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[Abhirup Chatterjee](#) *

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Article

Analysis of Gauge Invariant Geometric Phase for Optical Systems Undergoing Depolarization Modeled by SU(3) Polarization Scheme

Abhirup Chatterjee

Department of Physics, Indian Institute of Technology Hyderabad; ph24resch11005@iith.ac.in; chattopadhyayabhirup65@gmail.com

Abstract

Geometric Phase in Quantum Mechanics is generally formulated entirely in terms of geometric structure of the Complex Hilbert Space. We will exploit this fact in case of mixed states for three level open systems undergoing depolarization using the eight dimensional Poincare sphere and non unit vector rays in $\mathcal{H}^3$ within the limit of pure state approach may be found to be in agreement with the Pancharatnam Phase, Berry Phase and Aharonov-Anandan Phase. We will consider the Depolarization of the Three dimensional fields by introducing the stokes parameters and by redefining the definition of Degree of Polarization using the SU(3) algebra to describe the quantum light depolarization of Non-paraxial beams by minimal reservoir coupling models modeled by Lindblad Master equations and connect the solution of that with the geometric phase for systems undergoing depolarization.

Keywords: Geometric Phase; SU(3) depolarization; Lindblad master equation; pure dephasing process

1. Introduction

A Pure Quantum State retains a memory of its evolution in terms of Geometric Phase when it undergoes an evolution in the Parameter Space. The phase factor has its origin which is purely geometric in nature and it can arise even under the most general conditions where the system is undergoing an non-unitary evolution corresponding to the Hamiltonian which does not satisfy the criterion of adiabaticity and cyclicity. Although when geometric phase comes into the picture we talk about Berry's [1] where he assumed the evolution to be cyclic in some parameter space and the Hamiltonian obeys the adiabaticity and cyclicity condition but Pancharatnam's experimental work [2] on interference gives us the idea about the existence of such phase factor which is known as pancharatnam connection which states that if we consider any three mutually non-orthogonal vectors in a certain Hilbert space say $|\psi_1\rangle$, $|\psi_2\rangle$ and $|\psi_3\rangle$ so that, $\langle\psi_m|\psi_n\rangle \neq 0 \forall m, n = 1, 2, 3$ then if $|\psi_1\rangle$ is in phase with $|\psi_2\rangle$ and $|\psi_2\rangle$ is in phase with $|\psi_3\rangle$ then $|\psi_1\rangle$ may not be in phase with $|\psi_3\rangle$. The relative phase difference between any two non-orthogonal vectors $|\psi\rangle$ and $|\phi\rangle$ is defined as follows. If $\langle\psi|\phi\rangle = re^{i\theta}$ then, θ is the relative phase difference according to Pancharatnam. After Berry's discovery of the Geometric phase, there had been plenty of attempts for the further generalization of the Geometric phase, the first step towards the generalization was due to Aharonov-Anandan [3], who lifted up the condition of adiabaticity. Samuel and Bhandari [4] further generalised the notion of geometric phase and showed that the Geometric Phase can arise under the most general conditions lifting up the condition of cyclicity. Mukunda and Simon [5] described the Geometric Phase from the Kinematic approach and showed that Geometric phase can arise for a system undergoing and Non-unitary evolution even without the condition of adiabaticity and cyclicity.

Polarization is a fundamental ingredient of light, both in the quantum and in the classical domains. In the quantum regime this variable has been crucial in order to demonstrate experimentally fundamental properties and applications of the quantum theory such as entanglement, complementarity,

quantum cryptography.

The concept of the Polarization is broadly understood in the context of two mode fields for both single mode and multi mode fields where we can define the definite direction of propagation and the depolarization has been modelled through theoretical models coupling the field with the bath with different interaction Hamiltonian's.

Although the electric field is by definition a three dimensional (3D) magnitude, the polarization of strictly harmonic classical waves is locally a two-dimensional (2D) phenomenon [6], since the electric field at each spatial point describes an ellipse contained in a plane. Nevertheless, the plane of the ellipse may vary from point to point so that a three-dimensional analysis may be useful, especially for nonparaxial beams without a well-defined propagation direction.

Moreover, quantum fluctuations affect the three field components even when they are in the vacuum state, so that for every field state the quantum electric field varies unavoidably in a 3D region. Therefore, in the quantum domain polarization is always a 3D phenomenon.

Although the spin of the photon is one, it is found sufficient here, to use 2×2 matrices [7] in view of the fact that light is a transverse wave and consequently the longitudinal state of polarization is physically absent. However, in dealing with interactions between charged particles, it is well-known from quantum electrodynamics (Feynman 1962) that longitudinal state is also involved along with the two transverse states for the photons; in fact, the well-known Coulomb law between two charged particles is the result of an exchange of a longitudinal photon. So in this kind of situation one has to deal with the three mode polarization i.e. the polarization formulation for the three dimensional fields.

Here in the first section we will introduce the SU(3) polarization aspects for the monochromatic three dimensional fields along with the definition of the degree of polarization for the 3D case [8]. In the second section we will introduce the Models for the description of Depolarization Dynamics for the monochromatic plane polarized electromagnetic beam having three polarization modes. Specifically we will highlight the features in case of single photon states. In the third section we will introduce the Geometric Phase for three level quantum system and will do the calculations including the effect of Depolarization.

2. SU(3) Polarization Picture, Stokes operators and Degree of Polarization

Let us consider a plane polarized light beam which doesn't have a well defined propagation direction. Then the electric field associated with the electromagnetic beam can be treated as a three dimensional vector. For such non-paraxial beams the standard definition of the stokes operators for the two mode fields can be generalized to the three mode fields in the form,

$$\hat{S}_i = \hat{\mathbf{a}}^\dagger \lambda_i \hat{\mathbf{a}}, i = 0, 1, \dots, 8 \quad (2.1)$$

Where, we have associated three sets of bosonic creation and annihilation operators $\{\hat{a}_i, \hat{a}_i^\dagger\}$ for $i=1,2,3$ with the three polarization modes so that $\hat{\mathbf{a}} = \begin{pmatrix} \hat{a}_1 \\ \hat{a}_2 \\ \hat{a}_3 \end{pmatrix}$ and $\hat{\mathbf{a}}^\dagger = \begin{pmatrix} \hat{a}_1^\dagger & \hat{a}_2^\dagger & \hat{a}_3^\dagger \end{pmatrix}$. λ_i for $i=1,2,\dots,8$ are the standard Gell-Mann matrices with λ_0 being the 3×3 Unit Matrix given below,

$$\lambda_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}; \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}; \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}; \lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

$$\lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}; \lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}; \lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}; \lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

Gell-Mann Matrices $\lambda_i, i=1,2,\dots,8$ are the Generators of the SU(3) group which are traceless and hermitian and they follow some properties listed below,

$$\begin{aligned} \text{Tr}[\lambda_i] &= 0 \\ [\lambda_r, \lambda_s] &= 2if_{rst}\lambda_t \\ f_{123} &= 1, f_{458} = f_{678} = \frac{\sqrt{3}}{2}, f_{147} = f_{246} = f_{257} = f_{345} = f_{516} = f_{637} = \frac{1}{2}; \\ \{\lambda_r, \lambda_s\} &= \frac{4}{3}\delta_{rs} + 2d_{rst}\lambda_t \\ d_{118} &= d_{228} = d_{338} = -d_{888} = \frac{1}{\sqrt{3}}, d_{448} = d_{558} = d_{668} = d_{778} = -\frac{1}{2\sqrt{3}}, \\ d_{146} &= d_{157} = -d_{247} = d_{256} = d_{344} = d_{355} = -d_{366} = -d_{377} = \frac{1}{2}; \\ \text{Tr}[\lambda_r\lambda_s] &= 2\delta_{rs} \end{aligned} \quad (2.2)$$

After few steps of matrix multiplication we obtain,

$$\begin{aligned} \hat{S}_1 &= (\hat{a}_1^\dagger \hat{a}_2 + \hat{a}_2^\dagger \hat{a}_1) \\ \hat{S}_2 &= i(\hat{a}_2^\dagger \hat{a}_1 - \hat{a}_1^\dagger \hat{a}_2) \\ \hat{S}_3 &= (\hat{a}_1^\dagger \hat{a}_1 - \hat{a}_2^\dagger \hat{a}_2) \\ \hat{S}_4 &= (\hat{a}_1^\dagger \hat{a}_3 + \hat{a}_3^\dagger \hat{a}_1) \\ \hat{S}_5 &= i(\hat{a}_3^\dagger \hat{a}_1 - \hat{a}_1^\dagger \hat{a}_3) \\ \hat{S}_6 &= (\hat{a}_3^\dagger \hat{a}_2 + \hat{a}_2^\dagger \hat{a}_3) \\ \hat{S}_7 &= i(\hat{a}_3^\dagger \hat{a}_2 - \hat{a}_2^\dagger \hat{a}_3) \\ \hat{S}_8 &= \frac{1}{\sqrt{3}}(\hat{a}_1^\dagger \hat{a}_1 + \hat{a}_2^\dagger \hat{a}_2 - 2\hat{a}_3^\dagger \hat{a}_3) \\ \hat{S}_0 &= (\hat{a}_1^\dagger \hat{a}_1 + \hat{a}_2^\dagger \hat{a}_2 + \hat{a}_3^\dagger \hat{a}_3) \end{aligned} \quad (2.3)$$

In general we can write,

$$\langle \hat{S}_i \rangle = \text{Tr}[\Phi \sigma_i]$$

for $i=1,2,3$. where, Φ be the 3×3 coherence matrix or polarization matrix with elements $\Phi_{kl} = \langle \hat{a}_l^\dagger \hat{a}_k \rangle$, ($k,l=1,2,3$).

Upon inverting the above relation the coherence matrix matrix is found in terms of the expectation values of expectation values of stokes operators as follows,

$$\Phi_{3 \times 3} = \frac{1}{2} \begin{pmatrix} \frac{2}{3} \langle \hat{S}_0 \rangle + \langle \hat{S}_3 \rangle + \frac{1}{\sqrt{3}} \langle \hat{S}_8 \rangle & \langle \hat{S}_1 \rangle - i \langle \hat{S}_2 \rangle & \langle \hat{S}_4 \rangle - i \langle \hat{S}_5 \rangle \\ \langle \hat{S}_1 \rangle + i \langle \hat{S}_2 \rangle & \frac{2}{3} \langle \hat{S}_0 \rangle - \langle \hat{S}_3 \rangle + \frac{1}{\sqrt{3}} \langle \hat{S}_8 \rangle & \langle \hat{S}_6 \rangle - i \langle \hat{S}_7 \rangle \\ \langle \hat{S}_4 \rangle + i \langle \hat{S}_5 \rangle & \langle \hat{S}_6 \rangle + i \langle \hat{S}_7 \rangle & \frac{2}{3} \langle \hat{S}_0 \rangle - \frac{2}{\sqrt{3}} \langle \hat{S}_8 \rangle \end{pmatrix} \quad (2.4)$$

The Degree of polarisation in the SU(3) polarisation picture is defined as follows,

$$\mathbb{P} = \frac{\sqrt{3}}{2} \frac{\sqrt{\sum_{i=1}^8 \langle \hat{S}_i \rangle^2}}{\langle \hat{S}_0 \rangle} \quad (2.5)$$

The commutation and anti-commutation relations obeyed by the Stokes operator resembles the lie algebra of SU(3) Group given as follows,

$$\begin{aligned} [\hat{S}_r, \hat{S}_m] &= 2if_{rmt}\hat{S}_t \\ \{\hat{S}_r, \hat{S}_m\} &= \frac{4}{3}\delta_{rm} + 2d_{rmt}\hat{S}_t \\ \hat{S}_r\hat{S}_m &= \frac{2}{3}\delta_{rm}\hat{I} + if_{rmt}\hat{S}_t + d_{rmt}\hat{S}_t \end{aligned} \quad (2.6)$$

Where, f_{rmt}, d_{rmt} are the usual anti-symmetric and symmetric structure constant of the SU(3) group respectively. For the finite dimensional matrix representation of the operators the standard three mode Fock-Basis will be used which is the simultaneous eigenstate of the operators $\hat{S}_0, \hat{S}_3, \hat{S}_8$.

We have,

$$|n_1, n_2, n_3\rangle \equiv |n_1\rangle_1 \otimes |n_2\rangle_2 \otimes |n_3\rangle_3 \quad (2.7)$$

with, $n_1 + n_2 + n_3 = N$. Where, N be the total number of photons which are distributed in between three polarisation modes labelled by $\{1, 2, 3\}$ such that there are n_1, n_2, n_3 number of photons respectively in the polarisation modes $\{1, 2, 3\}$ respectively. We have the following relations,

$$\begin{aligned} \hat{S}_0 |n_1, n_2, n_3\rangle &= (n_1 + n_2 + n_3) |n_1, n_2, n_3\rangle \\ \hat{S}_3 |n_1, n_2, n_3\rangle &= (n_1 - n_2) |n_1, n_2, n_3\rangle \\ \hat{S}_8 |n_1, n_2, n_3\rangle &= \frac{1}{\sqrt{3}}(n_1 + n_2 - 2n_3) |n_1, n_2, n_3\rangle \end{aligned} \quad (2.8)$$

3. SU(3) Depolarisation Dynamics for single mode fields

The term depolarization has come to mean the effective decrease in the degree of polarization of a light traversing an optical system which is due to the interaction of the electromagnetic beam with the surroundings and the result is an effective anisotropy which leads to the decorrelation of the phase associated with the electric field vector. In the quantum theory the depolarization can be broadly understood in terms of decoherence which corresponds to the loss of coherence in the quantum system due to the interaction with the surroundings i.e. the appearance of uncontrollable quantum correlations due to the dissipation. When the system interacts with the surrounding there will be an exchange of energy due to the interaction which causes dissipation and gives rise to decoherence. Here, in the present context we will consider the effect of decoherence in such a way so that the net exchange of energy between the system and the surroundings is very negligible which is also known as the pure dephasing dynamics.

3.1. Master Equation for the Pure Dephasing Dynamics

First we begin by focusing our attention on the single mode field which is traversing through some optical medium i.e. here the system is the single mode (monochromatic) polarized electromagnetic wave coupled to a bath system so that the total Hamiltonian for the global system (system+bath) can be written as the sum of the system Hamiltonian or the field Hamiltonian, the bath Hamiltonian and the interaction Hamiltonian between the field and the bath.

$$\hat{H} = \hat{H}_{field} + \hat{H}_{bath} + \hat{H}_{int} \quad (3.1.1)$$

The field Hamiltonian will be identical to the two dimensional isotropic harmonic oscillator Hamiltonian apart from the constant term $\hbar\omega$, where ω being the frequency of the electromagnetic wave so that we have,

$$\hat{H}_{field} = \hbar\omega \sum_{s=1}^3 \hat{a}_s^\dagger \hat{a}_s \quad (3.1.2)$$

$\hat{H}_{bath}$ describes the free evolution of the environment, we are not making any assumption regarding the type of bath actually we don't need the precise knowledge of the bath, we assume that the bath with which the field is coupled with is so large that its statistical properties remain unaffected due to the interaction with the system. In, the interaction picture of quantum mechanics if we consider the global density operator which can be represented as $\hat{\rho}_{Tot} = \hat{\rho}_{field} \otimes \hat{\rho}_B$ then the partial trace of the global density operator with respect to the reservoir or, the bath states will give the actual density operator for the system i.e. field. Then $\hat{\rho}_{field} = Tr_B[\hat{\rho}_{Tot}]$ Under the weak coupling limit the Born-Markov master equation leads to,

$$\dot{\hat{\rho}}(t) = -\frac{1}{\hbar^2} \int_0^\infty d\tau Tr_B \{ [\hat{H}_{int}(t), [\hat{H}_{int}(t-\tau), \hat{\rho}(t) \otimes \hat{\rho}_B]] \} \quad (3.1.3)$$

Here, Tr_B indicates the partial trace with respect to the bath states and $\hat{\rho}(t)$ being the reduced density operator for the field. As, we can see that the master equation does not depend on the specific choice of the bath or, it's Hamiltonian rather it only depends on the interaction Hamiltonian i.e. $\hat{H}_{int}$.

As we have anticipated earlier that we are interested to study the pure dephasing dynamics so that the amount of energy exchange between the system and the bath is negligible. In mathematical terms we can say that the Interaction Hamiltonian must commute with the system Hamiltonian i.e. $[\hat{H}_{field}, \hat{H}_{int}] = 0$ which implies no exchange of energy and only the phase changes.

One way to model the pure dephasing dynamics is by choosing the interaction Hamiltonian[9] of the following form so that it commutes with the field Hamiltonian.

$$\hat{H}_{int} = \hbar \sum_{\lambda} \sum_{s=1}^3 (\kappa_{\lambda} \Gamma_{\lambda} \hat{a}_s^{\dagger} \hat{a}_s + \kappa_{\lambda}^* \Gamma_{\lambda}^{\dagger} \hat{a}_s^{\dagger} \hat{a}_s) \quad (3.1.4)$$

Here, κ_{λ} denotes the coupling constant between the electromagnetic wave with the λ th bath mode and $\{\hat{\Gamma}_{\lambda}, \hat{\Gamma}_{\lambda}^{\dagger}\}$ denotes the bath quanta annihilation and creation operators describing the random bath. We will consider zero temperature bath which is tantamount to neglect the process of stimulated emission. With this choice of the interaction term we obtain Lindblad type Master equation[10] from [3.1.3] becomes,

$$\dot{\hat{\rho}}(t) = \sum_{s=1}^3 \frac{\gamma_s}{2} \mathcal{L}[\hat{a}_s^{\dagger} \hat{a}_s] \hat{\rho}(t) \quad (3.1.5)$$

In the above equation γ_s for $s = 1, 2, 3$ defines the depolarisation rates for each polarisation mode. The interaction Hamiltonian can be viewed as a scattering process, in which a bath quanta is absorbed or, emitted but the number of photons in each polarization is preserved. We will solve the master equation for the single photon system.

We have solved the above master equation for the single photon system. For the single photon system the three possible orthonormal basis vectors are,

$$|1, 0, 0\rangle \equiv |1\rangle_1 \otimes |0\rangle_2 \otimes |0\rangle_3, |0, 1, 0\rangle \equiv |1\rangle_0 \otimes |1\rangle_2 \otimes |0\rangle_3, |0, 0, 1\rangle \equiv |0\rangle_1 \otimes |0\rangle_2 \otimes |1\rangle_3$$

So, the Hilbert space is three dimensional and it can be treated as a three level quantum system. Let,

the density matrix being $\hat{\rho}(t) = \begin{pmatrix} \rho_{11}(t) & \rho_{12}(t) & \rho_{13}(t) \\ \rho_{21}(t) & \rho_{22}(t) & \rho_{23}(t) \\ \rho_{31}(t) & \rho_{32}(t) & \rho_{33}(t) \end{pmatrix}$. Upon solving the Master equation for the single photon system we get,

$$\rho_{11}(t) = \rho_{11}(0) \quad (3.1.6)$$

$$\rho_{22}(t) = \rho_{22}(0) \quad (3.1.7)$$

$$\rho_{33}(t) = \rho_{33}(0) \quad (3.1.8)$$

$$\rho_{12}(t) = \rho_{12}(0) \exp\{-(\gamma_1 + \gamma_2)t/2\} \quad (3.1.9)$$

$$\rho_{21}(t) = \rho_{21}(0) \exp\{-(\gamma_1 + \gamma_2)t/2\} \quad (3.1.10)$$

$$\rho_{23}(t) = \rho_{23}(0) \exp\{-(\gamma_3 + \gamma_2)t/2\} \quad (3.1.11)$$

$$\rho_{32}(t) = \rho_{32}(0) \exp\{-(\gamma_3 + \gamma_2)t/2\} \quad (3.1.12)$$

$$\rho_{31}(t) = \rho_{31}(0) \exp\{-(\gamma_1 + \gamma_3)t\} \quad (3.1.13)$$

$$\rho_{13}(t) = \rho_{13}(0) \exp\{-(\gamma_1 + \gamma_3)t\} \quad (3.1.14)$$

4. Formalism of Geometric Phase for three level Quantum System

In order to introduce the Geometric phase first we address the Bloch sphere representation and its parametrisation for the three level system along with its formulation using the notion of state space and Ray Space.

4.1. Parametrisation of the Bloch Sphere for the Three Level System

For three level system the density matrix $\hat{\rho}$ for the three level system will be a 3×3 hermitian matrix and it can be written as a linear combination of 3×3 unit matrix and eight Gell Mann Matrices (λ_i), $i=1,2,...,8$. The density matrix can be written as,

$$\hat{\rho} = \frac{1}{3} \left(I + \frac{\sqrt{3}}{2} \vec{n} \cdot \hat{\lambda} \right) \quad (4.1.1)$$

The components of the Bloch vector ($\vec{n}$) can be written as,

$$n_i = \frac{\sqrt{3}}{2} \text{Tr}[\hat{\rho} \hat{\lambda}_i] = \frac{\sqrt{3}}{2} \langle \psi | \hat{\lambda}_i | \psi \rangle, i=1,2,...,8 \quad (4.1.2)$$

Where, $|\psi\rangle \in \mathcal{H}^3$. In case of three level system we need a 8D Bloch sphere which is represented by the set of eight parameters more specifically one radial parameter and seven angular parameters or, coordinates. Let, the set of parameters be $(r, \theta, \phi, \alpha, \beta, \gamma, \chi, \xi)$ with r being the radial coordinate and the others being the angular coordinates. It can be also viewed as the case that in order to designate the point on the surface of the sphere we require 2 angular coordinates respectively (θ, ϕ) in case of a 3D Bloch sphere here it requires seven angular coordinates to denote the position of a point on the surface of the eight dimensional Bloch Sphere i.e. $(\theta, \phi, \alpha, \beta, \gamma, \chi, \xi)$ respectively. The Bloch Sphere Parametrisation equations [11] are defined as follows,

$$\cos \theta = \frac{1}{\sqrt{3}} \left[1 - \frac{\sqrt{3}}{2r} (\rho_{11} + \rho_{22} - 2\rho_{33}) \right]^{\frac{1}{2}}, \quad (4.1.3a)$$

$$\cos \phi = \frac{1}{\sqrt{2}} \left[1 + \frac{\rho_{11} - \rho_{22}}{r \sin^2 \theta} \right]^{\frac{1}{2}}, \quad (4.1.3b)$$

$$\tan(\beta - \chi - \alpha + \gamma) = \frac{n_2}{n_1} = i \frac{\rho_{12} - \rho_{21}}{\rho_{12} + \rho_{21}}, \quad (4.1.3c)$$

$$\tan(\alpha - \gamma - \xi) = \frac{n_5}{n_4} = i \frac{\rho_{13} - \rho_{31}}{\rho_{13} + \rho_{31}}, \quad (4.1.3d)$$

$$\tan(\beta - \chi + \xi) = \frac{n_7}{n_6} = i \frac{\rho_{23} - \rho_{32}}{\rho_{23} + \rho_{32}}. \quad (4.1.3e)$$

The parametrisation relations are chosen in such a way that they satisfy the equation of the Bloch sphere. The next task is to express the Bloch vector components in terms of the Bloch Parameters which will satisfy the equation of the eight dimensional Bloch Sphere. Now, using the set of equations along with the Bloch Sphere Parametrisation equations we can express the Bloch vector components as follows,

$$n_1 = \sqrt{3}r \sin^2 \theta \sin \phi \cos \phi \cos (\beta - \chi - \alpha + \gamma) \quad (4.1.4a)$$

$$n_2 = \sqrt{3}r \sin^2 \theta \sin \phi \cos \phi \sin (\beta - \chi - \alpha + \gamma) \quad (4.1.4b)$$

$$n_3 = \sqrt{3}r \left\{ \frac{1}{2} (\cos^2 \phi - \sin^2 \phi) \right\} \sin^2 \theta \quad (4.1.4c)$$

$$n_4 = \sqrt{3}r \sin \theta \cos \theta \cos \phi \cos (\alpha - \gamma - \xi) \quad (4.1.4d)$$

and,

$$n_5 = -\sqrt{3}r \sin \theta \cos \theta \cos \phi \sin (\alpha - \gamma - \xi) \quad (4.1.5a)$$

$$n_6 = \sqrt{3}r \sin \theta \cos \theta \sin \phi \cos (\beta - \chi + \xi) \quad (4.1.5b)$$

$$n_7 = -\sqrt{3}r \sin \theta \cos \theta \sin \phi \sin (\beta - \chi + \xi) \quad (4.1.5c)$$

$$n_8 = \sqrt{3}r \frac{(1 - 3 \cos^2 \theta)}{2\sqrt{3}} \quad (4.1.5d)$$

the above set of equations given by satisfies the equation of the Bloch sphere $\sum_i n_i^2 = r^2 \leq 1$ and the components of the Bloch vector being real. Keeping an analogy with the result we have obtained in the case of two level quantum system that if the radius of the Bloch sphere is unity i.e. $r = 1$ we will call it a Poincare sphere and the three level system will be in the pure state. Thus the Bloch vector is pointed on the surface of the Bloch sphere. But if $r \leq 1$ then the Bloch vector $\mathbf{n}$ is in interior of this unit Poincare sphere. Therefore, the mixed states in the three level system may be identified with the interior points of this generalized unit Poincare sphere.

For the three level quantum system we need the non-unit state vector mapping i.e. the non-unit vector ray $|\psi\rangle$ will be a 3×1 column matrix which will obey the equation $n_i = \frac{\sqrt{3}}{2} \text{Tr}[\rho \lambda_i] = \frac{\sqrt{3}}{2} \langle \lambda_i \rangle = \frac{\sqrt{3}}{2} \langle \psi | \lambda_i | \psi \rangle$, giving back the Bloch vector components. For every bloch vector $(\vec{n})$ there exist an unique state vector $|\psi\rangle$ in $\mathcal{H}^3$. We begin with an ansatz that,

$$|\psi\rangle = \sqrt{r} \begin{pmatrix} e^{i(\alpha-\gamma)} \sin \theta \cos \phi \\ e^{i(\beta-\chi)} \sin \theta \sin \phi \\ e^{i\xi} \cos \theta \end{pmatrix} \quad (4.1.6)$$

It leads to the following equations after some steps of mathematical calculations.

$$n_1 = \sqrt{3}r \sin^2 \theta \sin \phi \cos \phi \cos (\beta - \chi - \alpha + \gamma) \quad (4.1.7)$$

$$n_2 = \sqrt{3}r \sin^2 \theta \sin \phi \cos \phi \sin (\beta - \chi - \alpha + \gamma) \quad (4.1.8)$$

$$n_3 = \sqrt{3}r \left\{ \frac{1}{2} (\cos^2 \phi - \sin^2 \phi) \right\} \sin^2 \theta \quad (4.1.9)$$

$$n_4 = \sqrt{3}r \sin \theta \cos \theta \cos \phi \cos (\alpha - \gamma - \xi) \quad (4.1.10)$$

$$n_5 = -\sqrt{3}r \sin \theta \cos \theta \cos \phi \sin (\alpha - \gamma - \xi) \quad (4.1.11)$$

$$n_6 = \sqrt{3}r \sin \theta \cos \theta \sin \phi \cos (\beta - \chi + \xi) \quad (4.1.12)$$

$$n_7 = -\sqrt{3}r \sin \theta \cos \theta \sin \phi \sin (\beta - \chi + \xi) \quad (4.1.13)$$

$$n_8 = \sqrt{3}r \frac{(1 - 3 \cos^2 \theta)}{2\sqrt{3}}. \quad (4.1.14)$$

5. Geometric Phase For Three Level Open Quantum System

Let us consider an open curve $\mathcal{C} = |\psi(t)\rangle$ defined in the manifold $\mathcal{B}$ which is also known as the state space and the curve is smoothly parameterised by some parameter t lies entirely inside the State space so that at each points on the curve corresponds to some $|\psi(t)\rangle$. Because of the mapping exist between State space and the Ray Space i.e. $\mathcal{B} \rightarrow \mathcal{R}$ the point on the curve $\mathcal{C}$ will be mapped to their corresponding density operators $\hat{\rho}(t) = |\psi(t)\rangle \langle \psi(t)| = \psi(t)\psi(t)^\dagger$ so that there will be an image curve in the Ray Space $\mathcal{R}$ defined by, $C = \{\hat{\rho}(t) = |\psi(t)\rangle \langle \psi(t)|\}$ on each point of the curve there is a density operator or, projection operator $\hat{\rho}(t) = \Pi |\psi(t)\rangle$ satisfying the conditions $\text{Tr}[\hat{\rho}(t)] = 1, \hat{\rho}^\dagger(t) = \hat{\rho}(t)$ and the inverse mapping leads to a lift of the curve C in the state space $\mathcal{B}$ which can be defined as,

$$\mathcal{C} \rightarrow C = |\psi(t)\rangle \rightarrow \hat{\rho}(t) = \Pi(|\psi(t)\rangle) = |\psi(t)\rangle \langle \psi(t)| \quad (5.1)$$

$$C \rightarrow \mathcal{C} = \{|\psi(t)\rangle = \Pi^{-1}\hat{\rho}(t)\} \quad (5.2)$$

Let the curve $\mathcal{C}$ describes the evolution of the quantum state $|\psi(t)\rangle$ parametrized by a smoothly varying parameter t although we can consider either an open curve or a closed curve. Let us divide the curve into small segments and the points of subdivisions are $t_0, t_1, t_2, \dots, t_N$. The state vectors at that points of subdivision are respectively $|\psi(t_0)\rangle, |\psi(t_1)\rangle, |\psi(t_2)\rangle, \dots, |\psi(t_N)\rangle$ or, in general $|\psi_i\rangle = |\psi(t = t_i)\rangle = |\psi(t_i)\rangle, i = 1, 2, 3, \dots, N$ and the corresponding density operators in the image curve be $\rho(t_i) = \rho(t = t_i) = |\psi(t_i)\rangle \langle \psi(t_i)|$ and $|\psi_i\rangle = |\psi(t = t_i)\rangle = \Pi^{-1}\rho(t_i) \in \mathcal{B}$. Then, each trajectory can be described by a discrete sequence of quantum states $\{|\psi_0\rangle, |\psi_1\rangle, |\psi_2\rangle, \dots, |\psi_N\rangle\}$. Now, in the continuum limit i.e. $\lim N \rightarrow \infty$ the geometric phase is given by,

$$\begin{aligned} \gamma_g &= -\mathcal{L}_{t_N \rightarrow \infty} \arg(\langle \psi_0 | \psi_1 \rangle \langle \psi_1 | \psi_2 \rangle \dots \langle \psi_{N-1} | \psi_N \rangle \langle \psi_N | \psi_0 \rangle) \\ &= \arg \langle \psi(t_0) | \psi(t) \rangle - \text{Im} \left(\int_{t_0}^t d\tau \frac{\langle \psi(\tau) | \frac{d}{d\tau} |\psi(\tau)\rangle}{\langle \psi(\tau) | \psi(\tau) \rangle} \right) \end{aligned} \quad (5.3)$$

Now, let us consider an interaction between the physical system and the surrounding, in this case we can't expect a cyclic evolution of the quantum system and as a result of the system is not undergoing a cyclic evolution the Bloch parameters, the components of the Bloch vector and the matrix elements of the density operator will be decaying function of time with the time evolution of the density operator of the system will be described by the Lindblad type of Master equation. The non-unit state vectors will also involve decaying exponent. When the system is isolated from the environment, it can be considered that the system is undergoing a quasicyclic evolution so that the total phase $\varphi_{tot} = \arg \langle \psi(t_0) | \psi(t) \rangle$ becomes 2π which can be dropped out in quantum computation as it's a constant term and in general we identify the geometric phase term by dropping the overall phase or, total phase and the geometric phase is found as the negative of the dynamic phase which becomes a closed integral over $\mathcal{C}$.

Thus we obtained the geometric phase under the quasicyclic evolution of the quantum system with $|\psi\rangle = \sqrt{r} \begin{pmatrix} e^{i(\alpha-\gamma)} \sin \theta \cos \phi \\ e^{i(\beta-\chi)} \sin \theta \sin \phi \\ e^{i\xi} \cos \theta \end{pmatrix}$ is given by,

$$\begin{aligned} \gamma_g(\mathcal{C}) &= -\Im \oint_{\mathcal{C}} \frac{\langle \psi | d | \psi \rangle}{\langle \psi | \psi \rangle} \\ &= -\oint_{\mathcal{C}} \left\{ \sin^2 \theta [d(\alpha - \gamma) \cos^2 \phi + d(\beta - \chi) \sin^2 \theta] + \cos^2 \theta d\xi \right\} \end{aligned} \quad (5.4)$$

Here, as a special case we have assumed the spherical symmetry due to which we have considered the variations of the Bloch parameters $\alpha, \gamma, \beta, \chi, \xi$ appearing in the exponents in $|\psi\rangle$ only with the other parameters fixed.

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