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Article

# Analytic Speed Limits and Extremal Rigidity for Strip-Holomorphic Monotone Profiles

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## Abstract

We study real-valued transition profiles on the real axis that admit holomorphic extension to a horizontal strip in the complex plane. The functions considered have a continuously differentiable and nondecreasing real trace and are normalized to take values strictly between zero and one. We assume that the associated conformal transform obtained by rescaling and shifting the profile extends holomorphically to the strip and maps it into the unit disk. Under these conditions strip analyticity imposes a sharp pointwise bound on the rate of change along the real axis. The bound depends only on the width of the analytic strip and is optimal. We further prove a rigidity result: if the bound is attained at any real point then the profile is uniquely determined up to translation and coincides with the logistic transition. The argument is purely analytic and follows from the Schwarz–Pick contraction principle applied to the strip geometry. No classification of non-saturating profiles is attempted.

**Keywords:** analytic speed limits; strip holomorphy; monotone transition profiles; Schwarz–Pick inequality; strip–disk conformal geometry; rigidity; extremal profiles

## Reader’s Guide

This paper studies real-valued transition profiles on the real axis that admit holomorphic extension to a horizontal strip in the complex plane. The real trace is assumed to be continuously differentiable, nondecreasing, and normalized so that the profile takes values strictly between zero and one. The associated conformal transform obtained by rescaling and shifting the profile extends holomorphically to the strip and maps the strip into the unit disk.

Two structural questions arise in this setting. First, how rapidly can such a profile change along the real axis? Second, when is that maximal rate of change realized?

The first main result establishes a sharp pointwise derivative bound along the real axis. The constant in this bound depends only on the width of the analytic strip. The second main result is a rigidity theorem. If the bound is attained at a real point, then the profile is uniquely determined up to translation and coincides with the logistic transition.

The argument proceeds through strip–disk conformal transfer together with the Schwarz–Pick contraction principle. No dynamical model is assumed. The paper focuses on the extremal structure imposed by strip analyticity and does not attempt to classify non-saturating profiles.

## 1. Introduction

This paper studies real-valued transition profiles on the real axis that admit holomorphic extension to a horizontal strip in the complex plane. The analytic setting may be viewed as the natural envelope of functions whose underlying generators possess bounded spectral support; in that case Paley–Wiener theory implies strip holomorphy together with controlled exponential growth. The analysis developed here is abstract and does not rely on a specific spectral realization.

We consider functions whose real trace is continuously differentiable, nondecreasing, and normalized so that the profile takes values strictly between zero and one. The associated conformal transform obtained by rescaling and shifting the profile extends holomorphically to the strip and maps the strip

into the unit disk. Within this class strip analyticity imposes a sharp constraint on the rate at which the profile can change along the real axis.

Two structural results are established. First, every admissible profile satisfies a sharp pointwise derivative bound whose constant depends only on the width of the analyticity strip. Second, if this bound is attained at a real point then the profile is uniquely determined up to translation and coincides with the logistic transition.

The analytic mechanism is classical and follows from the Schwarz–Pick contraction principle applied to strip geometry. The proof proceeds by transferring the strip domain to the unit disk through a conformal map and applying the Schwarz–Pick inequality in that setting. The contribution of this paper is structural: we isolate the differential envelope governing normalized strip-holomorphic monotone profiles and identify the unique extremizer.

The scope is intentionally narrow. No classification of non-saturating profiles is attempted; no statement is made regarding typicality or approximate saturation. The rigidity theorem applies only under the explicit extremality hypothesis described above.

## 2. Analytic Setting and Admissible Class

This section fixes notation and hypotheses. All subsequent results depend only on the conditions stated here.

### 2.1. Horizontal Strip

Let  $a > 0$  and define the horizontal strip

$$\mathcal{S}_a := \{ \tau \in \mathbb{C} : |\Im \tau| < a \}.$$

Real variables are identified with the real axis  $\mathbb{R} \subset \mathcal{S}_a$ .

### 2.2. Admissible Class

Define  $\mathcal{F}_a$  to be the class of functions  $F : \mathbb{R} \rightarrow (0, 1)$  satisfying the following conditions.

1. **Strip holomorphy.** The function  $F$  admits a holomorphic extension, still denoted  $F$ , to the strip  $\mathcal{S}_a$ .
2. **Boundary regularity.** The restriction  $F|_{\mathbb{R}}$  lies in  $C^1(\mathbb{R})$ .
3. **Monotone transition.** The function  $F$  is nondecreasing on  $\mathbb{R}$ .
4. **Normalized limits.**

$$\lim_{\tau \rightarrow -\infty} F(\tau) = 0, \quad \lim_{\tau \rightarrow +\infty} F(\tau) = 1.$$

5. **Disk-bounded conformal transform.** Define the conformal transform

$$G(\tau) := 2F(\tau) - 1.$$

The holomorphic extension of  $G$  to  $\mathcal{S}_a$  satisfies

$$G(\mathcal{S}_a) \subset \mathbb{D},$$

where  $\mathbb{D}$  denotes the unit disk.

6. **Exponential-type growth bound.** There exist constants  $C > 0$  and  $\Lambda \geq 0$  such that

$$|F(\tau)| \leq C e^{\Lambda |\Im \tau|}, \quad \text{for all } \tau \in \mathcal{S}_a.$$

These conditions define the admissible class  $\mathcal{F}_a$  studied in the remainder of the paper.

The disk-bounded condition ensures that the strip geometry may be transferred to the unit disk through a conformal map. This structure permits the use of the Schwarz–Pick contraction principle in the derivative bound proved below.

The exponential-type growth condition reflects the operator-theoretic origin of the strip domain. In applications arising from bounded spectral support, Paley–Wiener theory yields strip holomorphy together with exponential-type control.

### 2.3. Remark on Normalization

The strip half-width  $a$  and the exponential-type parameter  $\Lambda$  are linked in spectral applications through a Paley–Wiener constraint, with  $a$  inversely proportional to the spectral cutoff. Different choices of  $a$  correspond to affine reparameterizations of the real variable  $\tau$ . Since the results below are invariant under affine changes of  $\tau$ , the particular normalization of  $a$  does not affect the derivative bound or the rigidity statement.

## 3. Paley–Wiener Strip from Bounded Spectrum

This section records the standard implication from bounded spectral support to strip analyticity and exponential-type control. Its role is preparatory: it explains how the analytic strip used in the admissible class arises in spectral settings. No derivative bounds or rigidity statements are invoked here.

### 3.1. Strip Analyticity from Compact Spectral Support

**Lemma 1** (Strip analyticity and growth from bounded spectrum). *Let  $K$  be a self-adjoint operator on a Hilbert space  $\mathcal{H}$  with bounded spectrum*

$$\sigma(K) \subset [-\Lambda_K, \Lambda_K],$$

for some  $\Lambda_K > 0$ . Fix  $\psi \in \mathcal{H}$  and define

$$f(\tau) := \langle \psi, e^{iK\tau} \psi \rangle, \quad \tau \in \mathbb{R}.$$

Then  $f$  extends to a holomorphic function on the strip

$$\{ \tau \in \mathbb{C} : |\Im \tau| < a \}, \quad a = \frac{\pi}{2\Lambda_K},$$

and satisfies the exponential-type bound

$$|f(\tau)| \leq \|\psi\|^2 e^{\Lambda_K |\Im \tau|}, \quad \tau \in \mathcal{S}_a.$$

**Proof.** Let  $\tau = x + iy$  with  $x, y \in \mathbb{R}$ . By the spectral theorem there exists a projection-valued measure  $E(\lambda)$  such that

$$e^{iK\tau} = \int_{-\Lambda_K}^{\Lambda_K} e^{i\lambda(x+iy)} dE(\lambda) = \int_{-\Lambda_K}^{\Lambda_K} e^{i\lambda x} e^{-\lambda y} dE(\lambda).$$

Hence

$$f(\tau) = \int_{-\Lambda_K}^{\Lambda_K} e^{i\lambda x} e^{-\lambda y} d\mu_\psi(\lambda), \quad d\mu_\psi(\lambda) := \langle \psi, dE(\lambda) \psi \rangle,$$

where  $\mu_\psi$  is a finite positive measure with total mass

$$\mu_\psi(\mathbb{R}) = \|\psi\|^2.$$

For fixed  $y$  with  $|y| < a$  the integrand is continuous in  $x$  and bounded by an  $L^1(d\mu_\psi)$  function, so  $f$  is continuous on the strip. Holomorphy follows from differentiation under the integral sign on compact subsets of the strip, justified by uniform domination. Finally, for  $\tau = x + iy$ ,

$$|f(\tau)| \leq \int_{-\Lambda_K}^{\Lambda_K} |e^{i\lambda x} e^{-\lambda y}| d\mu_\psi(\lambda) \leq e^{\Lambda_K |y|} \int_{-\Lambda_K}^{\Lambda_K} d\mu_\psi(\lambda) = \|\psi\|^2 e^{\Lambda_K |\Im \tau|}.$$

□

### 3.2. Remark and Reference

Lemma 1 is a standard operator-theoretic form of the Paley–Wiener principle: bounded spectral support implies holomorphic extension to a strip together with exponential-type growth control. Classical references include Rudin [3]. The lemma is included only to motivate the appearance of the strip domain  $\mathcal{S}_a$  and the growth condition used in the admissible class.

## 4. Conformal Mapping and Schwarz–Pick Inequality

This section transfers the Schwarz–Pick inequality from the unit disk to the horizontal strip  $\mathcal{S}_a$  through an explicit conformal map. The resulting estimate produces the derivative bound that governs admissible monotone transition profiles on the real axis.

### 4.1. Strip-to-Disk Conformal Map

Define  $\Phi : \mathcal{S}_a \rightarrow \mathbb{D}$  by

$$\Phi(\tau) := \tanh\left(\frac{\pi}{4a}\tau\right).$$

Then  $\Phi$  is biholomorphic with inverse

$$\Phi^{-1}(z) = \frac{4a}{\pi} \operatorname{artanh}(z), \quad z \in \mathbb{D}.$$

Its derivative is

$$\Phi'(\tau) = \frac{\pi}{4a} (1 - \Phi(\tau)^2).$$

### 4.2. Schwarz–Pick on the Strip

**Lemma 2** (Schwarz–Pick inequality on  $\mathcal{S}_a$ ). *Let  $G : \mathcal{S}_a \rightarrow \mathbb{D}$  be holomorphic. Then for every  $\tau \in \mathcal{S}_a$ ,*

$$|G'(\tau)| \leq \frac{\pi}{4a} \frac{1 - |G(\tau)|^2}{1 - |\Phi(\tau)|^2} |1 - \Phi(\tau)^2|.$$

In particular, for  $\tau \in \mathbb{R}$ ,

$$|G'(\tau)| \leq \frac{\pi}{4a} (1 - |G(\tau)|^2).$$

**Proof.** Define  $H := G \circ \Phi^{-1} : \mathbb{D} \rightarrow \mathbb{D}$ , which is holomorphic. By the Schwarz–Pick inequality on the disk,

$$|H'(z)| \leq \frac{1 - |H(z)|^2}{1 - |z|^2}, \quad z \in \mathbb{D}.$$

Let  $z = \Phi(\tau)$ . By the chain rule,

$$G'(\tau) = H'(z) \Phi'(\tau).$$

Using  $H(z) = G(\tau)$ ,  $|z| = |\Phi(\tau)|$ , and  $\Phi'(\tau) = \frac{\pi}{4a} (1 - \Phi(\tau)^2)$ , we obtain

$$|G'(\tau)| \leq \frac{1 - |G(\tau)|^2}{1 - |\Phi(\tau)|^2} \frac{\pi}{4a} |1 - \Phi(\tau)^2|.$$

If  $\tau \in \mathbb{R}$  then  $\Phi(\tau) \in (-1, 1)$ , hence  $|\Phi(\tau)|^2 = \Phi(\tau)^2$  and

$$\frac{|1 - \Phi(\tau)^2|}{1 - |\Phi(\tau)|^2} = 1.$$

Therefore

$$|G'(\tau)| \leq \frac{\pi}{4a} (1 - |G(\tau)|^2),$$

which proves the lemma.  $\square$

### 4.3. Speed-Limit Inequality for Admissible Profiles

Let  $F \in \mathcal{F}_a$  and define

$$G(\tau) := 2F(\tau) - 1.$$

By condition (5) in the definition of the admissible class, the holomorphic extension of  $G$  maps  $\mathcal{S}_a$  into  $\mathbb{D}$ .

**Theorem 1** (Speed-limit inequality). *Let  $F \in \mathcal{F}_a$ . Then for every  $\tau \in \mathbb{R}$ ,*

$$0 \leq F'(\tau) \leq \frac{\pi}{a} F(\tau)(1 - F(\tau)).$$

**Proof.** Apply Lemma 2 on the real axis to the holomorphic map  $G : \mathcal{S}_a \rightarrow \mathbb{D}$ . Since  $F$  is nondecreasing on  $\mathbb{R}$  we have  $F'(\tau) \geq 0$ , hence  $G'(\tau) = 2F'(\tau) \geq 0$ . Using the real-axis estimate from Lemma 2 we obtain

$$2F'(\tau) = G'(\tau) \leq \frac{\pi}{4a} (1 - G(\tau)^2).$$

Since  $G(\tau) = 2F(\tau) - 1$ , we compute

$$1 - G(\tau)^2 = 4F(\tau)(1 - F(\tau)).$$

Substituting yields

$$2F'(\tau) \leq \frac{\pi}{4a} \cdot 4F(\tau)(1 - F(\tau)),$$

which simplifies to

$$F'(\tau) \leq \frac{\pi}{a} F(\tau)(1 - F(\tau)).$$

The inequality  $F'(\tau) \geq 0$  follows from monotonicity.  $\square$

## 5. Extremal Rigidity of the Saturating Profile

This section proves rigidity under an explicit extremality restriction. We consider admissible profiles that saturate the speed-limit inequality pointwise on  $\mathbb{R}$ . No rigidity statement is made for non-saturating profiles.

### 5.1. Extremality Restriction

**Definition 1** (Pointwise saturation). *An admissible profile  $F \in \mathcal{F}_a$  is called extremal if it saturates the speed-limit inequality of Theorem 1 pointwise on  $\mathbb{R}$ , meaning*

$$F'(\tau) = \frac{\pi}{a} F(\tau)(1 - F(\tau)), \quad \forall \tau \in \mathbb{R}.$$

### 5.2. Schwarz–Pick Equality and Automorphisms

We record the standard equality case of the Schwarz–Pick lemma; see for example Conway [2].

**Lemma 3** (Schwarz–Pick equality implies automorphism). *Let  $H : \mathbb{D} \rightarrow \mathbb{D}$  be holomorphic. If there exists  $z_0 \in \mathbb{D}$  such that equality holds in Schwarz–Pick at  $z_0$  with  $H'(z_0) \neq 0$ , that is,*

$$|H'(z_0)| = \frac{1 - |H(z_0)|^2}{1 - |z_0|^2},$$

*then  $H$  is a disk automorphism. Equivalently, there exist  $\theta \in \mathbb{R}$  and  $\alpha \in \mathbb{D}$  such that*

$$H(z) = e^{i\theta} \frac{z - \alpha}{1 - \bar{\alpha}z}, \quad z \in \mathbb{D}.$$



### 5.3. Rigidity of the Extremal Saturator

**Theorem 2** (Rigidity of the unique saturating profile). *Let  $F \in \mathcal{F}_a$  be extremal in the sense of Definition 1. Then there exists  $\tau_0 \in \mathbb{R}$  such that*

$$F(\tau) = \frac{1}{2} \left( 1 + \tanh\left(\frac{\pi}{2a}(\tau - \tau_0)\right) \right), \quad \tau \in \mathbb{R}.$$

*In particular, the extremal profile is unique up to translation of  $\tau$ .*

**Proof.** Define

$$G(\tau) := 2F(\tau) - 1, \quad H := G \circ \Phi^{-1} : \mathbb{D} \rightarrow \mathbb{D},$$

where  $\Phi(\tau) = \tanh\left(\frac{\pi}{4a}\tau\right)$  is the strip-to-disk conformal map from Section 4.1. Since  $F \in \mathcal{F}_a$ , the map  $G$  is holomorphic on  $\mathcal{S}_a$  and satisfies  $G(\mathcal{S}_a) \subset \mathbb{D}$ . Hence  $H$  is holomorphic from  $\mathbb{D}$  to  $\mathbb{D}$ .

Step 1: Saturation forces Schwarz–Pick equality.

The inequality in Theorem 1 is obtained by applying Schwarz–Pick on  $\mathbb{D}$  to  $H$  and transferring the estimate back through  $\Phi$ . Under the extremality assumption,

$$F'(\tau) = \frac{\pi}{a} F(\tau)(1 - F(\tau)) \quad \text{for all } \tau \in \mathbb{R},$$

the transferred Schwarz–Pick bound is saturated pointwise on  $\mathbb{R}$ . Since  $F$  is nondecreasing we have  $F'(\tau) \geq 0$ , and hence  $G'(\tau) = 2F'(\tau) \geq 0$ . Choosing any  $\tau_1 \in \mathbb{R}$  with  $G'(\tau_1) \neq 0$  and setting  $z_1 = \Phi(\tau_1) \in (-1, 1) \subset \mathbb{D}$ , equality holds in Schwarz–Pick at  $z_1$  with  $H'(z_1) \neq 0$ .

Step 2:  $H$  is a disk automorphism.

By Lemma 3,  $H$  is a disk automorphism. Therefore

$$H(z) = e^{i\theta} \frac{z - \alpha}{1 - \bar{\alpha}z}, \quad \theta \in \mathbb{R}, \alpha \in \mathbb{D}.$$

Step 3: Real-axis monotonicity restricts the automorphism.

For  $\tau \in \mathbb{R}$  we have  $\Phi(\tau) \in (-1, 1)$  and  $G(\tau) \in (-1, 1)$ . Hence  $H$  maps  $(-1, 1)$  into itself. This forces  $\alpha \in (-1, 1)$  and  $e^{i\theta} = 1$ . Consequently

$$H(z) = \frac{z - \alpha}{1 - \alpha z}, \quad \alpha \in (-1, 1).$$

Step 4: Recover the extremal form.

Substituting  $z = \Phi(\tau)$  yields

$$G(\tau) = \frac{\Phi(\tau) - \alpha}{1 - \alpha \Phi(\tau)}.$$

Let  $\beta := \operatorname{artanh}(\alpha) \in \mathbb{R}$ . Using the identity

$$\frac{\tanh(x) - \tanh(\beta)}{1 - \tanh(\beta) \tanh(x)} = \tanh(x - \beta),$$

we obtain

$$G(\tau) = \tanh\left(\frac{\pi}{4a}(\tau - \tau_0)\right), \quad \tau_0 := \frac{4a}{\pi} \beta.$$

Since  $F = (1 + G)/2$ , this yields

$$F(\tau) = \frac{1}{2} \left( 1 + \tanh\left(\frac{\pi}{2a}(\tau - \tau_0)\right) \right),$$

which proves the claim.

Uniqueness.

The parameter  $\alpha \in (-1, 1)$  determines a unique shift  $\tau_0 \in \mathbb{R}$ . Therefore the extremal profile is unique up to translation of the real variable.  $\square$

## 6. Geometric Origin of the Logistic Extremizer

This section explains why the logistic transition arises as the unique extremizer in the rigidity theorem. The argument introduces no new theorems. It interprets the preceding results in terms of the hyperbolic geometry of the strip and the unit disk.

### 6.1. Strip–Disk Correspondence

The horizontal strip

$$\mathcal{S}_a = \{\tau \in \mathbb{C} : |\Im \tau| < a\}$$

is conformally equivalent to the unit disk. A convenient biholomorphic map is

$$\Phi(\tau) = \tanh\left(\frac{\pi}{4a}\tau\right),$$

which sends  $\mathcal{S}_a$  onto  $\mathbb{D}$  and maps the real axis onto the real diameter  $(-1, 1) \subset \mathbb{D}$ .

Under this transformation a holomorphic map

$$G : \mathcal{S}_a \rightarrow \mathbb{D}$$

corresponds to a disk map

$$H := G \circ \Phi^{-1} : \mathbb{D} \rightarrow \mathbb{D}.$$

Derivative estimates on the strip therefore arise from the Schwarz–Pick contraction principle on the disk.

### 6.2. Extremals of Schwarz–Pick

The Schwarz–Pick inequality states that every holomorphic self-map of the disk contracts the hyperbolic metric. Equality occurs only for disk automorphisms. These maps have the form

$$H(z) = e^{i\theta} \frac{z - \alpha}{1 - \bar{\alpha}z}, \quad \alpha \in \mathbb{D}, \theta \in \mathbb{R}.$$

When the extremality condition of Section 5 holds, the Schwarz–Pick inequality is saturated for the transferred map  $H$ . Consequently  $H$  must be a disk automorphism.

### 6.3. Pullback to the Strip

The real-axis monotonicity of admissible profiles implies that  $H$  preserves the real diameter of the disk. Hence the automorphism reduces to

$$H(z) = \frac{z - \alpha}{1 - \alpha z}, \quad \alpha \in (-1, 1).$$

Pulling this map back to the strip gives

$$G(\tau) = H(\Phi(\tau)) = \frac{\Phi(\tau) - \alpha}{1 - \alpha \Phi(\tau)}.$$

Using the hyperbolic tangent identity

$$\frac{\tanh(x) - \tanh(\beta)}{1 - \tanh(\beta) \tanh(x)} = \tanh(x - \beta),$$



one obtains

$$G(\tau) = \tanh\left(\frac{\pi}{4a}(\tau - \tau_0)\right)$$

for some real parameter  $\tau_0$ .

Since  $F = (1 + G)/2$ , the real-axis profile becomes

$$F(\tau) = \frac{1}{2}\left(1 + \tanh\left(\frac{\pi}{2a}(\tau - \tau_0)\right)\right).$$

6.4. Geometric Interpretation

Through the conformal map  $\Phi$ , the strip  $\mathcal{S}_a$  inherits the Poincaré hyperbolic metric of the unit disk. The Schwarz–Pick lemma expresses that holomorphic maps are contractions with respect to this metric. Equality occurs only for hyperbolic isometries.

Under the extremality condition the transferred map  $H$  becomes a disk automorphism, hence a hyperbolic isometry. The corresponding trajectory in the strip therefore follows a hyperbolic geodesic of  $\mathcal{S}_a$ . The logistic transition obtained above is precisely the real-axis trace of this geodesic.

The parameter  $a$  controls the width of the analytic continuation region. Through the conformal equivalence with the disk it fixes the maximal rate at which a monotone analytic transition can occur along the real axis. The speed-limit inequality of Section 4 expresses this geometric constraint in differential form.

7. Discussion and Scope

This section summarizes the results and clarifies their scope.

7.1. Summary of Results

Under the hypotheses fixed in Section 2, two structural statements have been established.

First, every admissible profile in  $\mathcal{F}_a$  satisfies a sharp analytic derivative bound on the real axis. The constant in this bound depends only on the strip half-width  $a$ . The estimate follows directly from strip holomorphy together with the Schwarz–Pick contraction principle and does not rely on any dynamical model.

Second, rigidity holds under extremality. If an admissible profile saturates the derivative bound pointwise on  $\mathbb{R}$ , then the profile is uniquely determined up to translation of the real variable. The resulting extremal profile is the logistic transition obtained by conjugating a real disk automorphism through the strip–disk conformal map.

7.2. Limitations of the Analysis

The present work deliberately restricts its scope.

No classification of non-saturating profiles is attempted. Apart from the derivative bound, profiles that do not saturate the inequality remain unconstrained by the present analysis.

No statement is made regarding typicality, stability, or dynamical selection of extremal profiles. The rigidity theorem applies only under the explicit hypothesis of pointwise saturation.

7.3. Relation to Bounded Spectral Generators

The analytic hypotheses mirror conditions that arise when an underlying generator has bounded spectrum. In that setting, the strip width is fixed by the spectral cutoff through Paley–Wiener theory, and the derivative bound acquires a direct interpretation as a constraint imposed by analytic continuation width.

The present paper isolates this analytic structure independently of any specific realization. For a realization in the context of bounded modular flow, see [9].

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## Appendix A. Schwarz–Pick Rigidity and Standard References

This appendix records standard results concerning the Schwarz–Pick inequality and its equality case. The statements are classical and are included only for completeness.

### Appendix A.1. Schwarz–Pick Lemma

**Lemma A1** (Schwarz–Pick). *Let  $H : \mathbb{D} \rightarrow \mathbb{D}$  be holomorphic. Then for all  $z \in \mathbb{D}$ ,*

$$|H'(z)| \leq \frac{1 - |H(z)|^2}{1 - |z|^2}.$$

### Appendix A.2. Equality Case and Rigidity

**Lemma A2** (Schwarz–Pick rigidity). *Let  $H : \mathbb{D} \rightarrow \mathbb{D}$  be holomorphic. Suppose there exists  $z_0 \in \mathbb{D}$  such that*

$$|H'(z_0)| = \frac{1 - |H(z_0)|^2}{1 - |z_0|^2},$$

*and  $H'(z_0) \neq 0$ . Then  $H$  is a disk automorphism. Equivalently, there exist  $\theta \in \mathbb{R}$  and  $\alpha \in \mathbb{D}$  such that*

$$H(z) = e^{i\theta} \frac{z - \alpha}{1 - \bar{\alpha}z}, \quad z \in \mathbb{D}.$$

### Appendix A.3. Restriction to Real Automorphisms

**Lemma A3** (Real-axis preservation). *Let  $H : \mathbb{D} \rightarrow \mathbb{D}$  be a disk automorphism that maps the real diameter  $(-1, 1)$  into itself. Then  $H$  has real coefficients and is of the form*

$$H(z) = \frac{z - \alpha}{1 - \alpha z}, \quad \alpha \in (-1, 1).$$

*If  $H$  is strictly increasing on  $(-1, 1)$ , this form is uniquely determined.*

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