lorenz-Spreen, I. M. Sokolov, M. Starnini, Modeling echo chambers and polarization dynamics in social networks, Physical Review Letters 124 (2020) 048301.
2. H. Noorazar, Recent advances in opinion propagation dynamics: a 2020 survey, The European Physical Journal Plus 135 (2020) 1–20.
3. P. Clifford, A. Sudbury, A model for spatial conflict, Biometrika 60 (1973) 581–588.
4. J. Fernández-Gracia, K. Suchecki, J. J. Ramasco, M. San Miguel, V. M. Eguíluz, Is the voter model a model for voters?, Physical Review Letters 112 (2014) 158701.
5. H.-H. Jo, N. Masuda, Finite-size effects on the convergence time in continuous-opinion dynamics, Physical Review E 104 (2021) 014309.
6. K. Sznajd-Weron, J. Sznajd, Opinion evolution in closed community, International Journal of Modern Physics C 11 (2000) 1157–1165.
7. Q. Pan, Y. Qin, Y. Xu, M. Tong, M. He, Opinion evolution in open community, International Journal of Modern Physics C 28 (2017) 1750003.
8. K. Sznajd-Weron, J. Sznajd, T. Weron, A review on the Sznajd model20 years after, Physica A: Statistical Mechanics and its Applications 565 (2021) 125537.
9. T. Cheon, S. Galam, Dynamical Galam model, Physics Letters A 382 (2018) 1509–1515.
10. S. Galam, The Trump phenomenon: An explanation from sociophysics, International Journal of Modern Physics B 31 (2017) 1742015.
11. S. Galam, Will Trump win again in the 2020 election? An answer from a sociophysics model, Physica A: Statistical Mechanics and its Applications 570 (2021) 125835.
12. M. H. DeGroot, Reaching a consensus, Journal of the American Statistical Association 69 (1974) 118–121.
13. N. E. Friedkin, E. C. Johnsen, Social influence and opinions, Journal of Mathematical Sociology 15 (1990) 193–206.
14. H.-Y. Xu, Y.-P. Luo, J.-W. Wu, M.-C. Huang, Hierarchical centralities of information transmissions in reaching a consensus, Physics Letters A 383 (2019) 432–439.
15. G. Deffuant, D. Neau, F. Amblard, G. Weisbuch, Mixing beliefs among interacting agents, Advances in Complex Systems 3 (2000) 87–98.
16. R. Hegselmann, U. Krause, et al., Opinion dynamics and bounded confidence: models, analysis, and simulation, Journal of Artificial Societies and Social Simulation 5 (2002).
17. A. Bhattacharyya, M. Braverman, B. Chazelle, H. L. Nguyen, On the convergence of the Hegselmann–Krause system, in: Proceedings of the 4th conference on Innovations in Theoretical Computer Science, 2013, pp. 61–66.
18. G. Chen, W. Su, W. Mei, F. Bullo, Convergence properties of the heterogeneous Deffuant–Weisbuch model, Automatica 114 (2020) 108825.
19. M. Holzer, R. Khatri, Pattern formation, traveling fronts and consensus versus fragmentation in a model of opinion dynamics, Physics Letters A 381 (2017) 3197–3202.
20. W. Han, Y. Feng, X. Qian, Q. Yang, C. Huang, Clusters and the entropy in opinion dynamics on complex networks, Physica A: Statistical Mechanics and its Applications 559 (2020) 125033.
21. L. Shang, S. Chen, Opinion dynamics with decentralized proportional–integral control strategy, Physica A: Statistical Mechanics and Its Applications 533 (2019) 121916.

22. Y. Zhao, L. Zhang, M. Tang, G. Kou, Bounded confidence opinion dynamics with opinion leaders and environmental noises, Computers & Operations Research 74 (2016) 205–213.
23. X. Chen, S. Zhao, W. Li, Opinion dynamics model based on cognitive styles: Field-dependence and field-independence, Complexity 2019 (2019).
24. J. Hou, W. Li, M. Jiang, Opinion dynamics in modified expressed and private model with bounded confidence, Physica A: Statistical Mechanics and its Applications 574 (2021) 125968.
25. Y. Zhao, M. Xu, Y. Dong, Y. Peng, Fuzzy inference based Hegselmann–Krause opinion dynamics for group decision-making under ambiguity, Information Processing & Management 58 (2021) 102671.
26. W. Han, C. Huang, J. Yang, Opinion clusters in a modified Hegselmann–Krause model with heterogeneous bounded confidences and stubbornness, Physica A: Statistical Mechanics and its Applications 531 (2019) 121791.
27. R. Hegselmann, U. Krause, Opinion dynamics under the influence of radical groups, charismatic leaders, and other constant signals: A simple unifying model, Networks & Heterogeneous Media 10 (2015) 477.
28. H. Z. Brooks, M. A. Porter, A model for the influence of media on the ideology of content in online social networks, Physical Review Research 2 (2020) 023041.
29. M. Pineda, R. Toral, E. Hernández-García, The noisy Hegselmann-Krause model for opinion dynamics, The European Physical Journal B 86 (2013) 1–10.
30. B. Chazelle, Q. Jiu, Q. Li, C. Wang, Well-posedness of the limiting equation of a noisy consensus model in opinion dynamics, Journal of Differential Equations 263 (2017) 365–397.
31. W. Su, G. Chen, Y. Hong, Noise leads to quasi-consensus of Hegselmann–Krause opinion dynamics, Automatica 85 (2017) 448–454.
32. G. Chen, W. Su, S. Ding, Y. Hong, Heterogeneous Hegselmann–Krause dynamics with environment and communication noise, IEEE Transactions on Automatic Control 65 (2019) 3409–3424.
33. C. Huang, Q. Dai, W. Han, Y. Feng, H. Cheng, H. Li, Effects of heterogeneous convergence rate on consensus in opinion dynamics, Physica A: Statistical Mechanics and its Applications 499 (2018) 428–435.
34. G. Fu, W. Zhang, Z. Li, Opinion dynamics of modified Hegselmann–Krause model in a group-based population with heterogeneous bounded confidence, Physica A: Statistical Mechanics and its Applications 419 (2015) 558–565.
35. C. Cheng, C. Yu, Opinion dynamics with bounded confidence and group pressure, Physica A: Statistical Mechanics and its Applications 532 (2019) 121900.
36. L. Liu, X. Wang, X. Chen, S. Tang, Z. Zheng, Modeling confirmation bias and peer pressure in opinion dynamics, Frontiers in Physics 9 (2021) 120.
37. G. Chen, H. Cheng, C. Huang, W. Han, Q. Dai, H. Li, J. Yang, Deffuant model on a ring with repelling mechanism and circular opinions, Physical Review E 95 (2017) 042118.
38. S. Gao, C. Huang, W. Han, J. Yang, General consensus with circular opinion under attractive and repulsive mechanisms, International Journal of Modern Physics C 31 (2020) 2050068.
39. J. C. Dittmer, Consensus formation under bounded confidence, Nonlinear Analysis: Theory, Methods & Applications 47 (2001) 4615–4621.
40. F. Vasca, C. Bernardo, R. Iervolino, Practical consensus in bounded confidence opinion dynamics, Automatica 129 (2021) 109683.
41. Y. Liu, F. Xiong, J. Zhu, Y. Zhang, External activation promoting consensus formation in the opinion model with interest decay, Physics Letters A 377 (2013) 362–366.
42. X. Liu, C. Huang, H. Li, Q. Dai, J. Yang, The combination of pairwise and group interactions promotes consensus in opinion dynamics, Complexity 2021 (2021).
43. F. Battiston, G. Cencetti, I. Iacopini, V. Latora, M. Lucas, A. Patania, J.-G. Young, G. Petri, Networks beyond pairwise interactions: structure and dynamics, Physics Reports 874 (2020) 1–92.
44. F. Battiston, E. Amico, A. Barrat, G. Bianconi, G. Ferraz de Arruda, B. Franceschiello, I. Iacopini, S. Kéfi, V. Latora, Y. Moreno, et al., The physics of higher-order interactions in complex systems, Nature Physics 17 (2021) 1093–1098.
45. S. Majhi, M. Perc, D. Ghosh, Dynamics on higher-order networks: A review, Journal of the Royal Society Interface 19 (2022) 20220043.
46. I. Iacopini, G. Petri, A. Barrat, V. Latora, Simplicial models of social contagion, Nature Communications 10 (2019) 1–9.
47. G. F. de Arruda, G. Petri, Y. Moreno, Social contagion models on hypergraphs, Physical Review Research 2 (2020) 023032.

48. N. W. Landry, J. G. Restrepo, The effect of heterogeneity on hypergraph contagion models, Chaos: An Interdisciplinary Journal of Nonlinear Science 30 (2020) 103117.
49. G. Burgio, J. T. Matamalas, S. Gómez, A. Arenas, Evolution of cooperation in the presence of higher-order interactions: from networks to hypergraphs, Entropy 22 (2020) 744.
50. U. Alvarez-Rodriguez, F. Battiston, G. F. de Arruda, Y. Moreno, M. Perc, V. Latora, Evolutionary dynamics of higher-order interactions in social networks, Nature Human Behaviour 5 (2021) 586–595.
51. H. Guo, D. Jia, I. Sendiña-Nadal, M. Zhang, Z. Wang, X. Li, K. Alfaro-Bittner, Y. Moreno, S. Boccaletti, Evolutionary games on simplicial complexes, Chaos, Solitons & Fractals 150 (2021) 111103.
52. M. Perc, J. Gómez-Gardenes, A. Szolnoki, L. M. Floría, Y. Moreno, Evolutionary dynamics of group interactions on structured populations: a review, Journal of the Royal Society Interface 10 (2013) 20120997.
53. L. Neuhäuser, A. Mellor, R. Lambiotte, Multibody interactions and nonlinear consensus dynamics on networked systems, Physical Review E 101 (2020) 032310.
54. R. Sahasrabuddhe, L. Neuhäuser, R. Lambiotte, Modelling non-linear consensus dynamics on hypergraphs, Journal of Physics: Complexity 2 (2021) 025006.
55. A. Hickok, Y. Kureh, H. Z. Brooks, M. Feng, M. A. Porter, A bounded-confidence model of opinion dynamics on hypergraphs, SIAM Journal on Applied Dynamical Systems 21 (2022) 1–32.
56. L. Horstmeyer, C. Kuehn, Adaptive voter model on simplicial complexes, Physical Review E 101 (2020) 022305.