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Review

Problem Solving and Mathematical Modelling

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Abstract: Problem solving is one of the most important components of the human cognition that affects for ages the progress of the human society. Mathematical modelling is a special type of problem solving concerning problems related to science or everyday life situations. The present study is a review of the author's earlier works on problem solving and mathematical modelling from the scope of Education. Its real goal is that it presents in a systematic way and in a few pages only the results of many years research on the subject. This helps the reader to get a comprehensive idea about a very important topic belonging to the core of Mathematics Education, which is very useful to those wanting to study deeper the subject and get directions for further research in the area.

Keywords: mathematics education; problem solving; mathematical modelling; application-oriented teaching of mathematics

1. Introduction

Problem solving (PS) is one of the most important activities of the human cognition. Volumes of research have been written about the steps, the mechanisms and the difficulties of the PS process, which affects the everyday life from the time that humans appeared on the Earth.

Most authors ([1-4], etc.) agree that a problem could be considered as an obstacle to be overpassed in order to achieve a desired goal. A problem, however, is mainly characterized by the fact that you don't know exactly how to proceed about solving it. According to Schoenfeld [2], if a problematic situation can be overpassed by routine or familiar procedures (no matter how difficult!), it is not a problem, but simply an exercise of the individual's ability to tackle successfully this situation. The kind of a problem dictates the kind of the cognitive skills needed to solve it; e.g. linguistic skills are required to read and debate about a problem, memory skills to recall already existing knowledge being necessary to solve it, etc.

Mathematics by its nature is the subject whereby the PS process can be studied and analyzed in detail. *Mathematical modelling (MM)*, in particular, is a special kind of PS which formulates and solves mathematically real world problems connected to science and everyday life activities. The present review article studies the progress of research on PS from the time that Mathematics Education has been emerged as a self-sufficient science until recently, focusing in particular on the use of MM as a tool for teaching mathematics.

The rest of the article is organized as follows. The next Section describes the development of PS in Mathematics Education and the main modes of thinking needed for PS. Section 3 analyzes the basic principles of MM and its advantages and disadvantages as a tool for teaching mathematics. The article closes with the general conclusions, which are presented in Section 4.

2. Problem Solving in Mathematics Education

Mathematics Education has been emerged as a self-sufficient mathematical topic during the 1970's. The research methods applied on that time for the topic used to be almost exclusively statistical.

G. Polya (Figure 1), a Mathematics Professor of Hungarian origin at Stanford University, who introduced the use of *heuristic strategies* as the basic tool for tackling a problem's solution [5, 6], is considered to be the pioneer of the systematization of the PS process. Polya proposed also *Discovery* as a method for teaching mathematics [7], which is based on the idea that any new mathematical knowledge could be presented in the form of a suitably chosen problem related to already existing knowledge.

Figure 1. G. Polya (1887-1985)

Early work on PS focused mainly on the description and analysis of the PS process. The Schoenfeld's *expert performance model* [8] is an improved version of the Polya's framework for PS. Its real goal is that it provides a list of possible heuristics that could be used at each step of the PS process, which, according to Schoenfeld, are the analysis of the problem and the exploration, design, implementation and verification of its solution.

The assessment of the student PS skills and the effectiveness of the instructional treatments for improving those skills require measurement and several studies have been performed towards this direction ([9, 10], etc.). Voskoglou and Perdikaris [11] introduced a Markov chain on the steps of the Schoenfeld's expert performance model for PS and, by applying basic principles of the corresponding theory, obtained a measure of the student difficulties during the PS process. Voskoglou ([12], [13]: Chapter 7) used later principles of fuzzy logic too for modelling mathematically the PS process and for obtaining measures of student PS skills.

Much of the emphasis that has been placed during the 1980's on the use of heuristics for PS was based on observations that students are often unable to use their existing knowledge to solve problems. It was concluded, therefore, that they lack suitable general PS strategies. Several other explanations were also presented later disputing the effectiveness of the extensive teaching of heuristics and giving more emphasis to other factors, like the acquisition of the proper schemas, the automation of rules, etc. [14]. What it has been agreed by many authors ([3, 15, 16], etc.), however, is that a problem consists of three main parts: The starting state, the goal state and the obstacles, i.e. the existing restrictions which make difficult the access from the starting to the goal state. And while, during the solution of the same problem, the first two parts are more or less the same for all solvers, the last part may hide many ways of tackling it, which could differ from solver to solver.

As a result, more recent studies have focused mainly on solvers' behavior and required attributes during the PS process. The *Multidimensional PS Framework* of Carlson and Bloom [17] is based on the development of a broad taxonomy of PS attributes that have been identified as relevant to PS success. Schoenfeld [18], after a many years research for building a theoretical framework, concluded that PS is an example of a *goal-directed behavior*, under which a solver's "acting in the moment" can be explained and modelled by an architecture involving knowledge, goals, orientations and decision making which depends on subjective values that could differ from solver to solver.

In conclusion, PS is a complex cognitive action, directly related to the knowledge stored in the solver's mind. Therefore, it requires a combination of several modes of thinking in order to be successful. Apart from the very simple and often automated thought (e.g. when performing calculations), other modes of thinking required for PS include *critical, statistical, computational thinking* and *analogical reasoning*. The nature of the problem dictates the mode(s) of thinking required in order to be solved.

Critical thinking (CrT) [15] is a higher mode of thinking involving analysis, synthesis and evaluation of the existing data, actions which give rise to other ones, like predicting, estimating, inferring and generalizing the corresponding situations. When a complex problem is encountered, it has to be critically analyzed: What is the problem, what is the given information and so on. CrT, therefore, is involved in application of knowledge to solve the problem. CrT plays also an important role in the *transfer* of knowledge, i.e. the use of already existing knowledge for producing new knowledge.

Statistical thinking (ST) [19] is the ability to use properly existing statistical data for solving problems related to randomness. Consider, for example, the case of a high school employing 40 in total teachers, 38 of which are good teachers, whereas the other two are not good. A parent, who happens to know only the two not good teachers, concludes that the school is not good and decides to choose another school for his child. This is obviously a statistically wrong decision that could jeopardize the future of the child.

ST, however, must be combined with CrT for obtaining the correct solution. In fact, going back to the previous example, assume that another parent, who knows the 38 good teachers, decides to choose that school for his child. His child, however, happens to be interested only for the lessons taught by the two not good teachers and not for those taught by the 38 good teachers. In this case, therefore, the parent's decision is wrong again due to lack of CrT. In conclusion, CrT driven by logic and ST based on the rules of Probability and Statistics are necessary tools for PS.

If technology is added, however, those tools are not enough, since many technological problems are very complex. In such cases computational thinking (CT) becomes another prerequisite for PS. Although the term CT was introduced by S. Papert [20], it has been brought to the forefront by J.M. Wing [21], who describes it as "solving problems, designing systems and understanding human behavior by drawing on concepts fundamental to computer science". This, however, does not mean that CT proposes that problems must be necessarily solved in the way that computers tackle them. What it really does is that encourages the use of CrT with the help of computer science methods and techniques.

Voskoglou and Buckley [22] developed a theoretical framework explaining the relationship between CT and CrT in PS. According to it, if there exists sufficient background knowledge, the new, necessary for the solution of the problem, knowledge is obtained with the help of CrT and then CT is applied to find a solution that might not be forthcoming under other circumstances.

Computer science does not concern only programing, it is an entire way of thinking, which has become now part of our lives. All of today's students will go on to live a life heavily influenced by computing. Consequently, there is a need to be trained in thinking computationally as soon as possible, even before starting to learn programming [23].

A particular attention has been also placed by the experts on the use of analogical reasoning for PS [24]. In fact, a given problem (*target problem*) can be frequently solved by looking back and by properly adapting the solution of a previously solved similar problem (*source problem*). The use of computers, in particular, enables the creation and maintenance of a continuously increasing "library" of previously solved similar problems (*past cases*) and the retrieval of the proper one(s) for solving a new analogous problem. This approach, termed as *Case-Based Reasoning (CBR)*, is widely used nowadays in many sectors of the human activity including industry, commerce, healthcare, education, etc. [24].

Computers facilitate also the creation of *Communities of Practice (COPs)* for teaching and learning including students and teachers from different places and cultures [25]. In the area of PS in particular, such COPs could help the exchange of innovative ideas and techniques on PS and *problem-posing* [26].

More details about research and applications of PS can be found in earlier works of the present author ([27, 28], etc.).

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3. Mathematical Modelling in Classroom

A *model* is understood to be a simplified representation of a real system including only its characteristics which are related to a certain problem related to the system (*assumed real system*); e.g. maximizing the system's productivity, minimizing its functional costs, etc. The process of modelling is a fundamental principle of the systems' theory, since the experimentation on the real system is usually difficult (and even impossible sometimes) requiring a lot of money and time. Modelling a system involves a deep abstracting process, which is graphically represented in Figure 2 [29].

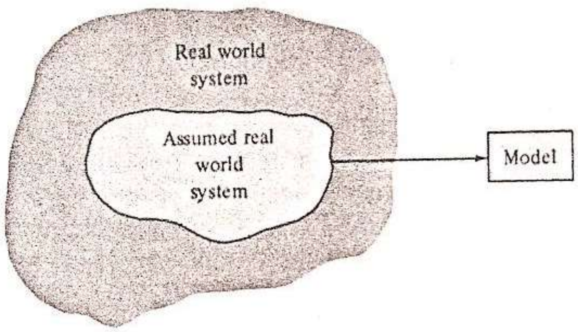

Figure 2. Representation of the modelling process

There are several types of models to be used according to the form of the system and the corresponding problem to be solved. In simple cases *iconic models* may be used, like maps, bas-relief representations, etc. *Analogical models,* such as graphs, diagrams, etc., are frequently used when the corresponding problem concerns the study of the relationship between two (only) of the system's variables; e.g. speed and time, temperature and pressure, etc. The *mathematical or symbolic models* use mathematical symbols and representations (functions, equations, inequalities, etc.) to describe the system's behavior. This is the most important type of models, because they provide accurate and general (holding even if the system's parameters are changed) solutions to the corresponding problems. In case of complex systems, however, like the biological ones, where the solution cannot be expressed in solvable mathematical terms or the mathematical solution requires laborious calculations, *simulation models* can be used. These models mimic the system's behavior over a period of time with the help of a well organized set of logical orders, usually expressed in the form of a computer program. Also, *heuristic models* are used for improving already existing solutions, obtained either empirically or by using other types of models.

Mathematical modelling (MM) until the middle of the 1970's used to be a tool mainly in hands of the scientists for solving problems related to their disciplines The failure of the introduction of the "new mathematics" to school education, however, turned the attention of the specialists to PS activities as a more effective way for teaching and learning mathematics. MM in particular, has been widely used for connecting mathematics to everyday life situations, on the purpose of increasing the student interest on the subject.

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One of the first who proposed the use of MM as a tool for teaching mathematics was H. O. Pollak [30], who presented during the ICME-3 Conference in Karlsruhe (1976) the scheme of Figure 3, known as the *circle of modelling.* In this scheme, given a problem of the everyday life or of a scientific topic different from mathematics (other world) for solution, the solver, following the direction of the arrows, is transferred to the "universe" of mathematics. There, the solver uses or creates suitable mathematics for the solution of the problen and then returns to the other world to check the validity of the mathematical solution obtained. If the verification of the solution is proved to be non compatible to the existing real conditions, the same circle is repeated one or more times.

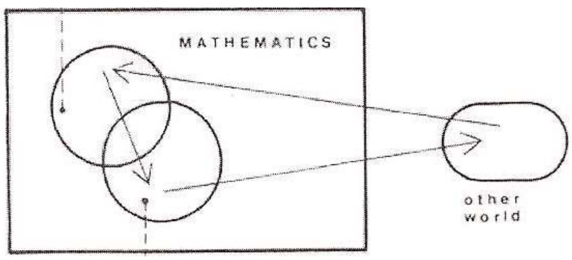

Figure 3. The Pollak's Circle of Modelling

Following the Pollak's presentation, much effort has been placed by several researchers of mathematics education to study and analyze in detail the process of MM on the purpose of using it for teaching mathematics. Several models have been developed towards this direction, a brief but comprehensive account of which can be found in [31], including a model of the present author (Figure 4).

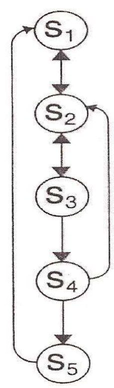

Figure 4. Flow-diagram of Voskoglou's model for the MM process

In fact, Voskoglou [32] described the MM process in terms of a Markov chain introduced on its main steps, which are: S_1 = Analysis of the problem, S_2 = Mathematization (formulation and construction of the model), S_3 = Solution of the model, S_4 = Validation of the solution and S_5 = Implementation of the solution to the real system. When the MM process is completed at step S_5, it is assumed that a new problem is given to the class, which implies that the process restarts again from step S_1.

Models like the previous one are useful for describing the solvers' ideal behavior when tackling MM problems. More recent researches [33-35], however, report that the reality is not like that. In fact, modellers follow individual routes related to their learning styles and the level of their cognition. Consequently, from the teachers' part there exists an uncertainty about the student way of thinking at each step of the MM process. Those findings inspired the present author to use principles of Fuzzy Logic for describing in a more realistic way the process of MM in the classroom on the purpose of understanding, and therefore treating better, the student reactions during the MM process [36]. The steps of the MM process in this model are represented as fuzzy sets on a set of linguistic labels characterizing the student performance in each step.

A complete methodology for teaching mathematics on the basis of MM has been eventually developed later, which is usually referred as the *application-oriented teaching of mathematics* [37]. However, as the present author underlines in [38], presenting also a representative example, teachers must be careful, because the extensive use of the application-oriented teaching as a general method for teaching mathematics could lead to far-fetched situations, in which more attention is given to the choice of the applications rather, than to the mathematical content!

More details about MM and representative examples can be found in earlier works of the author ([39, 40], etc.).

4. Conclusions

The discussion performed in this study leads to the following conclusions:

- The failure of the introduction of the "new mathematics" in school education turned, from the late 1970's, the attention of the specialists in Mathematics Education to the use of the problem as a tool for teaching and learning mathematics more effectively, with two components: Mathematical PS and MM.
- Polya, who proposed the use of the heuristic strategies, is the pioneer of the theoretical development of PS. The research on the subject was based for many years on his ideas, focusing mainly to the description of the PS process and the detailed analysis of its steps. More recent studies, however, have turned the attention mainly to the solvers' behavior and required attributes during the PS process, which depend upon their personal style, their cognitive level and their subjective values and beliefs.
- PS is a complex cognitive action that needs the use of a variety of modes of thinking, according to the form of each problem, in order to be successful. Those modes include critical and statistical thinking, computational thinking and analogical reasoning.
- MM is a special type of PS concerning the solution of problems related to scientific applications or to everyday life activities. An integrated didactic approach has been eventually developed based on MM and termed as the application-oriented teaching of mathematics.
- Markov chain and Fuzzy Logic models have been developed in earlier works of the present author for a more effective description of the processes of PS and MM and for evaluating the corresponding student skills. The former type of models describes the ideal student behavior in the classroom, whereas the latter type attempts the description of their real behavior, which differs from student to student.

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References

1. Polya, G., *How I solve it: A new aspect of mathematical method.* Princeton University Press: New Jersey, 1973.
2. Schoenfeld, A., The wild, wild, wild, wild world of problem solving: A review of sorts. *For the Learning of Mathematics*, **1983**, 3, 40-47.
3. Green, A.J.K.; Gillhooly, K., Problem Solving. In *Cognitive Psychology*, Braisby, N. and Gelatly, A. (Eds.); Oxford University Press: Oxford, 2005.
4. Martinez, M., What is metacognition? Teachers intuitively recognize the importance of metacognition, but may not be aware of its many dimensions. *Phi Delta Kappan*, **2007**, 87(9), 696-714.
5. Polya G., *How to solve it.* Princeton Univ. Press: Princeton, 1945.
6. Polya G., *Mathematics and Plausible Reasoning*. Princeton Univ. Press: Princeton, 1954.
7. Polya G., *Mathematical Discovery*. Wiley and Sons: New York, 1962/65 (2 Vols.).
8. Schoenfeld, A., Teaching Problem Solving Skills. *Amer. Math. Monthly*, **1980**, 87, 794-805.
9. Wright B. D., Solving measurement problems with the Rasch model. *J. of Educational Measurement*, **1977**, 14(2), 97-116.
10. Schoenfeld, A., Measures of problem-solving performance and of problem-solving instruction. *J. Res. Math. Educ.*, **1982**, 13, 31-49.
11. Voskoglou, M. G.; Perdikaris, S. C., A Markov chain model in problem solving. *Int. J. Math. Educ. Sci. Technol.*, **1991**, 22, 909-914.
12. Voskoglou, M. G., A Fuzzy Model for Problem Solving. *Turkish Journal of Fuzzy Systems*, **2012**, 3(1), 1-15.
13. Voskoglou, M. G., *Finite Markov Chains and Fuzzy Logic Assessment Models.* Amazon-Create Space Independent Publishing Platform: Columbia, SC, 2017.
14. Owen, E.; Sweller, J., Should problem solving be used as a learning device in Mathematics? *J. for Research in Mathematics Education*, **1989**, 20, 322-328.
15. Halpern, D., *Thought and `knowledge: An introduction to critical thinking*, Earlbaum: NJ, 4TH Edition, 2003.
16. Matlin, W.M., *Cognition*, Wiley & Sons: New York, 2005
17. Carlson, M.P.; Bloom, I., The cyclic nature of problem solving: An emergent multidimensional problem-solving framework *Educational Studies in Mathematics*, **2005,** 58, 47-75.
18. Schoenfeld, A., How we think: A theory of goal-oriented decision making and its educational applications, Routledge: New York, 2010.
19. Voskoglou, M. G.; Athanassopoulos, E. Statistical Thinking in Problem Solving, *American Journal of Educational Research*, **2020**, 8(10), 754-761.
20. Papert, S. An exploration in the space of Mathematics Education. *Int. J. of Computers for Mathematics*, **1996,** 1(1), 95-123.
21. Wing, J.M., Computational Thinking, *Communications of the ACM*, **2006**, 49, 33-35.
22. Voskoglou, M.G.; Buckley, S., Problem Solving and Computers in a Learning Environment. *Egyptian Computer Science Journal*, **2012**, 36(4), 28-46.
23. Kazimoglu, C.; Kiernan, M.; Bacon, L.; MacKinnon, L. Understanding Computational Thinking before Programming: Developing Guidelines for the Design of Games to Learn Introductory Programming through Game-Play, *Int. J. of Game-Based Learning*, **2011**, 1(3), 30-52.
24. Voskoglou, M. G.; Salem, A.-B.M. Analogy Based and Case Based Reasoning: Two Sides of the Same Coin, *Int. J. Appl. Fuzzy Sets Artif. Intell.*, **2014**, 4, 5-51.
25. Voskoglou, M. G., Communities of Practice for Teaching and Learning Mathe3matics, *American Journal of Educational Research*, **2019,** 7(6), 386-391.
26. Brown, S.I.; Walters, M.I. *The Art of Problem Posing*, Erlbaum: Hillsdale, NJ, 1990.
27. Voskoglou, M. G., Problem-Solving from Polya to Nowadays: A Review and Future Perspectives. In *Progress in Education*, R.V. Nata (Ed.), Nova Science Publishers: NY, Vol. 22, Chapter 4, 65-82, 2011.
28. Voskoglou, M. G. Problem Solving in the Forthcoming Era of the Third Industrial Revolution, *Int. J. Psychology Research*, **2016**, 10(4), 361-380.
29. Taha, H. A. *Operations Research – An Introduction*, Collier Macmillan: N.Y.-London, Second Edition, 1967.
30. Pollak H. O. The interaction between Mathematics and other school subjects, *New Trends in Mathematics Teaching*, Volume IV, Paris: UNESKO, 1979.
31. Haines, C.; Crouch, R. Remarks on a Modelling Cycle and Interpretation of Behaviours. In *Modelling Students' Mathematical Modelling Competencies (ICTMA 13)*, Lesh, R.A. et al. (Eds.), Springer: US, 145-154, Springer, 2010.
32. Voskoglou, M. G. A stochastic model for the modelling process. In *Mathematical Modelling: Education, Engineering and Economics (ICTMA 12)*, Chaines, C.; Galbraith, P.; Blum, W.; Khan, S. (Eds.), Horwood Publishing: Chichester, 149-157, 2007.
33. Galbraith, P.L.; Stillman, G., Assumptions and context: Pursuing their role in modeling activity. In *Modelling and Mathematics Education: Applications in Science and Technology (ICTMA 9)*, Matos, J.F. et al. (Eds.), Horwood Publishing: Chichester, 300-310, 2001.
34. Borroneo Ferri, R. Modelling problems from a cognitive perspective. In *Mathematical Modelling: Education, Engineering and Economics (ICTMA 12)*, Chaines, C.; Galbraith, P.; Blum, W.; Khan, S. (Eds.), Horwood Publishing: Chichester, 260-270, 2007.

 doi:10.20944/preprints202012.0672.v1

35. Doer, H.M. What knowledge do teachers need for teaching mathematics through applications and modeling? In *Modelling and Applications in Mathematics Education*, Blum, W. et al. (Eds.), Springer: NY, 69-78, 2007.
36. Voskoglou, M. G. Use of total possibilistic uncertainty as a measure of students' modeling capacities, *Int. J. Mathematical Education in Science and Technology*, **2010**, 41(8), 1051-1060.
37. Voskoglou, M.G. The Application Orientated Teaching of Mathematics, *Proceedings International Conference on Mathematics Education,* 8-90, Svishtov, Bulgaria, 2005.
38. Voskoglou, M.G. Teaching Mathematics or Mathematical Modelling? An Answer to a Comment, *ICTMA Newsletter*, **2015,** 8(1), 12-13.
39. Voskoglou, M. G. The use of mathematical modeling as a learning tool of mathematics, *Quaderni di Ricerca in Didactica*, **2006**, 16, 53-60.
40. Voskoglou, M. G. Mathematical Modelling as a Teaching Method of Mathematics, *Journal of Research in Innovative Teaching*, **2015**, 8(1), 35-50.