

Article

Not peer-reviewed version

Polar Codes For 6G And Beyond Wireless Quantum Optical Communications

[Peter Jung](#)*, [Kushtrim Dini](#), [Faris Abdel Rehim](#), [Hamza Almujaheed](#)

Posted Date: 18 August 2025

doi: 10.20944/preprints202508.1258.v1

Keywords: 6G; Kohonen maps based quantum data detection; log-likelihood ratio (LLR); optimum quantum data detection; polar codes; polar code design; polar sequence; reduced state quantum data detection; soft outputs; turbo equalization with polar codes; wireless quantum optical communications



Preprints.org is a free multidisciplinary platform providing preprint service that is dedicated to making early versions of research outputs permanently available and citable. Preprints posted at Preprints.org appear in Web of Science, Crossref, Google Scholar, Scilit, Europe PMC.

Copyright: This open access article is published under a Creative Commons CC BY 4.0 license, which permit the free download, distribution, and reuse, provided that the author and preprint are cited in any reuse.

Disclaimer/Publisher's Note: The statements, opinions, and data contained in all publications are solely those of the individual author(s) and contributor(s) and not of MDPI and/or the editor(s). MDPI and/or the editor(s) disclaim responsibility for any injury to people or property resulting from any ideas, methods, instructions, or products referred to in the content.

Article

Polar Codes for 6G And Beyond Wireless Quantum Optical Communications [†]

Peter Jung *, Kushtrim Dini, Faris Abdel Rehim and Hamza Almujaheed

Lehrstuhl für Kommunikationstechnik, Universität Duisburg-Essen, Duisburg, Germany

* Correspondence: peter.jung@kommunikationstechnik.org

[†] This work has been solely founded by the Universität Duisburg-Essen.

Abstract

Wireless communication applications above 300 GHz need careful analog electronics design that takes into account the frequency-dependent nature of ohmic resistance at these frequencies. The cumbersome development of electronics brings quantum optical communication solutions for the sixth generation (6G) THz band located between 300 GHz and 10 THz into focus. In this manuscript, the authors propose to replace the classical radio frequency based inner physical layer transceiver blocks used in classical channel coded short range wireless communication systems by wireless quantum optical communication concepts. Besides discussing the resulting generic concept of the wireless quantum optical communications and illustrating optimum quantum data detection schemes, novel reduced state quantum data detection and novel Kohonen maps based quantum data detection, will be addressed. All the considered quantum data detection schemes provide soft outputs required for the lowest possible block error ratio (BLER) at the output of the channel decoding. Furthermore, a novel polar codes design approach determining the polar sequence by appropriately combining already available polar sequences tailored for low BLER is presented for the first time after illustrating the basics of polar codes. In addition, turbo equalization for wireless quantum optical communications using polar codes will be presented, for the first time explicitly stating the generation of soft information associated with the codebits and introducing a novel scheme for the computation of extrinsic soft outputs to be used in the turbo equalization iterations. New simulation results emphasize the viability of the theoretical concepts.

Keywords: 6G; Kohonen maps based quantum data detection; log-likelihood ratio (LLR); optimum quantum data detection; polar codes; polar code design; polar sequence; reduced state quantum data detection; soft outputs, turbo equalization with polar codes; wireless quantum optical communications

1. Introduction

Recent studies [1] have illustrated that radio applications operating above 300 GHz, which is approximately equal to that particular natural frequency at the low end of the far infrared radiation [2][Table E1-1, p. 208], [3], require meticulous analog design. This is due to the frequency-dependent behavior of ohmic resistance at such high frequencies [1], which brings optical communication solutions for the sixth generation (6G) THz band [300 GHz, 10 THz] into sharp focus. Following the experimental demonstration of quantum supremacy [4], there has been a significant surge of interest in wireless quantum optical communication concepts. These aim to replace conventional radio frequency-based transceivers at the inner physical layer of classical channel coded wireless systems. Generally, quantum communications means the transmission of classical information over a quantum channel, where the information is encoded in quantum states [5–7]. In future generations of wireless communication systems, the aforementioned growing popularity may suggest the replacement of mixed signal and analog components in the transceivers for mobile radio systems by components relying on quantum optics.

The basic block diagram of the resulting physical layer transmission, originating at a single transmitter located e.g. at a base station, usually called eNB, and terminating at a single receiver located consequently at the user equipment (UE), is shown in Figure 1, cf. e.g. [8][Fig. 1.30, p. 289]. The transmitter consists of the classical channel encoder, which has been common in classical wireless communication systems, cf. e.g. [8], and a quantum transmitter, which represents the classical information generated by the classical channel encoder by a sequence quantum states. In this communication, we will focus on sophisticated channel coding based on polar codes, cf. e.g. [8][Chapter 4, pp. 336-388] for a brief introduction. The quantum transmitter can e.g. be realized by a quantum mechanical harmonic oscillator that provides coherent states, namely a laser [5][pp. 11, 80f.], [6][p. 1584], [9][pp. 125-128], which will be considered in this communication. The quantum channel, which conveys the information generated by the transmitter to the single receiver, is connected to the output of the quantum transmitter and to the input of the receiver, which consists of a quantum receiver followed by a classical channel decoder. The quantum receiver is supposed to be a soft output quantum data detector that consists of the measurement device, characterized by commuting measurement operators, and the soft output generation unit, which uses the measured probability values to create soft outputs in the form of log-likelihood ratios (LLRs). The mode of operation of the wireless quantum optical communication system depicted in Figure 1 will be addressed in Sect. 2.

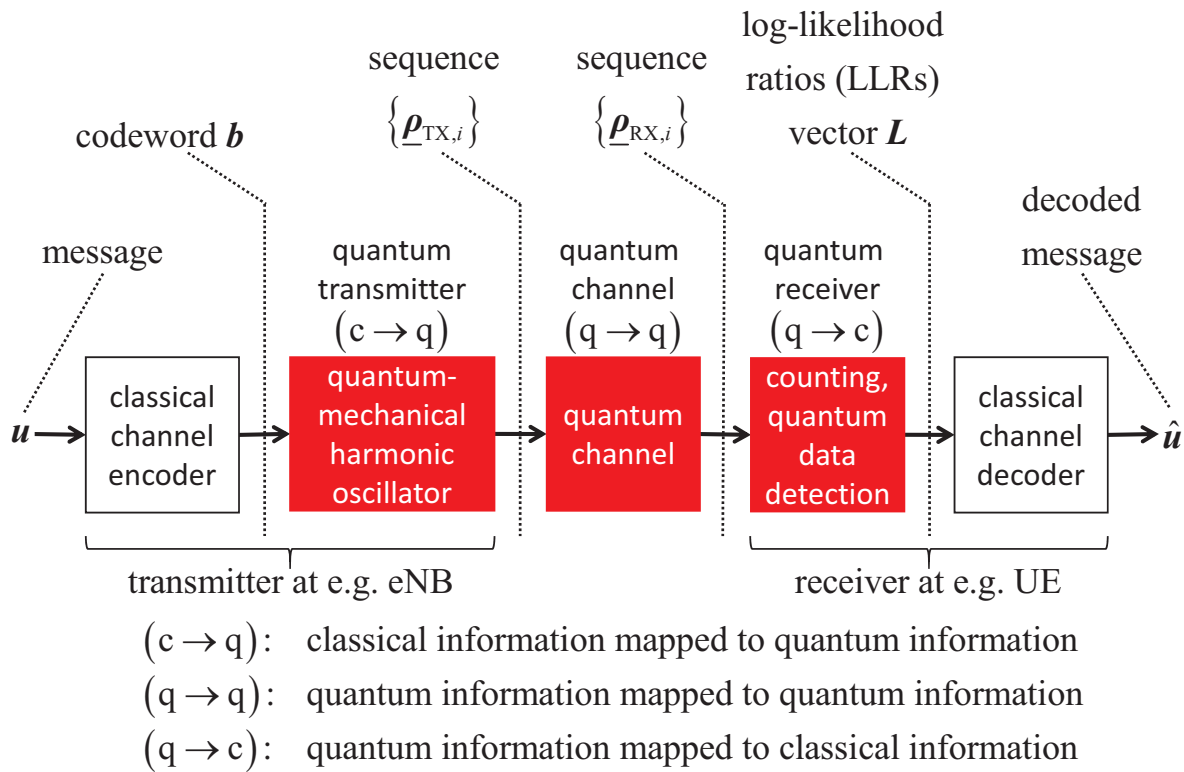


Figure 1. Wireless quantum optical communication system with classical channel coding, cf. also [8][Fig. 1.30, p. 289].

Every communication system must be tailored to the physical conditions presented by the transmission channel. In the context of short range communication links, which are typical for e.g. IEEE 802.11 “Wireless Local Area Network” (802.11/WLAN) standard [10], recently amended to accommodate light communications [11] in a similar manner discussed in e.g. [12], and for the anticipated short range wireless optical solutions, becoming increasingly important in the context of 6G wireless communications, cf. e.g. [13], we may suppose that the quantum channel has *one single path*, thus being memoryless, and the transmission is subject to little environmental interaction, i.e. negligible scattering and turbulence, therefore, neglecting any decoherence [14–16]. Since the received radiation field may contain a coherent signal, emitted by the quantum mechanical harmonic oscillator

in the case it is on and superimposed by the omnipresent thermal radiation field [6,9,17] the statistical operator of the received radiation field given in the basis of the number states may not be diagonal, having unvanishing *coherences*, i.e. interference terms, which are the off-diagonal elements of the matrix representation of the said statistical operator, cf. also [14][p. 34]. However, in e.g. the case of unknown random carrier phases [9][pp. 171f.], the said coherences will approach zero always yielding a diagonal statistical operator in the basis of the number states, cf. also [14][pp. 88-93].

In the context of mobile communications, however, wireless communication links typically suffer from multipath propagation with uncorrelated scattering, cf. e.g. [17,18], which will hold true for the quantum channel, as well [19–21]. Similar to mobile communications, intersymbol interference (ISI) at the quantum receiver depicted in Figure 1 must be expected, which had not been discussed earlier than [19]. We will illustrate this still novel corresponding quantum mechanical system model in Sect. 2, considerably extending the presentation of [19], by setting out from the quantum mechanical communication theory known for single path transmission [6,9,17]. Also, soft output quantum data detection, mitigating the effect of ISI will be treated in Sect. 2. Besides the discussion of optimum quantum data detection with soft outputs, a novel soft output reduced state quantum data detection approach will be introduced for the first time. In addition, the authors will propose a novel soft output Kohonen maps based quantum data detection for the first time, which extends a concept recently introduced for the data detection in classical mobile communications [22].

In Sect. 3 polar coding will be addressed, by briefly illustrating its general concept and also by introducing a detailed design strategy for the first time summarized in e.g. Algorithm 2, yielding an appropriately chosen polar sequence for wireless quantum optical communications. In addition, applying polar codes in turbo equalization, explicitly stating, seemingly for the first time, the creation of the required soft outputs for the codebits, and introducing two viable novel methods for the analytically closed generation of the extrinsic soft outputs to be used in the turbo equalization iterations, will be discussed for the first time in Sect. 3.

Sect. 4 reports obtained performance results, showing the feasibility and the viability of the concepts suggested in this manuscript. Sect. 5 concludes the manuscript.

In what follows, complex values are underlined and $(\cdot)^*$ denotes complex conjugation. Operators, matrices and vectors are in bold face italics. $\mathbb{F}_2$ equal to $\{0, 1\}$ denotes the Galois field with the two elements 0 and 1. $\mathbb{N}$ equal to $\{0, 1, 2, \dots\}$ is the set of nonnegative integers and $\mathbb{N}^*$ equal to $\{1, 2, \dots\}$ denotes the set of positive integers. $\mathbb{K}$ equal to $\{0, 1, \dots, (K-1)\}$, $K \in \mathbb{N}^*$, is the set of message bit indices and $\mathbb{B}$ equal to $\{0, 1, \dots, (N-1)\}$, $N \in \mathbb{N}^*$, $N \geq K$, is the set of codebit indices. We will apply the Dirac notation.

2. Quantum Mechanical System Model

2.1. Initial Considerations

Since the wireless quantum optical communication system model considered by the authors and sketched in e.g. Figure 1 bears considerable novelty, first being addressed in recent publications such as e.g. [20,21,23], the authors will illustrate its mode of operation in this Sect. 2 to facilitate an easy access by the reader.

2.2. Transmitter

Let us begin with the transmitter, cf. Figure 1. Suppose that messages $\mathbf{u}$ containing $K \in \mathbb{N}^*$ message bits $u_k \in \mathbb{F}_2$, $k \in \mathbb{K}$, are to be transmitted. To improve protection against transmission errors, the said messages $\mathbf{u}$ are encoded by the classical channel encoder, generating the codewords $\mathbf{b}$ with $N \in \mathbb{N}^*$, $N \geq K$, codebits $b_i \in \mathbb{F}_2$, $i \in \mathbb{B}$. T_s denotes the codebit period of a codebit b_i prevailing in a time interval $[iT_s, (i+1)T_s)$, leading to the total transmission period of NT_s .

Next, the N codebits b_i , $i \in \mathbb{B}$, are transmitted via a device converting this classical information into quantum states. Analog to conventional antennas that generate an electromagnetic field in a coherent state [6][p. 1584], we will use a laser [5][pp. 11, 80f.], [6][p. 1584], [9][pp. 125-128]. Any

codebit b_i , $i \in \mathbb{B}$, modulates the laser as follows. The laser light is turned on if the present codebit b_i is equal to 1 and off if the present codebit b_i is equal to 0. This simple modulation scheme is termed on-off keying (OOK) [5][pp. 11, 318-320], [17][pp. 435-446].

Suppose that only a single mode with a fixed angular frequency ω is used for communication purposes. With the number states $|n\rangle$, $n \in \mathbb{N}$, the coherent state of the radiation field emitted by the laser and associated with the i th codebit b_i , having the complex amplitude $b_i\mu \in \mathbb{C}$, $i \in \mathbb{B}$, is given by the coherent state vector [5][pp. 80, 479f.]

$$|b_i\mu\rangle = \exp\left\{-\frac{|b_i\mu|^2}{2}\right\} \sum_{n=0}^{\infty} \frac{(b_i\mu)^n}{\sqrt{n!}} |n\rangle. \quad (1)$$

If b_i equals 0, the emitted radiation field is in its ground state $|0\rangle$ and contains no signal photons. On the other hand, if b_i equals 1 the number of signal photons in the radiation field represented by the above coherent state given in (1) follows a Poisson distribution with the mathematical expectation $\bar{N}_\mu$ equal to $|\mu|^2$. It is assumed that the modulus $|\mu|$ remains constant throughout the entire transmission duration NT_s , see also [9, eq. (2.17), p. 125].

Although the coherent state $|b_i\mu\rangle$ described in (1) is not stationary over an observation time interval of duration T_s , any time dependence within the i th observation time interval, characterized by the angular frequency ω and the initial phase angle φ_i , is entirely captured by the phase factor $\exp\{j(\omega t + \varphi_i)\}$ in μ , because the magnitude $|\mu|$ is assumed to remain constant [9, pp. 125, 171]. Therefore, apart from a phase factor, the number state $|b_i N_{\mu,i}\rangle$ stays constant throughout the observation time interval $[iT_s, (i+1)T_s)$.

2.3. Short Range Transmission via a Single Path Quantum Channel

As previously noted in Sect. 1, we will first consider the transmission occurring via a single path quantum channel, where the received radiation field consists of the coherent signal emitted by the aforementioned laser, superimposed with the ever present thermal radiation field [6,9,17]. This situation was recently illustrated in [23, Sect. II], which forms the foundation of the following illustration. Due to the short range nature of the link and the relatively weak thermal radiation field, the interaction between the coherent signal and the surrounding environment remains minimal.

Let $\underline{\alpha} \in \mathbb{C}$ represent the complex amplitudes of the coherent states comprising the thermal radiation field, and let $\bar{N}_\alpha$ equal to $|\underline{\alpha}|^2$ denote the average number of thermal noise photons. Then, during the observation time interval $[iT_s, (i+1)T_s)$, $i \in \mathbb{B}$, the statistical operator in Glauber's "P-representation" is given by [9, eqs. (4.6) and (4.10), pp. 134f.]

$$\underline{\rho}_{\text{RX},b_i} = \int_{\underline{\alpha}} \underbrace{\frac{1}{\pi\bar{N}_\alpha} \exp\left\{-\frac{|\underline{\alpha} - b_i\mu|^2}{\bar{N}_\alpha}\right\}}_{=P(\underline{\alpha})} |\underline{\alpha}\rangle \langle \underline{\alpha}| d^2\underline{\alpha}. \quad (2)$$

Given the Boltzmann constant k_B , the temperature T , and the reduced Planck constant $\hbar$, the average number $\bar{N}_\alpha$ of thermal photons is determined by the Planck distribution law [9,17]

$$\bar{N}_\alpha = \left(\exp\left\{\frac{\hbar\omega}{k_B T}\right\} - 1 \right)^{-1}. \quad (3)$$

Let $L_n^{(\alpha)}(x)$ denote the associated Laguerre polynomial of degree n [24], and let the standard Laguerre polynomial of degree n , written as $L_n(x)$, be defined as $L_n^{(0)}(x)$ [24]. Also, let $n!$ denote the factorial of $n \in \mathbb{N}$. Assuming $\bar{N}_\mu$ is the average number of signal photons present in the coherent radiation field emitted by the laser when b_i equals 1, i.e., when the laser is active, and $\bar{N}_\alpha$ is the average

number of thermal photons present in the thermal radiation field, and computing in the number state basis $\{|0\rangle, |1\rangle, \dots\}$, the matrix element $\langle n|\underline{\rho}_{\text{RX},1}|m\rangle$ in the n th row and m th column of the matrix representation of $\underline{\rho}_{\text{RX},b_i}$ as defined in (2) is given by

$$\begin{aligned} \langle n|\underline{\rho}_{\text{RX},b_i}|m\rangle &= \frac{\bar{N}_\alpha^m}{(1 + \bar{N}_\alpha)^{m+1}} \cdot \sqrt{\frac{n!}{m!}} \cdot \left(\frac{b_i \mu^*}{\bar{N}_\alpha}\right)^{m-n} \\ &\cdot \exp\left\{-\frac{b_i \bar{N}_\mu}{1 + \bar{N}_\alpha}\right\} \\ &\cdot L_n^{(m-n)}\left(-\frac{b_i \bar{N}_\mu}{(1 + \bar{N}_\alpha) \bar{N}_\alpha}\right) \end{aligned} \quad (4)$$

in the case where $m \geq n$, that is, within the upper triangular submatrix including the main diagonal of the said matrix representation of $\underline{\rho}_{\text{RX},b_i}$, cf. e.g. [9,17] for a similar representation. For the case where $m < n$, corresponding to the strict lower triangular submatrix excluding the main diagonal of the matrix representation of $\underline{\rho}_{\text{RX},b_i}$, we obtain

$$\langle n|\underline{\rho}_{\text{RX},b_i}|m\rangle = \langle m|\underline{\rho}_{\text{RX},b_i}|n\rangle^*, \quad (5)$$

see also e.g. [9,17].

When a nonvanishing coherent signal is present in the received radiation field, that is, when b_i equals 1, equations (4) and (5) reveal the existence of nonzero interference terms called coherences, indicated by the nonzero off-diagonal elements of $\underline{\rho}_{\text{RX},1}$. Clearly, $\underline{\rho}_{\text{RX},1}$ corresponds to a scenario without decoherence and with a known carrier phase of the quantity μ^* appearing in (4). Since $\underline{\rho}_{\text{RX},b_i}$ is an observable and, thus, a Hermitian operator, it admits a diagonalization, yielding a complete orthonormal system of eigenkets that constitute a new basis of the Hilbert space, distinct from the number state basis.

If decoherence, an unknown carrier phase, or both dominate the transmission scenario, the phase of the quantity μ^* has to be revisited. The said phase, which is then appropriately modelled as a uniformly distributed random variable, must be averaged out during the data detection process. As a result, all interference terms, that is, all off-diagonal elements in the matrix representation of $\underline{\rho}_{\text{RX},1}$ with respect to the number state basis, vanish. The resulting statistical operator then becomes diagonal, having the matrix representation [9]

$$\begin{aligned} \tilde{\rho}_{\text{RX},1} &= \sum_{n=0}^{\infty} \frac{\bar{N}_\alpha^n}{(1 + \bar{N}_\alpha)^{n+1}} \cdot \exp\left\{-\frac{\bar{N}_\mu}{1 + \bar{N}_\alpha}\right\} \\ &\cdot L_n\left(-\frac{\bar{N}_\mu}{(1 + \bar{N}_\alpha) \bar{N}_\alpha}\right) |n\rangle\langle n|. \end{aligned} \quad (6)$$

When the coherent signal is absent, i.e., when b_i equals 0, equation (2) simplifies to [9,17]

$$\underline{\rho}_{\text{RX},0} = \sum_{n=0}^{\infty} \frac{\bar{N}_\alpha^n}{(1 + \bar{N}_\alpha)^{n+1}} |n\rangle\langle n|, \quad (7)$$

representing the Bose-Einstein distribution of thermal photons within the thermal radiation field. Clearly, $\underline{\rho}_{\text{RX},0}$ is diagonal when expressed in the number state basis, which therefore forms the set of its eigenkets. Then, no interference terms, i.e., no coherences, are present, indicating that the carrier phase of the thermal radiation is unknown and can be accurately modelled as a uniformly distributed random variable.

According to (4) as well as (5) and (7), the operators $\underline{\rho}_{RX,1}$ and $\underline{\rho}_{RX,0}$ possess different eigenstates and therefore do not commute. In contrast, $\underline{\rho}_{RX,1}$ from (6) and $\underline{\rho}_{RX,0}$ from (7) share the same complete orthonormal system of eigenkets, namely, the number states, and are thus compatible. These observations govern the design of the optimum soft output quantum data detector.

Since the considered quantum channel has one single path, no memory effects need to be taken into account when designing the optimum quantum data detector. Therefore, quantum data detection can be performed independently within each observation time interval.

We begin with the i th observation time interval $[iT_s, (i+1)T_s)$, supposing that b_i equal to 0 and b_i equal to 1 are equally probable. Using the Kronecker delta $\delta_{n,m}$, and defining the cost $C_{n,m}$ equal to $(1 - \delta_{n,m})$ for deciding in favor of hypothesis H_n when the transmitted codebit b_i had the value m , where $m, n \in \mathbb{F}_2$, under the maximum a-posteriori probability (MAP) concept [17, eq. (3.1181), p. 433], and considering the Hermitian measurement operator $\underline{M}_n$ corresponding to hypothesis H_n , the average cost of the measurement strategy during the i th observation time interval $[iT_s, (i+1)T_s)$ is given by [9, eq. (1.3), p. 90]

$$\begin{aligned}\bar{C} &= \sum_{m,n=0}^1 P\{b_i = m\} \underbrace{C_{n,m}}_{=(1-\delta_{n,m})} \text{tr}\left\{\underline{\rho}_{RX,b_i=m} \underline{M}_n\right\} \\ &= P\{1\} \left\{1 - \text{tr}\left\{\left[\underline{\rho}_{RX,1} - \frac{P\{0\}}{P\{1\}} \underline{\rho}_{RX,0}\right] \underline{M}_1\right\}\right\} \\ &= P\{0\} \left\{1 + \frac{P\{1\}}{P\{0\}} \text{tr}\left\{\left[\underline{\rho}_{RX,1} - \frac{P\{0\}}{P\{1\}} \underline{\rho}_{RX,0}\right] \underline{M}_0\right\}\right\}.\end{aligned}\quad (8)$$

Let $|\xi_n\rangle$ be the eigenkets and $\xi_n, n \in \mathbb{N}$, the corresponding real eigenvalues of the Hermitian detection operator

$$\underline{D} = \underline{\rho}_{RX,1} - [P\{0\}/P\{1\}] \underline{\rho}_{RX,0}. \quad (9)$$

The optimum measurement operators are then given by [6,9]

$$\underline{M}_0 = \sum_{\xi_n \leq 0} |\xi_n\rangle \langle \xi_n| \quad (10)$$

and

$$\underline{M}_1 = \sum_{\xi_n > 0} |\xi_n\rangle \langle \xi_n|. \quad (11)$$

If the measurement at the input of the quantum receiver, represented by the trace $\text{tr}\{\underline{D}\underline{M}_0\}$ of $\underline{D}\underline{M}_0$, yields the eigenstate $|\xi_n\rangle$ with eigenvalue $\xi_n > 0$, the detected codebit is $\hat{b}_i$ equal to 1. Otherwise, the detected codebit is $\hat{b}_i$ equal to 0. Similarly, if the measurement represented by the trace $\text{tr}\{\underline{D}\underline{M}_1\}$ of $\underline{D}\underline{M}_1$ yields the eigenstate $|\xi_n\rangle$ with eigenvalue $\xi_n > 0$, the detected codebit is $\hat{b}_i$ equal to 1. Otherwise, it is $\hat{b}_i$ equal to 0. Therefore, with

$$\underbrace{\langle \xi_n | \underline{D} | \xi_n \rangle}_{=\xi_n} = \langle \xi_n | \underline{\rho}_{RX,1} | \xi_n \rangle - \frac{P\{0\}}{P\{1\}} \langle \xi_n | \underline{\rho}_{RX,0} | \xi_n \rangle, \quad (12)$$

the optimum soft output in the i th observation time interval $[iT_s, (i+1)T_s)$ is hence represented by the LLR

$$L_i = \ln \left\{ \frac{P\{b_i = 1\} \langle \xi_n | \underline{\rho}_{RX,b_i=1} | \xi_n \rangle}{P\{b_i = 0\} \langle \xi_n | \underline{\rho}_{RX,b_i=0} | \xi_n \rangle} \right\}. \quad (13)$$

Using the sign function $\text{sgn}\{\cdot\}$, the MAP detection rule applied by the optimum soft output quantum data detector is expressed as

$$\hat{b}_i = \frac{\text{sgn}\{L_i\} + 1}{2} \quad (14)$$

$$= \arg \max_{b_i} \left\{ \ln\{P\{b_i\}\} + \ln\left\{ \langle \xi_n | \underline{\rho}_{\text{RX}, b_i} | \xi_n \rangle \right\} \right\}. \quad (15)$$

Nota bene that in the special case where $\tilde{\rho}_{\text{RX},1}$ is given by (6) and $\underline{\rho}_{\text{RX},0}$ by (7), equation (13) simplifies to the well-known expression for L_i

$$L_i = \ln \left\{ L_n \left(-\frac{\bar{N}_\mu}{(1 + \bar{N}_\alpha)\bar{N}_\alpha} \right) \right\} - \frac{\bar{N}_\mu}{1 + \bar{N}_\alpha} \quad (16)$$

for equiprobable bits, i.e. $P\{b_i = 0\}$ equal to $P\{b_i = 1\}$ [17].

2.4. Multipath Transmission and Reception with Decoherence

In particular, scenarios involving longer transmission links often experience multipath transmission and reception. In such cases, the quantum channel is characterized by multiple paths, through which the transmitted signals arrive at the receiver. This situation has been illustrated very recently for the first time, cf. e.g. [19–21].

Consider a quantum channel modelled as a linear multipath channel that causes uncorrelated scattering of the signal photons emitted by the laser. These scatterers are randomly located and statistically independent, forming a continuum of uncorrelated scatterers similar to what occurs in mobile propagation channels [18,19]. The channel impulse response (CIR) of this multipath channel, parameterized by the delay τ , is denoted as $\underline{h}(\tau)$ [19]. We assume that the CIR $\underline{h}(\tau)$ is nonzero only for τ within the delay time interval $[0, \tau_{\text{max}}]$, where $T_s < \tau_{\text{max}}$. Under these conditions, ISI arises at the quantum receiver. Defining

$$W_p \stackrel{\text{def}}{=} \left\lceil \frac{\tau_{\text{max}}}{T_s} \right\rceil \geq \frac{\tau_{\text{max}}}{T_s}, \quad W_p \in \{2, 3, \dots\}, \quad (17)$$

the said multipath channel is referred to as a W_p -path channel. The signal arriving at the receiver can be expressed by the ket vector [19]

$$|\nu(t)\rangle = \sum_{i=0}^{N-1} \underline{h}(t - iT_s) |b_i N_{\mu,i}\rangle. \quad (18)$$

The previously mentioned uncorrelated scattering induces decoherence [14–16,25], which contributes to the loss of phase coherence between the transmitter and the receiver, see e.g. [19–21]. As a result, we should assume that no phase coherence is preserved between the transmitter and receiver [6][p. 1590], [19], [9][p. 171]. Consequently, the zero phase angles φ_i appearing in the aforementioned phase factors mentioned in Sect. 2.2 are considered unknown and modelled as uniformly distributed random variables with the constant probability density function of $1/(2\pi)$. Moreover, the uncorrelated scattering inherent to the multipath quantum channel further contributes to the degradation of phase coherence, cf. e.g. [19] and the references therein.

In the absence of phase coherence, the optimum quantum receiver employs photon counting at its front end [6][p. 1590]. Assume that the transmitter and the receiver are synchronized in time, meaning that the quantum receiver operates over mutually disjoint and consecutive observation time intervals $[iT_s, (i+1)T_s)$ of duration T_s , each beginning with the arrival of the corresponding codebit b_i [19]. Using the number operator $\hat{\mathcal{N}}$ and denoting mathematical expectation by $E\{\cdot\}$, the average number of signal photons $\bar{N}_{\mu,i}$ received during the observation time interval $[iT_s, (i+1)T_s)$ is given by $\langle \underline{\nu}(t) | \hat{\mathcal{N}} | \underline{\nu}(t) \rangle$, after averaging over all uniformly distributed phase differences φ , and we obtain

$$\begin{aligned}
\bar{N}_{\mu,i} = & \bar{N}_{\mu} \cdot \sum_{n=0}^{N-1} b_n \cdot \underbrace{\frac{1}{T_s} \int_{iT_s}^{(i+1)T_s} \frac{1}{2\pi} \int_0^{2\pi} |\underline{h}(t - nT_s)|^2 d\varphi_i dt}_{\text{first integral}} \\
& + \bar{N}_{\mu} \cdot \sum_{n=0}^{N-1} \sum_{\substack{m=0 \\ n \neq m}}^{N-1} b_n b_m E\left\{ \delta_{N_{\mu,n}, N_{\mu,m}} \right\} \\
& \cdot \underbrace{\frac{1}{T_s} \int_{iT_s}^{(i+1)T_s} \frac{1}{2\pi} \int_0^{2\pi} \underline{h}^*(t - nT_s) \underline{h}(t - mT_s) d\varphi_i dt}_{\text{second integral}}.
\end{aligned} \tag{19}$$

Due to decoherence and averaging over φ_i , the second integral vanishes, while the first integral remains unaffected. Therefore, equation (19) simplifies to [19]

$$\bar{N}_{\mu,i} = \bar{N}_{\mu} \cdot \sum_{n=0}^{N-1} b_n \underbrace{\frac{1}{T_s} \int_{(i-n)T_s}^{(i-n+1)T_s} |\underline{h}(t)|^2 dt}_{=s_{i-n} \geq 0}, \tag{20}$$

which becomes

$$\bar{N}_{\mu,i} = \bar{N}_{\mu} b_i s_0 + \underbrace{\bar{N}_{\mu} \sum_{w=1}^{W_p-1} b_{i-w} s_w}_{\text{ISI term}}, \tag{21}$$

supposing that only W_p channel coefficients $s_w, w \in \{0, 1 \dots (W_p - 1)\}$, can assume nonzero values.

In the case of a single path channel where W_p equals 1, only the first term $\bar{N}_{\mu}, b_i, s_0$ of equation (21) remains. The scenario where s_0 equals 1 has already been discussed, for instance, in [6, pp. 1589f.] and [9, pp. 171-176]. Moreover, if s_0 is a nonnegative random variable with unit expected value, the result is a fading single path turbulent channel, also known as a turbulence channel, which was already investigated from a quantum communications perspective several decades ago [26][pp. 1630-1644].

For $W_p > 1$, the ISI term $\sum_{w=1}^{W_p-1} b_{i-w} s_w$ in equation (21) does not vanish [19]. As a result, the average number of received signal photons $\bar{N}_{\mu,i}$ of (21) depends not only on the current codebit b_i , but also on the $(W_p - 1)$ preceding codebits $b_{i-W_p+1}, b_{i-W_p+2} \dots b_{i-1}$. Therefore, equation (21) describes a finite state machine (FSM) that operates according to a Markov chain model [17,19,27].

In the context of Mealy modelling, cf. e.g. [27], the previously described FSM consists of 2^{W_p-1} states $\mathbf{s}_i$, defined as $(b_{i-W_p+1}, b_{i-W_p+2} \dots b_{i-1})$ at each time index $i \in \{0, 1 \dots (N + W_p - 1)\}$. Since the term $\bar{N}_{\mu} b_i s_0$ can take on either of two distinct values, 0 or $\bar{N}_{\mu} s_0$, each of the 2^{W_p-1} states $\mathbf{s}_i$ gives rise to two transitions, labelled by the hypothetical codebit value $b_i \in \mathbb{F}_2 = \{0, 1\}$. This results in a total of 2^{W_p} distinct transitions, which lead to the 2^{W_p-1} states $\mathbf{s}_{i+1}$ at the next time step $(i + 1)$ [19]. The behavior of the said FSM is conveniently represented by a trellis diagram, often abbreviated as trellis [17,19,28]. Accordingly, the 2^{W_p} possible transitions correspond to the 2^{W_p} different values that the average number of received signal photons $\bar{N}_{\mu,i}$, as defined in (21), can assume during each observation time interval $[iT_s, (i + 1)T_s)$, with $i \in \{0, 1 \dots (N + W_p - 2)\}$. For this reason, we will denote $\bar{N}_{\mu,i}$ as $\bar{N}_{\mu, [\mathbf{s}_i, b_i]}$ going forward, identifying these quantities as the 2^{W_p} distinct hypotheses employed in the quantum data detection scheme [19]. Note that the said hypotheses $\bar{N}_{\mu, [\mathbf{s}_i, b_i]}$ do not necessarily match the actual number of received signal photons $\hat{N}_{\mu,i}$ observed in the i th observation time interval [19].

Figure 2 shows the detail of a trellis representing a quantum channel with W_p equal to 3 possibly nonvanishing channel coefficients s_0, s_1 and s_2 , thus, when applying Mealy modelling, having 2^{W_p-1} equal to 4 states at each time instant. The two states $\mathbf{s}_i$ equal to $(0,0)$ and $\mathbf{s}_i$ equal to $(1,0)$ at time instant i are connected to the two states $\mathbf{s}_{i+1}$ equal to $(0,0)$ and $\mathbf{s}_{i+1}$ equal to $(0,1)$ at time instant $(i+1)$ by a total of 2^{W_p-1} equal to four transitions. The said states and the aforementioned four transitions, all represented in black color, form the first of the two butterflies of the considered trellis. The two states $\mathbf{s}_i$ equal to $(0,1)$ and $\mathbf{s}_i$ equal to $(1,1)$ at time instant i are connected to the two states $\mathbf{s}_{i+1}$ equal to $(1,0)$ and $\mathbf{s}_{i+1}$ equal to $(1,1)$ at time instant $(i+1)$ by a total of 2^{W_p-1} equal to four transitions. These states and the associated four transitions, all represented in green color, form the second of the two butterflies of the considered trellis.

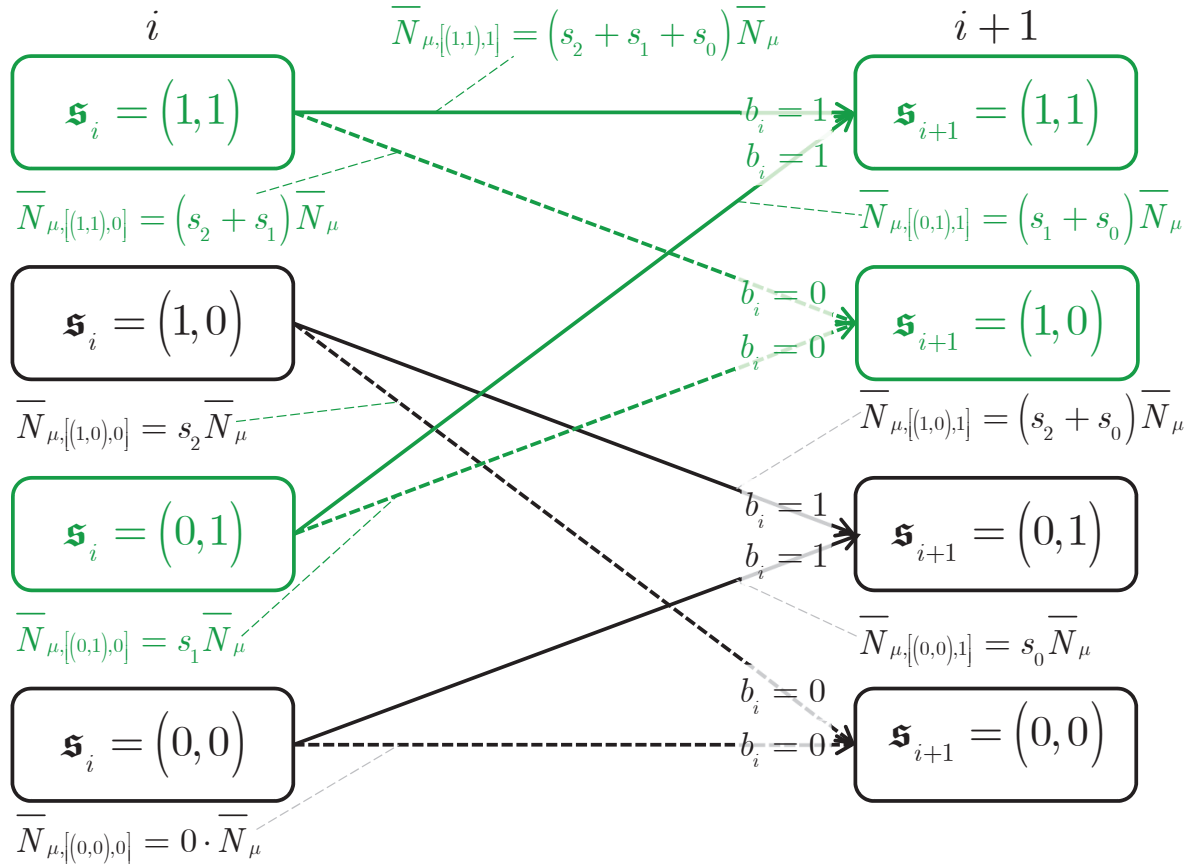


Figure 2. Detail of a trellis representing a quantum channel with W_p equal to 3 possibly nonvanishing channel coefficients s_0, s_1 and s_2 , thus, when applying Mealy modelling, having 2^{W_p-1} equal to 4 states at each time instant; the two states $\mathbf{s}_i$ equal to $(0,0)$ and $\mathbf{s}_i$ equal to $(1,0)$ at time instant i are connected to the two states $\mathbf{s}_{i+1}$ equal to $(0,0)$ and $\mathbf{s}_{i+1}$ equal to $(0,1)$ at time instant $(i+1)$ by a total of 2^{W_p-1} equal to four transitions, forming the first of the two butterflies; the two states $\mathbf{s}_i$ equal to $(0,1)$ and $\mathbf{s}_i$ equal to $(1,1)$ at time instant i are connected to the two states $\mathbf{s}_{i+1}$ equal to $(1,0)$ and $\mathbf{s}_{i+1}$ equal to $(1,1)$ at time instant $(i+1)$ by a total of 2^{W_p-1} equal to four transitions, forming the second of the two butterflies; on-off keying (OOK) is assumed.

To support trellis initialization in the state $\mathbf{s}_0$ equal to $(1,1 \dots 1)$ and trellis termination in the state $\mathbf{s}_{N+W_p-1}$ equal to $(1,1 \dots 1)$, a choice known to improve data detection performance, we introduce $(W_p - 1)$ initial and $(W_p - 1)$ terminal tailbits. Specifically, we set b_i equal to 1 for all $i \in \{(-W_p + 1) \dots -1, N \dots (N + W_p - 2)\}$. All other codebits b_i , for $i \leq -W_p$ and $i \geq (N + W_p - 1)$, are assumed to be 0.

The schematic representation of the transmitted bit vector used when applying the trellis based quantum data detectors, which set out from the Mealy modelling of the FSM representing the multipath quantum channel, is depicted in Figure 3. As pointed out above, the said transmitted bit vector consists

of N codebits, $(W_p - 1)$ initial tailbits and $(W_p - 1)$ terminal tailbits, hence, containing $(N + 2W_p - 2)$ bits.

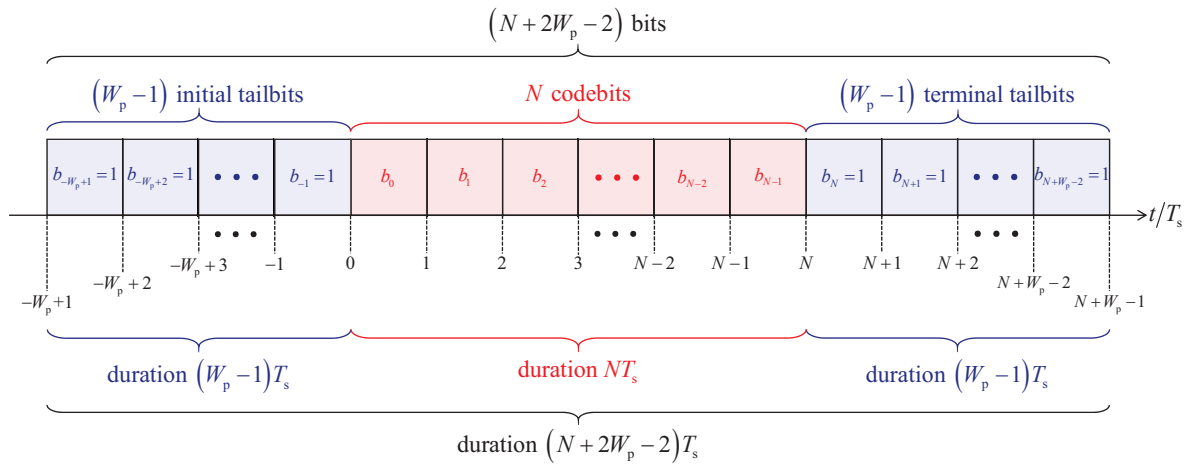


Figure 3. Schematic representation of the transmitted bit vector used when applying the trellis based quantum data detectors, which set out from the Mealy modelling of the FSM representing the multipath quantum channel; the transmitted bit vector consists of N codebits, $(W_p - 1)$ initial tailbits and $(W_p - 1)$ terminal tailbits, hence, containing $(N + 2W_p - 2)$ bits.

At the output of the quantum channel, the radiation field resulting from laser emission is linearly superimposed with the omnipresent thermal background radiation field, which is in a thermal state characterized by Bose-Einstein distributed thermal noise photons. As already mentioned above, $\bar{N}_\alpha$ denotes the average number of thermal noise photons and $\underline{\alpha} \in \mathbb{C}$ represents the complex amplitudes of the associated coherent states present in the thermal field. The said linear superposition is modelled by the map

$$\mathcal{E}\{\underline{\rho}_{\text{TX},i}\} = \underline{\rho}_{\text{RX},[\mathbf{s}_i, b_i]}. \quad (22)$$

Given

$$\underline{\mu}_{[\mathbf{s}_i, b_i]} = \sqrt{\bar{N}_{\mu, [\mathbf{s}_i, b_i]}} \exp\{j\theta_i\}, \quad (23)$$

where θ_i is a zero phase angle, the corresponding statistical operator in Glauber's "P-representation" with $P(\underline{\alpha})$ defined as [9, eqs. (4.6) and (4.10), pp. 134f.]

$$P(\underline{\alpha}) = \frac{1}{\pi \bar{N}_\alpha} \exp\left\{-\frac{|\underline{\alpha} - \underline{\mu}_{[\mathbf{s}_i, b_i]}|^2}{\bar{N}_\alpha}\right\} \quad (24)$$

is denoted by

$$\underbrace{\underline{\rho}_{\text{RX},[\mathbf{s}_i, b_i]}}_{=\underline{\rho}_{\text{RX},i}} = \int_{\underline{\alpha}} \underbrace{\frac{1}{\pi \bar{N}_\alpha} \exp\left\{-\frac{|\underline{\alpha} - \underline{\mu}_{[\mathbf{s}_i, b_i]}|^2}{\bar{N}_\alpha}\right\}}_{=P(\underline{\alpha})} |\underline{\alpha}\rangle \langle \underline{\alpha}| d^2 \underline{\alpha} \quad (25)$$

during the observation time interval $[iT_s, (i+1)T_s]$, $i \in \{0, 1, \dots, (N + W_p - 2)\}$.

Since the function $P(\underline{\alpha})$ of (24) used in (25) is always nonnegative, the associated laser light is often referred to as "classical." In contrast, light, for which $P(\underline{\alpha})$ takes on negative values in some regions is sometimes labelled "nonclassical" [29][p. 60]. However, this distinction may be misleading. For instance, a Fock state is entirely nonclassical by nature [29][p. 58], and, fundamentally, all states of

light are governed by quantum mechanics [29][pp. 60, 150]. Thus, the binary classification of light as either “classical” or “nonclassical” may not be entirely appropriate. Considering that Max Planck’s treatment of thermal radiation on December 14, 1900, marked the beginning of quantum mechanics, cf. e.g. [30][pp. 217, 277], and that laser light was theoretically introduced by Albert Einstein in 1917 on the basis of quantum principles, it seems inconsistent not to regard both thermal radiation and laser light as “nonclassical”.

With the regular Laguerre polynomial $L_{\hat{N}_i}(x)$ of degree $\hat{N}_i$, we find

$$\begin{aligned} \rho_{\text{RX},[\mathbf{s}_i,b_i]} = \sum_{\hat{N}_i=0}^{\infty} \frac{\bar{N}_{\alpha}^{\hat{N}_i}}{(1 + \bar{N}_{\alpha})^{\hat{N}_i+1}} \cdot \exp\left\{-\frac{\bar{N}_{\mu,[\mathbf{s}_i,b_i]}}{1 + \bar{N}_{\alpha}}\right\} \\ \cdot L_{\hat{N}_i}\left(-\frac{\bar{N}_{\mu,[\mathbf{s}_i,b_i]}}{\bar{N}_{\alpha}(1 + \bar{N}_{\alpha})}\right) |\hat{N}_i\rangle\langle\hat{N}_i|, \end{aligned} \quad (26)$$

neglecting any phase coherence, see also [19][p. 2], [9][pp. 136f.]. Equation (26) is based on the assumption that the quantum receiver front-end essentially consists of a photon counting detector. This photon counting detector measures the number $\hat{N}_i$ of photons present at the receiver during the observation time interval $[iT_s, (i+1)T_s]$ [19]. Using the trace operation, denoted by $\text{tr}\{\cdot\}$, and the projection operator $\Pi_{\hat{N}_i}$ defined as $|\hat{N}_i\rangle\langle\hat{N}_i|$,

$$\begin{aligned} P\{\hat{N}_i | \mathbf{s}_i, b_i\} &\stackrel{\text{def}}{=} \Gamma_{\hat{N}_i}(\mathbf{s}_i, \mathbf{s}_{i+1}) = \text{tr}\{\rho_{\text{RX},[\mathbf{s}_i,b_i]} \Pi_{\hat{N}_i}\} \\ &= \frac{\bar{N}_{\alpha}^{\hat{N}_i}}{(1 + \bar{N}_{\alpha})^{\hat{N}_i+1}} \cdot \exp\left\{-\frac{\bar{N}_{\mu,[\mathbf{s}_i,b_i]}}{1 + \bar{N}_{\alpha}}\right\} \\ &\quad \cdot L_{\hat{N}_i}\left(-\frac{\bar{N}_{\mu,[\mathbf{s}_i,b_i]}}{\bar{N}_{\alpha}(1 + \bar{N}_{\alpha})}\right) \end{aligned} \quad (27)$$

is the conditional probability of detecting $\hat{N}_i$ photons at the photon counting detector, given the transmission of the codeword $\mathbf{b}$. Equation (27) describes this probability as a Laguerre distribution, see also [5, eq. (4.71), p. 174]. Notably, $P\{\hat{N}_i | \mathbf{s}_i, b_i\}$ coincides with the transition probability $\Gamma_{\hat{N}_i}(\mathbf{s}_i, \mathbf{s}_{i+1})$ associated with the FSM introduced earlier [19]. It follows that equation (27) assumes the use of an ideal photon counting detector, such as the one discussed in [31][pp. 231f.], as the front-end component of the optimum quantum data detection scheme.

If the photon counting detector cannot precisely count the exact number of photons but is only able to differentiate among zero photons, one photon, or more than one photon, we may use $\Gamma_0(\mathbf{s}_i, \mathbf{s}_{i+1})$, $\Gamma_1(\mathbf{s}_i, \mathbf{s}_{i+1})$ and $\Gamma_{>1}(\mathbf{s}_i, \mathbf{s}_{i+1})$ equal to $(1 - \Gamma_0(\mathbf{s}_i, \mathbf{s}_{i+1}) - \Gamma_1(\mathbf{s}_i, \mathbf{s}_{i+1}))$ instead of $\Gamma_{\hat{N}_i}(\mathbf{s}_i, \mathbf{s}_{i+1})$ for arbitrary $\hat{N}_i$ [19]. Accounting for real-world effects such as dark counts with the mathematical expectation $\bar{N}_d$ and quantum efficiency $\eta \leq 1$, in equation (27) $\bar{N}_{\alpha}$ should be replaced by $(\eta\bar{N}_{\alpha} + \bar{N}_d)$ and $\bar{N}_{\mu,[\mathbf{s}_i,b_i]}$ has to be replaced by $(\eta\bar{N}_{\mu,[\mathbf{s}_i,b_i]} + \bar{N}_d)$ [19].

Based on the above discussion, the actual number of received signal photons $\hat{N}_{\mu,i}$ in the i th observation time interval $[iT_s, (i+1)T_s]$ results from a linear superposition of W_p Poisson distributed random variables $N_{\mu,i-w}b_{i-w}S_w$, each with mean $\bar{N}_{\mu}b_{i-w}S_w$. Consequently, $\hat{N}_{\mu,i}$ itself follows a Poisson distribution [19,32]. It is important to note that Poisson distributed random variables are, by definition, mutually independent [32]. Moreover, the thermal noise photons, governed by the Bose-Einstein distribution, follow a memoryless geometric distribution [19,32]. As a result, measurements taken across any two disjoint observation time intervals are statistically independent. This insight aligns with Forney’s classical considerations [28].

With reference to, e.g., [33][pp. 100-104], the likelihood function $P\{\hat{\mathbf{N}} | \mathbf{b}\}$, which denotes the conditional probability of observing the photon count vector

$$\hat{\mathbf{N}} = (\hat{N}_0, \hat{N}_1, \dots, \hat{N}_{N+W_p-2}) \quad (28)$$

given the transmission of the codeword $\mathbf{b}$ including the terminal tailbits, is defined as follows. Let $\bigotimes_{i=0}^{N+W_p-2} \rho_{\text{RX},[\mathbf{s}_i, b_i]}$ represent the tensor product of the corresponding $(N + W_p - 1)$ statistical operators $\rho_{\text{RX},[\mathbf{s}_i, b_i]}$, and $\bigotimes_{i=0}^{N+W_p-2} \Pi_{\hat{N}_i}$ the tensor product of the associated projection measurement operators $\Pi_{\hat{N}_i}$. Then, the likelihood function is given by:

$$\begin{aligned} P\{\hat{\mathbf{N}} | \mathbf{b}\} \\ = \text{tr} \left\{ \left(\bigotimes_{i=0}^{N+W_p-2} \rho_{\text{RX},[\mathbf{s}_i, b_i]} \right) \left(\bigotimes_{i=0}^{N+W_p-2} \Pi_{\hat{N}_i} \right) \right\}. \end{aligned} \quad (29)$$

This expression reflects the joint probability of measuring the specific photon count sequence $\hat{\mathbf{N}}$ conditioned on the transmission of the bit sequence $\mathbf{b}$, via the tensor product structure of the quantum states and measurements across all observation intervals. We obtain

$$P\{\hat{\mathbf{N}} | \mathbf{b}\} = \prod_{i=0}^{N+W_p-2} \underbrace{\text{tr} \left\{ \rho_{\text{RX},[\mathbf{s}_i, b_i]} \Pi_{\hat{N}_i} \right\}}_{=\Gamma_{\hat{N}_i}(\mathbf{s}_i, \mathbf{s}_{i+1})}. \quad (30)$$

When the codebits b_i are treated as outcomes of statistically independent random experiments, the a-priori probability $P\{\mathbf{b}\}$ of a specific codeword $\mathbf{b}$ including the fixed-value terminal tailbits set to 1 is given by the regular product of the individual a-priori probabilities

$$P\{\mathbf{b}\} = \prod_{i=0}^{N+W_p-2} P\{b_i\}. \quad (31)$$

Applying the definition of conditional probability as found in [17, Def. 3.25, p. 270], the corresponding a-posteriori probability is obtained by

$$P\{\mathbf{b} | \hat{\mathbf{N}}\} = \frac{P\{\hat{\mathbf{N}} | \mathbf{b}\} \cdot P\{\mathbf{b}\}}{P\{\hat{\mathbf{N}}\}}, \quad (32)$$

where $P\{\hat{\mathbf{N}}\}$ denotes the total probability of observing the photon count vector $\hat{\mathbf{N}}$, accounting for all possible codewords. Using the definition of the conditional probability [17, Def. 3.25, p. 270], [34, eq. (5), p. 5], we yield

$$P\{\mathbf{b} | \hat{\mathbf{N}}\} \propto \prod_{i=0}^{N+W_p-2} \underbrace{P\{b_i\} \cdot \text{tr} \left\{ \rho_{\text{RX},[\mathbf{s}_i, b_i]} \Pi_{\hat{N}_i} \right\}}_{=P\{b_i\} \cdot \Gamma_{\hat{N}_i}(\mathbf{s}_i, \mathbf{s}_{i+1}) \propto P\{b_i | \hat{\mathbf{N}}\}} \quad (33)$$

from (30) after multiplying by the joint a-priori probability $P\{\mathbf{b}\}$ of (31).

The likelihood function $P\{\hat{\mathbf{N}} | \mathbf{b}\}$ and its logarithmic counterpart

$$\sum_{i=0}^{N+W_p-2} \ln \left\{ \Gamma_{\hat{N}_i}(\mathbf{s}_i, \mathbf{s}_{i+1}) \right\} = \sum_{i=0}^{N+W_p-2} \gamma_{\hat{N}_i}(\mathbf{s}_i, \mathbf{s}_{i+1}), \quad (34)$$

as well as the a-posteriori probability $P\{\mathbf{b} | \hat{\mathbf{N}}\}$, or its logarithmic form

$$\begin{aligned} & \sum_{i=0}^{N+W_p-2} \ln\{P\{b_i\} \cdot \Gamma_{\hat{N}_i}(\mathbf{s}_i, \mathbf{s}_{i+1})\} \\ &= \sum_{i=0}^{N+W_p-2} \left(\ln\{P\{b_i\}\} + \gamma_{\hat{N}_i}(\mathbf{s}_i, \mathbf{s}_{i+1}) \right), \end{aligned} \quad (35)$$

constitute the foundation of the quantum data detection process that directly follows photon counting and serves to mitigate ISI [19]. The term $\gamma_{\hat{N}_i}(\mathbf{s}_i, \mathbf{s}_{i+1})$ is referred to as the *metric increment*. In summary, optimum quantum data detection leverages either the likelihood function $P\{\hat{\mathbf{N}} | \mathbf{b}\}$, as defined in (30), or the a-posteriori probability $P\{\mathbf{b} | \hat{\mathbf{N}}\}$, see (33). Since classical channel decoding follows the quantum detection step, cf. Figure 1, producing LLRs at the quantum data detector output, assembled into the LLR vector $\mathbf{L}$, is advantageous.

Referring to optimum quantum data detection as first presented in e.g. [19–21], like in Sect. 2.3, we set out from the i th observation time interval $[iT_s, (i+1)T_s)$. Again, with the Kronecker symbol $\delta_{n,m}$, with the cost $C_{n,m}$ equal to $(1 - \delta_{n,m})$ of the decision in favor of hypothesis H_n when the codebit b_i with value m was transmitted, $m, n \in \{0, 1\}$, and with the Hermitian measurement operator $\underline{\mathbf{M}}_n$ reflecting the hypothesis H_n , in the case of the MAP concept [17, eq. (3.1181), p. 433] the average cost of the observational strategy in the said i th observation time interval $[iT_s, (i+1)T_s)$ is

$$\begin{aligned} \bar{C} &= \sum_{m,n=0}^1 P\{b_i = m\} \underbrace{C_{n,m}}_{=(1-\delta_{n,m})} \text{tr}\{\underline{\rho}_{\text{RX},[\mathbf{s}_i, b_i=m]} \underline{\mathbf{M}}_n\} \\ &= P\{0\} \left\{ 1 - \text{tr}\left\{ \left[\underline{\rho}_{\text{RX},[\mathbf{s}_i, 0]} - \frac{P\{1\}}{P\{0\}} \underline{\rho}_{\text{RX},[\mathbf{s}_i, 1]} \right] \underline{\mathbf{M}}_0 \right\} \right\} \\ &= P\{1\} \left\{ 1 - \text{tr}\left\{ \left[\underline{\rho}_{\text{RX},[\mathbf{s}_i, 1]} - \frac{P\{0\}}{P\{1\}} \underline{\rho}_{\text{RX},[\mathbf{s}_i, 0]} \right] \underline{\mathbf{M}}_1 \right\} \right\}, \end{aligned} \quad (36)$$

which is a modified version of (36), replacing $\underline{\rho}_{\text{RX}, b_i=m}$ used in (36) by $\underline{\rho}_{\text{RX},[\mathbf{s}_i, b_i=m]}$, hence taking the FSM related to the ISI into account explicitly.

Each $\underline{\rho}_{\text{RX},[\mathbf{s}_i, b_i]}$ is diagonal in the Fock basis, with the Fock state vectors $|\hat{N}_i\rangle$ acting as eigenkets, and the associated eigenvalues given by $\Gamma_{\hat{N}_i}(\mathbf{s}_i, \mathbf{s}_{i+1})$, which are equal to $P\{\hat{N}_i | \mathbf{s}_i, b_i\}$. As an example, consider the Hermitian detection operator in equation (36), corresponding to $\underline{\mathbf{M}}_1$ and given by

$$\underline{\mathbf{D}} = \underline{\rho}_{\text{RX},[\mathbf{s}_i, 1]} - \frac{P\{0\}}{P\{1\}} \underline{\rho}_{\text{RX},[\mathbf{s}_i, 0]}, \quad (37)$$

with the eigenvalues

$$\zeta_{\hat{N}_i} = P\{\hat{N}_i | \mathbf{s}_i, b_i = 1\} - \frac{P\{0\}}{P\{1\}} P\{\hat{N}_i | \mathbf{s}_i, b_i = 0\}. \quad (38)$$

Accordingly, the optimum measurement operators that minimize $\bar{C}$ of (36) and are naturally realized by the ideal photon counter discussed in Sect. 2 are $\underline{\mathbf{M}}_0$ given by $\sum_{\hat{N}_i \forall \zeta_{\hat{N}_i} \leq 0} \underline{\Pi}_{\hat{N}_i}$ and $\underline{\mathbf{M}}_1$ given by $\sum_{\hat{N}_i \forall \zeta_{\hat{N}_i} > 0} \underline{\Pi}_{\hat{N}_i}$ [9].

At the conclusion of the measurement within the i th observation time interval $[iT_s, (i+1)T_s)$, the exact Fock state $|\hat{N}_i\rangle$ of the received radiation field, as present in the ideal photon counter, is known. Consequently, only the projector $\underline{\Pi}_{\hat{N}_i}$ within $\underline{\mathbf{M}}_0$ or $\underline{\mathbf{M}}_1$ contributes a nonzero eigenvalue $\zeta_{\hat{N}_i}$. This nonzero eigenvalue $\zeta_{\hat{N}_i}$ is then evaluated using the conventional classical likelihood ratio test. Nonetheless, due to the presence of ISI, the decision made in the i th observation time interval $[iT_s, (i+1)T_s)$ not only depends on earlier transmitted codebits but also influences the decisions in

following intervals. As a result, and as reflected in equation (33), the classical likelihood ratio test must be modified, accordingly, taking the following form

$$\frac{P\{b_i = 1 \mid \hat{\mathbf{N}}\}}{P\{b_i = 0 \mid \hat{\mathbf{N}}\}} = \frac{\sum_{\substack{\text{all codebits} \\ \text{except } b_i=1}} P\{\mathbf{b} \mid \hat{\mathbf{N}}\}_{b_i=1}}{\sum_{\substack{\text{all codebits} \\ \text{except } b_i=0}} P\{\mathbf{b} \mid \hat{\mathbf{N}}\}_{b_i=0}} \stackrel{\hat{b}_i=1}{\underset{\hat{b}_i=0}{\geq}} 1. \quad (39)$$

This result is an extended variant of Helstrom's theory [9], as expressed in equation (33). Hence, the optimum MAP symbol by symbol quantum data detection strategy is expressed as

$$\begin{aligned} \hat{b}_i &= \arg \max_{b_i} \{P\{b_i \mid \hat{\mathbf{N}}\}\} \\ &= \arg \max_{b_i} \{\ln\{P\{b_i \mid \hat{\mathbf{N}}\}\}\}, \end{aligned} \quad (40)$$

cf. e.g. [35][p. 216], being a classical likelihood ratio test, because we left the quantum regime at the end of the quantum measurement process. However, due to the occurrence of ISI, modelled by the FSM discussed above and, thus, by a Markov chain, (40) requires an appropriate detection scheme, e.g. the trellis based Bahl-Cocke-Jelinek-Raviv (BCJR) algorithm [17,19,35,36]. It seems that this approach has been first introduced by [19].

Correspondingly, also taking ISI and Markov chain modelling into account explicitly, with (33), the optimum MAP sequence quantum data detection strategy is presented by

$$\hat{\mathbf{b}} = \arg \max_{\mathbf{b}} \{P\{\mathbf{b} \mid \hat{\mathbf{N}}\}\} = \arg \max_{\mathbf{b}} \{\ln\{P\{\mathbf{b} \mid \hat{\mathbf{N}}\}\}\}, \quad (41)$$

cf. e.g. [35][p. 216], which can be implemented by the trellis based Viterbi algorithm [17,19,35].

Maximizing the likelihood function $\ln\{P\{\hat{\mathbf{N}} \mid \mathbf{b}\}\}$ over all possibly transmitted codewords $\mathbf{b}$ will yield the maximum-likelihood (ML) sequence detector, which can be realized by the soft output Viterbi algorithm (SOVA) [17,19,35]. Maximizing the a-posteriori probability $P\{\mathbf{b} \mid \hat{\mathbf{N}}\}$ over all possibly transmitted codewords $\mathbf{b}$ will yield the MAP sequence detector. Since a-priori information is processed in the latter case, the corresponding variant of the SOVA is termed APRI-SOVA [35,37]. Both the SOVA and the APRI-SOVA are considered optimum because they maximize the probability of correct sequence detection, which is equivalent to minimizing the sequence error probability, provided that the metric increments discussed above are used. Details of the mode of operation of the SOVA and the APRI-SOVA have been well-known [17,19,28,35,37].

ML symbol by symbol detection focuses on maximizing the conditional probabilities of the observed data given the symbol, $P\{\hat{\mathbf{N}} \mid b_i\}$, while MAP symbol by symbol detection aims to maximize the posterior probabilities of the symbol given the observed data, $P\{b_i \mid \hat{\mathbf{N}}\}$ [17,35]. To reduce computational complexity, both detection methods are typically implemented in the log domain, similar to optimum sequence detection. Accordingly, ML symbol by symbol detection employs metric increments $\gamma_{\hat{\mathbf{N}}_i}(\mathbf{s}_i, \mathbf{s}_{i+1})$, whereas MAP symbol by symbol detection uses metric increments of the form $\left(\ln\{P\{b_i\}\} + \gamma_{\hat{\mathbf{N}}_i}(\mathbf{s}_i, \mathbf{s}_{i+1})\right)$ [17,19]. Both approaches can be efficiently realized using the BCJR algorithm [17,19,36], which is considered optimum as it maximizes the probability of correct symbol detection, equivalently, minimizing symbol error probability, when the metric increments $\gamma_{\hat{\mathbf{N}}_i}(\mathbf{s}_i, \mathbf{s}_{i+1})$ are used. The operational details of the BCJR algorithm are well-established, see e.g. [17,36]. It is important to note that optimum symbol by symbol detection schemes implemented in the log domain apply the Jacobian logarithm [17][pp. 472f.], which leads to the so-called "log-ML" and "log-MAP" algorithms [17][p. 479]. Approximations of the Jacobian logarithm yield variants known as "max*-log-ML" and "max*-log-MAP", as well as "max-log-ML" and "max-log-MAP", respectively [17][pp. 472f., p. 479].

Optimum sequence quantum data detectors, as well as optimum symbol by symbol quantum data detectors, operate as “forward/backward” algorithms that conceptually consist of four key processing units. These are:

1. The *transition metric unit*, which computes the metric increments $\gamma_{\hat{N}_i}(\mathbf{s}_i, \mathbf{s}_{i+1})$ for ML detection, or $\left(\ln\{P\{b_i\}\} + \gamma_{\hat{N}_i}(\mathbf{s}_i, \mathbf{s}_{i+1})\right)$ for MAP detection.
2. The *forward recursion unit*, responsible for calculating the forward path metrics.
3. The *backward recursion unit*, which either computes the backward path metrics or performs traceback to identify the most likely path through the trellis.
4. The *LLR generation unit*, which produces the LLR vector $\mathbf{L}$ mentioned earlier.

Importantly, only the computations within the transition metric unit differ from those in e.g. classical mobile radio, such as mobile radio detection. The other three units can be reused without modification.

For the SOVA and the APRI-SOVA algorithms, only approximate LLRs can be generated, using three distinct LLR computation methods known as the “SIMPLE RULE (SR)”, the “HUBER RULE (HR)”, and the “BATTAIL RULE (BR)” [17][pp. 463-475]. In contrast, all optimum symbol by symbol quantum data detectors produce LLR vectors $\mathbf{L}$ containing exact LLRs when implemented as log-ML or log-MAP algorithms, and approximate, though very close to exact, LLRs in the cases of max*-log-ML, the max*-log-MAP, the max-log-ML and the max-log-MAP. implementations.

Trellis based reduced state quantum data detection sets out from the now well-known trellis based optimum sequence quantum data detection, respectively the trellis based optimum symbol by symbol quantum data detection discussed above. Like in the cases of these optimum data detection schemes, the reduced state quantum data detection continues until all LLRs to be determined and, consequently, all the corresponding data symbols have been detected. Trellis based reduced state quantum data detection requires small modifications and extensions of the transition metric unit and the forward recursion unit, which we will discuss in what follows. Since the said modifications and extensions only concern the transition metric unit and the forward recursion unit, which are identical in the case of the max-log-MAP symbol by symbol detection and the SOVA, the novelties presented are identically applicable to

- the trellis based reduced state sequence quantum data detection, which is formed by setting out from the SOVA, termed *rsSOVA* with the “prefix *rs*” indicating the reduced state version, respectively from the APRI-SOVA, termed *rsAPRI-SOVA*, and
- the trellis based reduced state symbol by symbol quantum data detection, which is formed by setting out from the max-log-MAP symbol by symbol detection, termed *rsmax-log-MAP*, the max-log-ML symbol by symbol detection, termed *rsmax-log-ML*, the max*-log-MAP symbol by symbol detection, termed *rsmax*-log-MAP*, the max*-log-ML symbol by symbol detection, termed *rsmax*-log-ML*, the log-MAP symbol by symbol detection, termed *rslog-MAP* and the log-ML symbol by symbol detection, termed *rslog-ML*, the latter two differing from the max*-log-ML symbol by symbol detection and the max-log-MAP symbol by symbol detection only in the used version of Jacobi’s logarithm but not in the processing of the trellis states and the trellis transitions.

Suppose that we require reduced state quantum data detection with only $2^{W_{p,\text{red}}-1}$ trellis states, $W_{p,\text{red}} < W_p$, $W_p \in \{2, 3, \dots\}$, $W_{p,\text{red}} \in \mathbb{N}^*$. Using $W_{p,\text{red}}$ explicitly, (21) can be written as follows

$$\begin{aligned} \overline{N}_{\mu, [\mathbf{s}_i, b_i]} = & \overline{N}_{\mu} b_i s_0 \\ & + \overline{N}_{\mu} \underbrace{\sum_{w=1}^{W_{p,\text{red}}-1} b_{i-w} s_w}_{\substack{\text{contains} \\ (W_{p,\text{red}}-1) \text{ binary data symbols} \\ b_{i-W_{p,\text{red}}+1}, b_{i-W_{p,\text{red}}+2} \dots b_{i-1}, \\ \text{governing the } 2^{W_{p,\text{red}}-1} \\ \text{distinct trellis states } \mathbf{s}_i}} \\ & + \overline{N}_{\mu} \underbrace{\sum_{w=W_{p,\text{red}}}^{W_p-1} b_{i-w} s_w}_{\substack{\text{contains} \\ (W_p - W_{p,\text{red}}) \text{ binary data symbols} \\ b_{i-W_p+1}, b_{i-W_p+2} \dots b_{i-W_{p,\text{red}}}, \\ \text{required to be preliminary decided} \\ \text{prior to the } 2^{W_{p,\text{red}}} \text{ trellis transitions } \mathbf{s}_i \rightarrow \mathbf{s}_{i+1}}} . \end{aligned} \quad (42)$$

Reduced state data detection requires that the $(W_p - W_{p,\text{red}})$ binary data symbols $b_{i-W_p+1}, b_{i-W_p+2} \dots b_{i-W_{p,\text{red}}-1}, b_{i-W_{p,\text{red}}}$ contained in the third term on the right side of (42) are known to the quantum data detector prior to the $2^{W_{p,\text{red}}}$ trellis transitions $\mathbf{s}_i \rightarrow \mathbf{s}_{i+1}$. This requirement calls for preliminary decisions $\hat{b}_{i-W_p+1}, \hat{b}_{i-W_p+2} \dots \hat{b}_{i-W_{p,\text{red}}-1}, \hat{b}_{i-W_{p,\text{red}}}$ needed for the transition metric computation and hence in the forward recursion. The said preliminary decisions can be accomplished by recursive processing and adds iteratively one single decision per forward recursion step. Therefore, reduced state quantum data detection requires the modification of the transition metrics computation by using $\overline{N}_{\mu, [\mathbf{s}_i, b_i]}$ of (42), which relies on the preliminary decided binary data symbols $\hat{b}_{i-W_p+1}, \hat{b}_{i-W_p+2} \dots \hat{b}_{i-W_{p,\text{red}}-1}, \hat{b}_{i-W_{p,\text{red}}}$ and the extension of the forward recursion by adding preliminary decision taking.

In the reduced state trellis, there are $2^{W_{p,\text{red}}-1}$ trellis transitions $\mathbf{s}_i \rightarrow \mathbf{s}_{i+1}$, which originate at those $2^{W_{p,\text{red}}-2}$ states

$$\mathbf{s}_i|_0 = \underbrace{(b_{i-W_{p,\text{red}}+1}, b_{i-W_{p,\text{red}}+2} \dots b_{i-1})}_{=0}, \quad (43)$$

leading to the $2^{W_{p,\text{red}}-1}$ possible forward recursion path metrics

$$Q(\mathbf{s}_i|_0) + \left\{ \ln\{P\{b_i\}\} + \gamma_{\hat{N}_i}(\mathbf{s}_i|_0, \mathbf{s}_{i+1}) \right\},$$

$Q(\mathbf{s}_i|_0)$ denoting the path metric of the state $\mathbf{s}_i|_0$ at the time instant i . Also, there are $2^{W_{p,\text{red}}-1}$ trellis transitions $\mathbf{s}_i \rightarrow \mathbf{s}_{i+1}$, which originate at those $2^{W_{p,\text{red}}-2}$ states

$$\mathbf{s}_i|_1 = \underbrace{(b_{i-W_{p,\text{red}}+1}, b_{i-W_{p,\text{red}}+2} \dots b_{i-1})}_{=1}, \quad (44)$$

leading to the $2^{W_{p,\text{red}}-1}$ possible forward recursion path metrics

$$Q(\mathbf{s}_i|_1) + \left\{ \ln\{P\{b_i\}\} + \gamma_{\hat{N}_i}(\mathbf{s}_i|_1, \mathbf{s}_{i+1}) \right\},$$

$Q(\mathbf{s}_i|_1)$ denoting the path metric of the state $\mathbf{s}_i|_1$ at the time instant i .

Figure 4 depicts a butterfly of the trellis with $2^{W_{p,\text{red}}-1}$ states per time instant, considered in trellis based reduced state quantum data detection, which comprises of the two states $\mathbf{s}_i|_0$ of (43) and $\mathbf{s}_i|_1$ of (44) at the time instant i and the two states $\mathbf{s}_{i+1}$ equal to $(b_{i-W_{p,\text{red}}+2}, b_{i-W_{p,\text{red}}+3} \dots 0)$ and $\mathbf{s}_{i+1}$ equal

to $(b_{i-W_{p,red}+2}, b_{i-W_{p,red}+3} \cdots 1)$ at the time instant $(i+1)$. The butterfly consists of two transitions originating at the state $\mathbf{s}_i|_0$ at time instant i and two transitions originating at the state $\mathbf{s}_i|_1$ at time instant i . Two of these four transitions terminate at the state $\mathbf{s}_{i+1}$ equal to $(b_{i-W_{p,red}+2}, b_{i-W_{p,red}+3} \cdots 0)$ at time instant $(i+1)$ and the remaining two of these four transitions terminate at the state $\mathbf{s}_{i+1}$ equal to $(b_{i-W_{p,red}+2}, b_{i-W_{p,red}+3} \cdots 1)$ at time instant $(i+1)$.

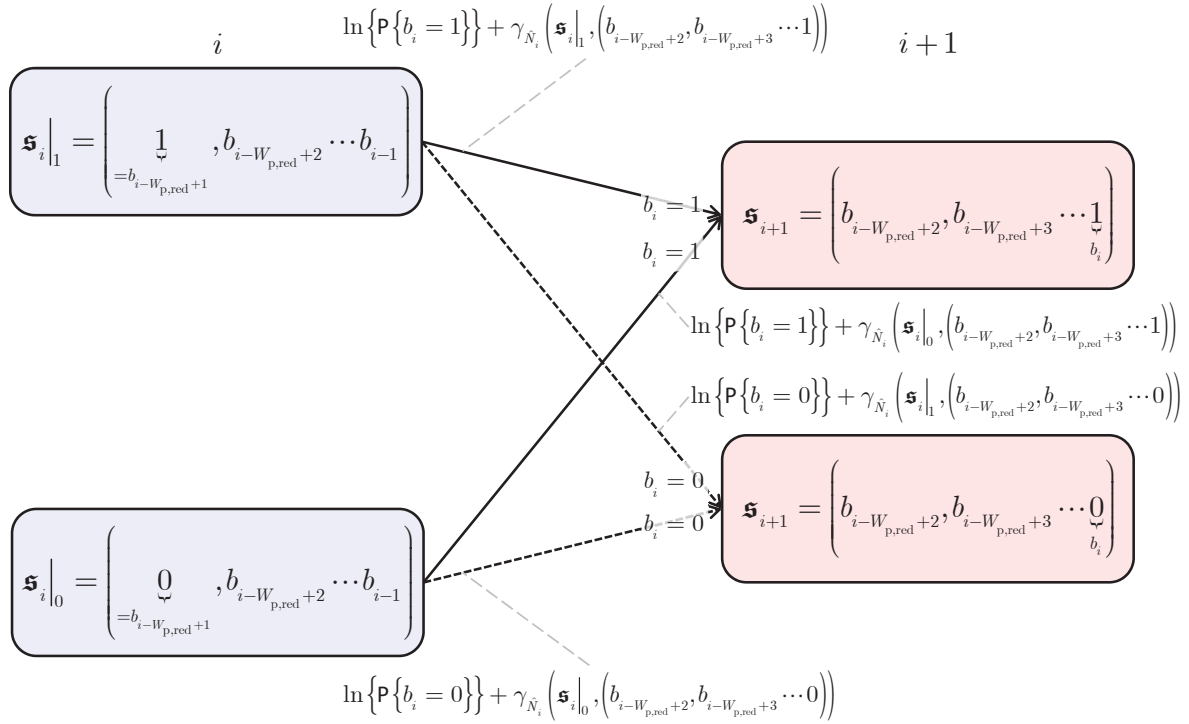


Figure 4. Butterfly of the trellis with $2^{W_{p,red}} - 1$ states per time instant considered in trellis based reduced state quantum data detection; on-off keying (OOK) is assumed.

When summing up all the $2^{W_{p,red}-1}$ above-mentioned possible forward recursion path metrics $Q(\mathbf{s}_i|_0) + \{\ln\{P\{b_i\}\} + \gamma_{N_i}(\mathbf{s}_i|_0, \mathbf{s}_{i+1})\}$ by iteratively applying Jacobi's logarithm, we yield the forward recursion estimate of the logarithmic probability $\ln\{P\{b_{i-W_{p,red}+1} = 0\}\}$. Next, when summing up all the $2^{W_{p,red}-1}$ possible forward recursion path metrics $Q(\mathbf{s}_i|_1) + \{\ln\{P\{b_i\}\} + \gamma_{N_i}(\mathbf{s}_i|_1, \mathbf{s}_{i+1})\}$ by iteratively applying Jacobi's logarithm, we yield the forward recursion estimate of the logarithmic probability $\ln\{P\{b_{i-W_{p,red}+1} = 1\}\}$. Subtracting $\ln\{P\{b_{i-W_{p,red}+1} = 0\}\}$ from $\ln\{P\{b_{i-W_{p,red}+1} = 1\}\}$ yields the forward recursion LLR of $b_{i-W_{p,red}+1}$, which is to be used for the preliminary decision $\hat{b}_{i-W_{p,red}+1}$. Since we now have the viable decision $\hat{b}_{i-W_{p,red}+1}$, we may continue with the next iteration step in the forward recursion.

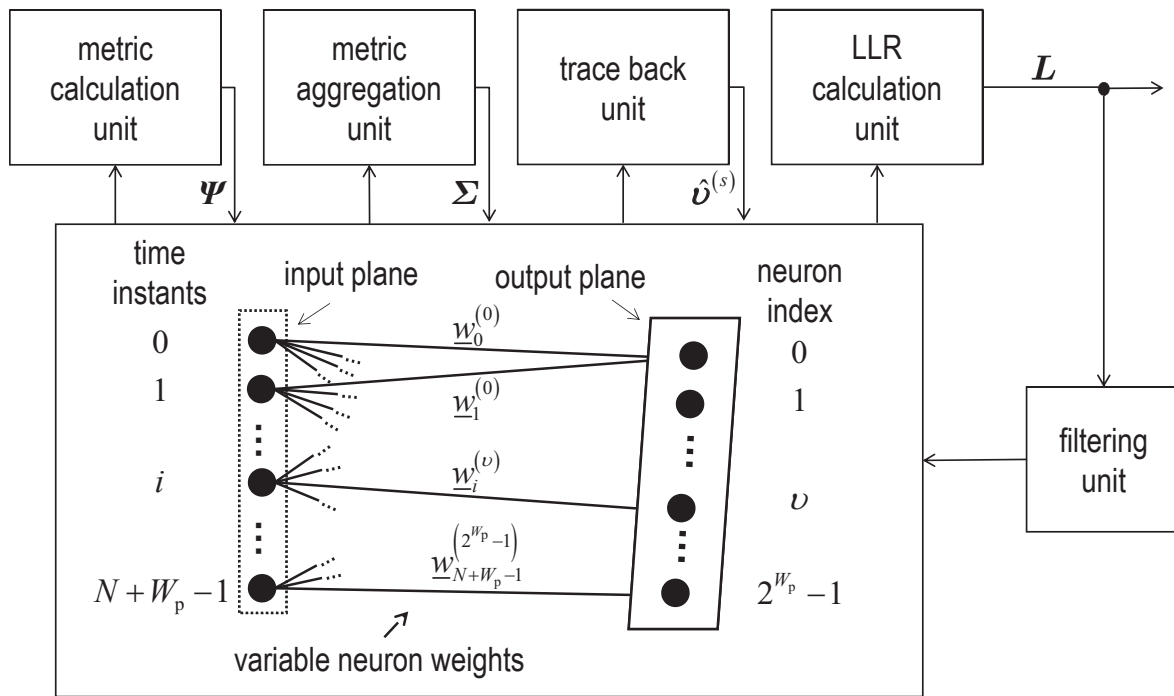


Figure 5. System design of the Kohonen maps based detector employing a neural network using variable neuron weights, cf. e.g. [22][Fig. 1].

While state reduction is an advantageous strategy for managing detection complexity, *Kohonen maps* can be employed for resource-efficient implementations, and near optimum performance in classical mobile communication systems [22]. The Kohonen maps based sequence detector, also referred to as “Kohonen maps based detector” or “Kohonen detector”, is also applicable to quantum channels. It consists of four processing units, namely the *metric calculation unit*, the *metric aggregation unit*, the *trace back unit* and the *LLR calculation unit* [22]. The combination of these four units follows the signal processing principle of a APRI-SOVA by applying a neural network with a one-dimensional linear structure in both the input plane and the output plane using distinct features of the Kohonen map by appropriately adjusting the weights and the distance function [22]. More specifically, the neural network adopted by the Kohonen maps based detector is a hybrid of learning vector quantization (LVQ) and Kohonen maps, both belonging to the family of soft competitive learning clustering algorithms [38]. This approach enables leveraging neural networks to perform near optimum sequence detection by efficiently utilizing existing hardware resources [22]. Training the Kohonen map by incorporating LLR feedback has been shown to improve the overall performance of the Kohonen maps based detector [22].

In order for the Kohonen maps based detector to be applied in quantum communication channels, the aforementioned FSM has to be modelled by means of a Moore machine. In the case of Moore modelling, the FSM contains 2^{W_p} unique states $\mathbf{s}_i$ equal to $(b_{i-W_p}, b_{i-W_p+1} \cdots b_{i-1})$ at each time instant $i \in \{0, 1 \cdots (N + W_p - 1)\}$. For this reason, we will use W_p initial and W_p terminal tailbits, setting b_i equal to 1 for all $i \in \{-W_p \cdots -1, N \cdots (N + W_p - 1)\}$.

The schematic representation of the transmitted bit vector used when applying the Kohonen map based quantum data detector, which sets out from the Moore modelling of the FSM representing the multipath quantum channel, is shown in Figure 6. As pointed out above, the said transmitted bit vector consists of N codebits, W_p initial tailbits and W_p terminal tailbits, hence, containing $(N + 2W_p)$ bits.

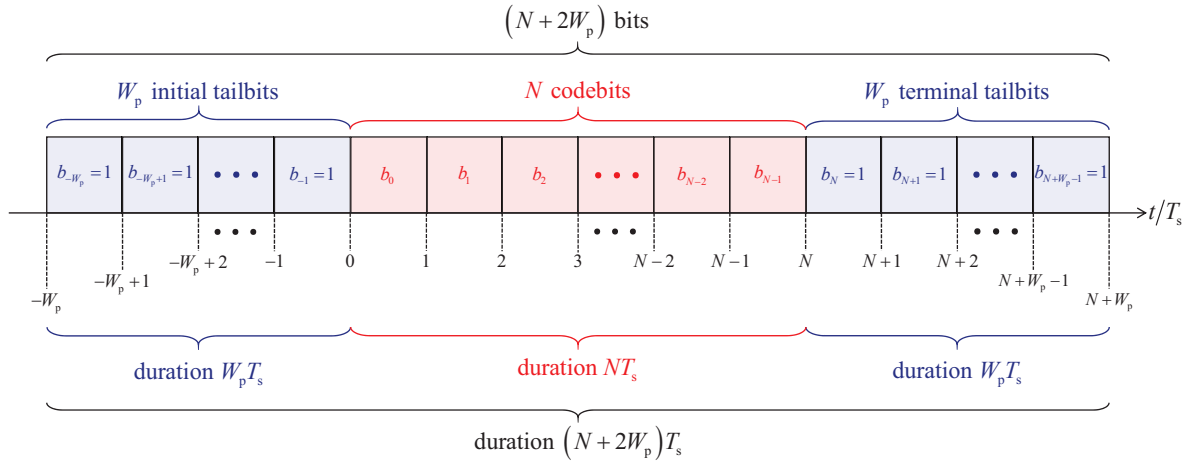


Figure 6. Schematic representation of the transmitted bit vector, used when applying the Kohonen map based quantum data detector, which sets out from the Moore modelling of the FSM representing the multipath quantum channel; the transmitted bit vector consists of N codebits, W_p initial tailbits and W_p terminal tailbits, hence, containing $(N + 2W_p)$ bits.

Each possible state at a given time instant i is associated with a state number, i.e. a state index $\nu \in \{0, 1, \dots, 2^{W_p} - 1\}$. Interpreting the FSM as a Moore machine involves a doubling of the number of states and a reinterpretation of the transition metrics, as they no longer depend on transitions, i.e., both the initial states and the target states, but solely on the target states. We further denote the set of possible states in the Mealy case by $\mathcal{S}_{\text{Mealy}}$ and the corresponding set in the Moore case by $\mathcal{S}_{\text{Moore}}$. The remodelling as a Moore machine necessitates the introduction of a mapping function in order to assign the previously determined transition metrics $(\ln\{P\{b_i\}\} + \gamma_{\hat{N}_i}(\mathbf{s}_i, \mathbf{s}_{i+1}))$, $\mathbf{s}_i, \mathbf{s}_{i+1} \in \mathcal{S}_{\text{Mealy}}$, for the Mealy case to transition metrics for the Moore case $(\ln\{P\{b_i\}\} + \gamma_{\hat{N}_i}(\mathbf{s}_i))$, respectively, $(\ln\{P\{b_{i+1}\}\} + \gamma_{\hat{N}_{i+1}}(\mathbf{s}_{i+1}))$, $\mathbf{s}_i, \mathbf{s}_{i+1} \in \mathcal{S}_{\text{Moore}}$. We define a surjective mapping

$$M : \mathcal{S}_{\text{Moore}} \rightarrow \mathcal{S}_{\text{Mealy}} \times \mathcal{S}_{\text{Mealy}}, \quad (45)$$

to assign the transition of a Mealy state pair to a Moore state. For each $\nu \in \{0, 1, \dots, 2^{W_p} - 1\}$ the mapping is defined as

$$\begin{aligned} M(\nu) &= (\mu_{\text{init}}, \mu_{\text{target}}), \\ \mu_{\text{init}} &= \left\lfloor \frac{\nu}{2} \right\rfloor, \\ \mu_{\text{target}} &= \{2\mu_{\text{init}} + (\nu \bmod 2)\} \bmod 2^{W_p-1}, \end{aligned} \quad (46)$$

where $\lfloor \cdot \rfloor$ denotes the floor function (in German “Gaußklammer”). Furthermore, μ_{init} denotes the state index of the initial state and μ_{target} the state index of the target state in the case of Mealy modelling.

Once the transition metrics according to Moore modelling are obtained and summarized in a $2^{W_p} \times (N + W_p)$ matrix Ψ , the Kohonen detector operates similarly to the case of classical wireless communication systems. At this stage, the Kohonen map serves as a state mapper allowing for an adequate add-compare-select (ACS) operation for the permissible state transitions [22]. In what follows, $\mathbf{s}_i^{(\kappa)}$ denotes a possible predecessor state with its state index $\kappa \in \{0, 1, \dots, 2^{W_p} - 1\}$ at the time instant i of the successor state $\mathbf{s}_{i+1}^{(\nu)}$ with state index $\nu \in \{0, 1, \dots, 2^{W_p} - 1\}$ at the time instant $(i + 1)$. The output neuron for the state $\mathbf{s}_{i+1}^{(\nu)}$ with state number $\nu \in \{0, 1, \dots, 2^{W_p} - 1\}$ at the time instant $(i + 1)$ is configured with the weight vector

$$\mathbf{w}_{i+1}^{(\nu)} = \left(w_{i+1}^{(0 \rightarrow \nu)}, w_{i+1}^{(1 \rightarrow \nu)} \dots w_{i+1}^{(2^{W_p}-1 \rightarrow \nu)} \right) \quad (47)$$

using

$$w_{i+1}^{(\kappa \rightarrow \nu)} = \begin{cases} Q(\mathbf{s}_i^{(\kappa)}) + \ln\{P\{b_{i+1}\}\} + \gamma_{\hat{N}_{i+1}}(\mathbf{s}_{i+1}^{(\nu)}) & \text{if transition } \kappa \rightarrow \nu \text{ is permissible,} \\ -\infty & \text{otherwise,} \end{cases} \quad (48)$$

where $Q(\mathbf{s}_i^{(\kappa)})$ denotes the path metric of the state with state number κ at the time instant i . Any state $\mathbf{s}_{i+1}^{(\nu)}$ can be reached by two distinct predecessors, $\mathbf{s}_i^{(\kappa)}$ and $\mathbf{s}_i^{(\bar{\kappa})}$ say. In order to determine the path metrics at the time instant $(i+1)$ for each possible successor state $\mathbf{s}_{i+1}^{(\nu)}$, solely the weights of the Kohonen map must be examined. We obtain

$$\begin{aligned} Q(\mathbf{s}_{i+1}^{(\nu)}) &= \max\{w_{i+1}^{(\nu)}\} \\ &= \max\{Q(\mathbf{s}_i^{(\kappa)}), Q(\mathbf{s}_i^{(\bar{\kappa})})\} \\ &\quad + \ln\{P\{b_{i+1}\}\} + \gamma_{\hat{N}_{i+1}}(\mathbf{s}_{i+1}^{(\nu)}), \end{aligned} \quad (49)$$

which can be arranged in the 2^{W_p} path metrics row vectors

$$\mathbf{Q}^{(\nu)} = (Q_0^{(\nu)}, Q_1^{(\nu)} \dots Q_i^{(\nu)} \dots Q_{N+W_p-1}^{(\nu)}). \quad (50)$$

We will illustrate the above discussions of the soft output Kohonen maps based quantum data detection in what follows. Let the states with the state indices α and $\bar{\alpha}$ refer to the predecessor states at time instant $(i-1)$ of the state $\mathbf{s}_i^{(\kappa)}$ with the state index κ at time instant i . Furthermore, let the state indices β and $\bar{\beta}$ refer to the predecessor states at time instant $(i-1)$ of the state $\mathbf{s}_i^{(\bar{\kappa})}$ with the state index $\bar{\kappa}$ at time instant i . As shown in Figure 7, the state

$$\mathbf{s}_i^{(\kappa)} = (b_{i-W_p+1}^{(\alpha)}, b_{i-W_p+2}^{(\alpha)} \dots b_{i-1}^{(\alpha)}, b_i^{(\kappa)}) \quad (51)$$

with the state index κ , $b_{i-W_p+1}^{(\alpha)}$, $b_{i-W_p+2}^{(\alpha)} \dots b_{i-1}^{(\alpha)}$ and $b_i^{(\kappa)}$ denoting the particular bit values associated with this state $\mathbf{s}_i^{(\kappa)}$, and the state

$$\mathbf{s}_i^{(\bar{\kappa})} = (1 \oplus b_{i-W_p+1}^{(\alpha)}, b_{i-W_p+2}^{(\alpha)} \dots b_{i-1}^{(\alpha)}, b_i^{(\kappa)}) \quad (52)$$

with the state index $\bar{\kappa}$, $(1 \oplus b_{i-W_p+1}^{(\alpha)})$, $b_{i-W_p+2}^{(\alpha)} \dots b_{i-1}^{(\alpha)}$ and $b_i^{(\kappa)}$ denoting the particular bit values associated with the state $\mathbf{s}_i^{(\bar{\kappa})}$, are the predecessor states at time instant i of the state

$$\mathbf{s}_{i+1}^{(\nu)} = (b_{i-W_p+2}^{(\alpha)} \dots b_{i-1}^{(\alpha)}, b_i^{(\kappa)}, b_{i+1}^{(\nu)}) \quad (53)$$

with the state index ν at time instant $(i+1)$. The aforementioned states $\mathbf{s}_i^{(\kappa)}$ of (51), $\mathbf{s}_i^{(\bar{\kappa})}$ of (52) and $\mathbf{s}_{i+1}^{(\nu)}$ of (53) are part of the trellis representing the multipath quantum channel with W_p possibly nonvanishing channel coefficients s_w , $w \in \{0, 1 \dots (W_p - 1)\}$, setting out from the Moore modelling

applied in the soft output Kohonen maps based quantum data detector. In Figure 7, the average numbers

$$\begin{aligned}\bar{N}_{\mu, [\mathbf{s}_i^{(\kappa)}]} &= \bar{N}_\mu s_{W_p-1} b_{i-W_p+1}^{(\alpha)} \\ &\quad + \bar{N}_\mu \sum_{w=1}^{W_p-2} s_w b_{i-w}^{(\alpha)} + \bar{N}_\mu s_0 b_i^{(\kappa)},\end{aligned}\quad (54)$$

$$\begin{aligned}\bar{N}_{\mu, [\mathbf{s}_i^{(\bar{\kappa})}]} &= \bar{N}_\mu s_{W_p-1} (1 \oplus b_{i-W_p+1}^{(\alpha)}) \\ &\quad + \bar{N}_\mu \sum_{w=1}^{W_p-2} s_w b_{i-w}^{(\alpha)} + \bar{N}_\mu s_0 b_i^{(\kappa)},\end{aligned}\quad (55)$$

and

$$\begin{aligned}\bar{N}_{\mu, [\mathbf{s}_{i+1}^{(\nu)}]} &= \bar{N}_\mu \sum_{w=1}^{W_p-2} s_{w+1} b_{i-w}^{(\alpha)} + \bar{N}_\mu s_1 b_i^{(\kappa)} \\ &\quad + \bar{N}_\mu s_0 b_{i+1}^{(\nu)}\end{aligned}\quad (56)$$

of signal photons at the photon counter are associated with the appropriate states. Also, the recursive computation of the path metrics

$$\begin{aligned}Q(\mathbf{s}_i^{(\kappa)}) &= \max \left\{ Q(\mathbf{s}_{i-1}^{(\alpha)}), Q(\mathbf{s}_{i-1}^{(\bar{\alpha})}) \right\} \\ &\quad + \ln \{P\{b_i\}\} + \gamma_{\hat{N}_i}(\mathbf{s}_i^{(\kappa)}),\end{aligned}\quad (57)$$

$$\begin{aligned}Q(\mathbf{s}_i^{(\bar{\kappa})}) &= \max \left\{ Q(\mathbf{s}_{i-1}^{(\beta)}), Q(\mathbf{s}_{i-1}^{(\bar{\beta})}) \right\} \\ &\quad + \ln \{P\{b_i\}\} + \gamma_{\hat{N}_i}(\mathbf{s}_i^{(\bar{\kappa})}),\end{aligned}\quad (58)$$

$$\begin{aligned}Q(\mathbf{s}_{i+1}^{(\nu)}) &= \max \left\{ Q(\mathbf{s}_i^{(\kappa)}), Q(\mathbf{s}_i^{(\bar{\kappa})}) \right\} \\ &\quad + \ln \{P\{b_{i+1}\}\} + \gamma_{\hat{N}_{i+1}}(\mathbf{s}_{i+1}^{(\nu)})\end{aligned}\quad (59)$$

by the soft output Kohonen map based quantum data detector is also illustrated.

The calculated path metrics are arranged in the $2^{W_p} \times (N + W_p)$ matrix

$$\mathbf{\Sigma} = \begin{pmatrix} Q^{(0)} \\ Q^{(1)} \\ \vdots \\ Q^{(2^{W_p}-1)} \end{pmatrix} \quad (60)$$

for further processing. The optimum path is found in the trace back unit by again utilizing the Kohonen map as a state mapper, yet in a reverse-oriented manner. The weights of the neurons representing the possible predecessor states $\mathbf{s}_i^{(\kappa)}$ are configured with

$$x_i^{(\kappa \rightarrow \nu)} = \begin{cases} Q(\mathbf{s}_i^{(\kappa)}) & \text{if the transition} \\ & \kappa \rightarrow \nu \\ & \text{is permissible,} \\ -\infty & \text{otherwise.} \end{cases} \quad (61)$$

Denoting the state index of the survivor state at time instant $(i + 1)$ by $\hat{v}_{i+1}^{(s)}$, its predecessor state is found according to

$$\hat{v}_i^{(s)} = \arg \max_{\kappa} \left(x_i^{(\kappa \rightarrow \hat{v}_{i+1}^{(s)})} \right). \quad (62)$$

The remaining predecessor states can be determined recursively. All predecessors along the survivor path can be arranged in the vector $\hat{v}^{(s)}$ of length $(N + W_p)$. Similarly to the case of SOVA, respectively the APRI-SOVA, approximate LLRs can be generated using the Kohonen detector [22].

Owing to its simplicity and promising performance, the authors choose to begin with the aforementioned “SIMPLE RULE” (SR). According to the SR, approximate LLRs are built based on the path metric differences of the survivor and concurrent predecessor states [17, p. 466]. In the case of Moore modelling of the FSM, the path metric difference at time instant $i + 1$ is given by

$$\Delta_{i+1}^{(s)} = Q \left(\mathbf{s}_i^{(\hat{v}_i^{(s)})} \right) - Q \left(\mathbf{s}_i^{(\hat{v}_i^{(c)})} \right), \quad (63)$$

where $Q \left(\mathbf{s}_i^{(\hat{v}_i^{(c)})} \right)$ denotes the path metric of the concurrent predecessor at time instant i . The structure of the Kohonen map employed in the generation of LLRs mirrors that of the trace back unit, with the sole modification being the inversion of the sign of the concurrent predecessor state’s path metric and the insertion of zeros instead of $-\infty$ for impermissible state transitions, i.e.

$$x_{i+1}^{(\kappa \rightarrow \nu)} = \begin{cases} Q \left(\mathbf{s}_i^{(\hat{v}_i^{(\kappa)})} \right) & \text{if } \kappa \text{ is predecessor} \\ & \text{of } \nu, \\ -Q \left(\mathbf{s}_i^{(\hat{v}_i^{(\kappa)})} \right) & \text{if } \kappa \text{ is concurrent} \\ & \text{predecessor of } \nu, \\ 0 & \text{otherwise.} \end{cases} \quad (64)$$

The path metric difference $\Delta_{i+1}^{(s)}$ represents the distinct metric deviation that allows to distinguish between whether the FSM was in the predecessor state $\mathbf{s}_i^{(\hat{v}_i^{(s)})}$ or in the concurrent predecessor state $\mathbf{s}_i^{(\hat{v}_i^{(c)})}$. Due to the structure of FSM modelling and the influence of ISI, these two states differ by only a single symbol, specifically, the one at time instant $(i - W_p + 1)$. Therefore, the path metric difference is interpreted as a reliability measure for that particular symbol at time instant $(i - W_p + 1)$. The LLRs are found recursively using

$$L_{i-W_p+1}^{(SR)} = \Delta_{i+1}^{(\hat{q}_\lambda^{(s)})} = \sum_{\kappa} \left(x_{i+1}^{(\kappa \rightarrow \hat{v}_i^{(s)})} \right). \quad (65)$$

The updates according to the “HUBER RULE” (HR) and the “BATTAIL RULE” (BR) can be performed in accordance with [17, pp. 469, 474].

In order to improve the adaptability of the Kohonen map based detector, a novel updating procedure was also introduced, where, instead of investigating spatial distribution of FSM states, a probability based neighborhood function is calculated [22]. These probabilities are calculated based on the LLRs, which can be filtered and fed back in order to update the Kohonen map. Since Kohonen maps follow a “winner takes most” approach, not only the weights of the winning output neuron are updated, but also those within a defined “winning neighborhood” Ω_s . The probability of each state $\nu \in \{0, 1 \dots 2^{W_p} - 1\}$ at each time instant $i \in \{0, 1 \dots (N + W_p - 1)\}$ is governed by the a-priori

probabilities of the W_p most recently received codebits, which are supposed to be *mutually independent*. It is thus obtained by

$$P\{\mathbf{s}_i^{(v)}\} = \prod_{w=0}^{W_p-1} P\{\hat{b}_{i-w}\}. \quad (66)$$

Given the state probability (66) and the probability threshold τ , which must be exceeded if the weights of a neuron shall be updated, the neighborhood of the winning output neuron is given by

$$\Omega_S = \{\mathbf{s}_i^{(v)} \in \mathcal{S}_{\text{Moore}} \mid P\{\mathbf{s}_i^{(v)}\} > \tau\}. \quad (67)$$

[22]. This purely local perspective can be improved by using a “forward/backward” approach to obtain the optimum state probabilities.

By combining two megatrends, namely artificial intelligence and quantum communications, soft output quantum data detectors using Kohonen maps may provide low-complexity, high-performance detection methods that enhance the scalability and efficiency of next-generation communication networks.

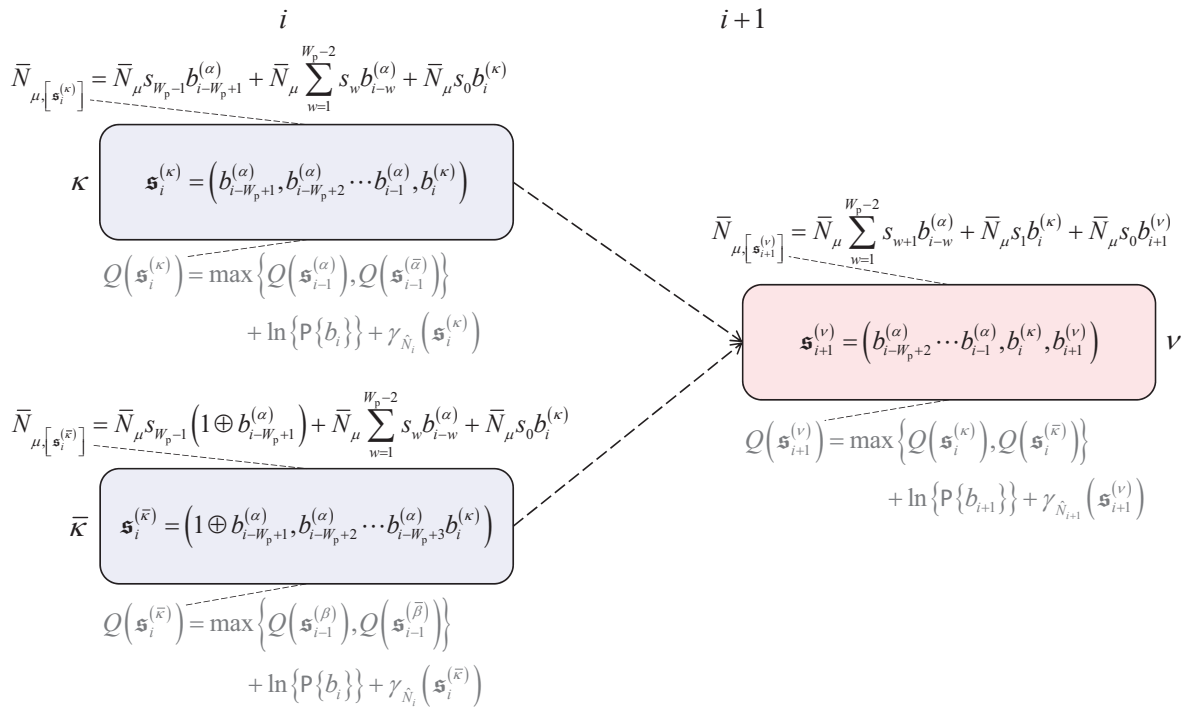


Figure 7. Schematic representation of the connections between the states $\mathbf{s}_i^{(\kappa)}$ and $\mathbf{s}_i^{(\bar{\kappa})}$ with the state indices κ and $\bar{\kappa}$ at the time instant i with the state $\mathbf{s}_{i+1}^{(\nu)}$ with the state index ν at the time instant $(i+1)$, which are part of the trellis representing the multipath quantum channel with W_p possibly nonvanishing channel coefficients s_w , $w \in \{0, 1, \dots, (W_p - 1)\}$, setting out from the Moore modelling applied in the soft output Kohonen maps based quantum data detector; the average numbers $\bar{N}_{\mu[\mathbf{s}_i^{(\kappa)}]}$, $\bar{N}_{\mu[\mathbf{s}_i^{(\bar{\kappa})}]}$ and $\bar{N}_{\mu[\mathbf{s}_{i+1}^{(\nu)}]}$ of signal photons at the photon counter are associated with the appropriate states; also, the recursive computation of the path metrics $Q(\mathbf{s}_i^{(\kappa)})$, $Q(\mathbf{s}_i^{(\bar{\kappa})})$ and $Q(\mathbf{s}_{i+1}^{(\nu)})$ by the soft output Kohonen map based quantum data detector is also illustrated.

3. Polar Coding for Wireless Quantum Optical Transmission

3.1. General Concept of Polar Codes

In 2009, Erdal Arikan introduced polar codes as a class of capacity-achieving error-correcting codes for binary memoryless symmetric channels [39]. The construction of polar codes is inspired by Reed-Muller codes and builds on the Plotkin construction proposed in 1960 [40], or alternatively, it can be described using a recursive formulation based on the Kronecker product [39].

Polar codes are notable for their deterministic construction and their ability to asymptotically achieve the capacity of binary memoryless symmetric channels under successive cancellation decoding. Understanding the polarization process requires some familiarity with information-theoretic concepts such as mutual information and entropy. For a more detailed theoretical background, the reader is referred to relevant literature, cf. e.g. [8, Chapter 4, pp. 336-388]. The key idea of channel polarization is the transformation of a set of identical channels into a set of virtual channels with split reliability levels, which allows a distinction between information bits and frozen bits.

For the construction of polar codes of even length

$$N = 2^n \Leftrightarrow n = \log_2\{N\}, \quad n \in \mathbb{N}, \quad (68)$$

$\mathbf{u}$ denotes the input vector to the encoder, containing $K \leq N$ information bits, and $\mathbf{b}$ denotes the encoded codeword at the output of the encoder, consisting of the K information bits and the $(N - K)$ frozen bits. Let $\mathbf{G}_N$ be the generator matrix. The polar code is then given by the linear transformation of $\mathbf{u}$ using $\mathbf{G}_N$ [39]

$$\mathbf{b} = \mathbf{u}\mathbf{G}_N. \quad (69)$$

Setting out from the smallest code length, i.e. N equal to 2, the basic generator matrix becomes [41]

$$\mathbf{G}_2 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \quad (70)$$

being capable of generating both parity and systematic codebits. The square $N \times N$ generator matrix of the 3GPP polar code is recursively defined by the multiple Kronecker product [8, Sect. 1.16, pp. 224-233]

$$\mathbf{G}_2^{\otimes n} = \mathbf{G}_2^{\otimes(n-1)} \otimes \mathbf{G}_2. \quad (71)$$

Arikan prefers to define the generator matrix in his work based on a permutation matrix, denoted here as $\mathbf{\Pi}_N$, and the identity matrices $\mathbf{I}_{N/2}$ and $\mathbf{I}_2$ [8,39]

$$\mathbf{G}_N = (\mathbf{I}_{N/2} \otimes \mathbf{G}_2)\mathbf{\Pi}_N(\mathbf{I}_2 \otimes \mathbf{G}_{N/2}). \quad (72)$$

The two variants of the linear transformation using $\mathbf{G}_N$, as defined by 3GPP and by Arikan, each require a dedicated decoder structure to correctly decode the encoded message. For N equal to 4, Figure 8 illustrates the decoding scheme according to the 3GPP standard and Figure 9 shows the decoding structure according to Arikan.

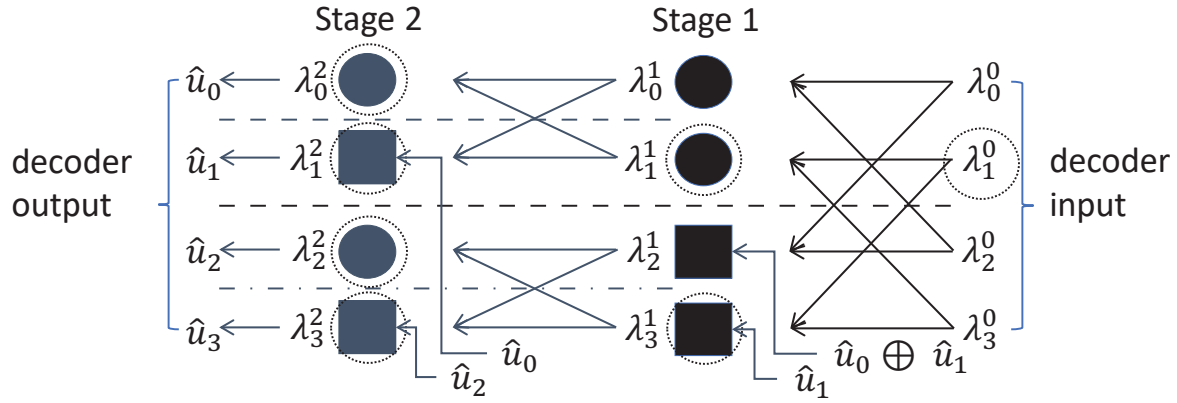


Figure 8. Illustration of the information flow in a successive cancellation decoder based on the 3GPP implementation for a polar code of length $N = 4$.

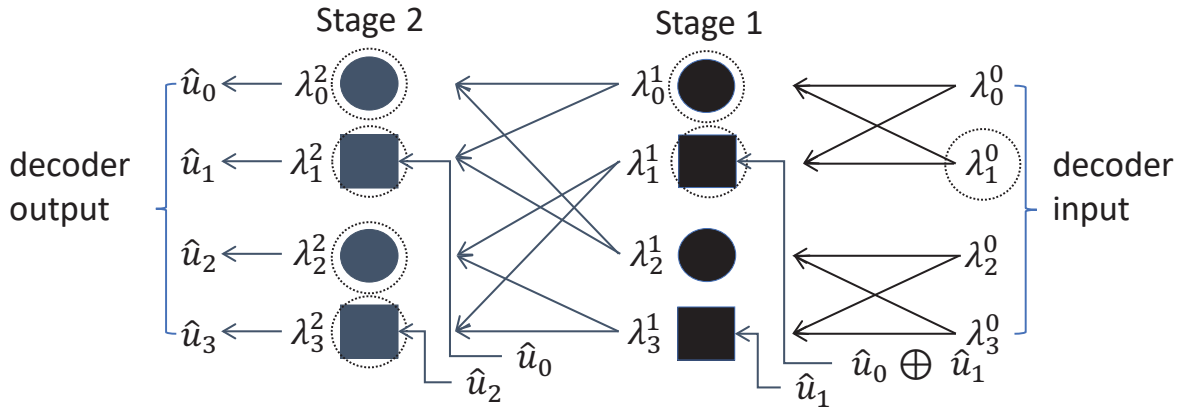


Figure 9. Illustration of the information flow in a successive cancellation decoder based on Arikan's original construction for a polar code of length $N = 4$.

The values λ_i represent the LLRs. In each decoding stage, two functions are applied, namely, the f function, represented by circles in Figure 8, and the g function, represented by squares in Figure 8. These functions are used to recursively update the LLRs based on previously decoded bits. The f and g functions operate on two input LLR values, typically denoted as λ_a and λ_b , and in the case of the g function, also on previously decoded bits $\hat{u}$. These inputs correspond to values from nodes in the decoding tree. The f and g functions can be defined as follows [8, Eq. (3.114), p. 364; Eqs. (4.260) and (4.261), p. 386]:

$$f(\lambda_a, \lambda_b) \approx \frac{|\lambda_a + \lambda_b| - |\lambda_a - \lambda_b|}{2}, \quad (73)$$

$$g(\lambda_a, \lambda_b, \hat{u}) = (1 - 2\hat{u}) \cdot \lambda_a + \lambda_b. \quad (74)$$

The expression for the f function represents an approximation of the exact soft decoding rules using the arctanh [8, Eq. (4.260), p. 386]. After a total of $\log_2(N)$ stages, the message $\hat{\mathbf{u}}$ is fully decoded using recursive decoding. This process starts with the least significant bit $\hat{u}_0$ and proceeds toward the most significant bit $\hat{u}_{N-1}$. Previously decoded bits are reused as intermediate results across the decoding stages.

A comparison of the 3GPP and the Arikan constructions of the polar code reveals that the indexing must be handled differently in the implementation of the respective decoder structures. This is due to the fact that the generator matrices used in the 3GPP and Arikan constructions are not identical and transform the message in different ways through linear operations. Another key difference between the 3GPP and the Arikan constructions lies in the structural symmetry present in every stage in the 3GPP decoder, as indicated by the horizontal dashed lines in Figure 8, which is not present in the Arikan construction. Furthermore, it becomes evident that a decoding error, e.g. at the position of

λ_1^0 , highlighted as dashed circles in Figures 8 and 9, effects the decoding process differently in each structure. The error propagation paths diverge between the two constructions, as illustrated by the dashed circled elements in the figures. Even under identical channel conditions, the decoders may produce similar but not identical results.

3.2. Design of Polar Codes: Finding the Polar Sequence

Even though the generator matrices G_N are known, constructing the strongest possible polar code for a given code rate R_c equal to K/N remains a laborious task. It requires identifying those virtual channels that should be frozen and thus excluded from carrying information [8,39]. This identification leads to a ranking list of all virtual channels, often referred to as the *reliability sequence* [8][pp. 347, 364, 378, 380], *polar sequence* [41], or *reliability order* [42]. The process of constructing this polar sequence is commonly known as *polar code definition* [39] or *polar code construction* [42].

For a combined channel W_N of length N , the process of channel combining and splitting results in a set of polarized subchannels $\{W_N^{(i)} : 0 \leq i \leq (N-1)\}$ [8, Chapter 4, pp. 336-388]. The task of polar code construction is to select the most reliable subchannels from this set, i.e., to determine an appropriate subset of virtual channels that will be used to carry the information bits. As Arikan has shown in his original work, the polar sequence depends on the transmission channel. For the binary erasure channel, he suggested the Bhattacharyya parameter $Z(W_N^{(i)})$ to determine the polar sequence [39]. Since the Bhattacharyya parameter of a binary erasure channel is equal to the erasure probability ε , the channel capacity C of a virtual channel approaches 1 when $Z(W_N^{(i)})$ approaches 0 [8].

Arikan's Bhattacharyya parameter $Z(W_N^{(i)})$ based approach is clearly dependent on the particular channel and its actual quality. Regrettably, it does not seem that the Bhattacharyya parameter $Z(W_N^{(i)})$ can be determined in analytic closed form for the quantum channels considered here. Therefore, other methods for determining the polar sequence when considering transmission channels other than the binary erasure channel must be used such as the Monte Carlo simulation approach [39] and the density evolution method [43]. In all these cases, the definition of the polar sequence will involve lengthy and cumbersome analyses and is likely not to yield a single optimum polar sequence for all circumstances.

The likelihood of never being able to define the optimum polar sequence in a closed form analytic expression and thus at the lowest possible cost has led to the abandonment of the strict requirement for an optimum solution. Instead, the focus has shifted toward a sufficiently good, best possible solution for the time being. In the case of 5G/New Radio (5G/NR), 3GPP delegates have succeeded in defining such a polar sequence [41][Table 5.3.1.2-1, pp. 17-19], which enables an impressively low block error ratio (BLER) performance in all required cases. The approach chosen in the 3GPP standardization of 5G/NR sets out from a closed form expression, which to the best of the authors' knowledge was first suggested in the technical document *R1-167209* presented at the TSG RAN WG1 Meeting #86. We will briefly summarize this approach in what follows:

- 1) Set out from the virtual channel with index $i \in \{0, 1 \dots (N-1)\}$ being a nonnegative integer and having the binary representation given by the values $B_j^{(i)} \in \{0, 1\}$ beginning from the least significant bit j equal to 0, i.e.

$$i = B_0^{(i)} \cdot 2^0 + B_1^{(i)} \cdot 2^1 + \dots + B_{n-1}^{(i)} \cdot 2^{n-1}, \quad (75)$$

$$n = \log_2\{N\}, \quad B_j^{(i)} \in \{0, 1\}.$$

- 2) Then, by replacing 2 by $\sqrt[4]{2}$ being approximately equal to 1.1892 in (75), define the quality factors called polar weights [42, Eq. (4), p. 2]

$$\tilde{W}_i \stackrel{\text{def}}{=} \sum_{j=0}^{n-1} B_j^{(i)} \cdot \left(\sqrt[4]{2}\right)^j. \quad (76)$$

Comparing (75) with (76), we immediately realize that (76) represents the generalization of the binary representation of (75).

- 3) Since a virtual channel with high quality is associated with a large value of the polar weight $\tilde{W}_i$ given by (76), establish the polar sequence as that particular sequence of the indices of $\tilde{W}_i$ in ascending order of the polar weights $\tilde{W}_i$.

The authors believe that the suggested approach of the technical document R1-167209 presented at the TSG RAN WG1 Meeting #86 presents a promising start. However, owing to the considerable difference between the mobile radio channel and the quantum channel faced in the case of wireless quantum optical communications, it must be studied whether there may exist better polar sequences for wireless quantum optical communications than the existing 3GPP 5G/NR polar sequence published in [41][Table 5.3.1.2-1, pp. 17-19]. In a first step, we will replace (76) by the following definition

$$\tilde{W}_i \stackrel{\text{def}}{=} \sum_{j=0}^{n-1} B_j^{(i)} \cdot p^j, \quad p \in (1, 2], \quad (77)$$

of the quality of the virtual channel with index i . The base p chosen in (77) determines the polar weights $\tilde{W}_i$ of the N virtual channels. Nota bene that the representation of (77) is sometimes attributed to Alfréd Rényi as a number theoretical concept published in 1957. However, its roots are more than half a century older as we will see in a moment.

Equation (77) is based on the p -adic expansion of integers illustrated in [45, p. VI; Eq. (2), pp. 49f.], for a finite number of n equal to $\log_2\{N\}$ terms and only nonnegative exponents $j \geq 0$, a concept well-known in number theory first introduced in 1897 by the German mathematician Kurt Hensel [44], see also [46], yielding a unique result for any nonnegative integer. In the p -adic expansion, the base p is a prime number, however, it is straight-forward and trivial to relax this requirement in favor of any base (in German “Grundzahl”) g [45, p. VI], such as e.g. g equal to 10, leading to the more general g -adic expansion [45, Eq. (7), p. 54]. In the case of the p -adic as well as the g -adic expansions, the coefficients $B_j^{(i)}$ are taken modulo p , respectively, g [45, p. 54]. Hensel also investigated the case of the general base g being a power p^k of the prime number p [45, p. 73]. Setting out from these insights, it is an immediate approach to obtain (77), neglecting the number theoretic burden of this step. Supposing p to be a general, preferably nonnegative, real number, the said p -adic expansion based approach (77), although according to the above discussion being suboptimum, is channel independent and provides a solution in analytic closed form.

The definition of the polar sequence by a closed expression of the type shown in (77) is elegant, efficient and implementation-friendly. Setting out from these facts, the authors will present further results on the selection of suitable polar sequences for quantum optical communication systems in what follows, proposing distinct polar sequences for wireless quantum optical communications with carrier phase knowledge versus random carrier phase scenarios as such a distinction facilitates a lowest possible BLER for the considered applications. In other words, the authors suggest using channel adaptive polar sequences in future applications, in particular, for wireless quantum optical communication systems.

In wireless quantum optical communications, the transmission is predominantly governed by unknown, random carrier phase most of the time, cf. e.g. Sect. 2.3 and [20]. In the absence of any channel state information (CSI) at the transmitters and the receivers, the polar sequences corresponding to

$$p = \sqrt[3]{\pi} \approx 1.1777 \quad (78)$$

and

$$p = \sqrt[4]{\Phi} = \sqrt[4]{\frac{1 + \sqrt{5}}{2}} \approx 1.1278 \quad (79)$$

are found to be good choices. Furthermore, if the transmission preserves the carrier phase, using (79) leads to the preferable polar sequences for any code rate R_c . These observations show that the suggested base p equal to $\sqrt[4]{2}$ used in the technical document *R1-167209* presented at the TSG RAN WG1 Meeting #86 differs from the choice of p in the case of wireless quantum optical communications. The following Algorithm 1 summarizes the approach.

Algorithm 1 Computation of Polar Sequences

Require: Code length N , positive real $g > 1$, e.g. Φ , 2, $\exp\{1\}$, π ; exponent $q \in (0, 1]$, e.g. the code rate R_c

- 1: Determine base $p = g^q$
 - 2: **for** $i = 0$ **to** $N - 1$ **do**
 - 3: Convert channel index i to binary:
 - 4: $i \rightarrow (B_0^{(i)}, B_1^{(i)} \dots B_{n-1}^{(i)})$
 - 5: Compute the weight: $\tilde{W}_i = \sum_{j=0}^{n-1} B_j^{(i)} \cdot p^j$
 - 6: **end for**
 - 7: Sort all channel indices i in ascending order of $\tilde{W}_i$
 - 8: **return** the resulting order as the polar sequence S
-

Since standardized implementations typically prefer a single, channel-independent polar sequence, the authors propose a polar sequence suitable for both coherent and incoherent transmission. This new polar sequence is derived from the combination of the two previously considered polar sequences found for p equal to $\sqrt[7]{\pi}$ and $\sqrt[4]{(1 + \sqrt{5})}/2$. To the best of the authors' knowledge, the approach of constructing a new polar sequence by combining multiple sequences has not been previously published and therefore seems to be novel in the context of constructing new polar sequences.

This work considers two analytical approaches for combining multiple reliability sequences. The first approach considers, at each position x_i of the polar sequence, only the largest channel index i among the available sequences. Therefore, the largest channel index i at that position x_i is selected as the winner. The second approach evaluates a data set consisting of at least three predefined reliability sequences. In this case, the selection is based on the absolute frequency of each channel index i at the position x_i . If no clear winner can be determined on the basis of frequency, the decision falls back to the selection of the larger channel index i according to Algorithm 1. In this manuscript, the first approach is preferred as it is more resource-efficient and easier to implement. The procedure to generate the aforementioned combined polar sequence is summarized in the novel Algorithm 2. The central idea of this algorithm is to resolve conflicts incrementally, position by position, by giving preference to the channel with the higher channel index i and filling the remaining gaps with unused indices, sorted by ascending weight. It is almost needless to mention that the iterative application of Algorithm 2 will facilitate to combine a multitude of available polar sequences.

Algorithm 2 Combination of Two Polar Sequences**Require:** Two polar sequences S_1, S_2 , each of length N **Ensure:** Combined polar sequence S_{combined} of length N

```

1: Sort all channel indices  $i$  of both sequences  $S_1, S_2$  in ascending order of channel weight  $W_i$ 
2: for  $i = 0$  to  $N - 1$  do
3:   if  $S_1[i] = S_2[i]$  then
4:     Assign  $S_{\text{combined}}[i] \leftarrow S_1[i]$ 
5:   else
6:     Assign  $S_{\text{combined}}[i] \leftarrow \max(S_1[i], S_2[i])$ 
7:     Store  $S_1[i]$  and  $S_2[i]$  (if not used)
8:       in temporary list
9:   end if
10: end for
11: Sort remaining unused channel indices  $i$ 
12:   in ascending order
13: for each unassigned position  $x_i$  in  $S_{\text{combined}}$  do
14:   Assign next smallest channel index  $i$ 
15:     from temporary list
16: end for
17: return  $S_{\text{combined}}$ 

```

3.3. Turbo Equalization with Polar Codes

Turbo equalization is an iterative scheme, which involves both data detection and channel decoding, striving for the iterative reduction of errors contained in the decoded message $\hat{u}$. Turbo equalization was first introduced in 1995 [47] in the case of additive white Gaussian noise (AWGN) channels with ISI and with as well as without fading. Both the data detector and the channel decoder applied in [47] were using the SOVA. A similar concept for source-controlled channel decoding was used in [37,48], which clearly demonstrates the necessity of providing soft information like LLRs from the source decoder to the channel decoder. None of these concept has yet been applied to polar coded wireless quantum optical communications via multipath quantum channels with uncorrelated scattering and ISI.

The principle of turbo equalization in polar coded wireless quantum optical communications via multipath channels with uncorrelated scattering and ISI is depicted in Figure 10. The generic receiver consists of the quantum data detector, the polar code decoder, the block error iteration control unit and the extrinsic information computer. Since in wireless communication systems interleaving is widely deployed and since the authors suggest the inclusion of quantum optics in future wireless communication systems, the optional introduction of an interleaver with the interleaver (permutation) matrix Π and its corresponding deinterleaver with the matrix Π^{-1} is also depicted in Figure 10. In this work, a simple square block interleaver will be considered when analyzing turbo equalization for wireless quantum optical communications. Suppose the square $N_I \times N_I$ interleaver matrix Π , yielding the interleaver size N_I^2 . Using the ceiling function (in German “obere Gaußklammer”) $\lceil \cdot \rceil$, choose N_I equal to $\lceil \sqrt{N} \rceil$ and form the *extended codeword*

$$\tilde{\mathbf{b}} = \left(b_0, b_1 \cdots b_{N-1}, \underbrace{0, 0 \cdots 0}_{\substack{(N_I^2 - N) \\ \text{zeros}}} \right) \quad (80)$$

of length $N_I^2 \geq N$. The said extended codeword $\tilde{\mathbf{b}}$ of (80) is then entered into the square block interleaver from left to right in a row by row fashion, beginning with the first row and the first column and the interleaved extended codeword is determined from top to bottom in a column by column fashion, beginning with the first row and the first, i.e. leftmost, column. The deinterleaving is accomplished by the reversed operation.

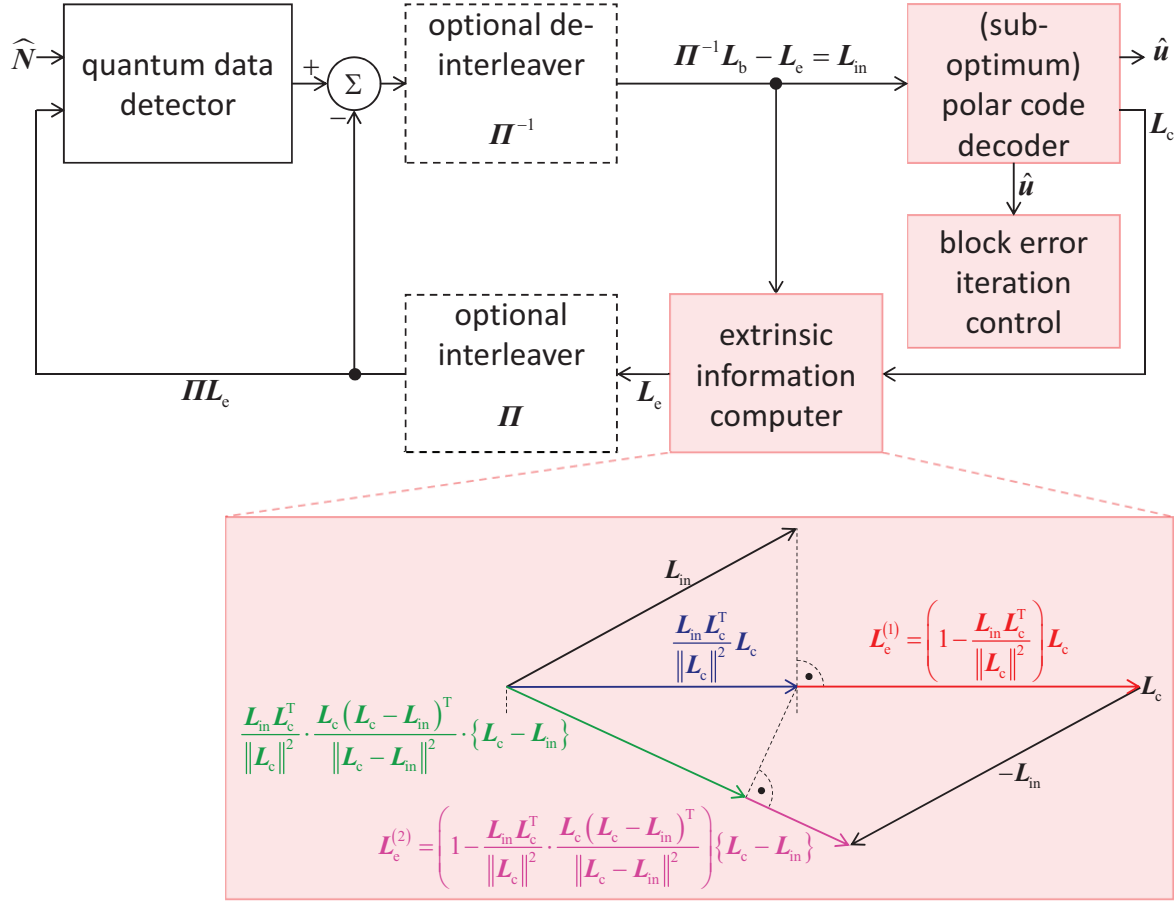


Figure 10. Generic receiver structure for data transmission in polar coded wireless quantum optical communications via multipath channels with uncorrelated scattering and intersymbol interference (ISI) including the turbo equalization concept.

Turbo equalization is only possible when the quantum data detector is capable of processing a-priori information. Regrettably, the said a-priori information is not known to the receiver prior to the quantum data detection. This fact requires the generation and continuous updating of some sort of information, which can serve as “a-priori information”, while the turbo equalization iterations are ongoing. Hence, there is no true a-priori information but some sort of an estimate. This estimate must be delivered by some receiver unit, which cannot be located in front of the input of the quantum data detector due to the above-mentioned prerequisite. In fact, a modified channel decoder, which in addition to providing the decoded message $\hat{u}$ is capable of determining the soft output vector L_c corresponding to the $N \geq K$, codebits $b_i \in \mathbb{F}_2$, $i \in \mathbb{B}$, provides the basis for the said estimate of the aforementioned “a-priori information”, which we will call the interleaved extrinsic soft output vector denoted by ΠL_e . In the context of the turbo equalization considered here, the expression “extrinsic” means “generated during the channel decoding in addition to the input vector L_{in} ”. In other words, the extrinsic soft output vector has not been part of the input to the channel decoder prior to the channel decoding.

According to Figure 10, after processing the photon count vector $\hat{N}$ introduced in (28) and the “a-priori information” ΠL_e , the quantum data detector delivers the soft output vector L_b corresponding to the $N \geq K$, codebits $b_i \in \mathbb{F}_2$, $i \in \mathbb{B}$. In order to optimize the mutual information governing the turbo equalization process just as in the case of turbo code decoding, the “a-priori information” ΠL_e is subtracted from L_b prior to the optional deinterleaving. At the output of the deinterleaver, the input vector

$$L_{in} = \Pi^{-1} L_b - L_e \quad (81)$$

prevails, which is obtained from e.g. (88) for γ equal to 1 and represents the input to the polar code decoder. Besides the decoded message $\hat{\mathbf{u}}$, the polar code decoder generates the vector $\mathbf{L}_c$, which is composed of soft outputs associated with the $N \geq K$, codebits $b_i \in \mathbb{F}_2$, $i \in \mathbb{B}$. In the block error iteration control unit, the decoded message $\hat{\mathbf{u}}$ is analyzed in order to identify block errors. If a block error is detected by e.g. evaluating a cyclic redundancy check (CRC) code, which is supposed to be part of $\mathbf{u}$ and, consequently, of $\hat{\mathbf{u}}$, according to the 5G/NR specification [49, Sect. 6.13.2.1, pp. 140-142], a further turbo equalization iteration is carried out. In the case $\hat{\mathbf{u}}$ successfully passes the analysis in the block error iteration control unit without any detected block error, the turbo equalization is terminated for the inspected message $\mathbf{u}$. As illustrated, the authors deploy adaptive iteration control of the turbo equalization, which facilitates considerable savings of processing efforts and, consequently, power consumption.

In the case of a required turbo equalization iteration, the soft output vector $\mathbf{L}_c$ is processed in the extrinsic information computer, generating the extrinsic soft output vector $\mathbf{L}_e$ at its output by exploiting the input vector $\mathbf{L}_{in}$ of (81). After the optional interleaving, which yields $\Pi\mathbf{L}_e$, the quantum data detection process can start again.

The success of turbo equalization based on polar coding relies on two prerequisites, namely

- 1) the availability of soft information per codebit $b_i \in \mathbb{F}_2$, $i \in \mathbb{B}$ collected in the soft output vector $\mathbf{L}_c$ and
- 2) the availability of extrinsic information of good quality, collected in the extrinsic soft output vector $\mathbf{L}_e$.

With respect to the first item, two alternative approaches have been mentioned in the open literature. The first concept is based on the sum product algorithm (SPA) [8, Sect. 3.2.3, pp. 332-335] used for the decoding of LDPC (low density parity check) codes when computing the LLRs of the check nodes, cf. e.g. [8, Sect. 3.2.2, pp. 327-332]. The second concept was suggested by Fayyaz and Barry as part of their soft cancellation (SCAN) polar decoding scheme [50]. Since the first concept provides correct LLRs, the authors do not pursue the second approach [50] further in this manuscript.

Nota bene that, although claimed, [50] does not provide any turbo equalization. It is merely a clever adaptation of the SPA to polar codes.

Furthermore, the rather little performance improvement of the SCAN polar code decoding reported by Fayyaz and Barry [50] demonstrates the strong correlation of the soft inputs vectors and the corresponding soft output vectors of the SCAN polar code decoder, which is governed by the code structure.

Regrettably, it seems that no one has explicitly stated the chosen approach in the context of polar codes. Hence, the authors are now making up for this omission in what follows:

Suppose that in the n th, $n \in \{0, 1 \dots (N-1)\}$, column of the generator matrix $\mathbf{G}_N$ of the polar code of length N only the elements $g_{i_k, n}$ in the rows denoted by i_k with i_k taken from a subset of $\{0, 1 \dots (N-1)\}$ are equal to one.

Let us denote the said subset by $\mathbb{O}_n \subseteq \{0, 1 \dots (N-1)\}$, having the cardinal number $|\mathbb{O}_n| \leq N$. Define

$$\begin{aligned} \mathbb{O}_n &= \{i_k | g_{i_k, n} = 1 \forall i_k \in \{0, 1 \dots (N-1)\}\} \\ &= \underbrace{\{i_1 \dots i_{|\mathbb{O}_n|-1}\}}_{\text{is equal to } \emptyset \text{ for } n=N-1} \cup \{N-1\}. \end{aligned} \quad (82)$$

Denoting the soft output associated with the N th message bit u_{N-1} by $L_{u, (N-1)}$, for every n , initialize $L \in \mathbb{R}$ by

$$\begin{aligned} B &\leftarrow L_{u, (N-1)}, \\ \mathbb{O}_n &\leftarrow \mathbb{O}_n \setminus \{N-1\}. \end{aligned} \quad (83)$$

Denoting the “box plus” operation by $\boxplus$, which realizes

$$B \boxplus L_{u,i} \stackrel{\text{def}}{=} \ln \left\{ \frac{1 + e^{B+L_{u,i}}}{e^B + e^{L_{u,i}}} \right\} \quad (84)$$

either exactly or by some approximation, cf. e.g. [51][pp. 357-366] and the references given therein, while $\mathbb{O}_n \neq \emptyset$ iterate

$$\begin{aligned} B &\leftarrow B \boxplus L_{u,i}, \quad i \in \mathbb{O}_n, \\ \mathbb{O}_n &\leftarrow \mathbb{O}_n \setminus \{i\}. \end{aligned} \quad (85)$$

The final output is

$$L_{c,n} = B. \quad (86)$$

Probably the most critical part of turbo equalization lies in the determination of the extrinsic soft output vector L_e , cf. the second item mentioned above. According to Douillard’s seminal paper [47, Fig. 5, p. 510], we obtain

$$L_e = L_c - L_{in} \quad (87)$$

$$= L_c - \left\{ \Pi^{-1} L_b - \gamma L_e \right\} \quad (88)$$

$$= \frac{L_c - \Pi^{-1} L_b}{1 - \gamma}, \quad \gamma \in \mathbb{R}_0^+. \quad (89)$$

Although Douillard et al. do not explicitly give the range of the coefficient γ , the reference to the variance on page 509 of [47] implicitly suggests that γ is nonnegative and real. It is also plausible to suppose that (89) implies $\gamma \in [0, 1)$. In 1997, Hagenauer [52, Fig. 4, p. 7] illustrated that (87) is optimum in the case of MAP based symbol by symbol data detection and MAP based channel decoding. A third known approach appeared in 1993 in the seminal paper [53, p. 1096] by Berrou et al. introducing turbo codes for the first time, suggesting the i th component $[L_e]_i$ of the extrinsic soft output vector L_e to be

$$[L_e]_i = \frac{[L_c - L_{in}]_i}{1 + \theta |[L_{in}]_i|}, \quad \theta = 0.15, \quad i \in \mathbb{B}, \quad (90)$$

although all constituent decoders applied in [53] are MAP symbol by symbol decoders.

Since optimum polar code decoders, which can be efficiently realized, are not known, the authors suppose that deployed polar code decoders are and will probably always be suboptimum. Therefore, one must raise the question, whether the above-mentioned schemes (87)-(90) are applicable. In what follows this concern will be addressed and the authors will suggest a viable solution for turbo equalization using polar codes.

The treatment of the availability of extrinsic information of good quality, collected in the extrinsic soft output vector L_e , seems still to be in its infancy as far as turbo equalization using polar codes is concerned. In fact, it seems that turbo equalization using polar codes has not been addressed explicitly prior to a paper [54] published in 2024 by Eltohamy, Hofmann, Knopp and Korb. Although not discussed in detail in [54], Eltohamy et al. either use the above stated approach for the generation of the soft output vector L_c or an equivalent one. With respect to the generation of the extrinsic soft output vector L_e , Eltohamy et al. apply [54, Fig. 6, pp. 146f.]

$$L_e = C_S(L_c - L_{in}), \quad C_S \in \mathbb{R}_0^+, \quad (91)$$

using the real nonnegative scaling factor $C_S \in \mathbb{R}_0^+$, which is “empirically optimized”, leaving a certain feeling of discomfort. Surprisingly, C_S may be arbitrarily chosen [54, Fig. 8, p. 147], which suggests a certain degree of novelty in comparison to (87)-(90). Although the number of turbo equalization

iterations was not stated in [54], the authors are convinced that this number was fixed in all cases, not being adaptive to the actual transmission. Furthermore, the performance gains reported in [54, Figs. 8 and 9, pp. 147f.] seem to be limited. This is a clear hint to the fact that notable correlations between L_c and L_{in} exist.

Regarding the availability of extrinsic information of good quality, collected in the extrinsic soft output vector L_e , the authors propose the following two realizations of the extrinsic soft output vector, shown as $L_e^{(1)}$ and $L_e^{(2)}$ in Figure 10. The said solutions proposed by the authors present an analytic way forward, avoiding any empirical optimizations. Correlations vanish when the vectors L_c and L_{in} are mutually orthogonal, at least on average. The demanded orthogonality is however unlikely, which yields a nonvanishing inner product $L_{in}L_c^T$, supposing that L_c and L_{in} are row vectors and denoting transposition by $(\cdot)^T$. Therefore, when subtracting that particular component $(L_{in}L_c^T / \|L_c\|^2)L_c$ of L_{in} , which is collinear to L_c , from L_c , the remainder $[1 - (L_{in}L_c^T / \|L_c\|^2)]L_c$ may serve as extrinsic information of desirable quality. The authors therefore propose the following first version of the extrinsic soft output vector

$$L_e = \left(1 - \frac{L_{in}L_c^T}{\|L_c\|^2}\right)L_c \propto L_c, \quad \frac{L_{in}L_c^T}{\|L_c\|^2} \in \mathbb{R}, \quad (92)$$

illustrated by $L_e^{(1)}$ in Figure 10.

The second version of the extrinsic soft output vector, identified by $L_e^{(2)}$ in Figure 10, sets out from the assertion that the extrinsic soft output vector L_e should be collinear with $(L_c - L_{in})$, which has been considered traditionally, cf. e.g. (87)-(90), and is also assumed in (91). As it was already observed by Eltohamy et al. [54], the real nonnegative scaling factor $C_S \in \mathbb{R}_0^+$ cannot be left equal to 1, hence (87) and its direct descendants do not seem viable. Extending the above derivation leading to (92) yields

$$\begin{aligned} L_e &= \left(1 - \frac{L_{in}L_c^T}{\|L_c\|^2} \cdot \frac{L_c(L_c - L_{in})^T}{\|L_c - L_{in}\|^2}\right)\{L_c - L_{in}\} \\ &\propto \{L_c - L_{in}\}, \quad \frac{L_{in}L_c^T}{\|L_c\|^2} \cdot \frac{L_c(L_c - L_{in})^T}{\|L_c - L_{in}\|^2} \in \mathbb{R}, \end{aligned} \quad (93)$$

cf. the geometrical illustration depicted in Figure 10. With (93), the real nonnegative scaling factor C_S of (91) becomes

$$C_S = 1 - \frac{L_{in}L_c^T}{\|L_c\|^2} \cdot \frac{L_c(L_c - L_{in})^T}{\|L_c - L_{in}\|^2}. \quad (94)$$

Nota bene that both approaches (92) and (93) suggested by the authors are fully adaptive with a clear theoretical foundation and without any empirical optimization. Furthermore, as opposed to the approaches presented in (87)-(91), the novel solutions (92) and (93) suggested by the authors use the real coefficients

- $L_{in}L_c^T / \|L_c\|^2$ and
 - $(L_{in}L_c^T / \|L_c\|^2)(L_c(L_c - L_{in})^T / \|L_c - L_{in}\|^2)$,
- respectively.

4. Selected Simulation Results

4.1. Uncoded Transmission and Reception

First, we will consider the short range transmission via a single path quantum channel, cf. Sect. 2.3. Consequently, the channel impulse response of the said quantum channel called CIR#1 in what follows has one single nonzero channel coefficient s_0 equal to 1, cf. Table 1. Both the scenario without decoherence and, thus, with known carrier phase as well as the scenario subject to decoherence and,

hence, with unknown carrier phase, the phase being a uniformly distributed random variable, have been evaluated by simulations in terms of the uncoded bit error ratio (BER) P_b , i.e. the BER obtained before the channel decoding. The obtained uncoded BER curves are depicted in Figure 11. In any of the two scenarios, the uncoded BER P_b decreases monotonically with increasing $\bar{N}_\mu$ for a given value of $\bar{N}_\alpha$. Similar to e.g. [6][Fig. 3, p. 1590], the average number of thermal noise photons $\bar{N}_\alpha$ was chosen from $\{0.01, 0.03, 0.05, 0.1\}$. Furthermore, the uncoded BER P_b increases with increasing thermal noise, i.e. with increasing $\bar{N}_\alpha$, for a given value of $\bar{N}_\mu$. Also, the steepness of the uncoded BER curves seems to decrease with increasing thermal noise, i.e. with increasing $\bar{N}_\alpha$. Remarkably, the uncoded BER P_b is always lower in the case of known carrier phase than in the case of unknown carrier phase for given values of $\bar{N}_\mu$ and $\bar{N}_\alpha$.

Table 1. Channel coefficients $s_w, w \in \{0, 1, 2\}$ of the selected channel impulse responses [55, Tab. 3.7, p. 662].

channel impulse response	number of paths W_p	channel coefficient s_0	channel coefficient s_1	channel coefficient s_2	$\sum_{w=0}^{W_p-1} s_w$
CIR#1	1	1	0	0	1
CIR#4	2	0.6667	0.3333	0	1
CIR#5	3	0.6250	0.3125	0.0625	1

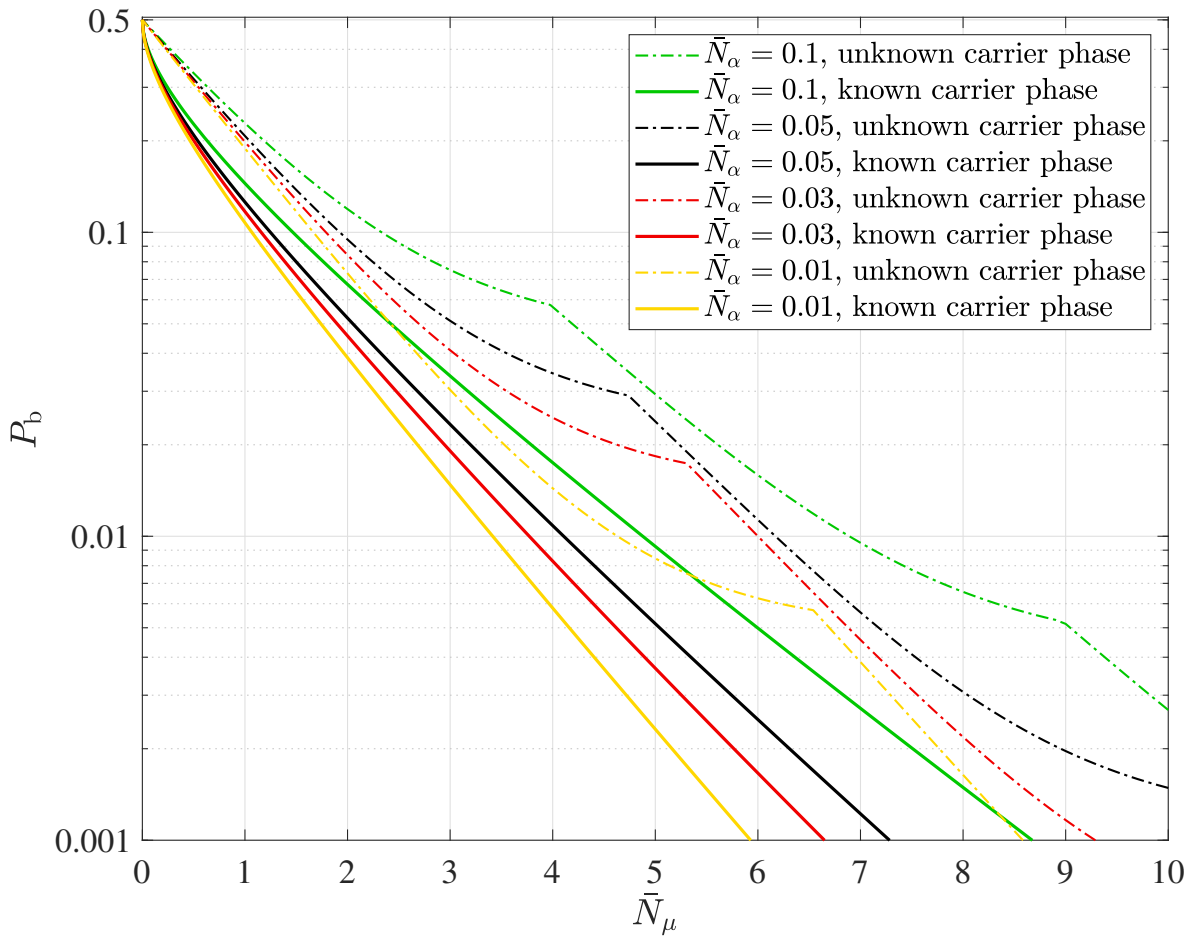


Figure 11. Detail of simulation results of the uncoded bit error ratio (BER) P_b performance versus the average number $\bar{N}_\mu$ of signal photons per message bit with the average number $\bar{N}_\alpha$ of thermal photons as the curve parameter for the transmission via the single path quantum channel CIR#1 with $W_p = 1$; the dashed curves refer to the case of unknown carrier phase, cf. e.g. [6][p. 1590], [9][pp. 171-174], [17][pp. 435-446] while the solid curves refer to the case of known carrier phase, cf. [5][pp. 390f.] for a similar result.

Next, we consider multipath transmission and reception under the presence of decoherence, see Sect. 2.4. The standardization of mobile radio systems has necessitated the development of widely accepted channel models, cf. e.g., [35,56,57]. Unfortunately, quantum communication systems, such as those discussed here, have not yet reached a comparable level of standardization. Consequently, commonly accepted multipath quantum channel models are not yet available. To proceed, we adopt multipath quantum channels with time-invariant channel impulse responses (CIRs), as first introduced in [55, Tab. 3.7, p. 662]. Specifically, we consider two distinct multipath quantum channels, referred to as CIR#4 and CIR#5 [55, Tab. 3.7, p. 662]. The corresponding CIR coefficients s_w , $w \in \{0, 1, 2\}$, are listed in Table 1 [55, Tab. 3.7, p. 662]. The selected multipath quantum channels are assumed to be energy-preserving, that is, they neither amplify nor attenuate the transmitted signals. The CIRs of both CIR#4 and CIR#5 are normalized accordingly by ensuring

$$\sum_{w=0}^{W_p-1} s_w = 1. \quad (95)$$

This guarantees that all depicted results are comparable in terms of average received energy. Consequently, if the laser remains continuously on, the average number of received signal photons is $\bar{N}_\mu$, regardless of the chosen channel model, see, e.g., (21).

Figure 12 shows a detail of simulation results of the uncoded BER P_b performance versus the average number $\bar{N}_\mu$ of signal photons per message bit with the average number $\bar{N}_\alpha$ of thermal photons as the curve parameter in the case of the transmission via the multipath quantum channel CIR#4, which has the two channel coefficients s_0 equal to 0.6667 and s_1 equal to 0.3333. Similar to e.g. [6][Fig. 3, p. 1590], the average number of thermal noise photons $\bar{N}_\alpha$ was chosen from $\{0, 0.1, 0.3, 0.5, 1.0\}$. Furthermore, Figure 13 illustrates a detail of simulation results of the uncoded BER P_b performance versus the average number $\bar{N}_\mu$ of signal photons per message bit with the average number $\bar{N}_\alpha$ of thermal photons as the curve parameter in the case of the transmission via the multipath quantum channel CIR#5, which has the three channel coefficients s_0 equal to 0.6250, s_1 equal to 0.3125 and s_2 equal to 0.0625. In both cases, the optimum quantum data detection schemes log-ML and SOVA are used. The uncoded BER curves obtained in the case of the log-ML are shown as solid lines while the uncoded BER curves determined in the case of the SOVA are depicted as dashed lines. As expected, the uncoded BER P_b decreases monotonically with increasing $\bar{N}_\mu$ for a given value of $\bar{N}_\alpha$. Furthermore, the uncoded BER P_b increases with increasing thermal noise, i.e. with increasing $\bar{N}_\alpha$, for a given value of $\bar{N}_\mu$. Also, the steepness of the uncoded BER curves seems to decrease with increasing thermal noise, i.e. with increasing $\bar{N}_\alpha$. Remarkably, for given values of $\bar{N}_\mu$ and $\bar{N}_\alpha$, the log-ML and the SOVA perform almost identically. In addition, compared with the multipath quantum channel CIR#4, the multipath quantum channel CIR#5 has a greater multipath spread. For given values of $\bar{N}_\mu$ and $\bar{N}_\alpha$, this leads to an increased uncoded BER P_b in the case of the multipath quantum channel CIR#5 versus the multipath quantum channel CIR#4.

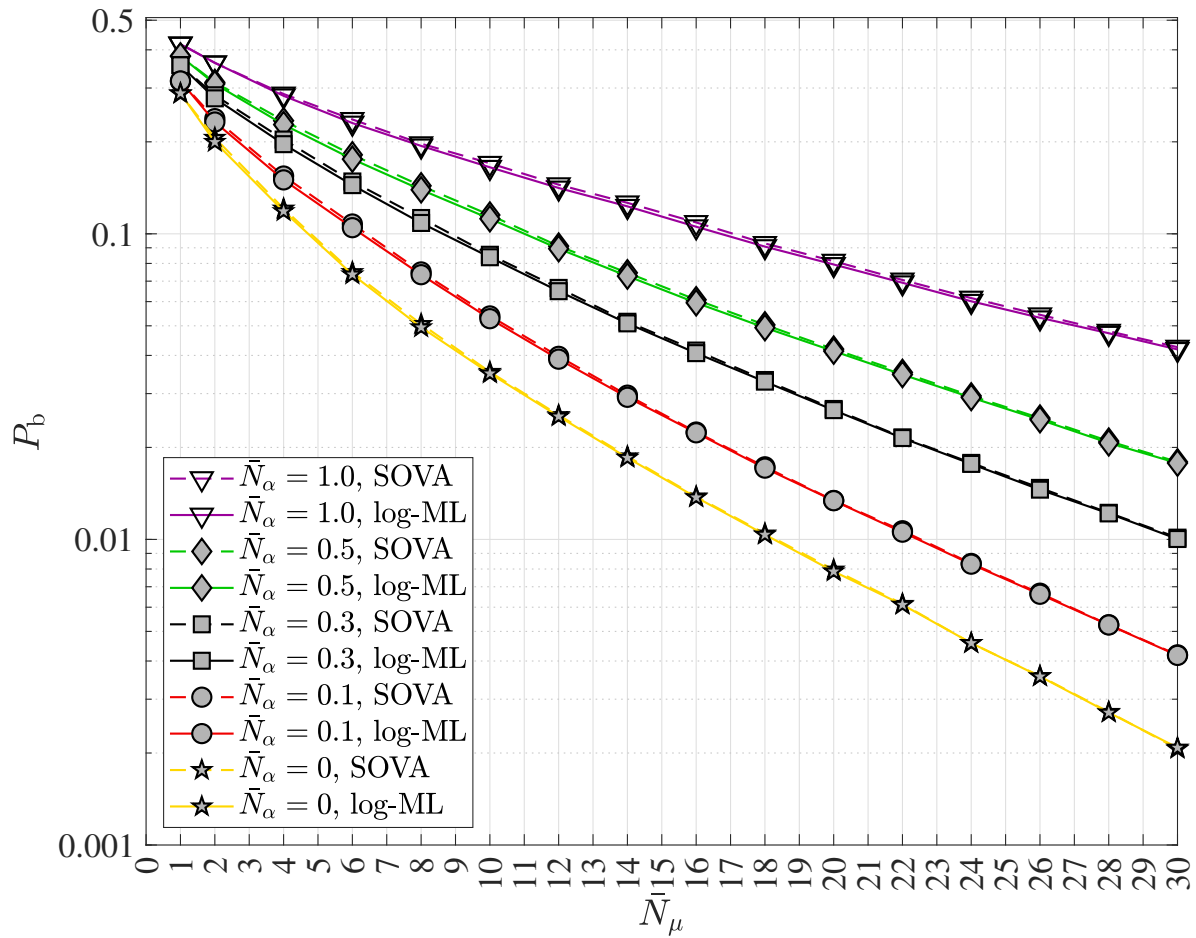


Figure 12. Detail of simulation results of the uncoded bit error ratio (BER) P_b performance versus the average number $\bar{N}_\mu$ of signal photons per message bit with the average number $\bar{N}_\alpha$ of thermal photons as the curve parameter for the transmission via the multipath quantum channel CIR#4 with $W_p = 2$; the optimum quantum data detection schemes log-ML (solid lines) and SOVA (dashed lines) are considered.

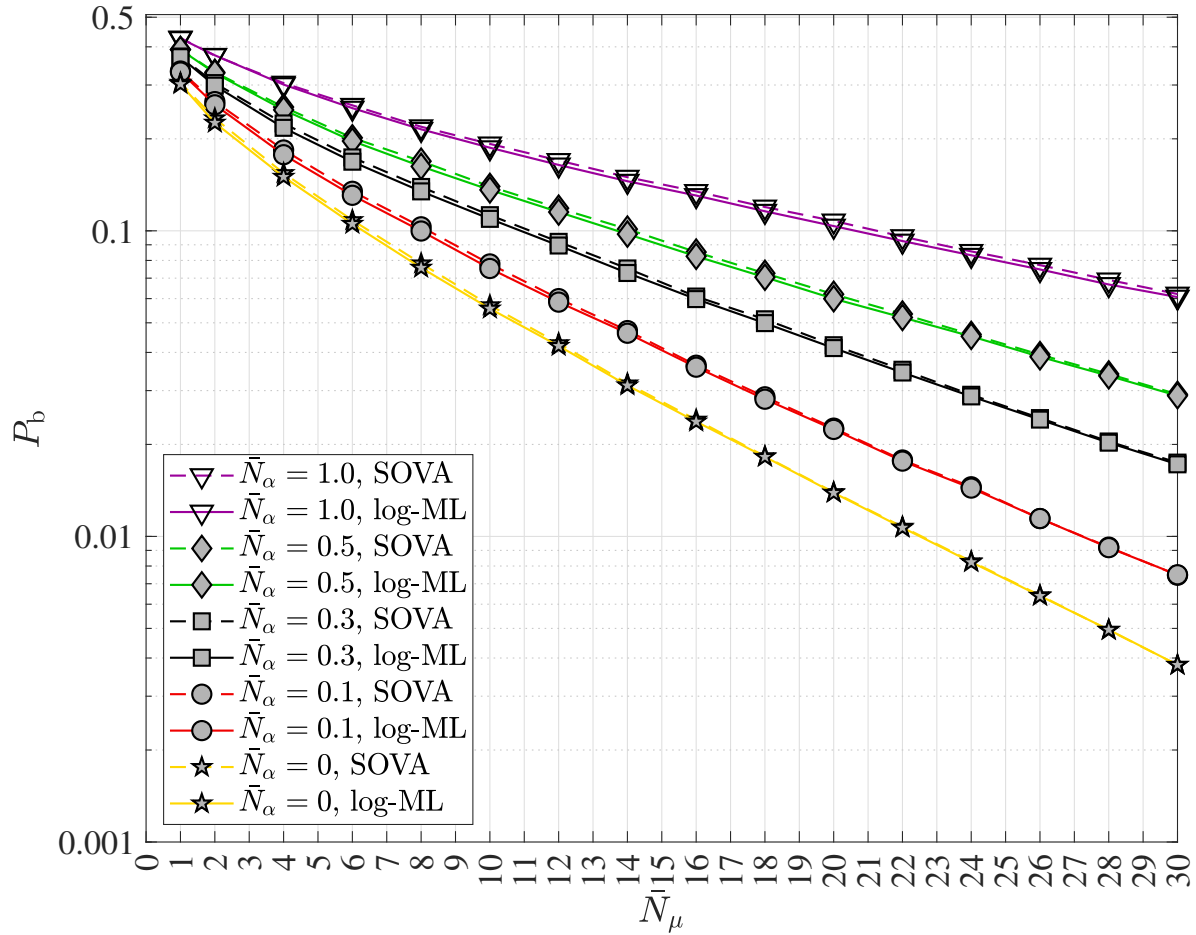


Figure 13. Detail of simulation results of the uncoded bit error ratio (BER) P_b performance versus the average number $\bar{N}_\mu$ of signal photons per message bit with the average number $\bar{N}_\alpha$ of thermal photons as the curve parameter for the transmission via the multipath quantum channel CIR#5 with $W_p = 3$; the optimum quantum data detection schemes log-ML (solid lines) and SOVA (dashed lines) are considered.

Figure 14 shows a detail of simulation results of the uncoded BER P_b performance versus the average number $\bar{N}_\mu$ of signal photons per message bit with the average number $\bar{N}_\alpha$ of thermal photons as the curve parameter in the case of the transmission via the multipath quantum channel CIR#5, having W_p equal to 3 nonzero channel coefficients s_0 equal to 0.6250, s_1 equal to 0.3125 and s_2 equal to 0.0625, hence, three distinct paths. The reduced state quantum data detection schemes $rslog$ -ML with $W_{p,red}$ equal to 2 and $rsSOVA$ with $W_{p,red}$ equal to 2 are applied. The uncoded BER curves obtained in the case of the $rslog$ -ML are shown as solid lines while the uncoded BER curves determined in the case of the $rsSOVA$ are depicted as dashed lines. As expected, the uncoded BER P_b decreases monotonically with increasing $\bar{N}_\mu$ for a given value of $\bar{N}_\alpha$. Furthermore, the uncoded BER P_b increases with increasing thermal noise, i.e. with increasing $\bar{N}_\alpha$, for a given value of $\bar{N}_\mu$. Also, the steepness of the uncoded BER curves seems to decrease with increasing thermal noise, i.e. with increasing $\bar{N}_\alpha$. Remarkably, for given values of $\bar{N}_\mu$ and $\bar{N}_\alpha$, the $rslog$ -ML and the $rsSOVA$ perform almost identically. When comparing the Figures 13 and 14, we realize that the performance degradation of the reduced state versions of the log-ML and the SOVA is small compared to the more complex log-ML and SOVA quantum data detectors.

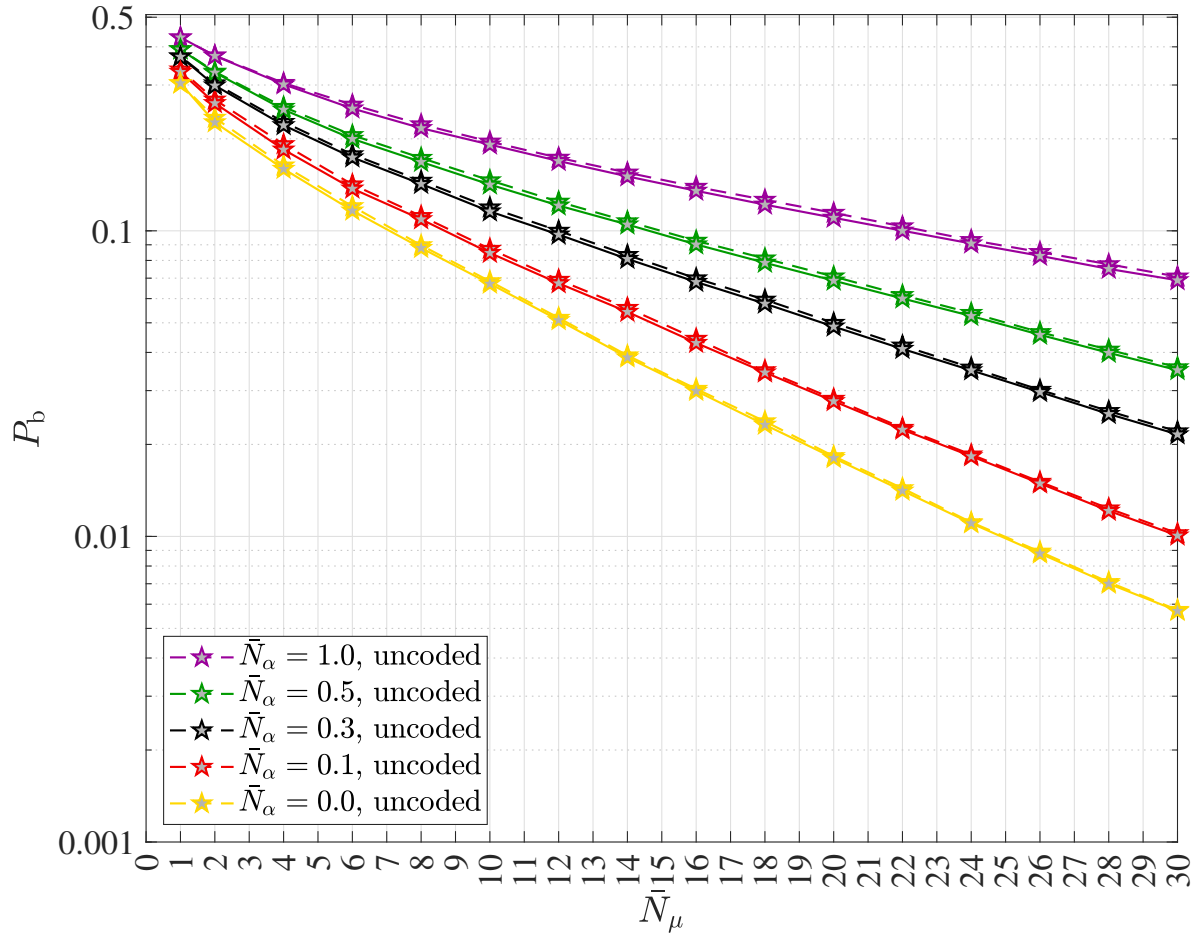


Figure 14. Detail of simulation results of the uncoded bit error ratio (BER) P_b performance versus the average number $\bar{N}_\mu$ of signal photons per message bit with the average number $\bar{N}_\alpha$ of thermal photons as the curve parameter for the transmission via the multipath quantum channel CIR#5 with $W_p = 3$; the reduced state quantum data detection schemes *rslog-ML* with $W_{p,\text{red}} = 2$ (solid lines) and *rsSOVA* with $W_{p,\text{red}} = 2$ (dashed lines) are considered.

Figure 15 illustrates a detail of simulation results of the uncoded BER P_b performance versus the average number $\bar{N}_\mu$ of signal photons per message bit with the average number $\bar{N}_\alpha$ of thermal photons as the curve parameter for the transmission via the multipath quantum channel CIR#5 having W_p equal to 3 nonzero channel coefficients s_0 equal to 0.6250, s_1 equal to 0.3125 and s_2 equal to 0.0625, hence, three distinct paths. The reduced state quantum data detection scheme *decision feedback equalizer (DFE)*, obtained for $W_{p,\text{red}}$ equal to 1 from the *rslog-ML*, the *rslog-MAP*, the *rsAPRI-SOVA* or the *rsSOVA*, is used. As expected, the uncoded BER P_b decreases monotonically with increasing $\bar{N}_\mu$ for a given value of $\bar{N}_\alpha$. Furthermore, the uncoded BER P_b increases with increasing thermal noise, i.e. with increasing $\bar{N}_\alpha$, for a given value of $\bar{N}_\mu$. Also, the steepness of the uncoded BER curves seems to decrease with increasing thermal noise, i.e. with increasing $\bar{N}_\alpha$. The *DFE* has the least implementation complexity, however, suffers from some performance degradation in comparison with the more elaborate log-ML, SOVA, *rslog-ML* with $W_{p,\text{red}}$ equal to 2 and *rsSOVA* with $W_{p,\text{red}}$ equal to 2, cf. Figures 13 and 14.

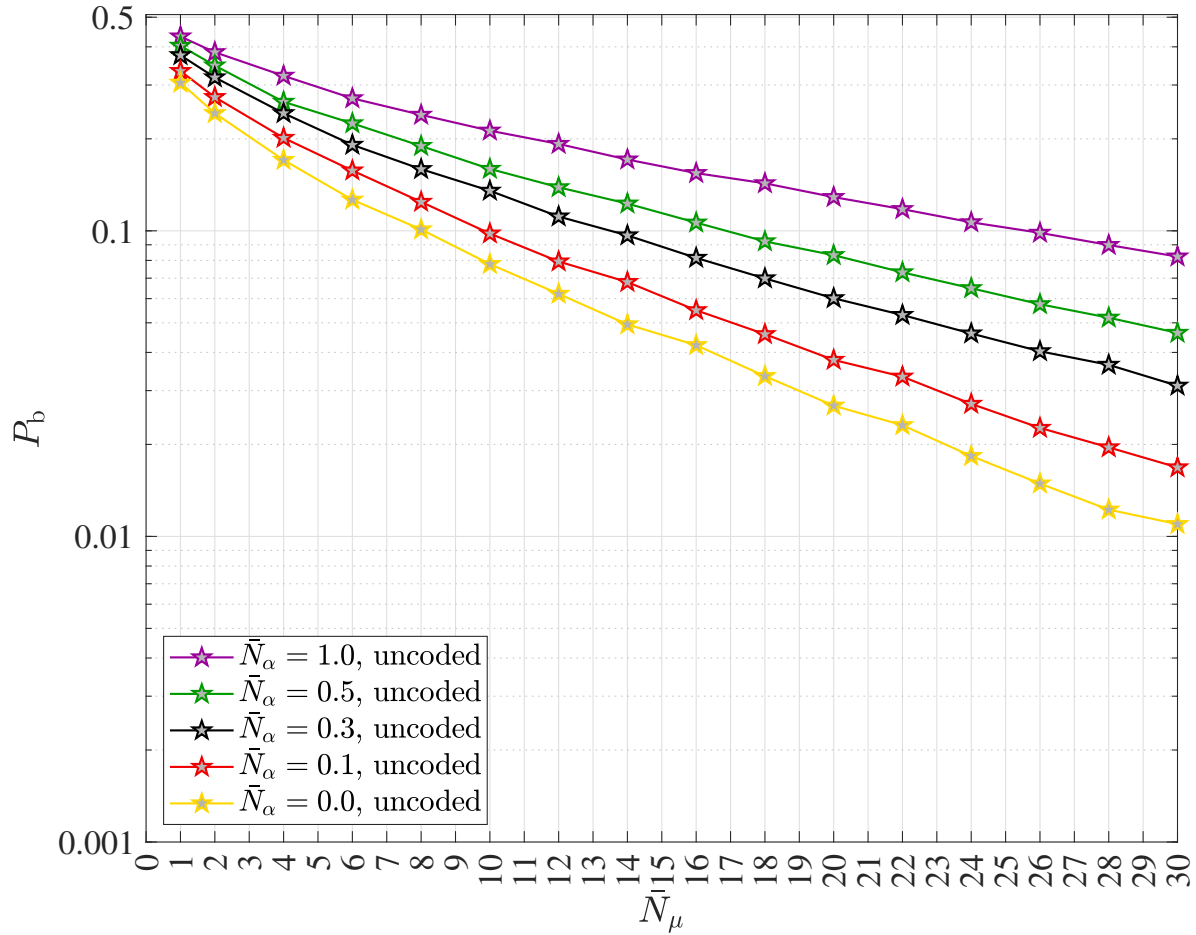


Figure 15. Detail of the simulation results of the uncoded bit error ratio (BER) P_b performance versus the average number $\bar{N}_\mu$ of signal photons per message bit with the average number $\bar{N}_\alpha$ of thermal photons as the curve parameter for the transmission via the multipath quantum channel CIR#5 with $W_p = 3$; the reduced state quantum data detection scheme *decision feedback equalizer (DFE)* with $W_{p,red} = 1$ is considered.

Figure 16 shows a detail of simulation results of the uncoded BER P_b performance versus the average number $\bar{N}_\mu$ of signal photons per message bit with the average number $\bar{N}_\alpha$ of thermal photons as the curve parameter for the transmission via the multipath quantum channel CIR#5 with W_p is equal to 3. The performance of the log-ML quantum data detector (solid lines), the SOVA quantum data detector (dashed lines) and the Kohonen map based quantum data detector (solid lines with marker x) are shown for comparison. Obviously, for a given average number $\bar{N}_\alpha$ of thermal photons, the uncoded BER P_b performance of the three considered quantum data detectors are almost identical.

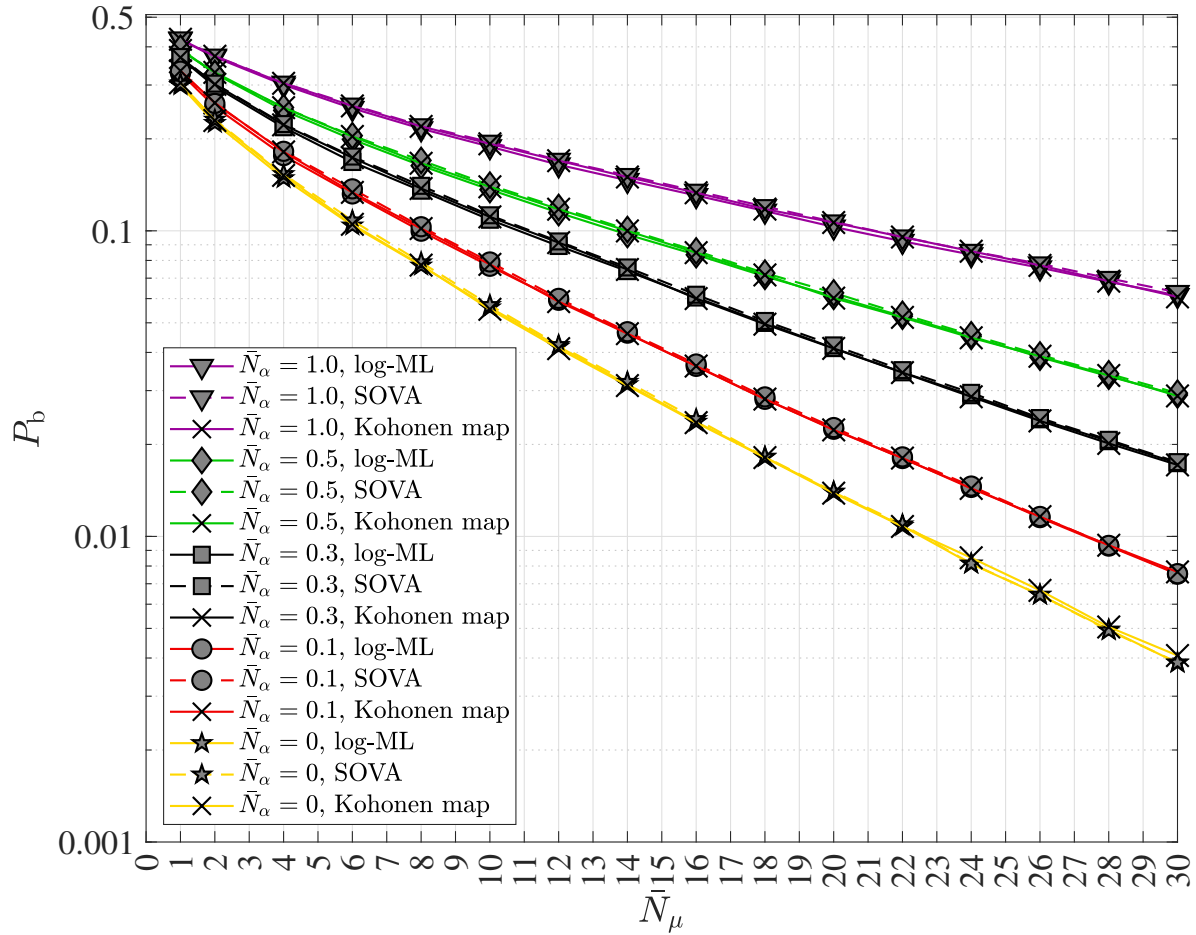


Figure 16. Detail of the simulation results of the uncoded bit error ratio (BER) P_b performance versus the average number $\bar{N}_\mu$ of signal photons per message bit with the average number $\bar{N}_\alpha$ of thermal photons as the curve parameter for the transmission via the multipath quantum channel CIR#5 with $W_p = 3$, comparing the performance of the log-ML quantum data detector (solid lines), the SOVA quantum data detector (dashed lines) and the Kohonen map based quantum data detector (solid lines with marker x).

4.2. Coded Transmission and Reception Using the 3GPP Polar Code and Arikan's Polar Code

Like in Sect. 4.1 we begin with the consideration of the short range transmission via a single path quantum channel CIR#1, cf. Table 1. Both the scenario without decoherence and, thus, with known carrier phase as well as the scenario subject to decoherence and, hence, with unknown carrier phase, the phase being a uniformly distributed random variable, have been evaluated by simulations in terms of

- the coded BER P_b , i.e. the BER obtained after the channel decoding, and
- the BLER P_{block} determined after the channel decoding.

The BLER P_{block} is used as the performance measure because it has become one of the most important figures of merit in mobile communications systems and, in addition, the considered 5G/NR polar code protects control channels, which require the least possible BLER. The obtained detail of the coded BER P_b performance is depicted in Figure 17 versus the average number $\bar{N}_\mu$ of signal photons per message bit with the average number $\bar{N}_\alpha$ of thermal photons as the curve parameter. The obtained detail of the coded BLER P_{block} performance versus the average number $\bar{N}_\mu$ of signal photons per message bit with the average number $\bar{N}_\alpha$ of thermal photons as the curve parameter, corresponding to Figure 17, is shown in Figure 18. The dashed curves refer to the case of unknown carrier phase while the solid curves refer to the case of known carrier phase. We applied the 3GPP 5G/NR polar code with the generator matrix of (71) and the polar sequence of [41][Table 5.3.1.2-1, pp. 17-19], relying on 24 CRC bits generated with polynomial "24C", code dimension K equal to 128, code length N

equal to 256 and code rate R_c equal to $(128 - 24)/256$, i.e. 0.40625. The average number $\bar{N}_\mu$ of signal photons per message bit includes the code rate R_c explicitly: $\bar{N}_\mu$ is hence $1/R_c$ times greater than the average number of signal photons per codebit. The polar code decoding is accomplished with a CRC aided successive cancellation list (CA-SCL) decoder with list length L equal to 8. Interleaving between the channel encoder and the quantum mechanical harmonic oscillator is not deployed. The observations made are similar to those stated in the context of Figure 11, cf. Sect. 4.1. In any of the two scenarios, the coded BER P_b , respectively the BLER P_{block} , decreases monotonically with increasing $\bar{N}_\mu$ for a given value of $\bar{N}_\alpha$. Furthermore, the coded BER P_b , respectively the BLER P_{block} , increases with increasing thermal noise, i.e. with increasing $\bar{N}_\alpha$, for a given value of $\bar{N}_\mu$. Remarkably, the coded BER P_b , respectively the BLER P_{block} , is always lower in the case of known carrier phase than in the case of unknown carrier phase for given values of $\bar{N}_\mu$ and $\bar{N}_\alpha$. Table 2 summarizes the estimated average number $\bar{N}_\mu$ of signal photons per message bit required to obtain P_{block} equal to 10^{-3} and P_b equal to 10^{-3} associated with the results shown in Figures 17 and 18.

Table 2. Estimated average number $\bar{N}_\mu$ of signal photons per message bit required to obtain $P_b = 10^{-3}$ and $P_{\text{block}} = 10^{-3}$ associated with the results shown in Figures 17 and 18.

	$\bar{N}_\alpha = 0.01$	$\bar{N}_\alpha = 0.03$	$\bar{N}_\alpha = 0.05$	$\bar{N}_\alpha = 0.1$
BER, unknown	$\bar{N}_\mu \approx 4.89$	$\bar{N}_\mu \approx 5.61$	$\bar{N}_\mu \approx 5.76$	$\bar{N}_\mu \approx 6.71$
BER, known carrier phase	$\bar{N}_\mu \approx 4.21$	$\bar{N}_\mu \approx 4.57$	$\bar{N}_\mu \approx 4.90$	$\bar{N}_\mu \approx 5.65$
BLER, unknown	$\bar{N}_\mu \approx 5.20$	$\bar{N}_\mu \approx 5.79$	$\bar{N}_\mu \approx 6.10$	$\bar{N}_\mu \approx 7.08$
BLER, known carrier phase	$\bar{N}_\mu \approx 4.59$	$\bar{N}_\mu \approx 4.96$	$\bar{N}_\mu \approx 5.35$	$\bar{N}_\mu \approx 6.09$

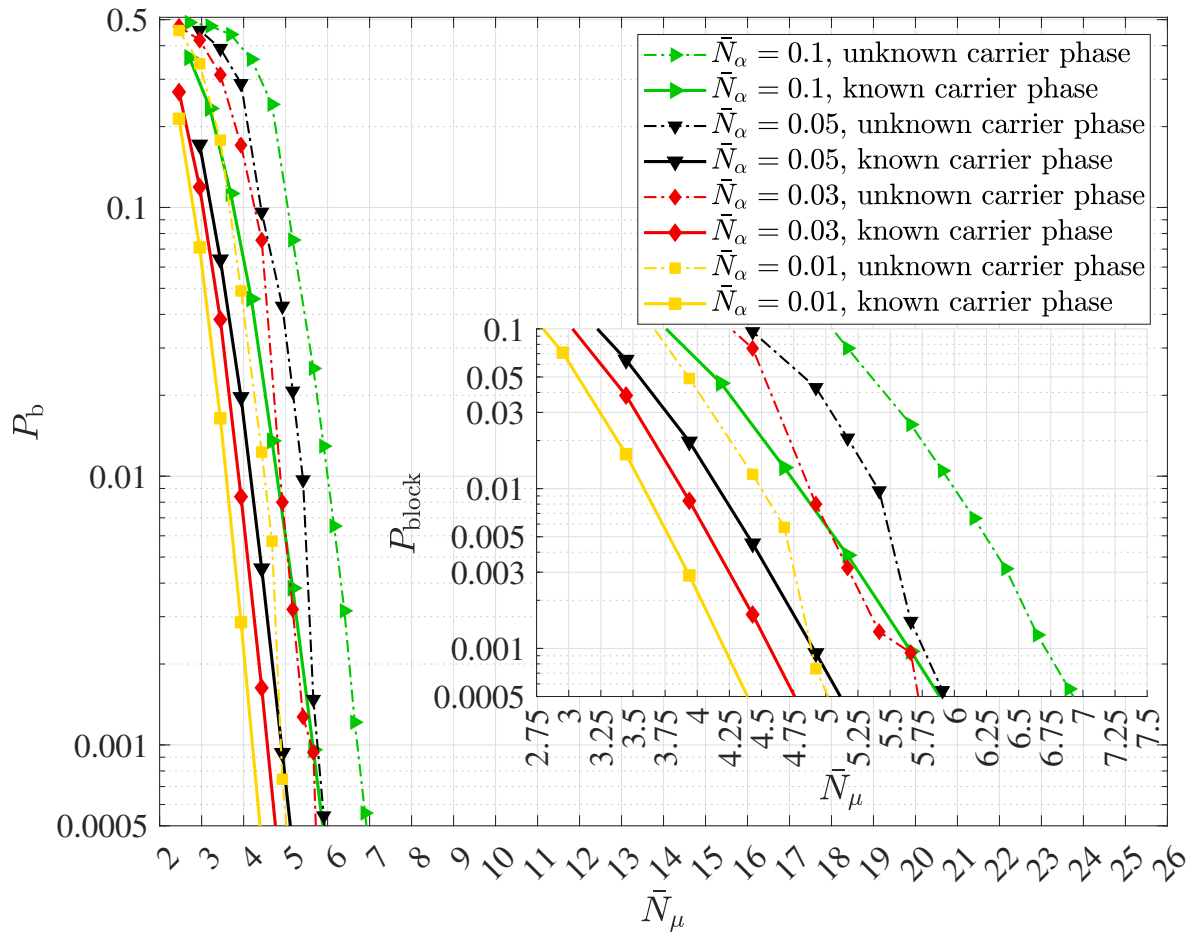


Figure 17. Detail of simulation results of the coded bit error ratio (BER) P_b performance versus the average number $\bar{N}_\mu$ of signal photons per message bit with the average number $\bar{N}_\alpha$ of thermal photons as the curve parameter; the dashed curves refer to the case of unknown carrier phase while the solid curves refer to the case of known carrier phase; the 3GPP 5G/NR polar code with the generator matrix of (71) and the polar sequence of [41][Table 5.3.1.2-1, pp. 17-19] is used with 24 CRC bits, code dimension $K = 128$, code length $N = 256$ and code rate $R_c = (128 - 24)/256 = 0.40625$; the code rate R_c is explicitly included in $\bar{N}_\mu$; the polar code decoding is accomplished with the CA-SCL decoder with list length $L = 8$; interleaving is not deployed.

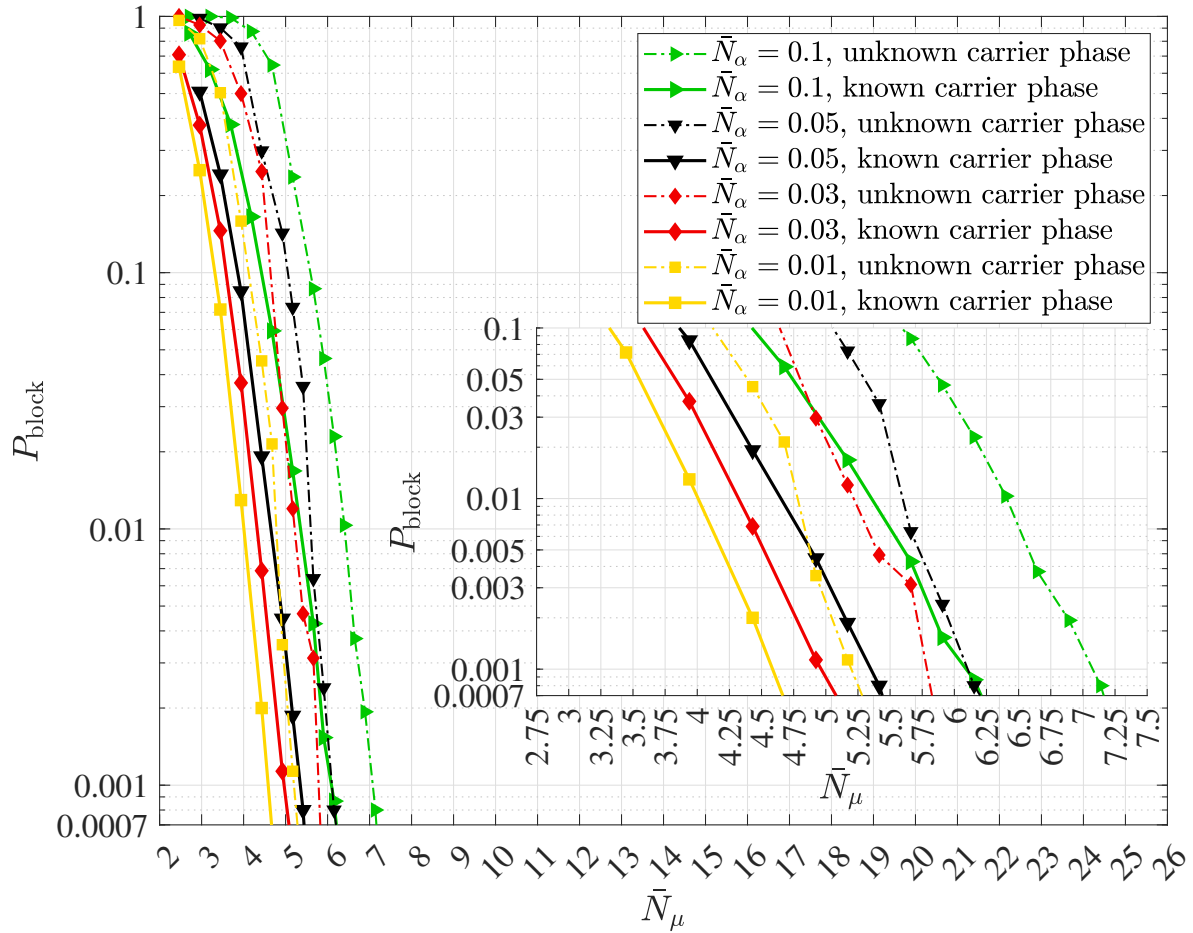


Figure 18. Detail of simulation results of the block error ratio (BLER) P_{block} performance versus the average number $\bar{N}_\mu$ of signal photons per message bit with the average number $\bar{N}_\alpha$ of thermal photons as the curve parameter; the dashed curves refer to the case of unknown carrier phase while the solid curves refer to the case of known carrier phase; the 3GPP 5G/NR polar code with the generator matrix of (71) and the polar sequence of [41][Table 5.3.1.2-1, pp. 17-19] is used with 24 CRC bits generated using the polynomial “24C”, code dimension $K = 128$, code length $N = 256$ and code rate $R_c = (128 - 24)/256 = 0.40625$; the code rate R_c is explicitly included in $\bar{N}_\mu$; the polar code decoding is accomplished with the CA-SCL decoder having the list length $L = 8$; interleaving is not deployed.

Figure 19 shows a detail of simulation results of the coded BER P_b performance and Figure 20 shows a detail of simulation results of the BLER P_{block} performance versus the average number $\bar{N}_\mu$ of signal photons per message bit with the average number $\bar{N}_\alpha$ of thermal photons as the curve parameter for the transmission via the multipath quantum channel CIR#4 when the 3GPP 5G/NR polar code with the generator matrix of (71) and the polar sequence of [41][Table 5.3.1.2-1, pp. 17-19] is used. Figure 21 shows a detail of simulation results of the coded BER P_b performance and Figure 22 shows a detail of simulation results of the BLER P_{block} performance versus the average number $\bar{N}_\mu$ of signal photons per message bit with the average number $\bar{N}_\alpha$ of thermal photons as the curve parameter for the transmission via the multipath quantum channel CIR#4 when the Arikan-style polar code with the generator matrix of (72) and the associated polar sequence generated by Arikan’s Bhattacharyya parameter $Z(W_N^{(i)})$ based approach setting out from the binary erasure channel with erasure probability ε equal to 0.5, is used.

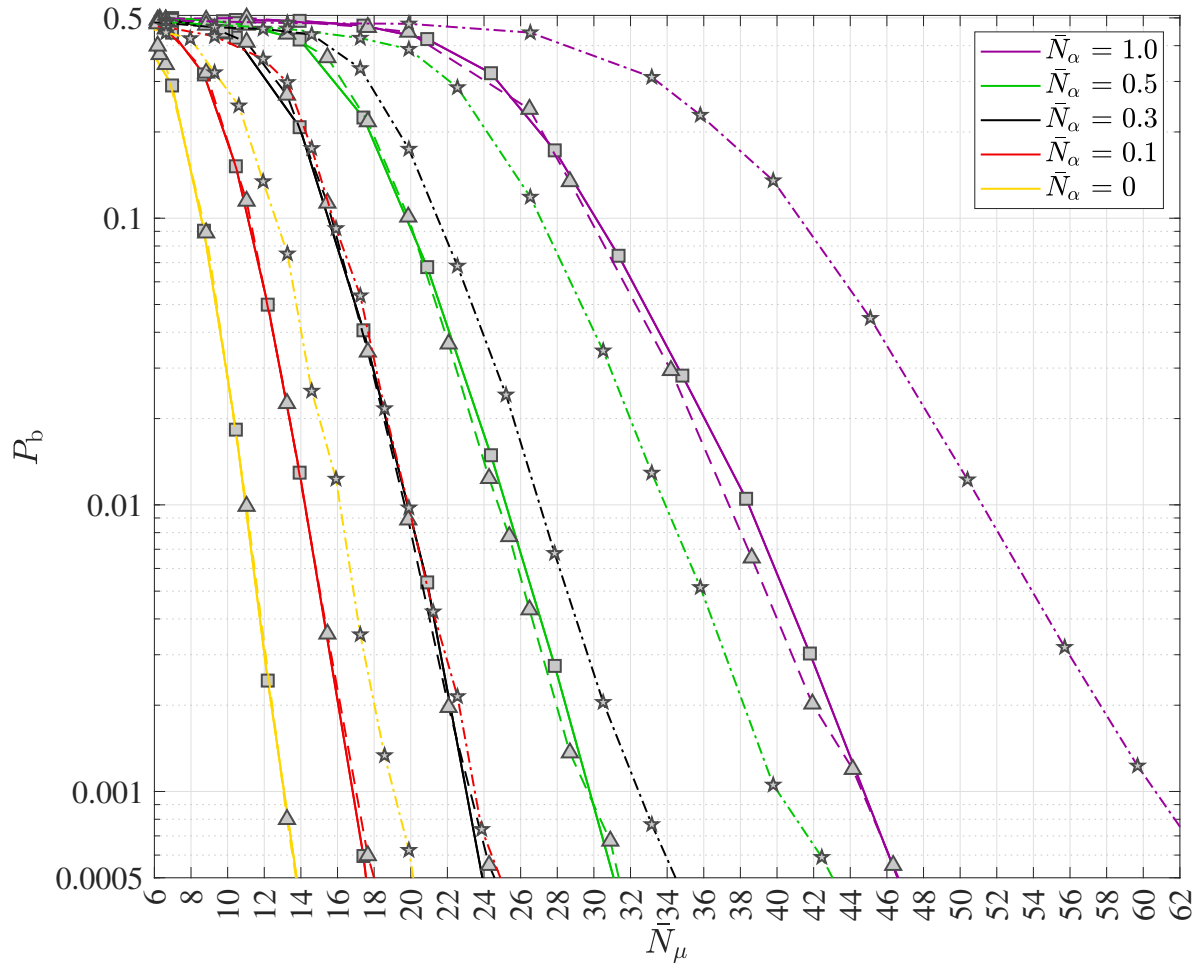


Figure 19. Detail of simulation results of the coded bit error ratio (BER) P_b performance versus the average number $\bar{N}_\mu$ of signal photons per message bit with the average number $\bar{N}_\alpha$ of thermal photons as the curve parameter for the transmission via the multipath quantum channel CIR#4; log-ML quantum data detection is applied; the 3GPP 5G/NR polar code with the generator matrix of (71) and the polar sequence of [41][Table 5.3.1.2-1, pp. 17-19] is used; 24 CRC bits generated with polynomial “24C” are used; the polar code decoding is accomplished with the SC decoder, hence, CRC bits are not processed by the SC decoder; the code length N is 512; three choices of the code dimension, $K = 171$ (—■—), $K = 256$ (---▲---) and $K = 410$ (-·-★-·-), leading to the three code rates $R_c = (171 - 24)/512 \approx 0.2871$ (—■—), $R_c = (256 - 24)/512 \approx 0.4531$ (---▲---) and $R_c = (410 - 24)/512 \approx 0.7539$ (-·-★-·-) are considered; the code rate R_c is explicitly included in $\bar{N}_\mu$; interleaving is deployed.

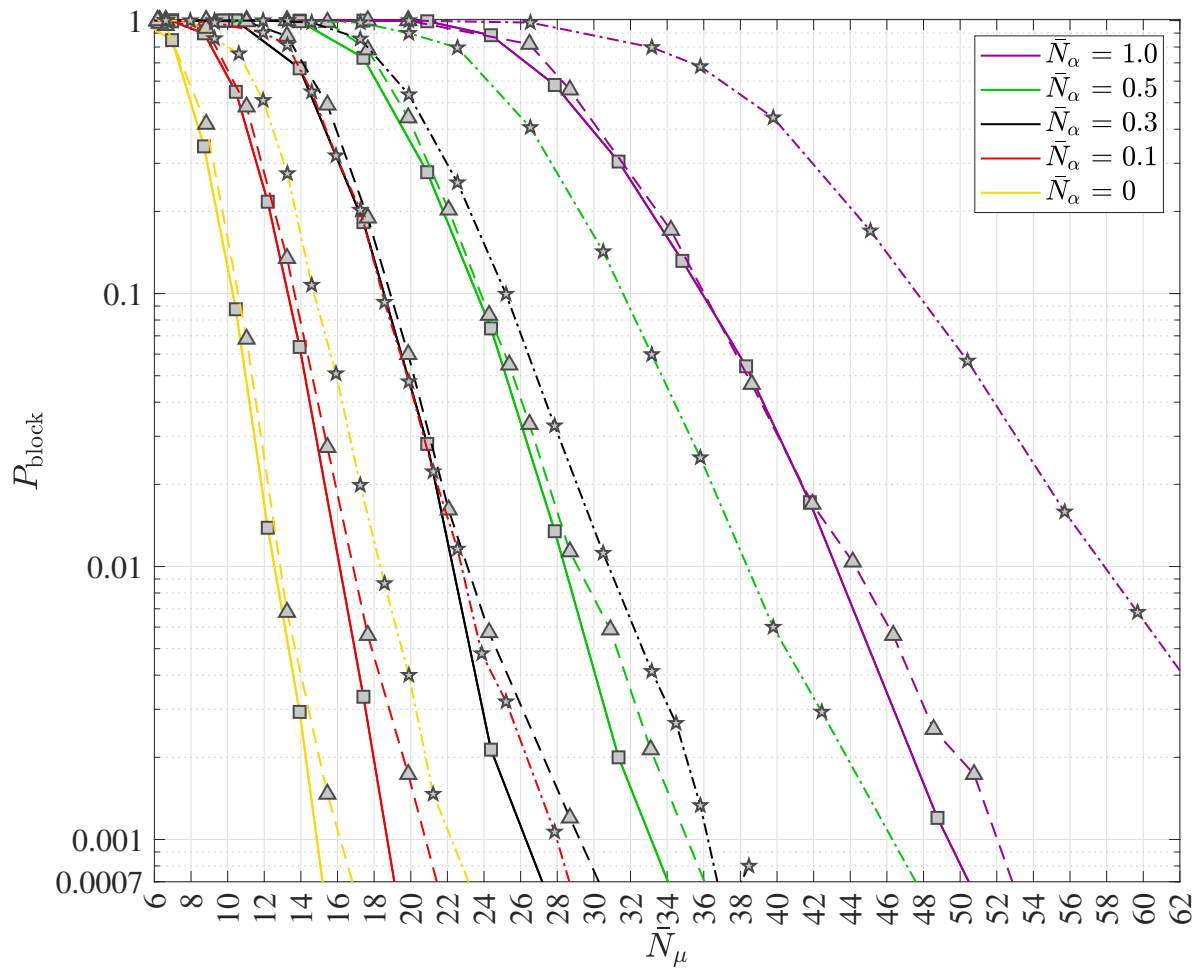


Figure 20. Detail of simulation results of the block error ratio (BLER) P_{block} performance versus the average number $\bar{N}_\mu$ of signal photons per message bit with the average number $\bar{N}_\alpha$ of thermal photons as the curve parameter for the transmission via the multipath quantum channel CIR#4; log-ML quantum data detection is applied; the 3GPP 5G/NR polar code with the generator matrix of (71) and the polar sequence of [41][Table 5.3.1.2-1, pp. 17-19] is used; 24 CRC bits generated with polynomial “24C” are used; the polar code decoding is accomplished with the SC decoder, hence, CRC bits are not processed by the SC decoder; the code length N is 512; three choices of the code dimension, $K = 171$ (—■—), $K = 256$ (---▲---) and $K = 410$ (-·-★-·-), leading to the three code rates $R_c = (171 - 24)/512 \approx 0.2871$ (—■—), $R_c = (256 - 24)/512 \approx 0.4531$ (---▲---) and $R_c = (410 - 24)/512 \approx 0.7539$ (-·-★-·-) are considered; the code rate R_c is explicitly included in $\bar{N}_\mu$; interleaving is deployed.

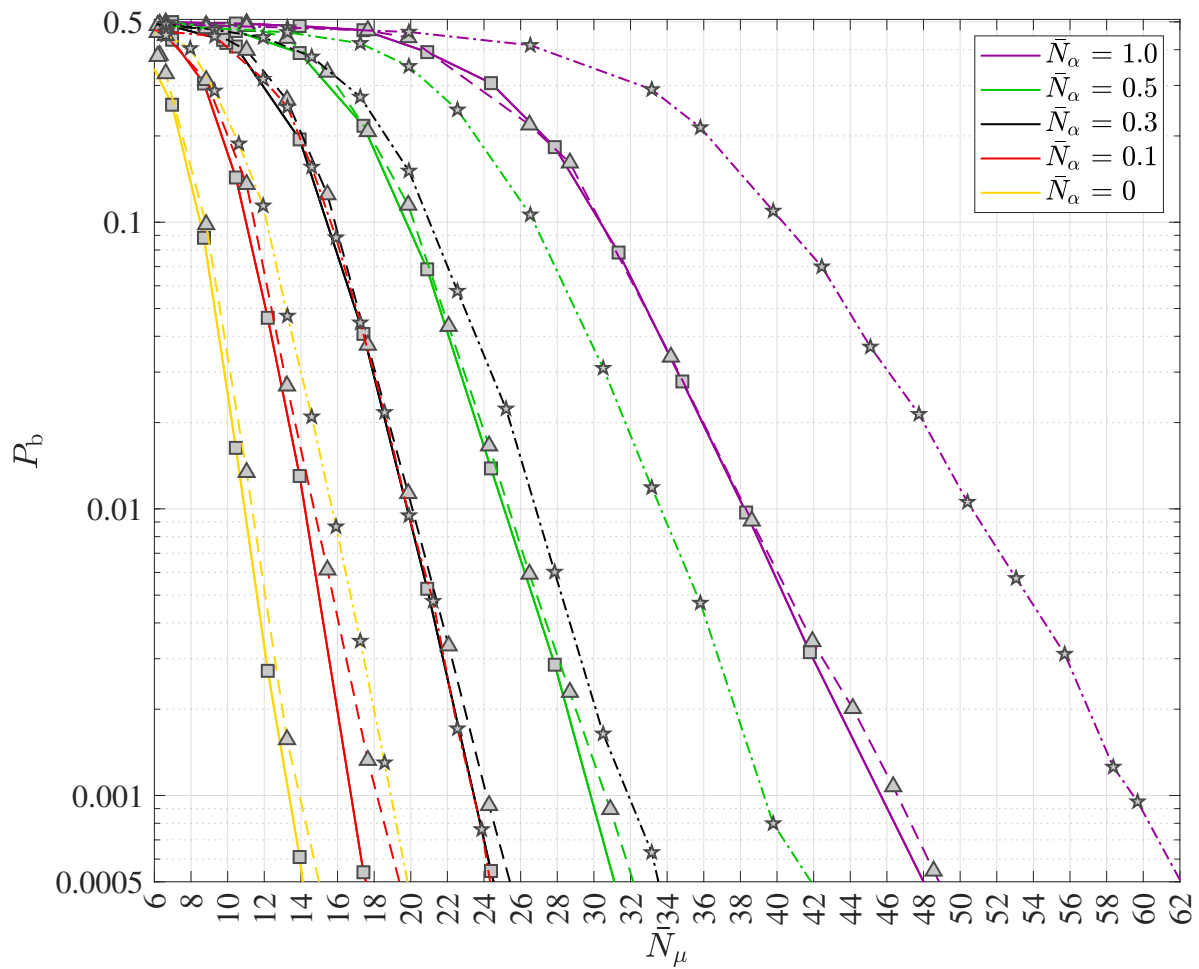


Figure 21. Detail of simulation results of the coded bit error ratio (BER) P_b performance versus the average number $\bar{N}_\mu$ of signal photons per message bit with the average number $\bar{N}_\alpha$ of thermal photons as the curve parameter for the transmission via the multipath quantum channel CIR#4; log-ML quantum data detection is applied; the Arikan-style polar code with the generator matrix of (72) and the associated polar sequence generated by Arikan's Bhattacharyya parameter $Z(W_N^{(i)})$ based approach setting out from the binary erasure channel with erasure probability $\varepsilon = 0.5$ is used; 24 CRC bits generated with polynomial "24C" are used; the polar code decoding is accomplished with the SC decoder, hence, CRC bits are not processed by the SC decoder; the code length N is 512; three choices of the code dimension, $K = 171$ (—■—), $K = 256$ (---▲---) and $K = 410$ (-·-★-·-), leading to the three code rates $R_c = (171 - 24)/512 \approx 0.2871$ (—■—), $R_c = (256 - 24)/512 \approx 0.4531$ (---▲---) and $R_c = (410 - 24)/512 \approx 0.7539$ (-·-★-·-) are considered; the code rate R_c is explicitly included in $\bar{N}_\mu$; interleaving is deployed.

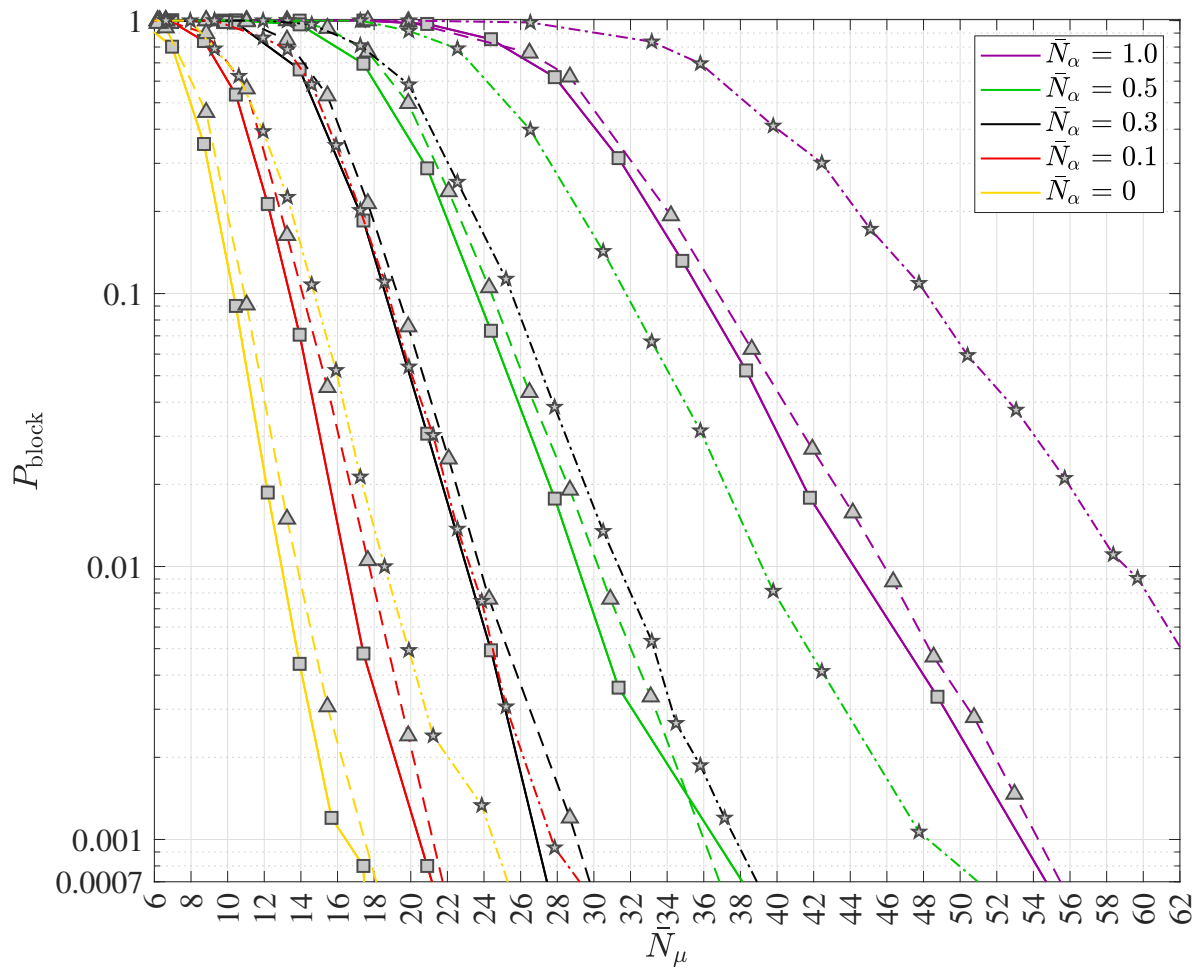


Figure 22. Detail of simulation results of the block error ratio (BLER) P_{block} performance versus the average number $\bar{N}_\mu$ of signal photons per message bit with the average number $\bar{N}_\alpha$ of thermal photons as the curve parameter for the transmission via the multipath quantum channel CIR#4; log-ML quantum data detection is applied; the Arikan-style polar code with the generator matrix of (72) and the associated polar sequence generated by Arikan's Bhattacharyya parameter $Z(W_N^{(i)})$ based approach setting out from the binary erasure channel with erasure probability $\varepsilon = 0.5$ is used; 24 CRC bits generated with polynomial "24C" are used; the polar code decoding is accomplished with the SC decoder, hence, CRC bits are not processed by the SC decoder; the code length N is 512; three choices of the code dimension, $K = 171$ (—■—), $K = 256$ (---▲---) and $K = 410$ (-·-★-·-), leading to the three code rates $R_c = (171 - 24)/512 \approx 0.2871$ (—■—), $R_c = (256 - 24)/512 \approx 0.4531$ (---▲---) and $R_c = (410 - 24)/512 \approx 0.7539$ (-·-★-·-) are considered; the code rate R_c is explicitly included in $\bar{N}_\mu$; interleaving is deployed.

Similar to e.g. [6][Fig. 3, p. 1590], the average number of thermal noise photons $\bar{N}_\alpha$ was chosen from $\{0, 0.1, 0.3, 0.5, 1.0\}$. In all cases, log-ML quantum data detection is applied. 24 CRC bits generated with polynomial "24C" are used. The polar code decoding is accomplished with the SC decoder, hence, CRC bits are not processed by the SC decoder. The code length N is equal to 512. Furthermore, three choices of the code dimension, K equal to 171 (—■—), K equal to 256 (---▲---) and K equal to 410 (-·-★-·-), leading to the three code rates R_c equal to $(171 - 24)/512$, i.e. ≈ 0.2871 (—■—), R_c equal to $(256 - 24)/512$, i.e. ≈ 0.4531 (---▲---) and R_c equal to $(410 - 24)/512$, i.e. ≈ 0.7539 (-·-★-·-) are considered. Of course, the code rate R_c is explicitly included in $\bar{N}_\mu$. Interleaving according to the illustration in Sect. 3.3 was deployed. Both the coded BER P_b , respectively the BLER P_{block} , decreases monotonically with increasing $\bar{N}_\mu$ for a given value of $\bar{N}_\alpha$. Furthermore, the coded BER P_b , respectively the BLER P_{block} , increases with increasing thermal noise, i.e. with increasing $\bar{N}_\alpha$, for a given value of $\bar{N}_\mu$. It can be seen that the considered polar code performs almost identical for code rates lower than or equal to 0.5. However, for higher code rates an increasing number of virtual

channels with low channel capacity are used, which leads to considerable performance degradations. The said degradations increase with increasing $\bar{N}_\alpha$. Table 3 summarizes the estimated average number $\bar{N}_\mu$ of signal photons required to obtain P_{block} equal to 10^{-3} and P_b equal to 10^{-3} associated with the results shown in Figures 19–22.

Table 3. Estimated average number $\bar{N}_\mu$ of signal photons per message bit required to obtain $P_b = 10^{-3}$ and $P_{\text{block}} = 10^{-3}$ associated with the results shown in Figures 19–22.

	R_c	polar code	$\bar{N}_\alpha = 0$	$\bar{N}_\alpha = 0.1$	$\bar{N}_\alpha = 0.3$	$\bar{N}_\alpha = 0.5$	$\bar{N}_\alpha = 1$
BER	≈ 0.2871	3GPP 5G/NR	$\bar{N}_\mu \approx 13.1$	$\bar{N}_\mu \approx 16.8$	$\bar{N}_\mu \approx 23.0$	$\bar{N}_\mu \approx 29.8$	$\bar{N}_\mu \approx 44.7$
BER	≈ 0.2871	Arikan	$\bar{N}_\mu \approx 13.4$	$\bar{N}_\mu \approx 16.7$	$\bar{N}_\mu \approx 23.4$	$\bar{N}_\mu \approx 29.8$	$\bar{N}_\mu \approx 45.7$
BER	≈ 0.4531	3GPP 5G/NR	$\bar{N}_\mu \approx 13.0$	$\bar{N}_\mu \approx 17.0$	$\bar{N}_\mu \approx 23.2$	$\bar{N}_\mu \approx 29.7$	$\bar{N}_\mu \approx 44.6$
BER	≈ 0.4531	Arikan	$\bar{N}_\mu \approx 13.9$	$\bar{N}_\mu \approx 18.2$	$\bar{N}_\mu \approx 24.1$	$\bar{N}_\mu \approx 30.6$	$\bar{N}_\mu \approx 46.6$
BER	≈ 0.7539	3GPP 5G/NR	$\bar{N}_\mu \approx 19.1$	$\bar{N}_\mu \approx 23.5$	$\bar{N}_\mu \approx 32.4$	$\bar{N}_\mu \approx 40.0$	$\bar{N}_\mu \approx 60.7$
BER	≈ 0.7539	Arikan	$\bar{N}_\mu \approx 18.9$	$\bar{N}_\mu \approx 23.4$	$\bar{N}_\mu \approx 31.9$	$\bar{N}_\mu \approx 39.3$	$\bar{N}_\mu \approx 59.5$
BLER	≈ 0.2871	3GPP 5G/NR	$\bar{N}_\mu \approx 14.9$	$\bar{N}_\mu \approx 18.7$	$\bar{N}_\mu \approx 26.3$	$\bar{N}_\mu \approx 33.1$	$\bar{N}_\mu \approx 49.3$
BLER	≈ 0.2871	Arikan	$\bar{N}_\mu \approx 16.5$	$\bar{N}_\mu \approx 20.5$	$\bar{N}_\mu \approx 26.9$	$\bar{N}_\mu \approx 36.6$	$\bar{N}_\mu \approx 53.3$
BLER	≈ 0.4531	3GPP 5G/NR	$\bar{N}_\mu \approx 16.2$	$\bar{N}_\mu \approx 20.8$	$\bar{N}_\mu \approx 29.2$	$\bar{N}_\mu \approx 35.1$	$\bar{N}_\mu \approx 52.0$
BLER	≈ 0.4531	Arikan	$\bar{N}_\mu \approx 17.5$	$\bar{N}_\mu \approx 21.2$	$\bar{N}_\mu \approx 29.1$	$\bar{N}_\mu \approx 36.0$	$\bar{N}_\mu \approx 54.3$
BLER	≈ 0.7539	3GPP 5G/NR	$\bar{N}_\mu \approx 22.2$	$\bar{N}_\mu \approx 28.0$	$\bar{N}_\mu \approx 36.2$	$\bar{N}_\mu \approx 46.3$	$\bar{N}_\mu \approx 69.5$
BLER	≈ 0.7539	Arikan	$\bar{N}_\mu \approx 24.5$	$\bar{N}_\mu \approx 27.7$	$\bar{N}_\mu \approx 37.7$	$\bar{N}_\mu \approx 48.2$	$\bar{N}_\mu \approx 70.7$

Like in Sect. 4.1, let us concentrate on multipath transmission and reception under the presence of decoherence, see Sect. 2.4. Again, we consider the two distinct multipath quantum channels CIR#4 and CIR#5 of Table 1 [55, Tab. 3.7, p. 662]. Figures 23 and 24 illustrate the details of simulation results of the BLER P_{block} performance versus the average number $\bar{N}_\mu$ of signal photons with the average number $\bar{N}_\alpha$ of thermal photons as the curve parameter for the transmission via the multipath quantum channels CIR#4 and CIR#5. Again, the 3GPP 5G/NR polar code with the generator matrix of (71) and the polar sequence of [41][Table 5.3.1.2-1, pp. 17-19] was used with 24 CRC bits generated with polynomial “24C”, having the code dimension K equal to 164, the code length N equal to 512 and, hence, the code rate R_c equal to $(164 - 24)/512$, i.e. ≈ 0.273 . The code rate R_c is explicitly included in $\bar{N}_\mu$, see above. Again, the polar code decoding is accomplished with the CA-SCL decoder having the list length L equal to 8. Interleaving between the channel encoder and the quantum mechanical harmonic oscillator is not deployed. The quantum data detectors log-ML (—■—), max*-log-ML (—▲—), max-log-ML (—★—), SOVA with the BATTAIL RULE (BR) (- - -●- - -), SOVA with the HUBER RULE (HR) (- - -▼- - -) and SOVA with the SIMPLE RULE (SR) (- - -*- - -) are considered.

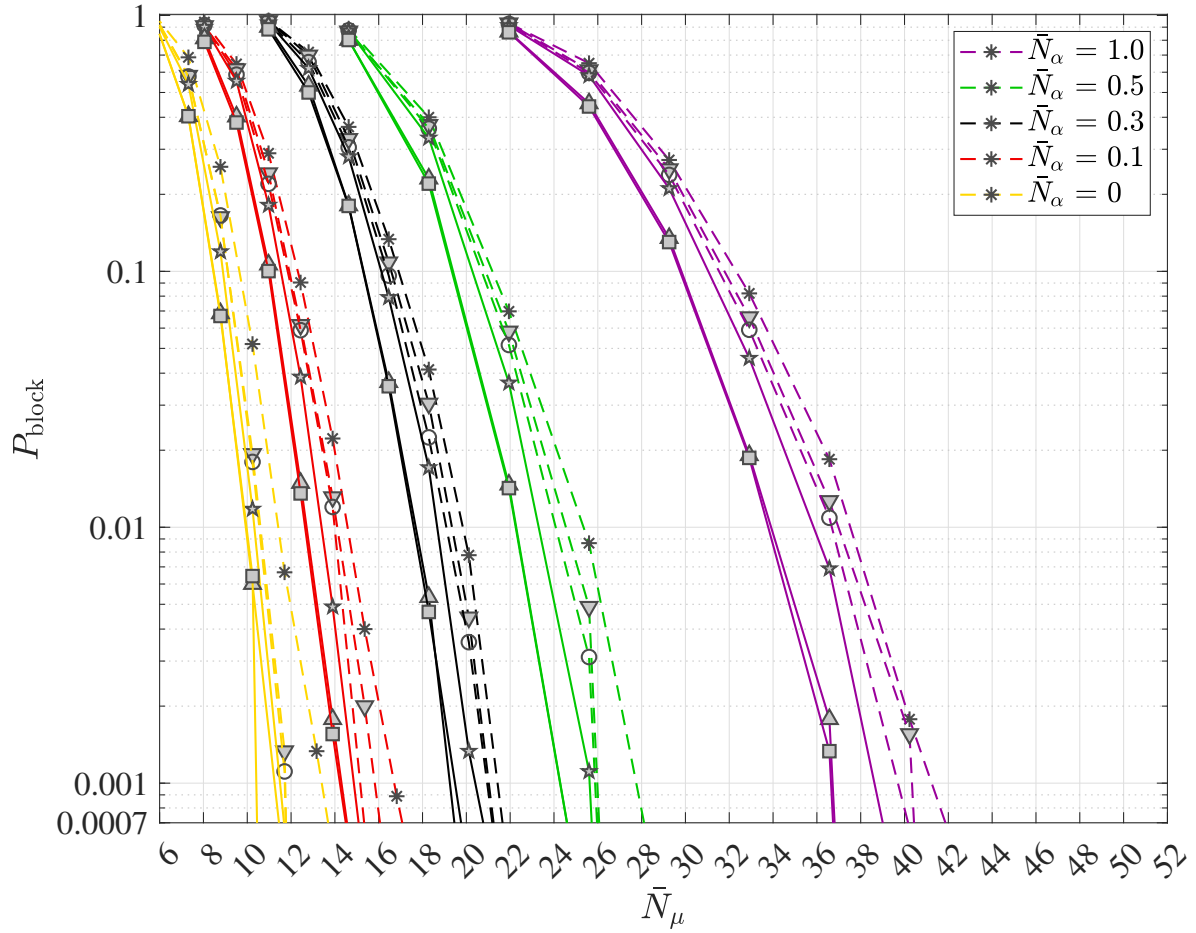


Figure 23. Detail of simulation results of the block error ratio (BLER) P_{block} performance versus the average number $\bar{N}_\mu$ of signal photons per message bit with the average number $\bar{N}_\alpha$ of thermal photons as the curve parameter for the transmission via the multipath quantum channel CIR#4; the 3GPP 5G/NR polar code with the generator matrix of (71) and the polar sequence of [41][Table 5.3.1.2-1, pp. 17-19] is used with 24 CRC bits generated with polynomial “24C”, code dimension $K = 164$, code length $N = 512$, code rate $R_c = (164 - 24)/512 \approx 0.273$; the code rate R_c is explicitly included in $\bar{N}_\mu$; the polar code decoding is accomplished with the CA-SCL decoder having the list length $L = 8$; interleaving is not deployed; associating the curves with the quantum data detectors is as follows: —■—: log-ML; —▲—: max*-log-ML; —★—: max-log-ML; - - -●- - -: SOVA w/ BATTAIL RULE (BR); - - -▼- - -: SOVA w/ HUBER RULE (HR); - - -*- - -: SOVA w/ SIMPLE RULE (SR).

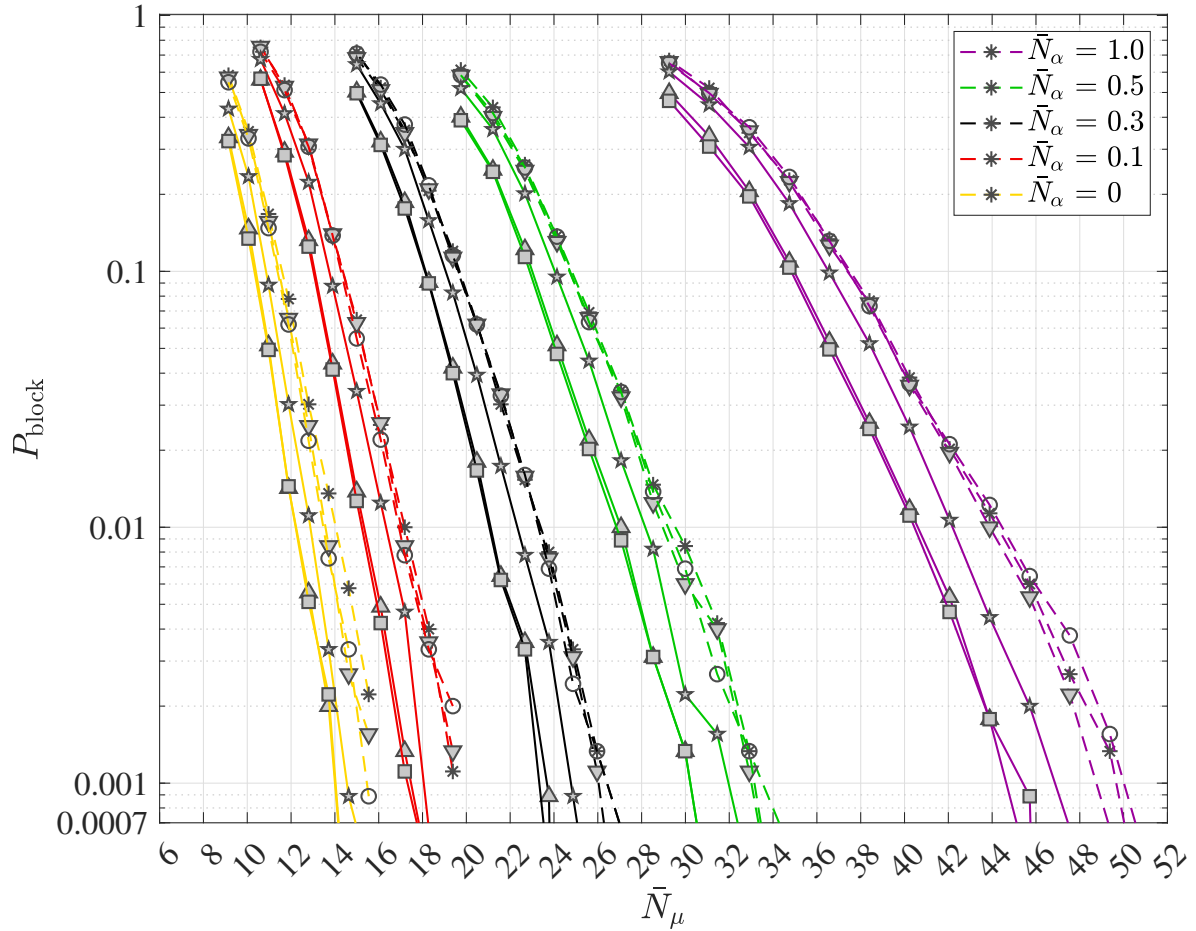


Figure 24. Detail of simulation results of the block error ratio (BLER) P_{block} performance versus the average number $\bar{N}_\mu$ of signal photons per message bit with the average number $\bar{N}_\alpha$ of thermal photons as the curve parameter for the transmission via the multipath quantum channel CIR#5; the 3GPP 5G/NR polar code with the generator matrix of (71) and the polar sequence of [41][Table 5.3.1.2-1, pp. 17-19] is used with 24 CRC bits generated with polynomial “24C”, code dimension $K = 164$, code length $N = 512$, code rate $R_c = (164 - 24)/512 \approx 0.273$; the code rate R_c is explicitly included in $\bar{N}_\mu$; the polar code decoding is accomplished with the CA-SCL decoder having the list length $L = 8$; interleaving is not deployed; associating the curves with the quantum data detectors is as follows: —■—: log-ML; —▲—: max*-log-ML; —★—: max-log-ML; - - -●- - -: SOVA w/ BATTAIL RULE (BR); - - -▼- - -: SOVA w/ HUBER RULE (HR); - - -*- - -: SOVA w/ SIMPLE RULE (SR).

Similar to e.g. [6][Fig. 3, p. 1590], the average number of thermal noise photons $\bar{N}_\alpha$ was chosen from $\{0, 0.1, 0.3, 0.5, 1.0\}$. The BLER performance obtainable with the six optimum quantum data detectors discussed in Sect. 2.4 for $P\{b_i\}$ equal to 1/2 is compared. It can be seen that for a given value of $\bar{N}_\alpha$, the log-ML usually performs best and the max*-log-ML exploiting [17][Fig. 3.83, p. 473] provides nearly the same performance. Next comes the max-log-ML. As expected, the three SOVA variants are worse than the log-ML quantum data detectors. Table 4 summarizes the estimated average number $\bar{N}_\mu$ of signal photons required to obtain P_{block} equal to 10^{-3} associated with the results shown in Figures 23 and 24.

Table 4. Estimated average number $\overline{N}_\mu$ of signal photons per message bit required to obtain $P_{\text{block}} = 10^{-3}$ associated with the results shown in Figures 23 and 24; the code rate R_c is approximately equal to 0.273; the symbol $\lesssim$ is used to indicate extrapolated values.

channel	detector	$\overline{N}_\alpha = 0$	$\overline{N}_\alpha = 0.1$	$\overline{N}_\alpha = 0.3$	$\overline{N}_\alpha = 0.5$	$\overline{N}_\alpha = 1$
CIR#4	SOVA, SR	$\overline{N}_\mu \approx 13.4$	$\overline{N}_\mu \approx 16.7$	$\overline{N}_\mu \approx 21.4$	$\overline{N}_\mu \approx 27.8$	$\overline{N}_\mu \approx 41.2$
CIR#4	SOVA, HR	$\overline{N}_\mu \approx 11.7$	$\overline{N}_\mu \approx 15.8$	$\overline{N}_\mu \approx 21.0$	$\overline{N}_\mu \approx 26.0$	$\overline{N}_\mu \approx 40.3$
CIR#4	SOVA, BR	$\overline{N}_\mu \approx 11.7$	$\overline{N}_\mu \approx 15.2$	$\overline{N}_\mu \approx 21.0$	$\overline{N}_\mu \approx 25.9$	$\overline{N}_\mu \approx 39.7$
CIR#4	max-log-ML	$\overline{N}_\mu \approx 11.5$	$\overline{N}_\mu \approx 14.9$	$\overline{N}_\mu \approx 20.4$	$\overline{N}_\mu \approx 25.6$	$\overline{N}_\mu \approx 38.6$
CIR#4	max*-log-ML	$\overline{N}_\mu \approx 11.2$	$\overline{N}_\mu \approx 14.3$	$\overline{N}_\mu \approx 19.2$	$\overline{N}_\mu \approx 24.3$	$\overline{N}_\mu \approx 36.7$
CIR#4	log-ML	$\overline{N}_\mu \approx 10.4$	$\overline{N}_\mu \approx 14.2$	$\overline{N}_\mu \approx 19.5$	$\overline{N}_\mu \approx 24.3$	$\overline{N}_\mu \approx 36.6$
CIR#5	SOVA, SR	$\overline{N}_\mu \lesssim 16.3$	$\overline{N}_\mu \lesssim 19.5$	$\overline{N}_\mu \approx 26.4$	$\overline{N}_\mu \approx 33.5$	$\overline{N}_\mu \approx 49.7$
CIR#5	SOVA, HR	$\overline{N}_\mu \lesssim 16.3$	$\overline{N}_\mu \lesssim 19.7$	$\overline{N}_\mu \approx 26.0$	$\overline{N}_\mu \approx 33.0$	$\overline{N}_\mu \approx 48.8$
CIR#5	SOVA, BR	$\overline{N}_\mu \approx 15.5$	$\overline{N}_\mu \lesssim 20.9$	$\overline{N}_\mu \approx 26.4$	$\overline{N}_\mu \approx 33.1$	$\overline{N}_\mu \approx 50.0$
CIR#5	max-log-ML	$\overline{N}_\mu \approx 14.5$	$\overline{N}_\mu \approx 18.1$	$\overline{N}_\mu \approx 24.8$	$\overline{N}_\mu \approx 32.0$	$\overline{N}_\mu \approx 46.9$
CIR#5	max*-log-ML	$\overline{N}_\mu \approx 14.0$	$\overline{N}_\mu \approx 17.5$	$\overline{N}_\mu \approx 23.7$	$\overline{N}_\mu \approx 30.2$	$\overline{N}_\mu \approx 44.6$
CIR#5	log-ML	$\overline{N}_\mu \approx 14.0$	$\overline{N}_\mu \approx 17.3$	$\overline{N}_\mu \approx 23.3$	$\overline{N}_\mu \approx 30.2$	$\overline{N}_\mu \approx 45.4$

Figures 25 and 26 illustrate selected simulation results showing the viability of the soft output Kohonen map based quantum data detection scheme (---x---) in comparison with the quantum data detectors log-ML (—■—), max*-log-ML (—▲—), max-log-ML (—★—), SOVA with the BATTAIL RULE (BR) (- - -●- - -), SOVA with the HUBER RULE (HR) (- - -▼- - -) and SOVA with the SIMPLE RULE (SR) (- - -*- - -). It is obvious that the soft output Kohonen map based quantum data detection scheme, which relies on the SR also applied in the SOVA with the SR, performs almost identical to this SOVA with the SR. Clearly, the soft output Kohonen map based quantum data detection scheme is an interesting alternative to the other quantum data detectors considered in this manuscript.

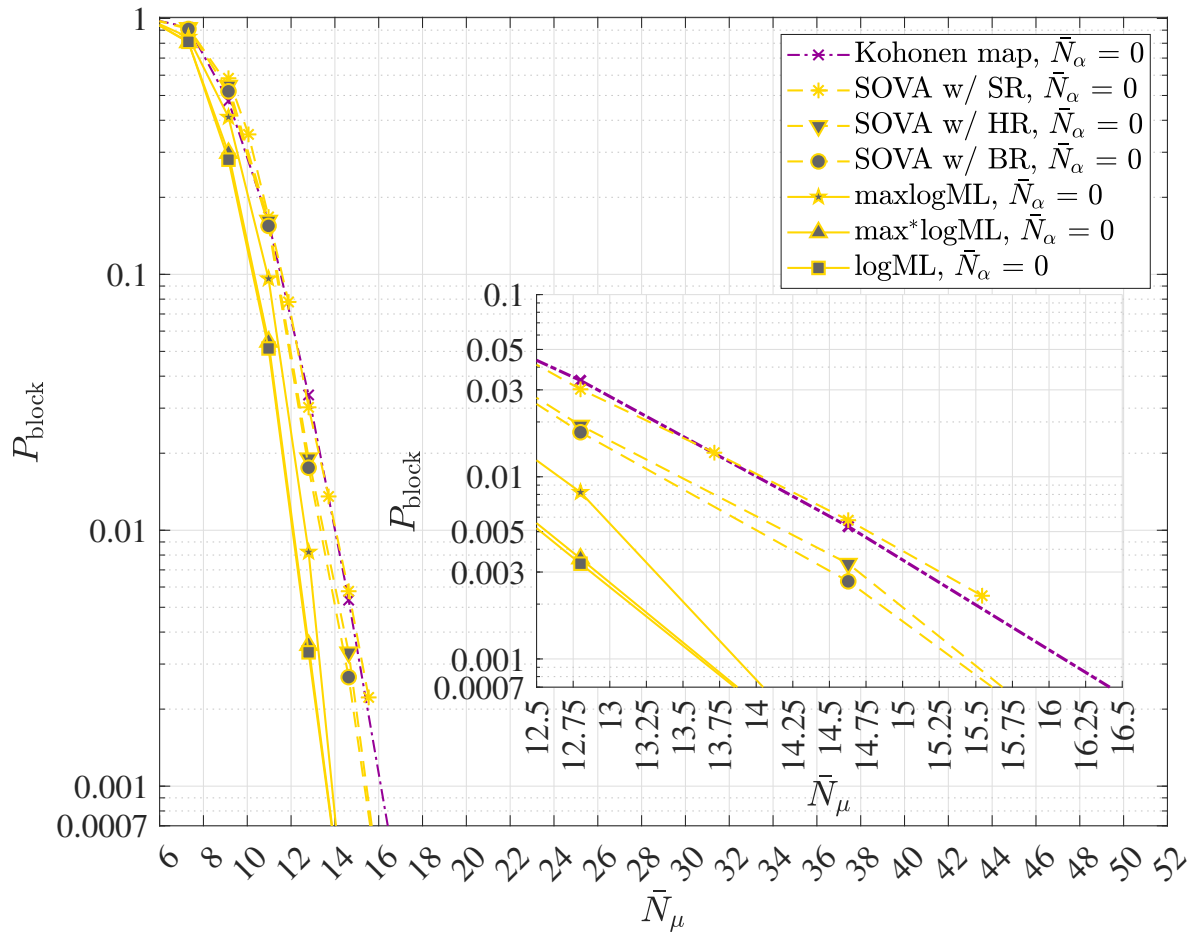


Figure 25. Detail of simulation results of the block error ratio (BLER) P_{block} performance versus the average number $\bar{N}_\mu$ of signal photons per message bit with the average number $\bar{N}_\alpha$ of thermal photons equal to 0 for the transmission via the multipath quantum channel CIR#5; the 3GPP 5G/NR polar code with the generator matrix of (71) and the polar sequence of [41][Table 5.3.1.2-1, pp. 17-19] is used with 24 CRC bits generated with polynomial "24C", code dimension $K = 164$, code length $N = 512$, code rate $R_c = (164 - 24)/512 \approx 0.273$; the code rate R_c is explicitly included in $\bar{N}_\mu$; the polar code decoding is accomplished with the CA-SCL decoder having the list length $L = 8$; interleaving is not deployed; associating the curves with the quantum data detectors is as follows: ---x---: Kohonen map; —■—: log-ML; —▲—: max*-log-ML; —★—: max-log-ML; ---●---: SOVA w/ BATTAIL RULE (BR); ---▼---: SOVA w/ HUBER RULE (HR); ---*---: SOVA w/ SIMPLE RULE (SR).

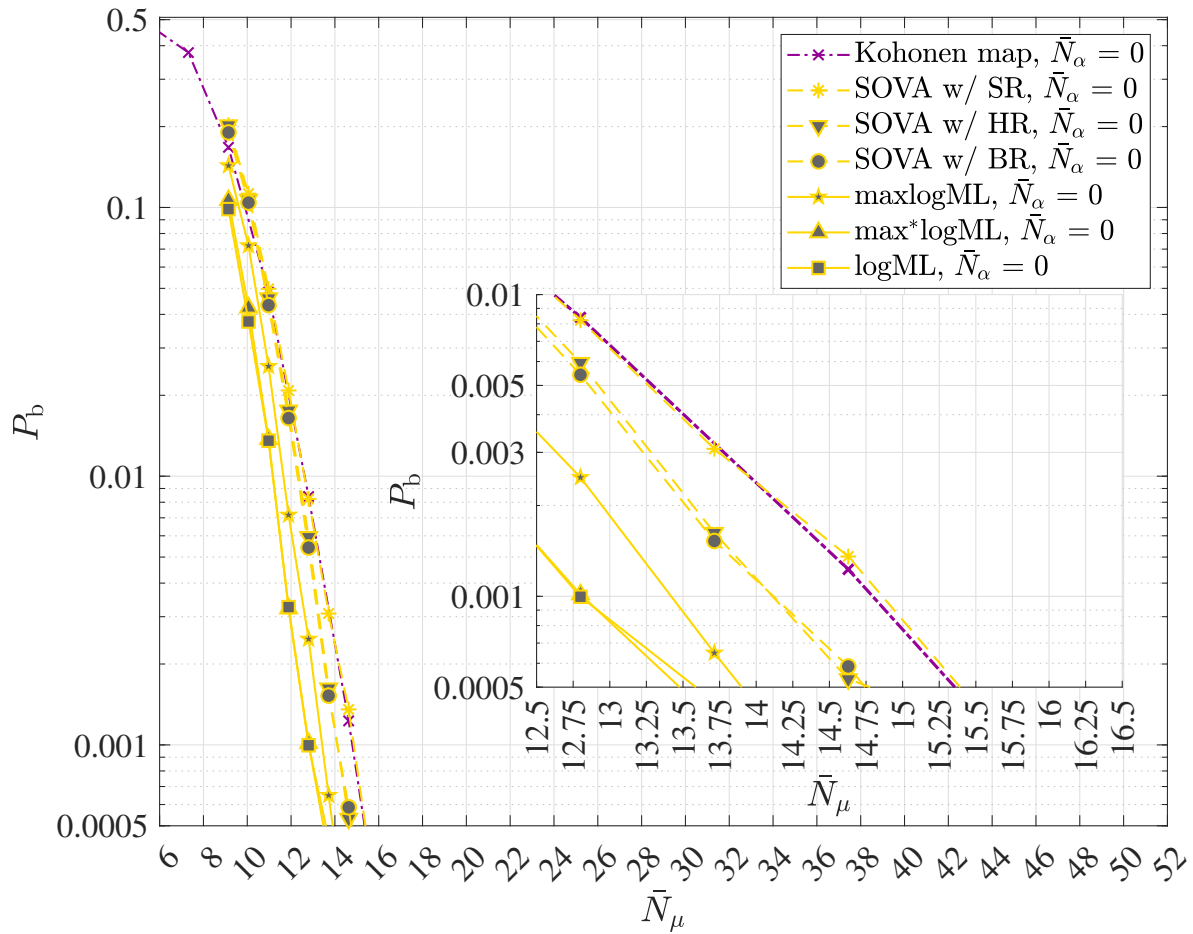


Figure 26. Detail of simulation results of the coded bit error rate (BER) P_b performance versus the average number $\bar{N}_\mu$ of signal photons per message bit with the average number $\bar{N}_\alpha$ of thermal photons equal to 0 for the transmission via the multipath quantum channel CIR#5; the 3GPP 5G/NR polar code with the generator matrix of (71) and the polar sequence of [41][Table 5.3.1.2-1, pp. 17-19] is used with 24 CRC bits generated with polynomial “24C”, code dimension $K = 164$, code length $N = 512$, code rate $R_c = (164 - 24)/512 \approx 0.273$; the code rate R_c is explicitly included in $\bar{N}_\mu$; the polar code decoding is accomplished with the CA-SCL decoder having the list length $L = 8$; interleaving is not deployed; associating the curves with the quantum data detectors is as follows: ---x---: Kohonen map; —■—: log-ML; —▲—: max*-log-ML; —★—: max-log-ML; ---●---: SOVA w/ BATTAIL RULE (BR); ---▼---: SOVA w/ HUBER RULE (HR); ---*---: SOVA w/ SIMPLE RULE (SR).

4.3. Optimized Polar Sequence

Table 5 presents the estimated average number $\bar{N}_\mu$ of signal photons per message bit received via the single path quantum channel CIR#1 required to obtain the BLER P_{block} equal to 10^{-3} . The 3GPP 5G/NR polar code with the generator matrix of (71) and either the 3GPP 5G/NR polar sequence of [41][Table 5.3.1.2-1, pp. 17-19], the polar sequence using p equal to $\sqrt[4]{2}$, the newly generated polar sequence using p equal to $\sqrt[4]{\Phi}$ (---▲---), cf. Algorithm 1, the newly generated polar sequence using p equal to $\sqrt[3]{\pi}$ (---★---), cf. Algorithm 1, or the newly generated polar sequence combining the polar sequences using p equal to $\sqrt[4]{\Phi}$ and p equal to $\sqrt[3]{\pi}$ (---●---), cf. Algorithm 2, is used. Since SC decoding is applied, no CRC bits are required. The code length was N equal to 256. The three code dimensions K equal to 86, K equal to 128 and K equal to 205, yielding the three code rates R_c equal to $86/256$, i.e. ≈ 0.3359 , R_c equal to $128/256$, i.e. 0.5 , and R_c equal to $205/256$, i.e. ≈ 0.8008 are selected. Of course, the code rate R_c is explicitly included in $\bar{N}_\mu$. We find that although the 3GPP polar sequence of [41][Table 5.3.1.2-1, pp. 17-19] facilitates an excellent BLER P_{block} performance, already, improvements are still possible.

Table 5. Estimated average number $\bar{N}_\mu$ of signal photons per message bit received via the single path quantum channel CIR#1 required to obtain $P_{\text{block}} = 10^{-3}$; the 3GPP 5G/NR polar code with the generator matrix of (71) is used.

carrier phase	R_c	polar sequence	$\bar{N}_\alpha = 0$	$\bar{N}_\alpha = 0.1$	$\bar{N}_\alpha = 0.3$	$\bar{N}_\alpha = 0.5$	$\bar{N}_\alpha = 1$
known	≈ 0.3359	3GPP 5G/NR	$\bar{N}_\mu \approx 5.1$	$\bar{N}_\mu \approx 6.7$	$\bar{N}_\mu \approx 9.1$	$\bar{N}_\mu \approx 12.0$	$\bar{N}_\mu \approx 19.3$
known	≈ 0.3359	$p = \sqrt[4]{2}$	$\bar{N}_\mu \approx 5.0$	$\bar{N}_\mu \approx 6.6$	$\bar{N}_\mu \approx 9.6$	$\bar{N}_\mu \approx 12.3$	$\bar{N}_\mu \approx 18.8$
known	≈ 0.3359	$p = \sqrt[4]{\Phi}$	$\bar{N}_\mu \approx 4.5$	$\bar{N}_\mu \approx 6.5$	$\bar{N}_\mu \approx 9.1$	$\bar{N}_\mu \approx 12.0$	$\bar{N}_\mu \approx 18.6$
known	≈ 0.3359	$p = \sqrt[4]{\pi}$	$\bar{N}_\mu \approx 4.6$	$\bar{N}_\mu \approx 6.9$	$\bar{N}_\mu \approx 9.9$	$\bar{N}_\mu \approx 12.8$	$\bar{N}_\mu \approx 18.8$
known	≈ 0.3359	$p = \sqrt[4]{\Phi} \& p = \sqrt[4]{\pi}$	$\bar{N}_\mu \approx 4.8$	$\bar{N}_\mu \approx 6.8$	$\bar{N}_\mu \approx 9.7$	$\bar{N}_\mu \approx 12.5$	$\bar{N}_\mu \approx 19.0$
known	≈ 0.5	3GPP 5G/NR	$\bar{N}_\mu \approx 5.1$	$\bar{N}_\mu \approx 6.8$	$\bar{N}_\mu \approx 9.9$	$\bar{N}_\mu \approx 12.6$	$\bar{N}_\mu \approx 18.3$
known	≈ 0.5	$p = \sqrt[4]{2}$	$\bar{N}_\mu \approx 5.2$	$\bar{N}_\mu \approx 7.3$	$\bar{N}_\mu \approx 10.0$	$\bar{N}_\mu \approx 12.7$	$\bar{N}_\mu \approx 19.6$
known	≈ 0.5	$p = \sqrt[4]{\Phi}$	$\bar{N}_\mu \approx 4.9$	$\bar{N}_\mu \approx 6.2$	$\bar{N}_\mu \approx 9.2$	$\bar{N}_\mu \approx 12.0$	$\bar{N}_\mu \approx 18.2$
known	≈ 0.5	$p = \sqrt[4]{\pi}$	$\bar{N}_\mu \approx 5.2$	$\bar{N}_\mu \approx 7.1$	$\bar{N}_\mu \approx 9.8$	$\bar{N}_\mu \approx 12.6$	$\bar{N}_\mu \approx 18.4$
known	≈ 0.5	$p = \sqrt[4]{\Phi} \& p = \sqrt[4]{\pi}$	$\bar{N}_\mu \approx 5.2$	$\bar{N}_\mu \approx 7.2$	$\bar{N}_\mu \approx 10.1$	$\bar{N}_\mu \approx 12.7$	$\bar{N}_\mu \approx 19.2$
known	≈ 0.8008	3GPP 5G/NR	$\bar{N}_\mu \approx 6.9$	$\bar{N}_\mu \approx 9.5$	$\bar{N}_\mu \approx 13.0$	$\bar{N}_\mu \approx 16.2$	$\bar{N}_\mu \approx 24.2$
known	≈ 0.8008	$p = \sqrt[4]{2}$	$\bar{N}_\mu \approx 7.2$	$\bar{N}_\mu \approx 10.0$	$\bar{N}_\mu \approx 13.2$	$\bar{N}_\mu \approx 16.6$	$\bar{N}_\mu \approx 23.9$
known	≈ 0.8008	$p = \sqrt[4]{\Phi}$	$\bar{N}_\mu \approx 6.5$	$\bar{N}_\mu \approx 9.2$	$\bar{N}_\mu \approx 12.3$	$\bar{N}_\mu \approx 15.6$	$\bar{N}_\mu \approx 22.7$
known	≈ 0.8008	$p = \sqrt[4]{\pi}$	$\bar{N}_\mu \approx 6.6$	$\bar{N}_\mu \approx 9.5$	$\bar{N}_\mu \approx 12.7$	$\bar{N}_\mu \approx 15.7$	$\bar{N}_\mu \approx 23.5$
known	≈ 0.8008	$p = \sqrt[4]{\Phi} \& p = \sqrt[4]{\pi}$	$\bar{N}_\mu \approx 6.7$	$\bar{N}_\mu \approx 9.4$	$\bar{N}_\mu \approx 12.7$	$\bar{N}_\mu \approx 15.7$	$\bar{N}_\mu \approx 23.3$
unknown	≈ 0.3359	3GPP 5G/NR	$\bar{N}_\mu \approx 4.6$	$\bar{N}_\mu \approx 7.5$	$\bar{N}_\mu \approx 11.0$	$\bar{N}_\mu \approx 14.6$	$\bar{N}_\mu \approx 22.4$
unknown	≈ 0.3359	$p = \sqrt[4]{2}$	$\bar{N}_\mu \approx 4.6$	$\bar{N}_\mu \approx 8.0$	$\bar{N}_\mu \approx 11.7$	$\bar{N}_\mu \approx 14.2$	$\bar{N}_\mu \approx 22.0$
unknown	≈ 0.3359	$p = \sqrt[4]{\Phi}$	$\bar{N}_\mu \approx 4.7$	$\bar{N}_\mu \approx 7.9$	$\bar{N}_\mu \approx 11.2$	$\bar{N}_\mu \approx 14.6$	$\bar{N}_\mu \approx 22.3$
unknown	≈ 0.3359	$p = \sqrt[4]{\pi}$	$\bar{N}_\mu \approx 4.6$	$\bar{N}_\mu \approx 7.5$	$\bar{N}_\mu \approx 11.2$	$\bar{N}_\mu \approx 14.0$	$\bar{N}_\mu \approx 21.8$
unknown	≈ 0.3359	$p = \sqrt[4]{\Phi} \& p = \sqrt[4]{\pi}$	$\bar{N}_\mu \approx 5.1$	$\bar{N}_\mu \approx 7.9$	$\bar{N}_\mu \approx 10.6$	$\bar{N}_\mu \approx 14.5$	$\bar{N}_\mu \approx 21.8$
unknown	≈ 0.5	3GPP 5G/NR	$\bar{N}_\mu \approx 5.2$	$\bar{N}_\mu \approx 7.8$	$\bar{N}_\mu \approx 11.0$	$\bar{N}_\mu \approx 13.9$	$\bar{N}_\mu \approx 21.8$
unknown	≈ 0.5	$p = \sqrt[4]{2}$	$\bar{N}_\mu \approx 5.2$	$\bar{N}_\mu \approx 7.4$	$\bar{N}_\mu \approx 11.1$	$\bar{N}_\mu \approx 14.0$	$\bar{N}_\mu \approx 21.7$
unknown	≈ 0.5	$p = \sqrt[4]{\Phi}$	$\bar{N}_\mu \approx 5.1$	$\bar{N}_\mu \approx 7.6$	$\bar{N}_\mu \approx 10.9$	$\bar{N}_\mu \approx 13.9$	$\bar{N}_\mu \approx 19.2$
unknown	≈ 0.5	$p = \sqrt[4]{\pi}$	$\bar{N}_\mu \approx 5.2$	$\bar{N}_\mu \approx 7.7$	$\bar{N}_\mu \approx 11.0$	$\bar{N}_\mu \approx 14.0$	$\bar{N}_\mu \approx 21.0$
unknown	≈ 0.5	$p = \sqrt[4]{\Phi} \& p = \sqrt[4]{\pi}$	$\bar{N}_\mu \approx 5.2$	$\bar{N}_\mu \approx 7.8$	$\bar{N}_\mu \approx 10.8$	$\bar{N}_\mu \approx 13.9$	$\bar{N}_\mu \approx 21.4$
unknown	≈ 0.8008	3GPP 5G/NR	$\bar{N}_\mu \approx 6.6$	$\bar{N}_\mu \approx 9.2$	$\bar{N}_\mu \approx 13.1$	$\bar{N}_\mu \approx 16.4$	$\bar{N}_\mu \approx 25.3$
unknown	≈ 0.8008	$p = \sqrt[4]{2}$	$\bar{N}_\mu \approx 7.3$	$\bar{N}_\mu \approx 9.6$	$\bar{N}_\mu \approx 13.5$	$\bar{N}_\mu \approx 16.4$	$\bar{N}_\mu \approx 25.7$
unknown	≈ 0.8008	$p = \sqrt[4]{\Phi}$	$\bar{N}_\mu \approx 6.5$	$\bar{N}_\mu \approx 8.8$	$\bar{N}_\mu \approx 11.5$	$\bar{N}_\mu \approx 16.2$	$\bar{N}_\mu \approx 24.9$
unknown	≈ 0.8008	$p = \sqrt[4]{\pi}$	$\bar{N}_\mu \approx 6.4$	$\bar{N}_\mu \approx 8.8$	$\bar{N}_\mu \approx 13.0$	$\bar{N}_\mu \approx 16.2$	$\bar{N}_\mu \approx 24.4$
unknown	≈ 0.8008	$p = \sqrt[4]{\Phi} \& p = \sqrt[4]{\pi}$	$\bar{N}_\mu \approx 6.5$	$\bar{N}_\mu \approx 8.9$	$\bar{N}_\mu \approx 13.5$	$\bar{N}_\mu \approx 16.3$	$\bar{N}_\mu \approx 25.1$

4.4. Turbo Equalization with Polar Codes

In this Sect. 4.4, the performance that can be achieved by applying turbo equalization discussed in Sect. 3.3 will be illustrated.

Figure 27 shows the detail of simulation results of the coded BER P_b performance and Figure 28 illustrates the detail of simulation results of the BLER P_{block} performance versus the average number $\bar{N}_\mu$ of signal photons per message bit with the average number $\bar{N}_\alpha$ of thermal photons equal to 0 for the transmission via the multipath quantum channel CIR#5 when turbo equalization with at most 20 iterations is taken into account, alternatively using the extrinsic soft output vector of (92), denoted by $L_e^{(1)}$ in the legend of Figure 27, respectively Figure 28, and according to (93), denoted by $L_e^{(2)}$ in the legend of Figure 27, respectively Figure 28. The expression “1st iteration” in the legends of Figures 27 and 28 refers to a single pass beginning at the quantum data detector via the optional deinterleaver and terminating at the polar code decoder, decoding the message $\hat{u}$, which is passed on to later receiver stages, cf. Figure 10. The polar code decoding is accomplished with the original SC decoder proposed by Arikan [39], hence, no CRC bits are required for the decoding. However, the CRC bits are used for the termination control of the turbo equalization. The original Arikan polar code is applied with the generator matrix of (72) and the polar sequence generated by applying Arikan’s Bhattacharyya parameter $Z(W_N^{(i)})$ based approach, setting out from the binary erasure channel with erasure probability ε equal to 0.5. The code dimension K equals 88, the code length N is equal to 128, thus, the code rate R_c equals 88/512, i.e. 0.6875. In addition, block interleaving according to

the illustration in Sect. 3.3 was deployed. As usual, the code rate R_c is explicitly included in $\bar{N}_\mu$. As expected, the coded BER P_b , respectively the BLER P_{block} , decreases monotonically with increasing $\bar{N}_\mu$ for a given value of $\bar{N}_\alpha$. Remarkably, the coded BER P_b , respectively the BLER P_{block} , after at most 20 turbo equalization iterations is always lower than in the case of a single pass. In addition, it becomes obvious that both approaches (92) and (93) show an almost identical performance. Therefore, in what follows, the authors will concentrate on the application of (92), which is less computationally demanding than (93).

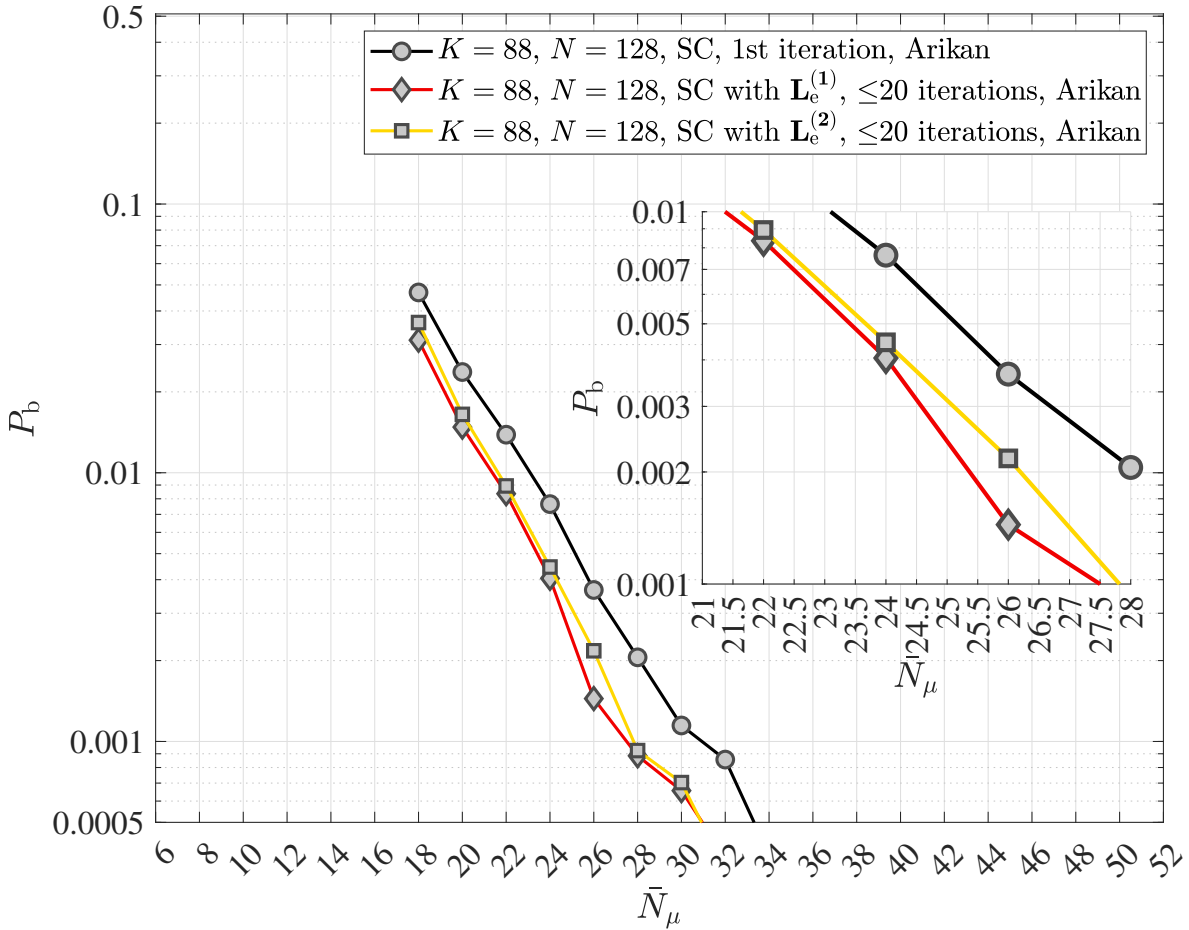


Figure 27. Detail of simulation results of the coded bit error ratio (BER) P_b performance versus the average number $\bar{N}_\mu$ of signal photons per message bit with the average number $\bar{N}_\alpha$ of thermal photons equal to 0 for the transmission via the multipath quantum channel CIR#5 when turbo equalization with at most 20 iterations is taken into account, alternatively using the extrinsic soft output vector of (92), denoted by $L_e^{(1)}$ in the legend of this Figure 27, and according to (93), denoted by $L_e^{(2)}$ in the legend of this Figure 27; the polar code decoding is accomplished with the original SC decoder proposed by Arikan, hence, no CRC bits are required for the decoding but only for the termination control of the turbo equalization; the original Arikan polar code is applied with the generator matrix of (72) and the polar sequence generated by applying Arikan's Bhattacharyya parameter $Z(W_N^{(i)})$ based approach, setting out from the binary erasure channel with erasure probability $\varepsilon = 0.5$, code dimension $K = 88$, code length $N = 128$, code rate $R_c = 88/512 = 0.6875$; the code rate R_c is explicitly included in $\bar{N}_\mu$; block interleaving according to the illustration in Sect. 3.3 is deployed.

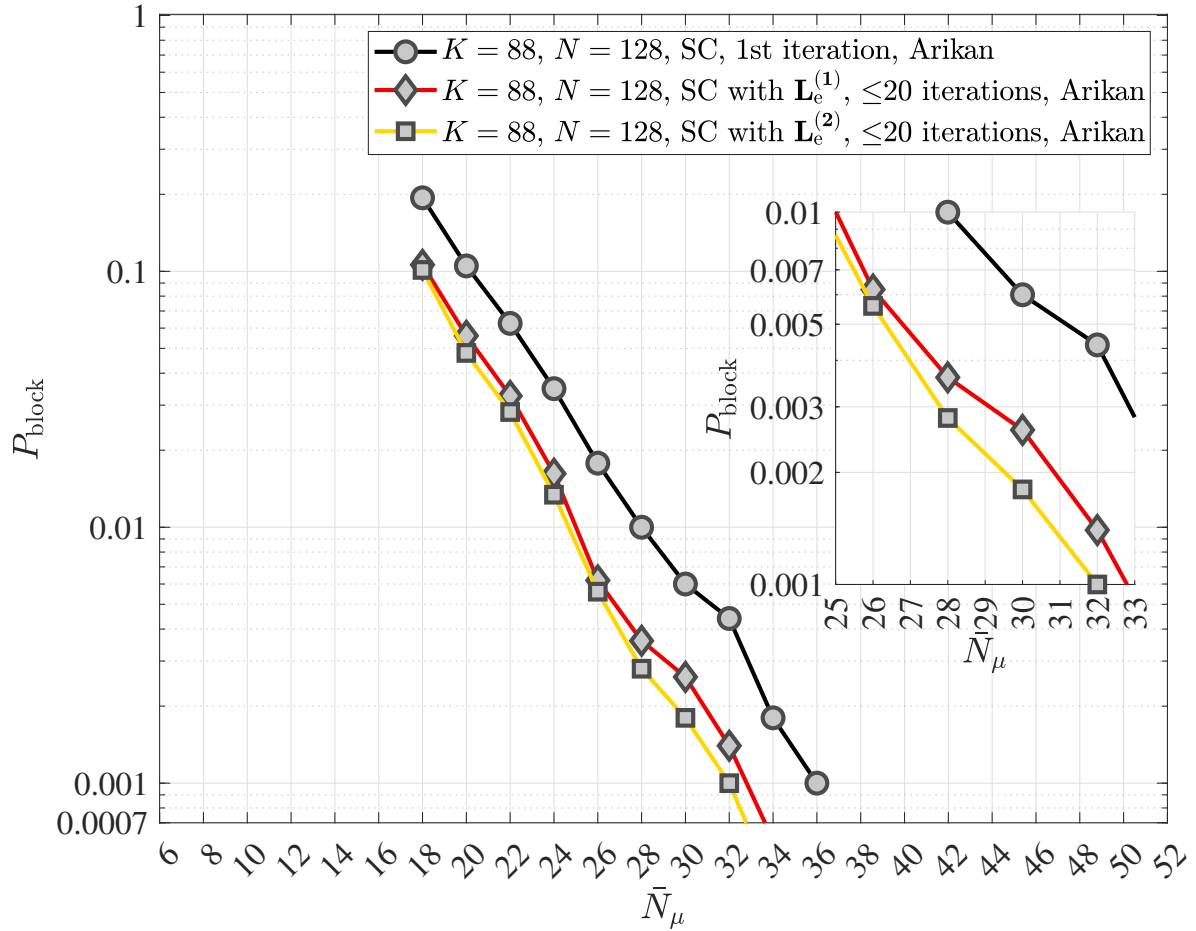


Figure 28. Detail of simulation results of the block error ratio (BLER) P_{block} performance versus the average number $\bar{N}_\mu$ of signal photons per message bit with the average number $\bar{N}_\alpha$ of thermal photons equal to 0 for the transmission via the multipath quantum channel CIR#5 when turbo equalization with at most 20 iterations is taken into account, alternatively using the extrinsic soft output vector of (92), denoted by $L_e^{(1)}$ in the legend of this Figure 28, and according to (93), denoted by $L_e^{(2)}$ in the legend of this Figure 28; the polar code decoding is accomplished with the original SC decoder proposed by Arikan, hence, no CRC bits are required for the decoding but only for the termination control of the turbo equalization; the original Arikan polar code is applied with the generator matrix of (72) and the polar sequence generated by applying Arikan's Bhattacharyya parameter $Z(W_N^{(i)})$ based approach, setting out from the binary erasure channel with erasure probability $\varepsilon = 0.5$, code dimension $K = 88$, code length $N = 128$, code rate $R_c = 88/512 = 0.6875$; the code rate R_c is explicitly included in $\bar{N}_\mu$; block interleaving according to the illustration in Sect. 3.3 is deployed.

Figure 29 depicts the average number of turbo equalization iterations versus the average number of signal photons $\bar{N}_\mu$ per message bit required to obtain the simulation results shown in Figures 27 and 28. In the considered scenario, on average, fewer than two turbo equalization iterations are sufficient to achieve the coded BER P_b of less than 10^{-2} , cf. Figure 27, and, correspondingly, the BLER P_{block} of less than $3 \cdot 10^{-2}$, cf. Figure 28, which occurs for $\bar{N}_\mu \geq 22$.

As expected, the real nonnegative scaling factor C_S of (94) varies adaptively. Moreover, the real nonnegative scaling factor C_S depends on the average number of signal photons $\bar{N}_\mu$ per message bit required to obtain the simulation results shown in Figures 27 and 28. In the case of $\bar{N}_\mu \geq 22$, the average real nonnegative scaling factor $\bar{C}_S$ assumes values around or below 0.6.

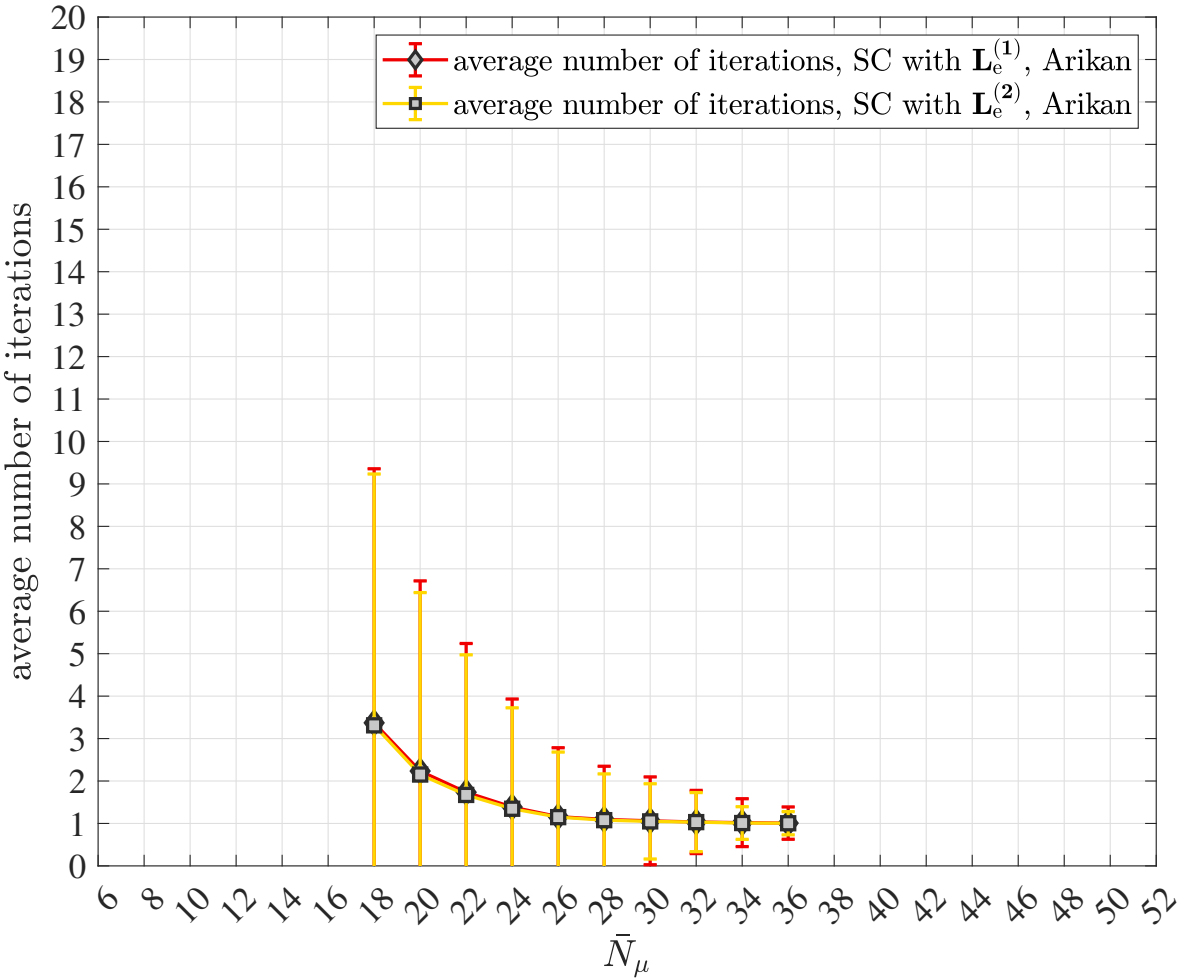


Figure 29. Average number of turbo equalization iterations versus the average number of signal photons $\bar{N}_\mu$ per message bit required to obtain the simulation results shown in Figures 27 and 28; the error bars indicate the estimated standard deviation in both the positive and negative directions.

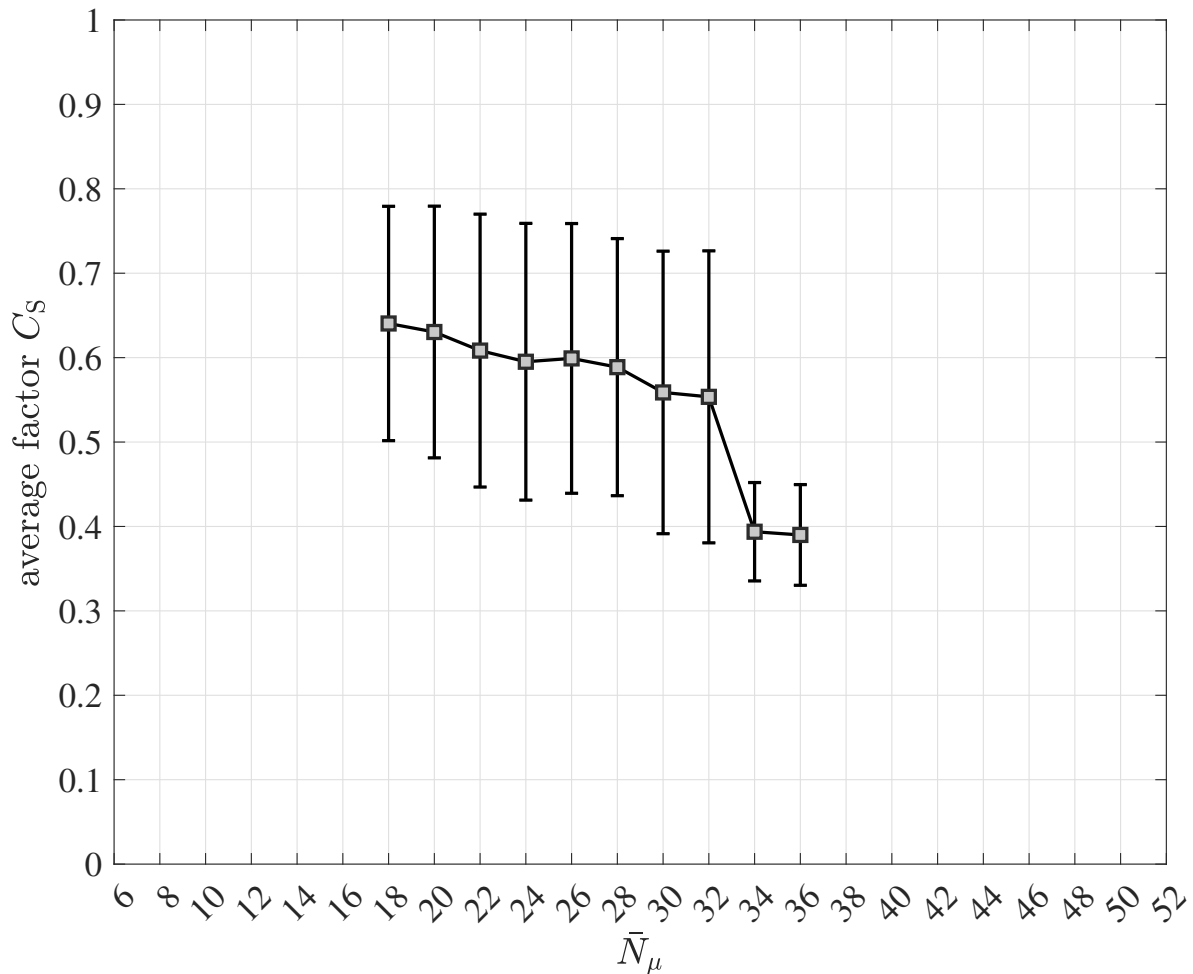


Figure 30. Real nonnegative scaling factor C_S of (94) versus the average number of signal photons $\bar{N}_\mu$ per message bit required to obtain the simulation results shown in Figures 27 and 28; the error bars indicate the estimated standard deviation in both the positive and negative directions.

Figure 31 illustrates the detail of simulation results of the BLER P_{block} performance versus the average number $\bar{N}_\mu$ of signal photons per message bit with the average number $\bar{N}_\alpha$ of thermal photons equal to 0 for the transmission via the multipath quantum channel CIR#4 when turbo equalization with at most 16 iterations is taken into account. In the case of curves labelled by “Arikan”, the polar code decoding is accomplished with the original SC decoder proposed by Arikan [39] or the soft cancellation (SCAN) decoder introduced by Fayyaz and Barry [50], hence, no CRC bits are required for the decoding. However, the CRC bits are used for the termination control of the turbo equalization. The original Arikan polar code was applied with the generator matrix of (72) and the polar sequence generated by applying Arikan’s Bhattacharyya parameter $Z(W_N^{(i)})$ based approach, setting out from the binary erasure channel with erasure probability ε equal to 0.5. In the case of the curve labelled by “5G Polar Code”, the 3GPP 5G/NR polar code with the generator matrix of (71) and the polar sequence of [41][Table 5.3.1.2-1, pp. 17-19] was used with 24 CRC bits generated with polynomial “24C”. The code dimension K is equal to 64, the code length N is equal to 128 and the code rate R_c is equal to $(64 - 24)/128$, i.e. 0.3125. The polar code decoding is accomplished with the CA-SCL decoder with the list length L equal to 8. In all cases, the code rate R_c is explicitly included in $\bar{N}_\mu$. Block interleaving according to the illustration in Sect. 3.3 was deployed.

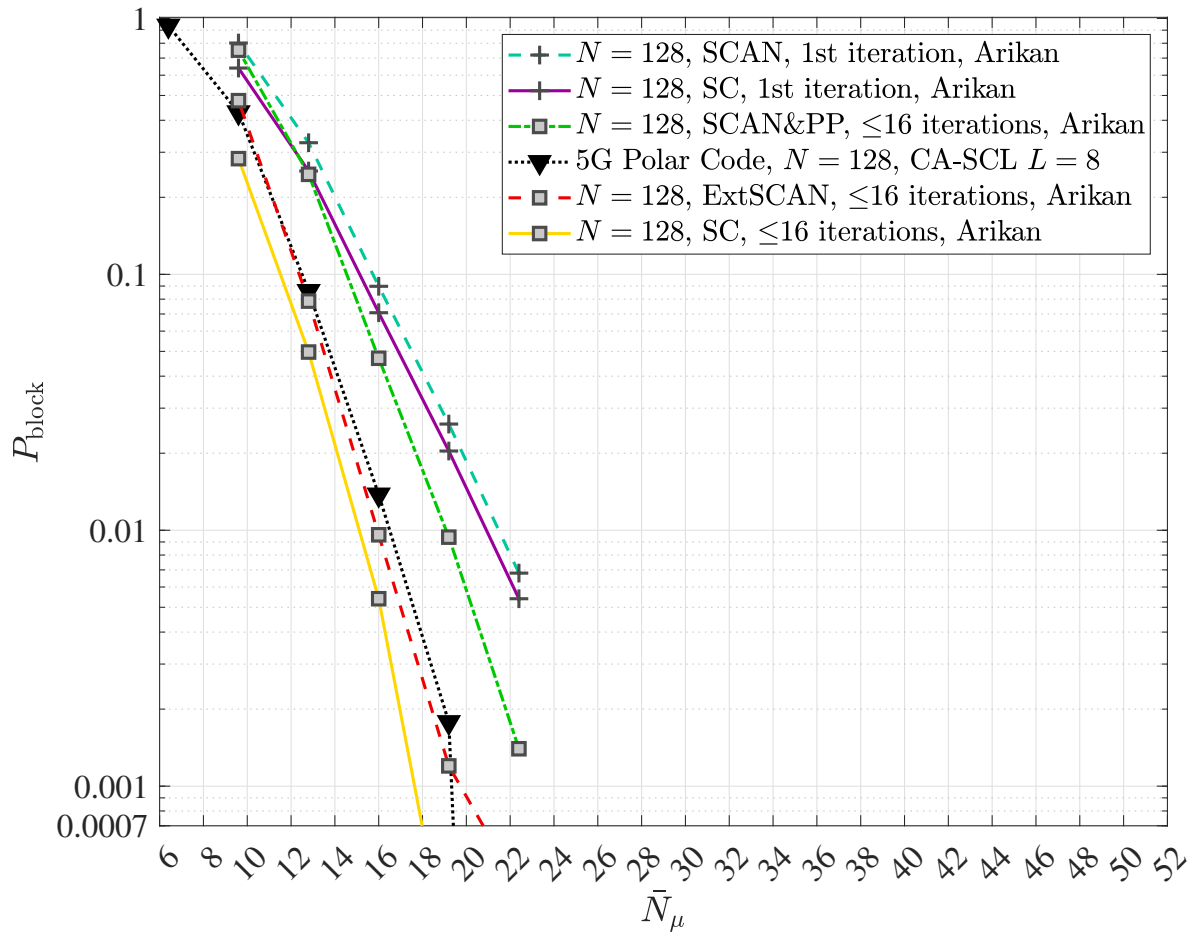


Figure 31. Detail of simulation results of the block error ratio (BLER) P_{block} performance versus the average number $\bar{N}_\mu$ of signal photons per message bit with the average number $\bar{N}_\alpha$ of thermal photons equal to 0 for the transmission via the multipath quantum channel CIR#4 when turbo equalization with at most 16 iterations is taken into account; in the case of curves labelled by “Arikan”, the polar code decoding is accomplished with the original SC decoder proposed by Arikan [39] or the soft cancellation (SCAN) decoder introduced by Fayyaz and Barry [50], hence, no CRC bits are required for the decoding but only for the termination control of the turbo equalization; the original Arikan polar code was applied with the generator matrix of (72) and the polar sequence generated by applying Arikan’s Bhattacharyya parameter $Z(W_N^{(i)})$ based approach, setting out from the binary erasure channel with erasure probability $\varepsilon = 0.5$; in the case of the curve labelled by “5G Polar Code”, the 3GPP 5G/NR polar code with the generator matrix of (71) and the polar sequence of [41][Table 5.3.1.2-1, pp. 17-19] is used with 24 CRC bits generated with polynomial “24C”; code dimension $K = 64$, code length $N = 128$, code rate $R_c = (64 - 24)/128 = 0.3125$; the polar code decoding is accomplished with the CA-SCL decoder with the list length $L = 8$; the code rate R_c is explicitly included in $\bar{N}_\mu$; block interleaving according to the illustration in Sect. 3.3 is deployed.

In Figure 31 “SCAN&PP” (SCAN and post processing) refers to the application of the SCAN decoder combined with the generation of the extrinsic soft output vector of (92). In the case of “ExtSCAN”, the extrinsic soft output vector is generated natively by the SCAN decoder as described in [50]. As expected, the BLER P_{block} decreases monotonically with increasing $\bar{N}_\mu$ for a given value of $\bar{N}_\alpha$. Remarkably, the BLER P_{block} after at most 16 turbo equalization iterations is always lower than in the case of a single pass. The SC decoder always provides an improved performance over the SCAN decoder. Furthermore, turbo equalization is capable of outperforming the 5G/NR polar code scheme using the CA-SCL decoder with the list length L equal to 8. In order to obtain the BLER $P_{\text{block}} < 10^{-1}$, the required average number of signal photons $\bar{N}_\mu$ is lower than 12.

Finally, Figure 32 shows the average number of turbo equalization iterations versus the average number of signal photons $\bar{N}_\mu$ per message bit required to obtain the simulation results labelled

by “ $N = 128$, SC, ≤ 16 iterations, Arikan” depicted in Figure 31. On average fewer than 5 turbo equalization iterations are required for $P_{\text{block}} < 10^{-1}$.

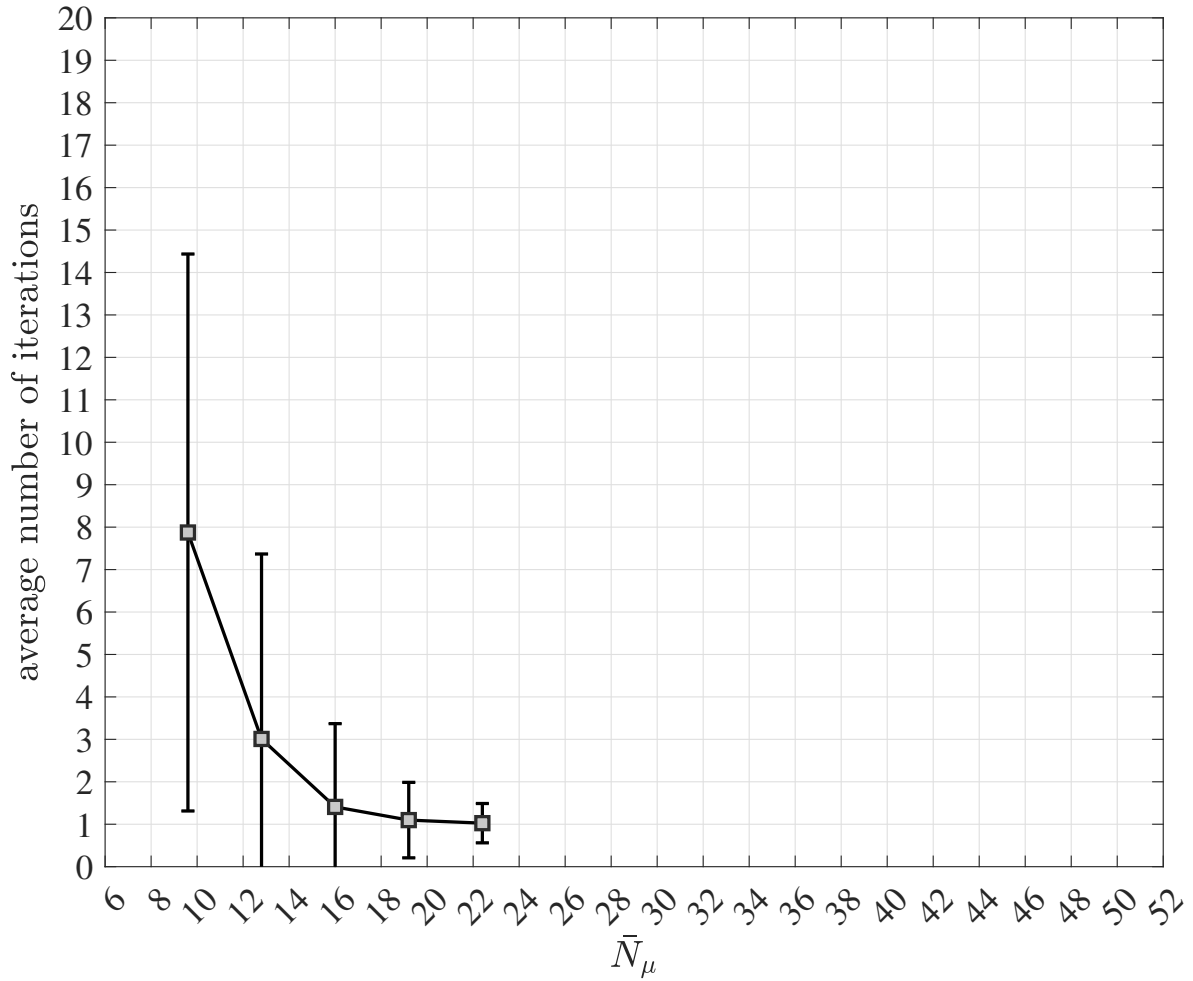


Figure 32. Average number of turbo equalization iterations versus the average number of signal photons $\bar{N}_\mu$ per message bit required to obtain the simulation results labelled by “ $N = 128$, SC, ≤ 16 iterations, Arikan”, shown in Figure 31; the error bars indicate the estimated standard deviation in both the positive and negative directions.

Figures 33 and 34 depict the details of simulation results of the coded BER P_b performance, respectively, the BLER P_{block} performance versus the average number $\bar{N}_\mu$ of signal photons per message bit with the average number $\bar{N}_\alpha$ of thermal photons as the curve parameter for the transmission via the multipath quantum channel CIR#4, cf. the Figures 19 and 20, when turbo equalization with at most 16 iterations is taken into account and the 3GPP 5G/NR polar code with the generator matrix of (71) associated with the polar sequence of [41][Table 5.3.1.2-1, pp. 17-19] is used. Figures 35 and 36 depict the details of simulation results of the coded BER P_b performance, respectively, the BLER P_{block} performance versus the average number $\bar{N}_\mu$ of signal photons per message bit with the average number $\bar{N}_\alpha$ of thermal photons as the curve parameter for the transmission via the multipath quantum channel CIR#4, cf. the Figures 19 and 20, when turbo equalization with at most 16 iterations is taken into account and the Arikan-style polar code with the generator matrix of (72) associated with the polar sequence generated by Arikan’s Bhattacharyya parameter $Z(W_N^{(i)})$ based approach, setting out from the binary erasure channel with erasure probability ε equal to 0.5, is used.

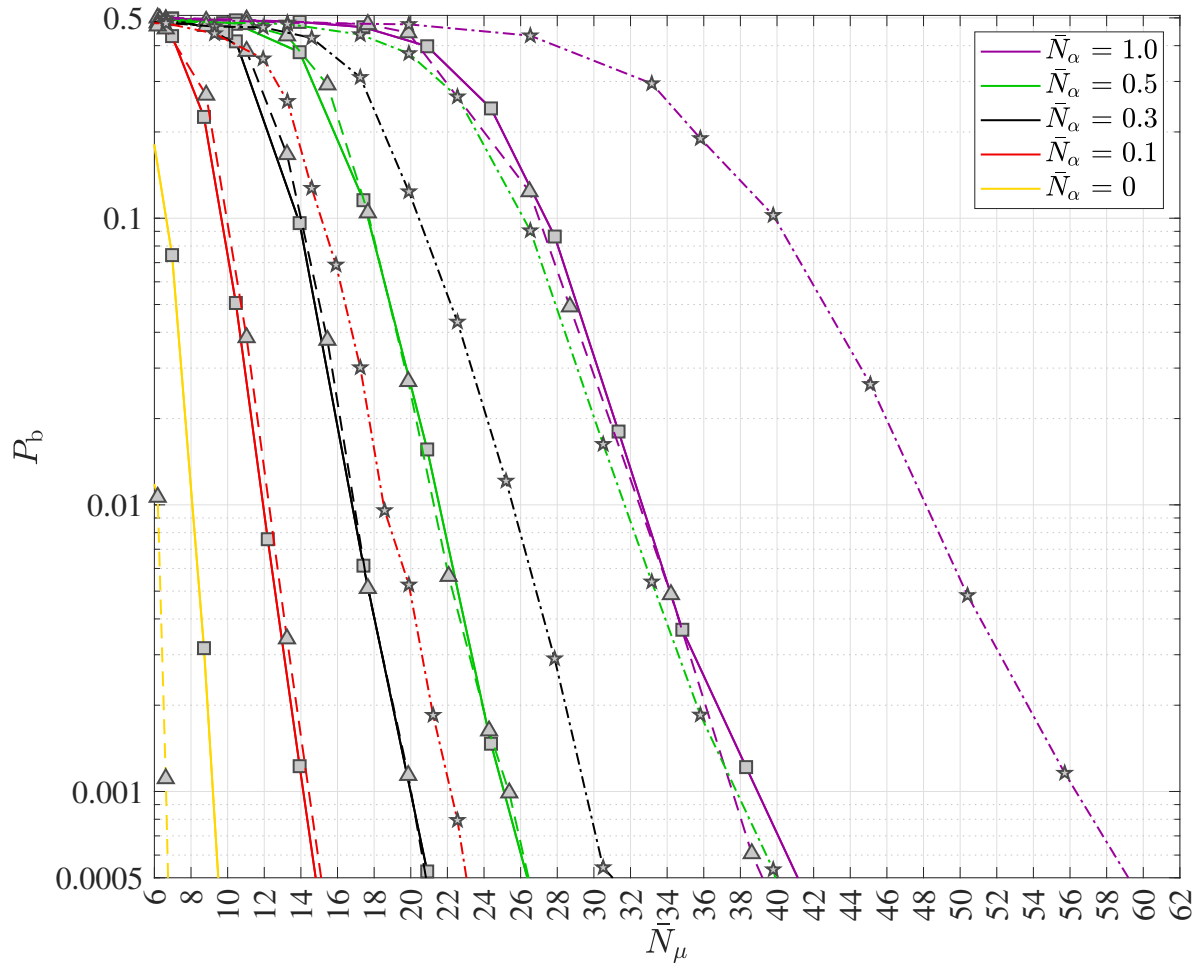


Figure 33. Detail of simulation results of the coded bit error ratio (BER) P_b performance versus the average number $\bar{N}_\mu$ of signal photons per message bit with the average number $\bar{N}_\alpha$ of thermal photons as the curve parameter for the transmission via the multipath quantum channel CIR#4, cf. Figure 19, when turbo equalization with at most 16 iterations is taken into account; log-MAP quantum data detection is applied; the 3GPP 5G/NR polar code with the generator matrix of (71) and the polar sequence of [41][Table 5.3.1.2-1, pp. 17-19] is used; 24 CRC bits generated with polynomial “24C” are used; the polar code decoding is accomplished with the SC decoder, hence, CRC bits are not processed by the SC decoder; the code length N is 512; three choices of the code dimension, $K = 171$ (—■—), $K = 256$ (---▲---) and $K = 410$ (-·-★-·-), leading to the three code rates $R_c = (171 - 24)/512 \approx 0.2871$ (—■—), $R_c = (256 - 24)/512 \approx 0.4531$ (---▲---) and $R_c = (410 - 24)/512 \approx 0.7539$ (-·-★-·-) are considered; the code rate R_c is explicitly included in $\bar{N}_\mu$; interleaving is deployed.

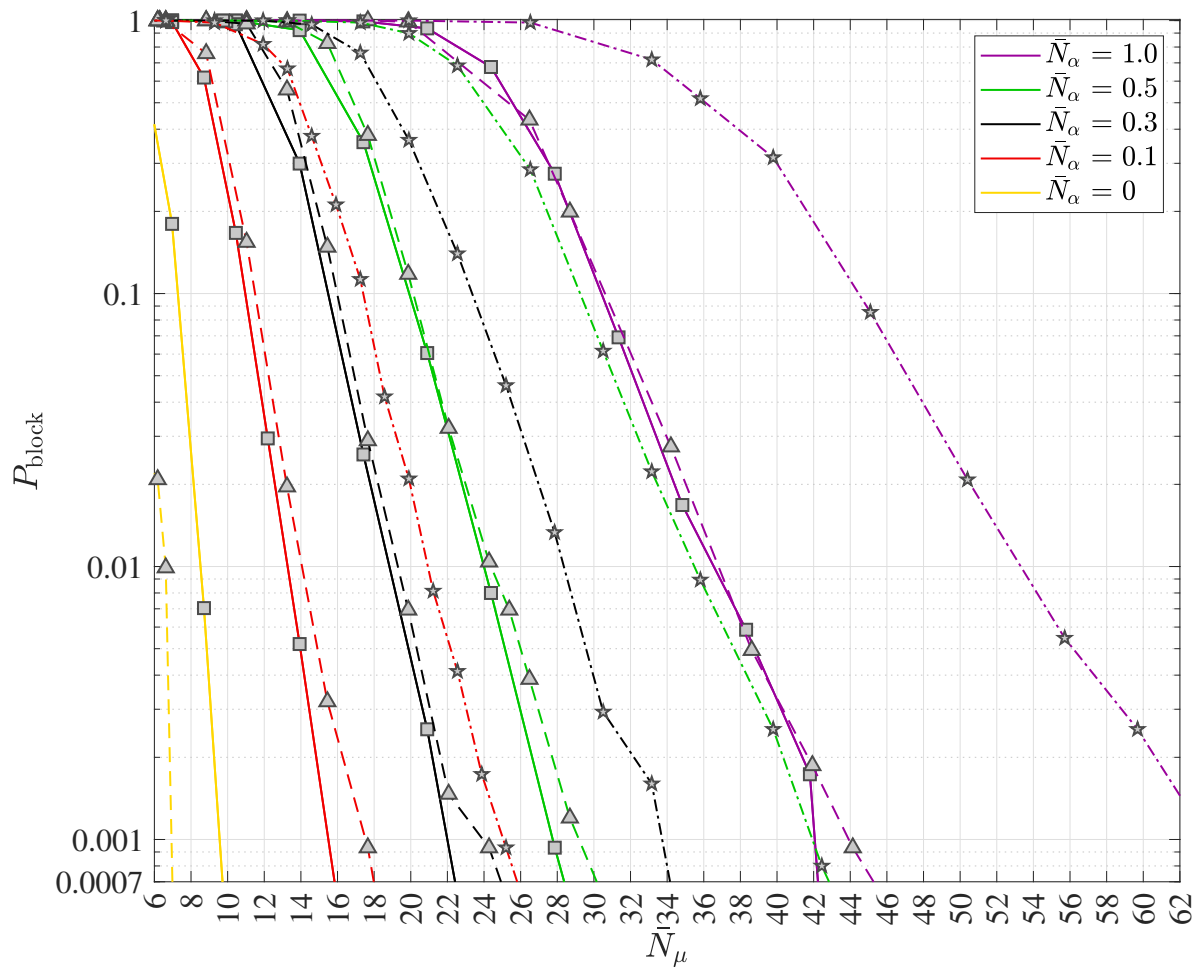


Figure 34. Detail of simulation results of the block error rate (BLER) P_{block} performance versus the average number $\bar{N}_\mu$ of signal photons per message bit with the average number $\bar{N}_\alpha$ of thermal photons as the curve parameter for the transmission via the multipath quantum channel CIR#4, cf. Figure 20, when turbo equalization with at most 16 iterations is taken into account; log-MAP quantum data detection is applied; the 3GPP 5G/NR polar code with the generator matrix of (71) and the polar sequence of [41][Table 5.3.1.2-1, pp. 17-19] is used; 24 CRC bits generated with polynomial “24C” are used; the polar code decoding is accomplished with the SC decoder, hence, CRC bits are not processed by the SC decoder; the code length N is 512; three choices of the code dimension, $K = 171$ (—■—), $K = 256$ (---▲---) and $K = 410$ (-·-★-·-), leading to the three code rates $R_c = (171 - 24)/512 \approx 0.2871$ (—■—), $R_c = (256 - 24)/512 \approx 0.4531$ (---▲---) and $R_c = (410 - 24)/512 \approx 0.7539$ (-·-★-·-) are considered; the code rate R_c is explicitly included in $\bar{N}_\mu$; interleaving is deployed.

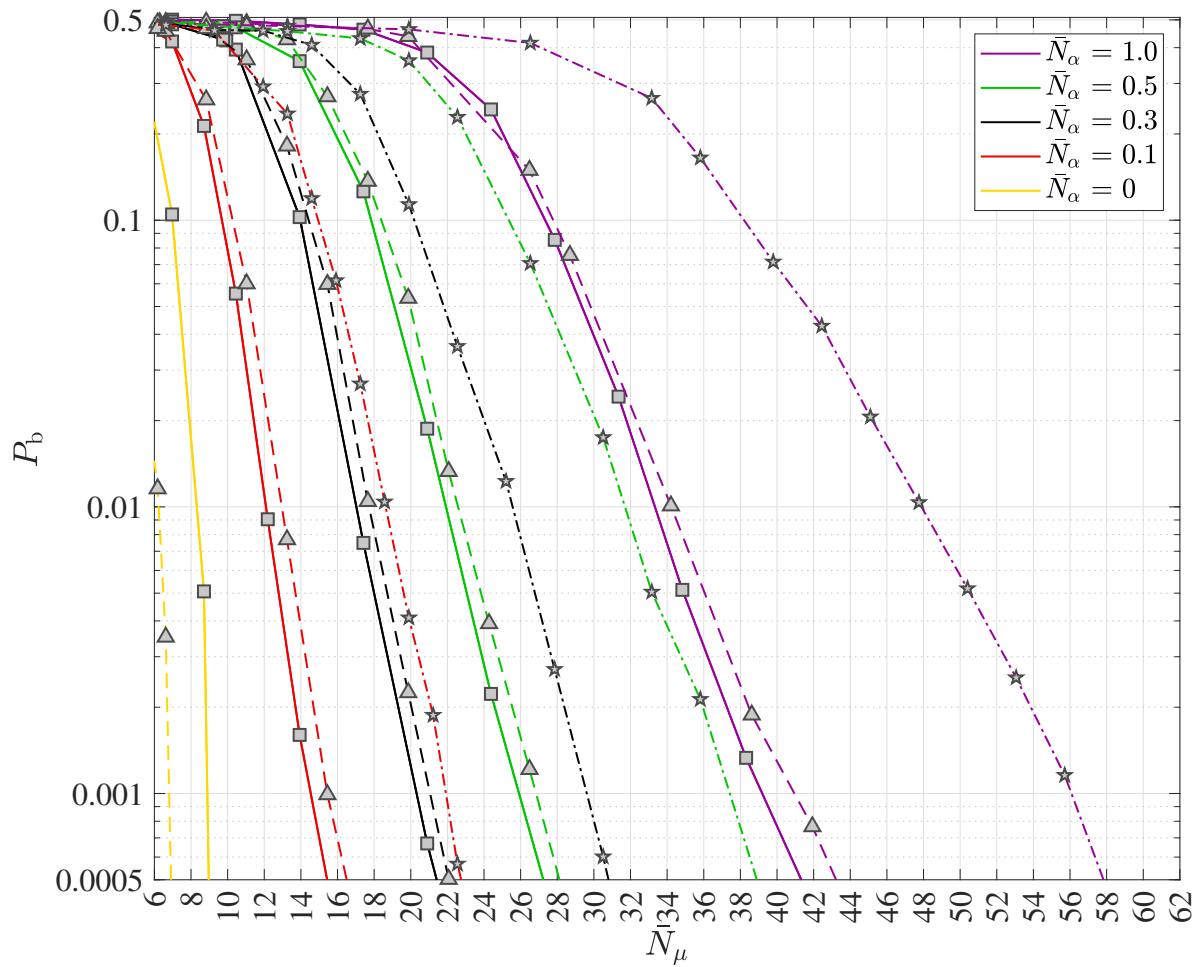


Figure 35. Detail of simulation results of the coded bit error ratio (BER) P_b performance versus the average number $\bar{N}_\mu$ of signal photons per message bit with the average number $\bar{N}_\alpha$ of thermal photons as the curve parameter for the transmission via the multipath quantum channel CIR#4, cf. Figure 19, when turbo equalization with at most 16 iterations is taken into account; log-MAP quantum data detection is applied; the Arikan-style polar code with the generator matrix of (72) and the associated polar sequence generated by Arikan's Bhattacharyya parameter $Z(W_N^{(i)})$ based approach setting out from the binary erasure channel with erasure probability $\varepsilon = 0.5$ is used; 24 CRC bits generated with polynomial "24C" are used; the polar code decoding is accomplished with the SC decoder, hence, CRC bits are not processed by the SC decoder; the code length N is 512; three choices of the code dimension, $K = 171$ (—■—), $K = 256$ (- - -▲- - -) and $K = 410$ (- · -★- · -), leading to the three code rates $R_c = (171 - 24)/512 \approx 0.2871$ (—■—), $R_c = (256 - 24)/512 \approx 0.4531$ (- - -▲- - -) and $R_c = (410 - 24)/512 \approx 0.7539$ (- · -★- · -) are considered; the code rate R_c is explicitly included in $\bar{N}_\mu$; interleaving is deployed.

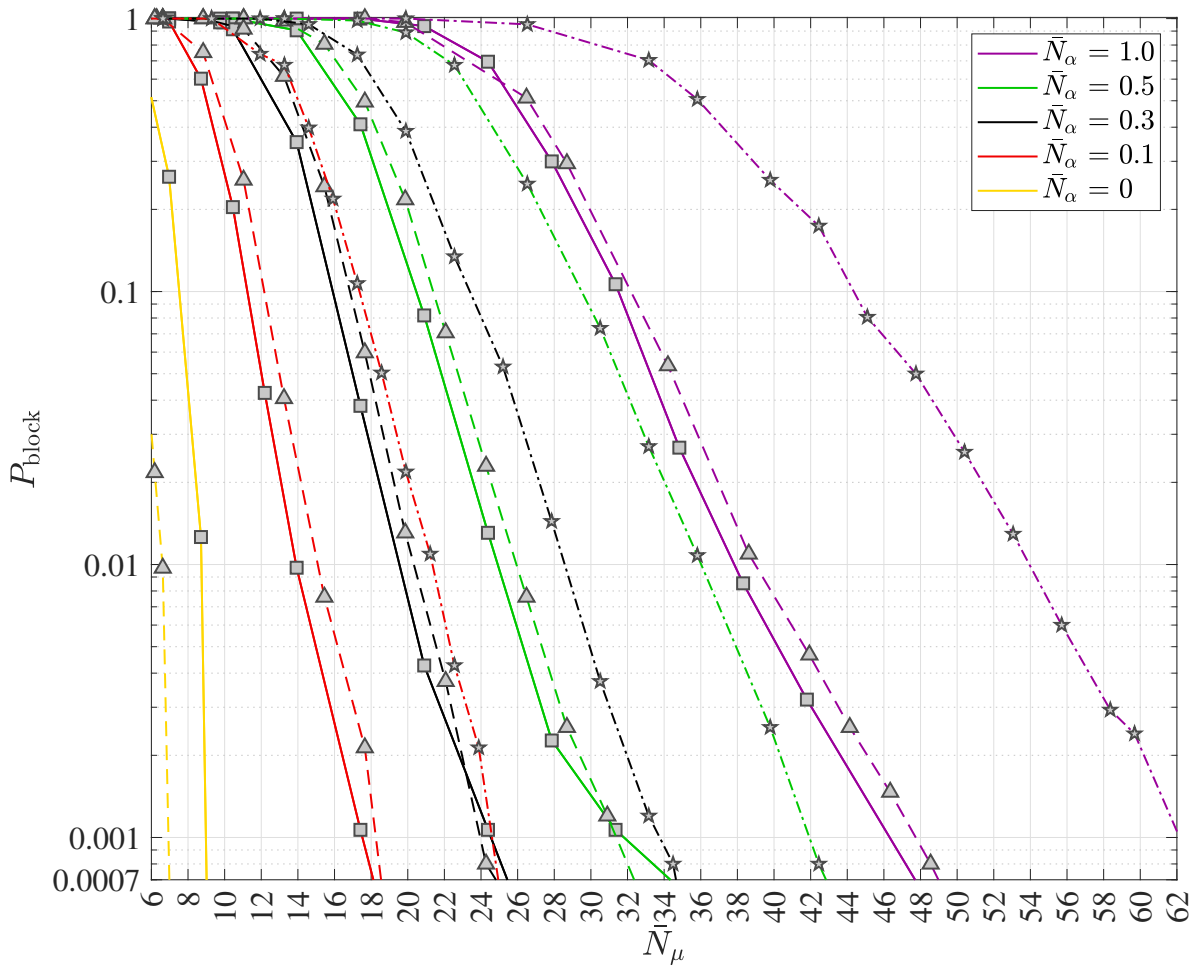


Figure 36. Detail of simulation results of the block error rate (BLER) P_{block} performance versus the average number $\bar{N}_\mu$ of signal photons per message bit with the average number $\bar{N}_\alpha$ of thermal photons as the curve parameter for the transmission via the multipath quantum channel CIR#4, cf. Figure 20, when turbo equalization with at most 16 iterations is taken into account; log-MAP quantum data detection is applied; the Arikan-style polar code with the generator matrix of (72) and the associated polar sequence generated by Arikan's Bhattacharyya parameter $Z(W_N^{(i)})$ based approach setting out from the binary erasure channel with erasure probability $\varepsilon = 0.5$ is used; 24 CRC bits generated with polynomial "24C" are used; the polar code decoding is accomplished with the SC decoder, hence, CRC bits are not processed by the SC decoder; the code length N is 512; three choices of the code dimension, $K = 171$ (—■—), $K = 256$ (- - -▲- - -) and $K = 410$ (- · -★- · -), leading to the three code rates $R_c = (171 - 24)/512 \approx 0.2871$ (—■—), $R_c = (256 - 24)/512 \approx 0.4531$ (- - -▲- - -) and $R_c = (410 - 24)/512 \approx 0.7539$ (- · -★- · -) are considered; the code rate R_c is explicitly included in $\bar{N}_\mu$; interleaving is deployed.

In all shown cases, log-MAP quantum data detection is applied. 24 CRC bits generated with polynomial "24C" are used. The polar code decoding is accomplished with the SC decoder, hence, CRC bits are not processed by the SC decoder. The code length N is equal to 512. Furthermore, three choices of the code dimension, K equal to 171 (—■—), K equal to 256 (- - -▲- - -) and K equal to 410 (- · -★- · -), leading to the three code rates R_c equal to $(171 - 24)/512$, i.e. ≈ 0.2871 (—■—), R_c equal to $(256 - 24)/512$, i.e. ≈ 0.4531 (- - -▲- - -) and R_c equal to $(410 - 24)/512$, i.e. ≈ 0.7539 (- · -★- · -) are considered. Of course, the code rate R_c is explicitly included in $\bar{N}_\mu$. In addition, interleaving according to the illustration in Sect. 3.3 was deployed. Both the coded BER P_b , respectively the BLER P_{block} , decreases monotonically with increasing $\bar{N}_\mu$ for a given value of $\bar{N}_\alpha$. Furthermore, the coded BER P_b , respectively the BLER P_{block} , increases with increasing thermal noise, i.e. with increasing $\bar{N}_\alpha$, for a given value of $\bar{N}_\mu$. Comparing the Figures 33 and 19 as well as 34 and 20, the beneficial effect of turbo equalization on the respective performance becomes obvious. For instance, in the case of R_c equal to $171/512$, i.e. ≈ 0.33 , no errors were measured in the simulations. Table 6 summarizes the

estimated average number $\bar{N}_\mu$ of signal photons required to obtain P_{block} equal to 10^{-3} and P_b equal to 10^{-3} associated with the results shown in Figures 19, 20, 21, 22, 33, 34, 35 and 36.

Table 6. Estimated average number $\bar{N}_\mu$ of signal photons per message bit required to obtain $P_{\text{block}} = 10^{-3}$ and $P_b = 10^{-3}$ associated with the results shown in Figures 19, 20, 21, 22, 33, 34, 35 and 36.

	R_c	polar code	iteration	$\bar{N}_\alpha = 0$	$\bar{N}_\alpha = 0.1$	$\bar{N}_\alpha = 0.3$	$\bar{N}_\alpha = 0.5$	$\bar{N}_\alpha = 1$
BER	≈ 0.2871	3GPP 5G/NR	1	$\bar{N}_\mu \approx 13.1$	$\bar{N}_\mu \approx 16.8$	$\bar{N}_\mu \approx 23.0$	$\bar{N}_\mu \approx 29.8$	$\bar{N}_\mu \approx 44.7$
BER	≈ 0.2871	3GPP 5G/NR	16	$\bar{N}_\mu \approx 9.2$	$\bar{N}_\mu \approx 14.1$	$\bar{N}_\mu \approx 20.0$	$\bar{N}_\mu \approx 25.1$	$\bar{N}_\mu \approx 38.9$
BER	≈ 0.2871	Arikan	1	$\bar{N}_\mu \approx 13.4$	$\bar{N}_\mu \approx 16.7$	$\bar{N}_\mu \approx 23.4$	$\bar{N}_\mu \approx 29.8$	$\bar{N}_\mu \approx 45.7$
BER	≈ 0.2871	Arikan	16	$\bar{N}_\mu \approx 8.9$	$\bar{N}_\mu \approx 14.5$	$\bar{N}_\mu \approx 20.3$	$\bar{N}_\mu \approx 25.9$	$\bar{N}_\mu \approx 39.2$
BER	≈ 0.4531	3GPP 5G/NR	1	$\bar{N}_\mu \approx 13.0$	$\bar{N}_\mu \approx 17.0$	$\bar{N}_\mu \approx 23.2$	$\bar{N}_\mu \approx 29.7$	$\bar{N}_\mu \approx 44.6$
BER	≈ 0.4531	3GPP 5G/NR	16	$\bar{N}_\mu \approx 6.2$	$\bar{N}_\mu \approx 14.4$	$\bar{N}_\mu \approx 20.0$	$\bar{N}_\mu \approx 25.4$	$\bar{N}_\mu \approx 37.6$
BER	≈ 0.4531	Arikan	1	$\bar{N}_\mu \approx 13.9$	$\bar{N}_\mu \approx 18.2$	$\bar{N}_\mu \approx 24.1$	$\bar{N}_\mu \approx 30.6$	$\bar{N}_\mu \approx 46.6$
BER	≈ 0.4531	Arikan	16	$\bar{N}_\mu \approx 6.8$	$\bar{N}_\mu \approx 15.4$	$\bar{N}_\mu \approx 21.1$	$\bar{N}_\mu \approx 26.8$	$\bar{N}_\mu \approx 41.0$
BER	≈ 0.7539	3GPP 5G/NR	1	$\bar{N}_\mu \approx 19.1$	$\bar{N}_\mu \approx 23.5$	$\bar{N}_\mu \approx 32.4$	$\bar{N}_\mu \approx 40.0$	$\bar{N}_\mu \approx 60.7$
BER	≈ 0.7539	3GPP 5G/NR	16	error-free	$\bar{N}_\mu \approx 22.2$	$\bar{N}_\mu \approx 29.5$	$\bar{N}_\mu \approx 37.8$	$\bar{N}_\mu \approx 56.3$
BER	≈ 0.7539	Arikan	1	$\bar{N}_\mu \approx 18.9$	$\bar{N}_\mu \approx 23.4$	$\bar{N}_\mu \approx 31.9$	$\bar{N}_\mu \approx 39.3$	$\bar{N}_\mu \approx 59.5$
BER	≈ 0.7539	Arikan	16	error-free	$\bar{N}_\mu \approx 21.9$	$\bar{N}_\mu \approx 29.6$	$\bar{N}_\mu \approx 37.4$	$\bar{N}_\mu \approx 56.1$
BLER	≈ 0.2871	3GPP 5G/NR	1	$\bar{N}_\mu \approx 14.9$	$\bar{N}_\mu \approx 18.7$	$\bar{N}_\mu \approx 26.3$	$\bar{N}_\mu \approx 33.1$	$\bar{N}_\mu \approx 49.3$
BLER	≈ 0.2871	3GPP 5G/NR	16	$\bar{N}_\mu \approx 9.6$	$\bar{N}_\mu \approx 15.5$	$\bar{N}_\mu \approx 22.0$	$\bar{N}_\mu \approx 27.8$	$\bar{N}_\mu \approx 42.1$
BLER	≈ 0.2871	Arikan	1	$\bar{N}_\mu \approx 16.5$	$\bar{N}_\mu \approx 20.5$	$\bar{N}_\mu \approx 26.9$	$\bar{N}_\mu \approx 36.6$	$\bar{N}_\mu \approx 53.3$
BLER	≈ 0.2871	Arikan	16	$\bar{N}_\mu \approx 9.0$	$\bar{N}_\mu \approx 17.5$	$\bar{N}_\mu \approx 24.5$	$\bar{N}_\mu \approx 31.8$	$\bar{N}_\mu \approx 46.3$
BLER	≈ 0.4531	3GPP 5G/NR	1	$\bar{N}_\mu \approx 16.2$	$\bar{N}_\mu \approx 20.8$	$\bar{N}_\mu \approx 29.2$	$\bar{N}_\mu \approx 35.1$	$\bar{N}_\mu \approx 52.0$
BLER	≈ 0.4531	3GPP 5G/NR	16	$\bar{N}_\mu \approx 6.2$	$\bar{N}_\mu \approx 17.5$	$\bar{N}_\mu \approx 23.9$	$\bar{N}_\mu \approx 29.2$	$\bar{N}_\mu \approx 43.9$
BLER	≈ 0.4531	Arikan	1	$\bar{N}_\mu \approx 17.5$	$\bar{N}_\mu \approx 21.2$	$\bar{N}_\mu \approx 29.1$	$\bar{N}_\mu \approx 36.0$	$\bar{N}_\mu \approx 54.3$
BLER	≈ 0.4531	Arikan	16	$\bar{N}_\mu \approx 6.9$	$\bar{N}_\mu \approx 18.3$	$\bar{N}_\mu \approx 24.0$	$\bar{N}_\mu \approx 31.4$	$\bar{N}_\mu \approx 47.7$
BLER	≈ 0.7539	3GPP 5G/NR	1	$\bar{N}_\mu \approx 22.2$	$\bar{N}_\mu \approx 28.0$	$\bar{N}_\mu \approx 36.2$	$\bar{N}_\mu \approx 46.3$	$\bar{N}_\mu \approx 69.5$
BLER	≈ 0.7539	3GPP 5G/NR	16	error-free	$\bar{N}_\mu \approx 25.1$	$\bar{N}_\mu \approx 33.7$	$\bar{N}_\mu \approx 41.9$	$\bar{N}_\mu \approx 63.1$
BLER	≈ 0.7539	Arikan	1	$\bar{N}_\mu \approx 24.5$	$\bar{N}_\mu \approx 27.7$	$\bar{N}_\mu \approx 37.7$	$\bar{N}_\mu \approx 48.2$	$\bar{N}_\mu \approx 70.7$
BLER	≈ 0.7539	Arikan	16	error-free	$\bar{N}_\mu \approx 24.6$	$\bar{N}_\mu \approx 33.8$	$\bar{N}_\mu \approx 41.9$	$\bar{N}_\mu \approx 62.1$

Extrinsic information transfer (EXIT) charts [58] have been widely regarded as a valuable tool for analyzing iterative reception processes. Building on this notion, we adopt a similar approach, but without attempting to separate the intrinsic information represented by the photon count vector $\hat{N}$ in (28) from the pure extrinsic information generated by the quantum data detector in Figure 10. Let $f^{([L_{\text{in}}]_i)}(\xi|b_i)$ denote the conditional probability density function (pdf) of the i th component $[L_{\text{in}}]_i$ of the input vector L_{in} , cf. (81), given the transmitted codebit $b_i \in \mathbb{F}_2$, $i \in \mathbb{B}$. Similarly, let $f^{([L_{\text{e}}]_i)}(\xi|b_i)$ denote the conditional pdf of the i th component $[L_{\text{e}}]_i$ of the extrinsic soft output vector denoted by L_{e} , given the transmitted codebit $b_i \in \mathbb{F}_2$, $i \in \mathbb{B}$. The conditional pdfs $f^{([L_{\text{in}}]_i)}(\xi|b_i)$ and $f^{([L_{\text{e}}]_i)}(\xi|b_i)$ are approximated via histograms obtained from simulations.

Following [58, Eqs. (12) and (19), pp. 1728f.] and using these approximated conditional pdfs $f^{([L_{\text{in}}]_i)}(\xi|b_i)$ and $f^{([L_{\text{e}}]_i)}(\xi|b_i)$, we estimate the mutual information between the transmitted codebits $b_i \in \mathbb{F}_2$, $i \in \mathbb{B}$, supposed to assume the values 0 and 1 with the same a-priori probability 1/2, and the components $[L_{\text{in}}]_i$ of the input vector L_{in} , cf. (81), as given by

$$I_{\text{QDD}} = \frac{1}{2} \sum_{b_i=0}^1 \int_{-\infty}^{+\infty} d\xi f^{([L_{\text{in}}]_i)}(\xi|b_i) \cdot \log_2 \left\{ \frac{2 \cdot f^{([L_{\text{in}}]_i)}(\xi|b_i)}{f^{([L_{\text{in}}]_i)}(\xi|b_i=0) + f^{([L_{\text{in}}]_i)}(\xi|b_i=1)} \right\} \quad (96)$$

as well as the mutual information between the transmitted codebits $b_i \in \mathbb{F}_2$, $i \in \mathbb{B}$, and the components $[L_e]_i$ of the extrinsic soft output vector denoted by L_e , given by

$$I_{PC} = \frac{1}{2} \sum_{b_i=0}^1 \int_{-\infty}^{+\infty} d\xi f^{([L_e]_i)}(\xi|b_i) \cdot \log_2 \left\{ \frac{2 \cdot f^{([L_e]_i)}(\xi|b_i)}{f^{([L_e]_i)}(\xi|b_i=0) + f^{([L_e]_i)}(\xi|b_i=1)} \right\} \quad (97)$$

for each iteration of the turbo equalization.

Figure 37 shows simulation results of the estimated mutual information I_{QDD} , cf. (96), versus the estimated mutual information I_{PC} , cf. (97), in the case of six different values of $\bar{N}_\mu$. Like in the original EXIT charts, each horizontal line refers to a distinct iteration of the turbo equalization, the leftmost line associated with a particular $\bar{N}_\mu$ referring to the respective first iteration. The multipath quantum channel CIR#5 is considered and the average number $\bar{N}_\alpha$ of thermal photons is equal to 0. The Arikan-style polar code with the generator matrix of (72) and the associated polar sequence generated by Arikan's Bhattacharyya parameter $Z(W_N^{(i)})$ based approach setting out from the binary erasure channel with erasure probability ϵ equal to 0.5 is used. As usual for the simulations discussed in this Sect. 4.4, 24 CRC bits generated with polynomial "24C" are applied for the termination control of the turbo equalization. The code dimension K is equal to 88 and the code length N is equal to 128, yielding the code rate R_c equal to $(88 - 24)/128$, i.e. 0.5. SC decoding and log-MAP quantum data detection are used. Also, block interleaving according to the illustration in Sect. 3.3 is deployed.

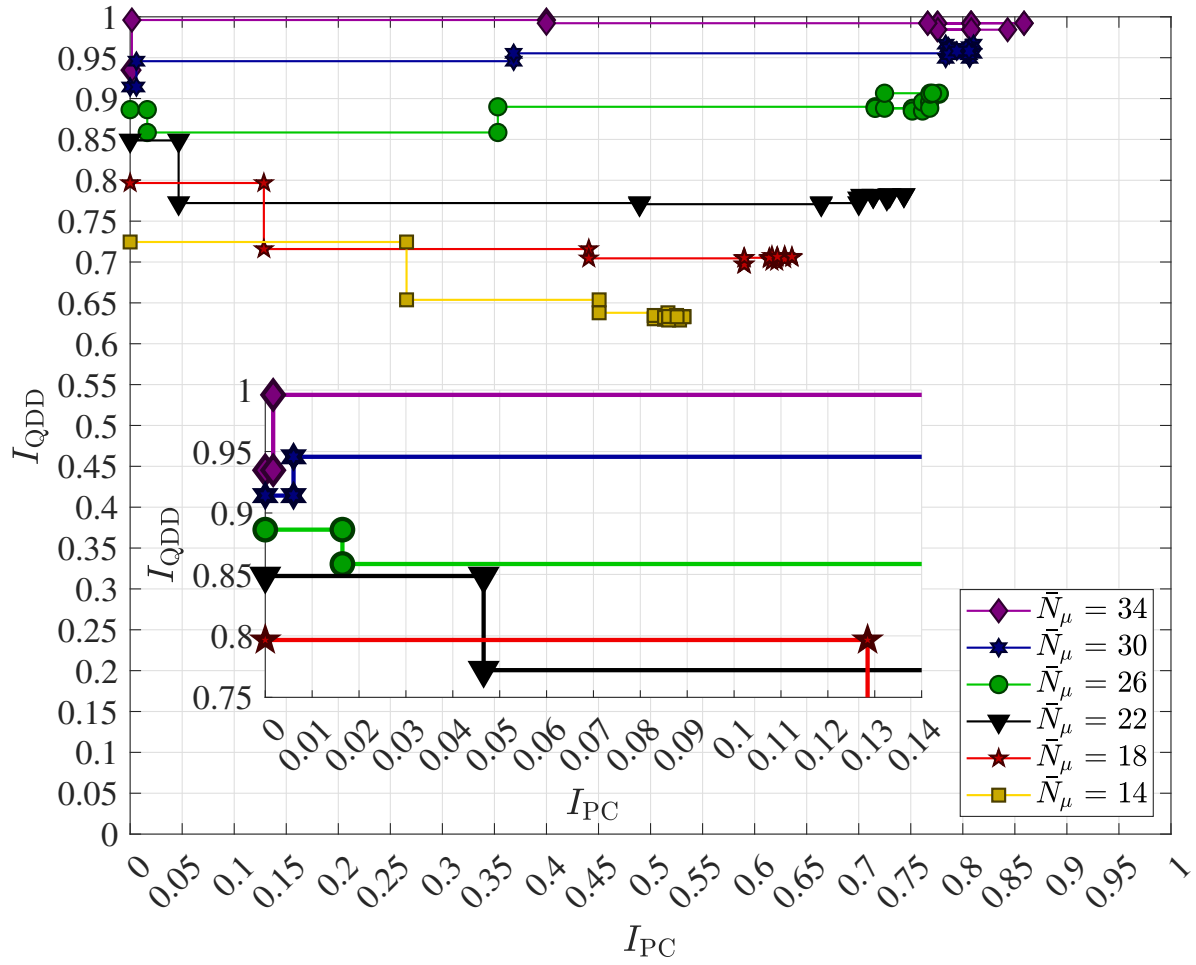


Figure 37. Simulation results of the estimated mutual information I_{QDD} , cf. (96), versus the estimated mutual information I_{PC} , cf. (97); each horizontal line refers to a distinct iteration of the turbo equalization; the multipath quantum channel CIR#5 is considered and the average number $\bar{N}_\mu$ of thermal photons is equal to 0; the Arikan-style polar code with the generator matrix of (72) and the associated polar sequence generated by Arikan's Bhattacharyya parameter $Z(W_N^{(i)})$ based approach setting out from the binary erasure channel with erasure probability $\varepsilon = 0.5$ is used; 24 CRC bits generated with polynomial "24C" are applied for the termination control of the turbo equalization; code dimension $K = 88$, code length $N = 128$, code rate $R_c = (88 - 24)/128 = 0.5$; log-MAP quantum data detection and SC decoding are used; block interleaving according to the illustration in Sect. 3.3 is deployed.

Obviously I_{PC} increases from 0 to positive values during the turbo equalization process. The maximum increase of I_{PC} is greater for high values of $\bar{N}_\mu$ than for low values of $\bar{N}_\mu$. Furthermore, after about the first three iterations, the change of I_{PC} depicted in Figure 37 diminishes when further iterations occur. Remarkably, I_{QDD} sets out from high values, greater than 0.7 in Figure 37, and the change of I_{QDD} during the turbo equalization is rather small. Similar observations for turbo equalization applied in other transmission schemes are found in e.g. [59, Fig. 3, p. 2; Fig. 6, p. 4]. Also, it seems that improvements of I_{QDD} are only possible for high values of $\bar{N}_\mu$, higher than 22 in Figure 37.

5. Conclusions

Future wireless communication systems, such as 6G and beyond, will operate at carrier frequencies above 100 GHz and will therefore increasingly need to employ optical transmission technologies. Motivated by this certainty, the authors have explored concepts in wireless quantum optical communications and demonstrated that such concepts are well-suited for implementation within the physical

layer. Of particular interest is the suitability of sophisticated channel coding, especially polar coding, in combination with such wireless quantum optical communication concepts.

To improve clarity and provide a foundation for the later discussions on polar codes, the authors first presented a brief overview of optimum soft output quantum data detection, a novel near-optimum soft output quantum data detection based on Kohonen maps, introduced here for the first time, and a likewise novel suboptimum soft output quantum data detection scheme using state reduction, also introduced here for the first time. Following a concise explanation of the fundamental differences between Arikan-style polar codes and those used in 3GPP 5G/NR, the authors describe a novel design method for generating the polar sequence, introduced here for the first time. They also discuss turbo equalization, including a detailed and explicit method for computing the soft output vector L_c associated with the codebits as well as the novel fully analytical generation of the extrinsic soft output vector L_e , without resorting to any time-consuming optimization procedures.

Finally, the achievable transmission quality when using polar codes in combination with wireless quantum optical communication concepts is thoroughly investigated and discussed in terms of BER P_b and BLER P_{block} in numerous scenarios. The studies consistently demonstrate the feasibility of the mentioned combination and confirm the long-term viability of suitably designed systems.

The authors point out that the laser as a light source can also be replaced by thermal light sources that produce chaotic light. The explanations presented here can be readily adapted. The authors intend to demonstrate this elsewhere.

The authors also make no secret of the fact that topics often associated with quantum communications, such as entanglement and quantum teleportation, were not explicitly considered. The above explanations have shown that these aspects of quantum physics are not fundamental prerequisites for the proper functioning of the considered concept of wireless quantum optical communication technology, and can therefore be added later as needed.

Furthermore, the authors did not address so-called “nonclassical” light sources, which, for example, produce squeezed light through the use of nonlinear optics. The authors are concerned with presenting a simple, robust, and yet functional technology. Due to the effort involved in generating squeezed light alone, it is by no means supporting this asset.

Based on the experience gathered over several years, some of which is presented here, the authors are confident that future analyses will help turn their conviction into a broadly shared certainty among many engineers. Therefore, the authors are convinced that the path they have taken will lead to both robust and economically viable solutions.

References

1. Jung, P.; Dini, K.: Frequency dependence of the ohmic resistance in the sixth generation terahertz band. Proceedings of the IEEE International Mediterranean Conference on Communications and Networking (MeditCom), Nice, France, 2025.
2. Rietschel, H.; Esdorn, H.: Raumklimatechnik — Band 1: Grundlagen. Sixteenth edition, Heidelberg: Springer, 2008.
3. Deutsches Institut für Normung (DIN): Strahlungsphysik im optischen Bereich und Lichttechnik; Benennung der Wellenlängenbereiche. DIN 5031 Part 7, January 1984.
4. Arute, F. et al.: Quantum supremacy using a programmable superconducting processor. *Nature*, vol. 574 (2019), pp. 505-511.
5. Cariolaro, G.: Quantum communications. Cham: Springer, 2015.
6. Helstrom, C.W.; Liu, J.W.S.; Gordon, J.P.: Quantum-mechanical communication theory. Proceedings of the IEEE, vol. 58 (1970) no. 10, pp. 1578-1598.
7. Jung, P.: Einführung in die Quantenkommunikation. Düren: Shaker, 2022.
8. Jung, P.: Advanced mobile communications — sophisticated channel codes. Berlin: Walter de Gruyter, 2024.
9. Helstrom, C.W.: Quantum detection and estimation theory. New York: Academic Press, 1976.
10. ISO/IEC/IEEE 8802-11: Telecommunications and information exchange between systems — specific requirements for local and metropolitan area networks — part 11: Wireless LAN medium access control (MAC) and physical layer (PHY) specifications. Third edition, July 2022.

11. IEEE Std 802.11bb-2023: Telecommunications and information exchange between systems — local and metropolitan area networks — specific requirements — part 11: Wireless LAN medium access control (MAC) and physical layer (PHY) specifications — amendment 6: Light communications. June 2023.
12. Jung, P.; Bruck, G.H.: Transmitter and receiver for transceiving optical signals. European Patent Specification EP 2 478 653 B1 (filed 17 Sept. 2009), 23 July 2014.
13. Shehata, M.; Wang, Y.; He, J.; Kandeepan, S.; Wang, K.: Optical and terahertz wireless technologies: The race to 6G communications. *IEEE Wireless Communications*, vol. 30 (2023) no. 5, pp. 10-18.
14. Schlosshauer, M.: Decoherence and the quantum-to-classical transition. Berlin: Springer, 2007.
15. Buchleitner, A.; Viviescas, C.; Tiersch, M. (Eds.): Entanglement and decoherence — foundations and modern trends. *Lecture Notes in Physics* 768, Berlin: Springer, 2009.
16. Joos, E.; Zeh, H.D.; Kiefer, C.; Giulini, D.; Kupsch, J.; Stamatescu, I.-O.: Decoherence and the appearance of a classical world in quantum theory. Second edition, Berlin: Springer, 2010.
17. Jung, P.: Advanced mobile communications — inner physical layer transceiver. Berlin: Walter de Gruyter, 2024.
18. Bello, P.A.: Characterization of randomly time-variant linear channels. *IEEE Transactions on Communications Systems*, vol. 11 (1963) no. 4, pp. 360-393.
19. Jung, P.; Bruck, G.H.: Wireless quantum optical communications with uncorrelated scattering multipath reception. *IET Electronics Letters*, vol. 60 (Sept. 2024) no. 17, pp. 1-3.
20. Jung, P.; Dini, K.; Almujaheed, H.: Applying turbo codes to wireless quantum optical communications via multipath channels causing intersymbol interference. *Proceedings of the 2025 IEEE International Conference on Quantum Communications, Networking, and Computing (QCNC)*, Nara, Japan, pp. 24-31, 2025.
21. Jung, P.; Dini, K.; Almujaheed, H.: Polar encoded wireless quantum optical communications with intersymbol interference caused by multipath propagation. *Proceedings of the 2025 IEEE International Conference on Communications (ICC)*, Montreal, Canada, 2025.
22. Abdel Rehim, F.; Jung, P.: Kohonen maps based near optimum sequence detection. *Proceedings of the IEEE International Mediterranean Conference on Communications and Networking (MeditCom)*, Nice, France, 2025.
23. Dini, K.; Jung, P.; Almujaheed, H.: Connecting polar codes and 6G short range wireless quantum optical communications. *Proceedings of the IEEE International Mediterranean Conference on Communications and Networking (MeditCom)*, Nice, France, 2025.
24. Gradshteyn, I.S.; Ryzhik, I.M.: Table of integrals, series, and products. Eighth edition, Amsterdam: Elsevier, 2015.
25. Schwabl, F.: *Quantenmechanik — eine Einführung*, Band 1. Seventh edition, Berlin: Springer, 2007.
26. Hoversten, E.V.; Harger, R.O.; Halme, S.J.: Communication theory for the turbulent atmosphere. *Proceedings of the IEEE*, vol. 58 (1970) no. 10, pp. 1626-1650.
27. Sauer-Greff, W.: Optimale und suboptimale Empfänger für verzerrte und gestörte Datensignale unter besonderer Berücksichtigung der aufwandsreduzierten Detektion mittels M-Algorithmus. Doctoral Dissertation, FB Elektrotechnik, Universität Kaiserslautern, Germany, 1989.
28. Forney, D.G.: Maximum-likelihood sequence estimation of digital sequences in the presence of intersymbol interference. *IEEE Transactions on Information Theory*, vol. IT-18 (1972) no. 3, pp. 363-378.
29. Gerry, C.; Knight, P.: *Introductory quantum optics*. Cambridge: Cambridge University Press, 2005.
30. Shih, Y.: *An introduction to quantum optics*. Boca Raton: CRC Press, 2021.
31. Takahashi, H.: Information theory of quantum-mechanical channels. In: *Advances in Communication Systems*, vol. 1 (1965), pp. 227-310.
32. Henze, N.: *Stochastik für Einsteiger*. Eleventh edition, Wiesbaden: Springer, 2017.
33. Jung, P.: *Einführung in die Quanteninformationstheorie*. Düren: Shaker, 2022.
34. Kolmogorov, A.N.: *Foundations of the theory of probability*. Mineola: Dover, 2018.
35. Jung, P.: *Analyse und Entwurf digitaler Mobilfunksysteme*. Stuttgart: Teubner, 1997.
36. Bahl, L.R.; Cocke, J.; Jelinek, F.; Raviv, J.: Optimal decoding of linear codes for minimizing symbol error rate. *IEEE Transactions on Information Theory*, vol. 20 (1974) no. 2, pp. 284-287.
37. Hagenauer, J.: Source-controlled channel decoding. *IEEE Transactions on Communications*, vol. 43 (1995) no. 9, pp. 2449-2457.
38. Kohonen, T.: *Self-organizing maps*. Third edition, Berlin: Springer, 2001.

39. Arikan, E.: Channel polarization: A method for constructing capacity-achieving codes for symmetric binary-input memoryless channels. *IEEE Transactions on Information Theory*, vol. 55 (2009) no. 7, pp. 3051-3073.
40. Plotkin, M.: Binary codes with specified minimum distance. *IRE Transactions on Information Theory*, vol. 6 (1960) no. 4, pp. 445-450.
41. 3GPP TS 38.212 V17.5.0 (2023-03): Multiplexing and channel coding. March 2023.
42. Zhou, Y.; Li, R.; Zhang, H.; Luo, H.; Wang, J.: Polarization weight family methods for polar code construction. *Proceedings of the 2018 IEEE 87th Vehicular Technology Conference (VTC Spring)*, Porto, Portugal, 2018.
43. Mori, R.; Tanaka, T.: Performance and construction of polar codes on symmetric binary-input memoryless channels. *Proceedings of the 2009 IEEE International Symposium on Information Theory*, Seoul, South Korea, 2009.
44. Hensel, K.: Über eine neue Begründung der Theorie der algebraischen Zahlen. *Jahresbericht der Deutschen Mathematiker-Vereinigung*, vol. 6 (1897), pp. 83-88.
45. Hensel, K.: *Zahlentheorie*. Berlin: G.J. Göschen'sche Verlagshandlung G.m.b.H., 1913.
46. Koblitz, N.: *p-adic numbers, p-adic analysis, and zeta-functions*. Second edition, New York: Springer, 1984.
47. Douillard, C.; Jezequel, M.; Berrou, C.; Picart, A.; Didier, P.: Iterative correction of intersymbol interference: Turbo-equalization. *European Transactions on Telecommunications*, vol. 6 (1995) no. 5, pp. 507-512.
48. Hagenauer, J.: Source-controlled channel decoding. *Proceedings of the 1994 IEEE International Symposium on Information Theory*, Trondheim, Norway, p. 165, 1994.
49. Holma, H.; Toskala, A.; Nakamura, T.: *5G technology — 3GPP evolution to 5G-Advanced*. Second edition, Chichester: Wiley, 2020.
50. Fayyaz, U.U.; Barry, J.R.: Low-complexity soft-output decoding of polar codes. *IEEE Journal on Selected Areas in Communications*, vol. 32 (2014) no. 5, pp. 958-966.
51. Jung, P.: *Mobilkommunikation — anspruchsvolle Kanalcodes*. Berlin: Walter de Gruyter, 2025.
52. Hagenauer, J.: The turbo principle: Tutorial introduction and state of the art. *Proceedings of the International Symposium on Turbo Codes and Related Topics*, Brest, France, pp. 1-11, 1997.
53. Berrou, C.; Glavieux, A.; Thitimajshima, P.: Near Shannon limit error-correcting coding and decoding: Turbo-codes (1). *Proceedings of the IEEE International Conference on Communications (ICC'93)*, Geneva, Switzerland, pp. 1064-1070, 1993.
54. Eltohamy, A.; Hofmann, C.A.; Knopp, A.; Korb, M.: Turbo receiver for short-block-length polar-coded direct-access satellite-IoT systems. *Proceedings of the IEEE International Mediterranean Conference on Communications and Networking (MeditCom)*, Madrid, Spain, pp. 143-148, 2024.
55. Jung, P.: *Mobilkommunikation — innere physikalische Schicht*. Berlin: Walter de Gruyter, 2025.
56. Pätzold, M.: *Mobile radio channels*. Second edition, Chichester: Wiley, 2012.
57. Failli, M. (chairman): *Digital land mobile radio communications*. European Cooperation in Science and Technology Action 207 (COST 207), Brussels/Luxemburg: ECSC-EEC-EAEC, 1989.
58. ten Brink, S.: Convergence behavior of iteratively decoded parallel concatenated codes. *IEEE Transactions on Communications*, vol. 49 (2001) no. 10, pp. 1727-1737.
59. Schorr, T.; Matuszak, M.; Sauer-Greff, W.; Urbansky, R.: LDPC code optimization for diversity reception and turbo equalization. *Proceedings of the IEEE International Mediterranean Conference on Signal Processing Algorithms, Architectures, Arrangements, and Applications (SPA 2007)*, Poznan, Poland, pp. 63-68, 2007.

Disclaimer/Publisher's Note: The statements, opinions and data contained in all publications are solely those of the individual author(s) and contributor(s) and not of MDPI and/or the editor(s). MDPI and/or the editor(s) disclaim responsibility for any injury to people or property resulting from any ideas, methods, instructions or products referred to in the content.