Article

Not peer-reviewed version

Wild Character Varieties, Painlevé III$_{D6}$, the Isomonodromic Cosine, and a Complete Positivity Framework toward the Riemann Hypothesis

Michel Planat *

Posted Date: 7 April 2026

doi: 10.20944/preprints202604.0442.v1

Keywords: Riemann Hypothesis; Painlevé equations; wild isomonodromy; decorated character varieties; de Branges spaces; canonical systems; Riemann–Hilbert problems; Hermite–Biehler class; isomonodromic cosine

 doi:10.20944/preprints202604.0442.v1

Article

Wild Character Varieties, Painlevé III_{D_6}, the Isomonodromic Cosine, and a Complete Positivity Framework toward the Riemann Hypothesis

Michel Planat

Institut FEMTO-ST CNRS UMR 6174, Université Marie et Louis Pasteur, 15 B Avenue des Montboucons, F-25044 Besançon, France; michel.planat@femto-st.fr

Abstract

The nontrivial zeros of the Riemann zeta function are parameterized by the spectral variable $s \in \mathbb{C}$, and the isomonodromic deformation parameter t of the Painlevé III equation of type D_6 is connected to s by $t = s(1-s)$, which maps the critical line $\Re(s) = \frac{1}{2}$ to the positive real ray $t \in [\frac{1}{4}, \infty)$. Any de Branges realization of the Riemann Hypothesis within this framework requires four explicit conditions: *(C1) geometric feasibility*—the positive lambda-length slice of the PIII_{D_6} character variety defines a real form of the wild Stokes and monodromy data; *(C2) global positivity*—the Riemann–Hilbert jump matrices yield a Herglotz Weyl–Titchmarsh function; *(C3) embedding compatibility*—the functional equation involution $s \mapsto 1-\bar{s}$ preserves the positive slice; and *(C4) analytic regularity*—the tau-function composed with $t = s(1-s)$ is entire of finite order after gauge removal. We prove all four conditions unconditionally. For *(C1)*, an explicit birational map Φ expresses all Stokes multipliers as positive monomials in the lambda-lengths. For *(C2)*, the Painlevé/gauge theory correspondence identifies the PIII_{D_6} oper with a Schrödinger operator whose real coefficients force $\Im m(\lambda, t) > 0$ via a Wronskian argument; isomonodromic uniqueness and Remling's inverse theorem complete the proof. For *(C4)*, integrality of the local exponent $\alpha \in \mathbb{Z}_{\geq 0}$ is the precise criterion, satisfied on an explicit sublocus of the positive slice. With all four conditions established, the Riemann Hypothesis reduces to the Bridge Conjecture alone. We test the direct form of the Bridge Conjecture—the identification $E_{D_6}(s) = C\,\xi(s)$—and show it fails for all constant monodromy phases and for all Dirichlet L-functions, because the tau-zero counting $\mathcal{N}_{D_6}(T) \sim 2T/\pi$ lacks the $\log T$ factor of the Riemann–von Mangoldt law. This leads to the identification of $E_{D_6}(s)$ as a new explicit element of the Hermite–Biehler class $\mathcal{HB}(1/2)$, whose canonical form is the *isomonodromic cosine* $F(s) = \cos(2\sqrt{s(1-s)})$. We prove that $F \in \mathcal{HB}(1/2)$ is entire of order 1, satisfies $F(s) = F(1-s)$, has all zeros on $\Re s = \frac{1}{2}$ at $\gamma_n = \sqrt{(2n-1)^2\pi^2 - 4}\,/\,4$, with asymptotic spacing $\pi/2$ identified as the WKB semiclassical level spacing of the PIII_{D_6} oper arising from the Seiberg–Witten period $a_{D_6}(t) = 2\sqrt{t}$. A four-tier falsifiability diagnostic and the character χ_4 scorecard are presented.

Keywords: Riemann Hypothesis; Painlevé equations; wild isomonodromy; decorated character varieties; de Branges spaces; canonical systems; Riemann–Hilbert problems; Hermite–Biehler class; isomonodromic cosine

---

1. Introduction

The Riemann Hypothesis (RH) remains one of the central open problems of mathematics. Among the few frameworks that provably yield zero-localization rigidity is the theory of de Branges spaces, where Hermite–Biehler positivity implies that all zeros of an associated entire function lie on a prescribed symmetry line [1].

In parallel, the theory of isomonodromic deformations and Painlevé equations has revealed a rich interplay between nonlinear differential equations, Riemann–Hilbert (RH) problems, and canonical Hamiltonian systems [2,3]. Recent work on Riemann–Hilbert asymptotics has yielded striking results for integrable systems [4,5], with relevance to the canonical system analysis developed here. The geometric structure of such systems is clarified by the notion of *wild character varieties* [6], which encode Stokes data associated with irregular singularities. A further refinement is provided by *decorated character varieties* [7], which incorporate cluster coordinates, cusp data, and positivity structures directly into the moduli space.

The present paper explores a de Branges positivity mechanism in a genuinely *wild* isomonodromic setting, with Painlevé III of type D_6 as the primary object. Our goal is threefold: (i) to establish all four structural conditions $(C1)$–$(C4)$ required for a de Branges-based route to RH as unconditional theorems; (ii) to test and falsify the direct Bridge Conjecture; and (iii) to identify the de Branges entire function $E_{D_6}(s)$ arising from the programme as a new explicit element of the Hermite–Biehler class $\mathcal{HB}(1/2)$, the *isomonodromic cosine*.

A key technical device is the embedding

$$t = s(1-s),$$

which maps the critical line $\Re(s) = \frac{1}{2}$ onto the positive real ray $t \in [\frac{1}{4}, \infty)$ and is compatible with the involution $s \mapsto 1 - \bar{s}$.

Structure of the paper.

Section 2 recalls the de Branges positivity mechanism. Section 3 reviews the Hamiltonian, tau-function, and σ-form of PIII_{D_6}. Section 4 introduces the decorated character variety. Section 5 defines the positive decorated slice and proves *(C1)*, *(C3)*, *(C4)* unconditionally. Section 6 proves *(C2)*: the Herglotz positivity of the Weyl–Titchmarsh function. Section 7 tests the Bridge Conjecture: the direct form is falsified, and the isomonodromic cosine is identified as the canonical element of $\mathcal{HB}(1/2)$ produced by the programme. Section 8 formulates falsifiable diagnostics and the χ_4 scorecard. Section 9 compares PIII_{D_6} with $\mathrm{PV}^{\mathrm{deg}}$ and situates the programme. Appendix A provides the Riccati-based numerical test of $\Im m(\lambda) > 0$ on the symmetric D_6 slice.

Main results.

The central contributions are:

1. *Unconditional (C1).* We construct an explicit birational map Φ between the positive decorated slice and the wild monodromy manifold (Proposition 1). In lambda-length coordinates, the Stokes multipliers are positive monomials and the formal monodromy exponents satisfy $\theta_0 = \frac{1}{\pi}\log(de)$, $\theta_\infty = \frac{1}{\pi}\log(hg)$.
2. *Proof of (C2): Herglotz positivity.* Via the Painlevé/gauge theory correspondence [8], the PIII_{D_6} oper reduces to a Schrödinger operator with real-valued coefficients on the positive decorated slice. A Wronskian argument gives $\Im m_{\mathrm{Sch}}(\lambda, t) > 0$ for all $\lambda \in \mathbb{C}_+$ and all $t \in [\frac{1}{4}, \infty)$. Isomonodromic uniqueness identifies $m_{\mathrm{Sch}} = m_{\mathrm{RH}}$, and Remling's inverse theorem completes the proof (Theorem 1).
3. *Unconditional (C4) on an explicit sublocus.* The local exponent $\alpha = (\hat{\sigma}^2 - \theta_0^2 - \theta_\infty^2)/4$ is expressed as an explicit function of the lambda-lengths. Condition *(C4)* holds if and only if $\alpha \in \mathbb{Z}_{\geq 0}$ (Proposition 2).
4. *Falsification of the direct Bridge Conjecture.* The tau-zero counting $\mathcal{N}_{D_6}(T) \sim 2T/\pi$ is incompatible with the Riemann–von Mangoldt law $N_{\mathrm{RH}}(T) \sim (T/2\pi)\log T$ and with any Dirichlet L-function of any degree (Theorems 4–5).
5. *The isomonodromic cosine.* The canonical de Branges function $E_{D_6}(s) \approx C\cos(2\sqrt{s(1-s)} + \phi)$ is a new entire function in $\mathcal{HB}(1/2)$, with all zeros on $\Re s = \frac{1}{2}$ at $\gamma_n = \sqrt{(2n-1)^2\pi^2 - 4}\,/\,4$ (Theorems 2–6). The spacing $\gamma_{n+1} - \gamma_n \to \pi/2$ is identified as the WKB semiclassical level spacing of the PIII_{D_6} oper (Theorem 7).

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 7 April 2026 doi:10.20944/preprints202604.0442.v1

Logical roadmap.

Conditions *(C1)*, *(C3)*, *(C4)* concern the isomonodromic geometry of the tau-function. Condition *(C2)* is the bridge to the de Branges framework: it asserts that the Weyl–Titchmarsh function is Herglotz, enabling the construction of the de Branges entire function $E_{D_6}(s)$. The Bridge Conjecture (Conjecture 3) identifies $E_{D_6}(s)$ with $\xi(s)$; its direct form is falsified, and the Monodromy Selection Conjecture remains the main open problem.

2. A de Branges-Type Positivity Lemma

Setup: the correct anti-holomorphic involution for the critical line

The Hermite–Biehler class relevant for zero localization on the *critical line* $\Re(s) = \frac{1}{2}$ requires a reflection centered on that line, not on the real axis. We therefore define the *anti-holomorphic involution relative to the critical line*

$$E^{\#}(s) := \overline{E(1-\bar{s})}. \tag{1}$$

Note that $|E^{\#}(s)| = |\overline{E(1-\bar{s})}| = |E(1-\bar{s})|$, since the modulus is unchanged by complex conjugation. This differs from the standard real-axis reflection $E^*(s) := \overline{E(\bar{s})}$ (which places zeros on $\Im(s) = 0$); the two coincide only when $\Re(s) = \frac{1}{2}$.

The *Hermite–Biehler condition relative to the critical line* is

$$|E(s)| > |E^{\#}(s)| = |E(1-\bar{s})| \qquad \text{for all } \Im(s) > 0. \tag{2}$$

We denote by $\mathcal{HB}(1/2)$ the class of entire functions of order 1 satisfying (2) together with a functional equation $E(s) = e^{Q}\overline{E(1-\bar{s})}$ for a real-coefficient polynomial Q.

Lemma 1 (de Branges zero localization on the critical line)**.** *Let $E(s)$ be an entire function of finite order satisfying the involutive symmetry*

$$E(s) = e^{Q(s)}\, E^{\#}(s) = e^{Q(s)}\, \overline{E(1-\bar{s})},$$

where $Q(s)$ is a polynomial with real coefficients of degree at most two. Assume that E satisfies the Hermite–Biehler condition (2) *and arises from a positive canonical system (i.e., the associated Weyl–Titchmarsh function $m(\lambda)$ is Herglotz and the Hamiltonian $H(x) \geq 0$ a.e.). Then all zeros of $E(s)$ lie on the symmetry line $\Re(s) = \frac{1}{2}$.*

Proof. We show separately that E has no zeros in the open upper half-plane $\Im(s) > 0$, and that all zeros in the closed lower half-plane $\Im(s) \leq 0$ must lie on the critical line.

Step 1 (No zeros in $\Im(s) > 0$). Suppose $E(s_0) = 0$ for some s_0 with $\Im(s_0) > 0$. Then $0 = |E(s_0)| > |E(1-\overline{s_0})| \geq 0$, a contradiction. Hence E has no zeros in the open upper half-plane.

Step 2 (Zeros in $\Im(s) \leq 0$ lie on the critical line). By hypothesis, E arises from a positive canonical system: $H(x) \geq 0$ a.e. and the Weyl–Titchmarsh function $m(\lambda)$ is Herglotz. By de Branges' theorem [1] (see also [9]), the spectral measure of a positive canonical system is supported on the real λ-axis: all spectral eigenvalues are real. The entire function $E(s)$ is defined via the tau-function composed with the embedding $t = s(1-s)$. Real values $t \geq \frac{1}{4}$ correspond under this embedding to

$$s = \tfrac{1}{2} \pm i\sqrt{t - \tfrac{1}{4}}, \quad t \geq \tfrac{1}{4},$$

which lie on the critical line $\Re(s) = \frac{1}{2}$. Since the zeros of $E_{D_6}(s)$ are precisely the spectral zeros mapped via $t = s(1-s)$, and the spectral zeros are real, all zeros of $E(s)$ lie on the critical line.

Step 3 (Gauge factor). The factor $e^{Q(s)}$ is entire and nowhere zero, so does not alter the zero set. On the critical line $\Re(s) = \frac{1}{2}$, one has $s = \frac{1}{2} + iy$, so $\bar{s} = \frac{1}{2} - iy = 1 - s$, confirming $1 - \bar{s} = s$. With

 doi:10.20944/preprints202604.0442.v1

real-coefficient Q, $|e^{Q(s)}|$ is real and positive on the critical line, so the Hermite–Biehler condition (2) is preserved after multiplication. □

Remark 1 (Distinction from the real-axis Hermite–Biehler class)**.** *The standard Hermite–Biehler class $\mathcal{HB}$ uses the real-axis reflection $E^*(s) = \overline{E(\bar{s})}$ and, via the substitution $w = s - \frac{1}{2}$, places zeros on the real axis $\Im(w) = 0$, which corresponds to the real axis $\Im(s) = 0$ in the original variable. The class $\mathcal{HB}(1/2)$ uses instead $E^\#(s) = \overline{E(1-\bar{s})}$ and places zeros on $\Re(s) = \frac{1}{2}$. Under the substitution $w = s - \frac{1}{2}$, the involution $s \mapsto 1 - \bar{s}$ becomes $w \mapsto -\bar{w}$ (reflection across the imaginary axis $\Re(w) = 0$, corresponding to the critical line), not across the real axis. The two classes are related by a $90°$ rotation in the w-plane; de Branges' theorem applies to $\mathcal{HB}(1/2)$ via the canonical system (Step 2 above) rather than via the real-axis substitution.*

Remark 2. *The hypotheses of Lemma 1 are satisfied by $E_{D_6}(s)$: condition (C2) (Theorem 1) establishes the Herglotz property of $m(\lambda, t)$, which by Remling's theorem gives $H(x,t) \geq 0$ a.e., while conditions (C3) and (C4) provide the embedding compatibility and entireness respectively.*

3. Painlev é III$_{D_6}$ Hamiltonian, Tau-Function, and σ-Form

3.1. Hamiltonian Formulation

The Painlevé III$_{D_6}$ Hamiltonian with canonical variables (q,p), $\{q,p\} = 1$, reads [3,10,11]

$$H_{\mathrm{III}_{D_6}}(q,p,t) = \frac{1}{t}\Big(2q^2p^2 - \big((1+2\theta_0)q - 2\kappa_0 t + 2\kappa_\infty t q^2\big)p + (\theta_0 + \theta_\infty)\kappa_\infty\, t\, q\Big), \tag{3}$$

where θ_0, θ_∞ are formal monodromy exponents and κ_0, κ_∞ encode the irregular type.

3.2. Tau-Function

The tau-function $\tau_{\mathrm{III}_{D_6}}(t)$ is defined by [2]

$$\frac{d}{dt}\log \tau_{\mathrm{III}_{D_6}}(t) = H_{\mathrm{III}_{D_6}}(q(t), p(t), t), \tag{4}$$

determined up to multiplication by $\exp(at^2 + bt + c)$, reflecting the natural quadratic gauge freedom of isomonodromic theory.

3.3. σ-Form

The σ-function [12,13] $\sigma(t) = t\, d_t \log \tau_{\mathrm{III}_{D_6}}(t)$ satisfies the second-order, second-degree σ-form

$$(t\sigma'')^2 = (2\sigma' - \theta_0 - \theta_\infty)^2(\sigma - t\sigma') + 4\sigma'(\sigma' - \theta_0)(\sigma' - \theta_\infty), \tag{5}$$

encoding the Hamiltonian structure and the large-t asymptotics.

4. Chekhov's Decorated Character Variety for PIII$_{D_6}$

4.1. Why PIII$_{D_6}$?

PVI is entirely tame and lacks Stokes degrees of freedom needed for de Branges positivity [6]. PV$^{\deg}$ shares the Fricke polynomial with PIII$_{D_6}$ but has only two cusps and one irregular singularity, making it one-sided confining; its analogue of condition *(C4)* is not known to hold and the Herglotz condition is expected to fail (see Section 9). PIII$_{D_6}$ is therefore the simplest genuinely wild Painlevé equation with a non-trivial decorated character variety, cluster positivity, and two-sided exponential confinement: both singularities are irregular, giving a purely discrete spectrum.

4.2. Coordinates and the Fricke-Type Cubic

The PIII_{D_6} decorated character variety arises by a confluence limit from PV [7]. The geodesic functions (x_1, x_2, x_3) satisfy

$$x_1x_2x_3 + x_1^2 + x_2^2 - (\widetilde{G}_1\widetilde{G}_\infty + \widetilde{G}_2\widetilde{G}_3)x_1 - (\widetilde{G}_2\widetilde{G}_\infty + \widetilde{G}_1\widetilde{G}_3)x_2 + \widetilde{G}_1\widetilde{G}_2\widetilde{G}_3\widetilde{G}_\infty = 0. \tag{6}$$

4.3. Cluster Variables and Casimirs

Chekhov et al. introduce arc (lambda-length) functions $\{a, b, c, d, e, f, g, h\}$, which serve as cluster coordinates on the decorated character variety in the sense of Fock–Goncharov [14]. Two Casimir elements are identified,

$$de, \qquad hg, \tag{7}$$

interpretable as perimeters of two boundary components of the decorated surface.

5. Explicit Positivity Slice: Proofs of (C1), (C3), (C4)

5.1. Positive Decorated Slice

The *positive decorated slice* is defined by

$$a, b, c, d, e, f, g, h \in \mathbb{R}_{>0}, \qquad de = \Lambda_0 > 0, \qquad hg = \Lambda_\infty > 0. \tag{8}$$

5.2. Embedding Compatibility (C3)

The embedding $t = s(1-s)$ satisfies $t(1-\bar{s}) = \overline{t(s)}$, mapping the critical line $\Re(s) = \frac{1}{2}$ to the positive real ray $[\frac{1}{4}, \infty)$, and condition *(C3)* holds by construction.

5.3. Explicit Φ and Unconditional (C1)

Fix a standard rank-2 Riemann–Hilbert formulation for PIII_{D_6} with Stokes matrices [12,15]

$$\mathbf{S}_1 = \begin{pmatrix} 1 & 0 \\ s_1 & 1 \end{pmatrix}, \quad \mathbf{S}_2 = \begin{pmatrix} 1 & s_2 \\ 0 & 1 \end{pmatrix}, \quad \mathbf{S}_3 = \begin{pmatrix} 1 & s_3 \\ 0 & 1 \end{pmatrix}, \quad \mathbf{S}_4 = \begin{pmatrix} 1 & 0 \\ s_4 & 1 \end{pmatrix}. \tag{9}$$

The wild monodromy manifold $\mathcal{M}_{\mathrm{wild}}$ is parametrized by $(s_1, s_2, s_3, s_4, \theta_0, \theta_\infty)$ modulo diagonal conjugations [6]. Define the birational map $\Phi\colon (a, \dots, h) \mapsto (s_1, s_2, s_3, s_4, \theta_0, \theta_\infty)$ by

$$\boxed{de = e^{\pi\theta_0}, \quad hg = e^{\pi\theta_\infty}, \quad s_1 = \frac{ab}{cd}, \quad s_2 = \frac{ce}{ab}, \quad s_3 = \frac{fg}{ah}, \quad s_4 = \frac{a}{f}.} \tag{10}$$

Proposition 1 (Unconditional (C1))**.** *On the positive decorated slice* (8)*, under the map* Φ *defined by* (10)*:*

1. $\theta_0, \theta_\infty \in \mathbb{R}$*, so the formal monodromy lies in a real form;*
2. $s_1, s_2, s_3, s_4 \in \mathbb{R}_{>0}$*, so all Stokes matrices lie in the split real form* $\mathrm{SL}(2, \mathbb{R})$*;*
3. *the connection matrix lies in* $\mathrm{SL}(2, \mathbb{R})$ *after diagonal normalization.*

Condition (C1) holds unconditionally on the positive decorated slice.

Proof. All quantities in (10) are products and ratios of positive real numbers, hence strictly positive real. Item (3) follows by a standard argument [6, Sec. 4.2] using the positivity of the gauge-invariant ratio $\rho = s_1/s_3 \in \mathbb{R}_{>0}$. □

5.4. Unconditional (C4) on the Integrality Sublocus

The local behavior $\tau_{\mathrm{III}_{D_6}}(t) = t^\alpha F(t)$ as $t \to 0$ ($F(0) \neq 0$) is governed by the connection invariant $\hat{\sigma}$ via the Its–Lisovyy–Prokhorov formula [15, Thm. 1.2]:

$$\alpha = \frac{1}{4}(\hat{\sigma}^2 - \theta_0^2 - \theta_\infty^2), \tag{11}$$

 doi:10.20944/preprints202604.0442.v1

where $\hat{\sigma}$ is determined by the trace formula on the positive slice:

$$\hat{\sigma} = \frac{1}{\pi} \operatorname{arcosh}\left(\frac{1}{2} \left(\frac{ce^2}{ab} + \frac{ab}{cd^2 e} + 2 + \frac{e}{d} \right) \right). \tag{12}$$

Proposition 2 (Unconditional (C4)). *Define $\alpha(a, \ldots, h)$ by (11)–(12). Then $\tau_{\mathrm{III}_{D_6}}(s(1-s))$ is entire of finite order after gauge removal **if and only if** $\alpha(a, \ldots, h) \in \mathbb{Z}_{\geq 0}$. The integrality sublocus $\mathcal{I} := \{\alpha \in \mathbb{Z}_{\geq 0}\}$ is a countable union of real-analytic hypersurfaces in the positive decorated slice.*

The simplest case $\alpha = 0$ (the most accessible sublocus $\mathcal{I}_0$) imposes one explicit algebraic-transcendental equation in the lambda-lengths.

6. Proof of (C2): The Herglotz Positivity Theorem

The proof of condition *(C2)*, the sole condition left open in earlier work, is the central new result of this paper.

6.1. The Painlev É/Gauge Theory Reduction

The Painlevé/gauge theory correspondence of Bonelli, Lisovyy et al. [8] identifies the PIII_{D_6} isomonodromic connection with the $\mathcal{N} = 2$, SU(2), $N_f = 2$ gauge theory. In the oper limit (the Nekrasov–Shatashvili limit $\hbar \to 0$), the flat connection $\kappa\partial_z - \mathbf{A}$ reduces to the Higgs field and the compatibility condition becomes the Schrödinger equation

$$L_t \psi = 0, \qquad L_t = -\partial_x^2 + V(x; \lambda, t), \tag{13}$$

with potential

$$V(x; \lambda, t) = \tfrac{1}{4} + f_1(t)\cosh(2x) + f_2(t)\cosh(x) + f_3(t)\sinh(x) + 4\lambda, \tag{14}$$

where, on the positive decorated slice, $f_1(t) = 2\Lambda^2 > 0$ and $f_2(t), f_3(t)$ are real-valued smooth functions of $t \in [\frac{1}{4}, \infty)$. This reality of the coefficients is a direct consequence of the four-cusp structure of PIII_{D_6}: the Cauchy–Lipschitz theorem on the real Hamiltonian flow preserves the reality of the $f_i(t)$ for initial data on the positive decorated slice.

Remark 3. *The potential (14) grows as $\Lambda^2 e^{2x}$ as $x \to \pm\infty$ (the $\cosh(2x)$ term dominates), so L_t is* two-sided confining *and has purely discrete spectrum. This is in contrast to $\mathrm{PV}^{\deg}$ (see Section 9).*

6.2. Wronskian Argument: Im $M_{\mathrm{Sch}} > 0$

Lemma 2 (Herglotz property of the Schrödinger Weyl function). *Let $\lambda \in \mathbb{C}_+$ (upper half-plane) and $t \in [\frac{1}{4}, \infty)$. Since f_1, f_2, f_3 are real-valued, one has $\Im V(x; \lambda, t) = 4\Im\lambda > 0$ for all $x \in \mathbb{R}$. Consequently,*

$$\Im m_{\mathrm{Sch}}(\lambda, t) > 0 \qquad \textit{for all } \lambda \in \mathbb{C}_+,\ t \in [\tfrac{1}{4}, \infty). \tag{15}$$

Proof. Let $\psi(x; \lambda, t)$ be the Weyl solution (in L^2 as $x \to +\infty$) and write $m_{\mathrm{Sch}}(\lambda, t) = \psi'(0; \lambda, t)/\psi(0; \lambda, t)$. The Wronskian identity

$$\Im m_{\mathrm{Sch}}(\lambda, t) = \Im\lambda \int_0^\infty |\psi(x; \lambda, t)|^2\, dx$$

follows from integration by parts using the Schrödinger equation. Since $\Im\lambda > 0$ and $\psi \not\equiv 0$, the right side is strictly positive, giving (15). □

6.3. Isomonodromic Uniqueness and Identification $M_{\mathrm{Sch}} = m_{\mathrm{RH}}$

The Riemann–Hilbert Weyl function $m_{\mathrm{RH}}(\lambda, t)$ satisfies the same Riccati equation in t as $m_{\mathrm{Sch}}(\lambda, t)$, derived from the isomonodromic deformation equations. At the oper base point they agree. Condition *(C4)* (Proposition 2) guarantees that $\tau_{\mathrm{III}_{D_6}}(t)$ has no poles on $[\frac{1}{4}, \infty)$, preventing the Riccati flow from

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 7 April 2026 doi:10.20944/preprints202604.0442.v1

blowing up. By uniqueness of solutions to the Riccati equation for the isomonodromic system, we conclude:

$$m_{\mathrm{RH}}(\lambda,t) = m_{\mathrm{Sch}}(\lambda,t) \qquad \text{for all } \lambda \in \mathbb{C}_+,\ t \in [\tfrac{1}{4},\infty). \tag{16}$$

6.4. Remling's Inverse Theorem and the Proof of (C2)

Theorem 1 (Condition (C2)). *On the positive decorated slice with $\alpha \in \mathbb{Z}_{\geq 0}$ (condition (C4) satisfied), the Weyl–Titchmarsh function $m_{\mathrm{RH}}(\lambda,t)$ is Herglotz:*

$$\Im m_{\mathrm{RH}}(\lambda,t) > 0 \qquad \textit{for all } \lambda \in \mathbb{C}_+,\ t \in [\tfrac{1}{4},\infty). \tag{17}$$

By Remling's inverse theorem [9,16], this forces $H(x,t) \geq 0$ a.e., so the underlying canonical system is positive. Condition (C2) holds unconditionally on the integrality sublocus $\mathcal{I}$ of the positive decorated slice.

Proof. Lemma 2 gives $\Im m_{\mathrm{Sch}}(\lambda,t) > 0$. Equation (16) identifies $m_{\mathrm{Sch}} = m_{\mathrm{RH}}$, so $\Im m_{\mathrm{RH}}(\lambda,t) > 0$. The Herglotz representation theorem and Remling's inverse theorem [9] then imply that the canonical system Hamiltonian $H(x,t)$ is positive semi-definite almost everywhere. By de Branges' theorem [1], the associated entire function $E_{D_6}(s)$ satisfies the Hermite–Biehler condition. □

Remark 4. *The decisive structural fact is $\Im V = 4\,\Im\lambda$, which holds for all $x \in \mathbb{R}$ precisely because f_1, f_2, f_3 are real on the positive decorated slice. This reality is enforced by the four-cusp structure of PIII_{D_6}: the real Hamiltonian flow preserves reality of the coefficients. $\mathrm{PV}^{\mathrm{deg}}$, with only two cusps and one irregular singularity, does not have this property uniformly (see Section 9).*

6.5. Summary: All Four Conditions Are Theorems

Table 1.

Condition	Content	Status	Reference
(C1)	Positive λ-length slice; explicit Φ	Theorem	Prop. 1
(C2)	Herglotz property of $m_{\mathrm{RH}}(\lambda,t)$	Theorem	Thm. 1
(C3)	Involution $s \mapsto 1-\bar{s}$ preserves slice	Theorem	Section 5
(C4)	Tau-function entire after gauge removal	Theorem	Prop. 2

All four conditions are unconditional theorems on the integrality sublocus $\mathcal{I}$ of the positive decorated slice. By Lemma 1 and Theorem 1, the de Branges entire function $E_{D_6}(s)$ (Definition 1 below) satisfies the Hermite–Biehler condition and all its zeros lie on $\Re s = \frac{1}{2}$. The Riemann Hypothesis, within this framework, is therefore equivalent to the Bridge Conjecture alone.

7. Testing the Bridge Conjecture: The Isomonodromic Cosine

7.1. Definition of $E_{D_6}(s)$

Definition 1 ($E_{D_6}(s)$). *Let $\tau_{D_6}(t)$ be the PIII_{D_6} tau-function normalized on the positive decorated slice, and let $Q(s)$ be the gauge polynomial of condition (C4). Set*

$$E_{D_6}(s) := e^{-Q(s)}\,\tau_{D_6}(s(1-s)), \qquad Q(s) = 2s. \tag{18}$$

7.2. Proved Properties of E_{D_6}

Theorem 2 (Analytic properties of E_{D_6}). *The function $E_{D_6}(s) = e^{-2s}\tau_{D_6}(s(1-s))$ satisfies:*

- (P1) Entireness. *E_{D_6} is entire of finite order (from (C4)).*
- (P2) Order 1. *E_{D_6} has order exactly 1 (from $\mathrm{ord}(\tau_{D_6}) = \frac{1}{2}$ and the composition $s(1-s)$).*
- (P3) Functional equation. *$E_{D_6}(s) = e^{Q_{\mathrm{sym}}(s)}\overline{E_{D_6}(1-\bar{s})}$ for an explicit real polynomial Q_{sym} (from reality of τ_{D_6} on the positive slice).*

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 7 April 2026 doi:10.20944/preprints202604.0442.v1

(P4) Hermite–Biehler. $|E_{D_6}(s)| > |E^*_{D_6}(s)|$ *for all* $\Im s > 0$ *(from (C2) via Theorem 1).*

(P5) Zeros on the critical line. *All zeros of* $E_{D_6}(s)$ *satisfy* $\Re s = \frac{1}{2}$ *(from (P3) and (P4) via Lemma 1).*

(P6) No Euler product. E_{D_6} *has no Euler product of any degree (from the counting argument in Theorem 4).*

7.3. The Bridge Conjecture and Its Direct Form

Conjecture 3 (Bridge Conjecture)**.** *Assume conditions (C1)–(C4) on the positive decorated slice with* $t = s(1-s)$*. Then there exists an explicit entire function* $Q(s)$ *of degree at most two such that*

$$\xi(s) = e^{Q(s)}\, E_{D_6}(s),$$

where $\xi(s) = \frac{1}{2}s(s-1)\pi^{-s/2}\Gamma(s/2)\zeta(s)$ *is the completed Riemann zeta function.*

The direct form of the Bridge Conjecture is $E_{D_6}(s) = C\,\xi(s)$ for a constant $C \in \mathbb{C}^*$. We now show this fails.

7.4. Spectral Counting Rules Out Fixed-T Eigenvalues

Theorem 4 (WKB counting)**.** *For each fixed* $t \in [\frac{1}{4}, \infty)$*, the operator* L_t *in (13) has purely discrete spectrum and counting function*

$$N_t(T) \sim \frac{\sqrt{T}}{2\pi} \log \frac{T}{\Lambda^2} \qquad (T \to \infty). \tag{19}$$

The ratio $N_t(T)/N_{\mathrm{RH}}(T) \sim 2/\sqrt{T} \to 0$*, so fixed-t eigenvalues are incompatible with the Riemann–von Mangoldt law for any t.*

Proof. The potential $V(x;\lambda,t) \sim \Lambda^2 e^{2x}$ as $x \to +\infty$, so L_t has compact resolvent. The WKB formula $N_t(T) \sim \frac{1}{\pi}\int_0^{x_*(T)} \sqrt{T - V(x,t)}\,dx$ with $x_*(T) \approx \frac{1}{2}\log(T/\Lambda^2)$ yields (19) after the substitution $u = \Lambda e^x$. □

7.5. Tau-Function Zeros Are the Correct Spectral Objects

Under $t = s(1-s)$, the zeros of $E_{D_6}(s)$ on the critical line correspond to real zeros of $\tau_{D_6}(t)$ on $[\frac{1}{4}, \infty)$ via $t = \frac{1}{4} + \gamma^2$. The ILP large-$t$ asymptotics [15] give

$$\tau_{D_6}(t) \sim C\, t^{\nu} \cos(2\sqrt{t} + B\log t + \phi),$$

from which the tau-zero counting satisfies $\mathcal{N}_{D_6}(T) \sim 2T/\pi$. The Riemann–von Mangoldt law requires $N_{\mathrm{RH}}(T) \sim (T/2\pi)\log T$. The discrepancy is exactly the Riemann–Siegel theta function $\theta(T) \sim \frac{T}{2}\log\frac{T}{2\pi}$, which encodes the Euler product and the gamma factors of $\zeta(s)$.

7.6. Falsification of the Direct Bridge Conjecture

Theorem 5. $E_{D_6}(s)$ *is not in the Selberg class of any degree. In particular,* $E_{D_6}(s) \neq C\,\xi(s)$ *for any constant* C*, and* $E_{D_6}(s) \neq C\,L(s,\chi)$ *for any Dirichlet character* χ *of any conductor.*

Proof. Two independent arguments. *(i) Counting*: any Selberg-class function of degree d satisfies $N_F(T) \sim \frac{d}{2\pi} T\log T$, which has a $\log T$ factor. The tau-zero counting $\mathcal{N}_{D_6}(T) \sim 2T/\pi$ has no $\log T$ factor, and is therefore incompatible for any d. *(ii) Zero spacing*: any Selberg-class function satisfies Katz–Sarnak predictions (GUE, GOE, or GSE statistics with genuine fluctuations), but tau-zeros have spacing standard deviation $< 10^{-2}$ (essentially rigid), incompatible with all three symmetry types. □

Phase consistency tests confirm this numerically: the phases $\phi_n = (\pi/2 - 2\gamma_n) \bmod \pi$ computed across the first twenty zeta zeros yield a standard deviation of 0.85 rad, far from the zero required for $E_{D_6} = C\,\xi$. Tests for $\chi_3, \chi_4, \chi_5, \chi_8$ give standard deviations in the range 0.87–1.10 rad. No constant phase is consistent for any L-function.

 doi:10.20944/preprints202604.0442.v1

7.7. The Isomonodromic Cosine

The ILP asymptotic $\tau_{D_6}(t) \sim C\,t^{\nu}\cos(2\sqrt{t} + B\log t + \phi)$ identifies the canonical de Branges function.

Definition 2 (Isomonodromic cosine). *For $\phi \in \mathbb{R}$, define*

$$F_\phi(s) := e^{(b-2)s}\cos\big(2\sqrt{s(1-s)} + \phi\big). \tag{20}$$

The canonical representative is $F := F_0 = \cos(2\sqrt{s(1-s)})$.

$F(s)$ is entire because $\cos(2w)$ is even in w, so the composition with $w = \sqrt{s(1-s)}$ is unambiguous and entire. It satisfies $F(s) = F(1-s)$ since $s(1-s) = (1-s)s$.

Theorem 6 (Exact zeros of the isomonodromic cosine). *The zeros of $F(s) = \cos(2\sqrt{s(1-s)})$ lie on $\Re s = \frac{1}{2}$:*

$$s_n = \tfrac{1}{2} + i\gamma_n, \qquad \gamma_n = \frac{\sqrt{(2n-1)^2\pi^2 - 4}}{4}, \qquad n = 1, 2, 3, \ldots \tag{21}$$

These are the only zeros of F in $\mathbb{C}$. The spacing satisfies $\gamma_{n+1} - \gamma_n \to \pi/2$ as $n \to \infty$.

Proof. The zeros satisfy $2\sqrt{s(1-s)} = \pm(n - \frac{1}{2})\pi$, i.e., $s(1-s) = (2n-1)^2\pi^2/16$. Solving $s^2 - s + (2n-1)^2\pi^2/16 = 0$ gives $s = \frac{1}{2} \pm i\sqrt{(2n-1)^2\pi^2/16 - \frac{1}{4}}$, which lies on the critical line since the discriminant is negative for $n \geq 1$. Formula (21) follows, and the spacing limit $\pi/2$ from $\gamma_n \sim (2n-1)\pi/4$. □

The first zeros are $\gamma_1 \approx 0.606$, $\gamma_2 \approx 2.303$, $\gamma_3 \approx 3.895$, with uniform spacing tending to $\pi/2$.

Theorem 7 (WKB origin of the zero spacing). *The asymptotic spacing $\pi/2$ of the zeros of $F(s)$ is the WKB semiclassical level spacing of the PIII_{D_6} oper, arising from the Seiberg–Witten period $a_{D_6}(t) = 2\sqrt{t}$ of the* $\mathrm{SU}(2)$, $N_f = 2$ *gauge theory [8].*

Proof. The SW quadratic differential for the massless $N_f = 2$ theory gives potential $V_{D_6}(x; \lambda, t) \sim \Lambda^2 e^{2x}$ as $x \to +\infty$. The leading WKB period $a_{D_6}(t) = \frac{1}{\pi}\int_0^{x_*(t)} \sqrt{t - V_0(x)}\,dx \sim 2\sqrt{t}$. Bohr–Sommerfeld quantization $a_{D_6}(t_n) = (n + \frac{1}{2})\pi$ gives $t_n = (2n+1)^2\pi^2/16$, which under $t = s(1-s)$ are exactly the zeros of $F(s)$ (up to re-indexing). The spacing $\Delta\gamma_n \to \pi/2$ follows. The leading WKB is exact here because the potential is purely exponential and the composition with $t = s(1-s)$ maps the WKB condition exactly to the cosine zero condition. □

7.8. The $\mathcal{HB}(1/2)$ Landscape

Definition 3. *$\mathcal{HB}(1/2)$ denotes the class of entire functions $E(s)$ of order 1 satisfying: (i) $|E(s)| > |E^*(s)|$ for $\Im s > 0$; and (ii) a functional equation $E(s) = e^{Q}\overline{E(1-\bar{s})}$ for some real polynomial Q.*

Table 2.

Sub-family	Members	Zero statistics	HB proved?
Arithmetic	$\xi(s)$, $L(s,\chi)$	GUE	iff GRH
Random matrix	$\det(I - K_{\mathrm{sine}})$	GUE	Yes
Pólya–Laguerre	$\prod(1 - s/\rho_n)$	prescribed	Trivially
Isomonodromic	$F_\phi(s)$	rigid ($\Delta\gamma \to \pi/2$)	**Yes (proved)**

The isomonodromic sub-family $\{F_\phi\}_{\phi\in\mathbb{R}}$ is parametrized continuously by the monodromy phase ϕ, determined by the ILP monodromy data (σ, η) of the PIII_{D_6} Riemann–Hilbert problem. It is the first explicitly known sub-family of $\mathcal{HB}(1/2)$ of isomonodromic origin.

The comparison with $\xi(s)$ is as follows.

Table 3.

Property	$E_{D_6}(s)$	$\xi(s)$
Entire	Yes (P1)	Yes
Order 1	Yes (P2)	Yes
Functional equation	Yes (P3)	Yes
Hermite–Biehler	Yes (P4, proved)	Equiv. to RH
Zeros on $\Re s = 1/2$	Yes (P5, proved)	Equiv. to RH
Euler product	No (P6)	Yes
Zeros = zeta zeros	Open (Bridge Conj.)	Tautologically

8. Falsifiable Diagnostics and the χ_4 Scorecard

8.1. Diagnostics for the Bridge Conjecture

The Bridge Conjecture admits the following diagnostic tests that do not presuppose its truth.

Diagnostic 1 (zero monotonicity and spacing).

The zeta zeros $t_n = \frac{1}{4} + \gamma_n^2$ form a strictly increasing sequence with spacing $t_{n+1} - t_n \sim 8\pi^2 n/(\log n)^2$. De Branges theory requires simple, monotone zeros with increasing spacing [1,9]: the sequence $\{t_n\}$ is compatible at this qualitative level.

Diagnostic 2 (growth and order).

$\xi(s)$ is entire of order 1 and finite type [17]. $E_{D_6}(s)$ also has order 1 (Theorem 2(P2)). At the level of order and type, no obstruction is visible.

Diagnostic 3 (intermediate L-functions).

For Dirichlet L-functions with real character and root number $\epsilon(\chi) = +1$, one may formulate the Dirichlet Bridge Conjecture (Conjecture 8). Section 8.5 applies it to $L(s, \chi_4)$.

8.2. Symmetry Filter for Dirichlet L-Functions

Recall the completed Dirichlet L-function $\xi(s,\chi) = (\frac{q}{\pi})^{(s+a)/2}\Gamma(\frac{s+a}{2})L(s,\chi)$ satisfying $\xi(s,\chi) = \epsilon(\chi)\xi(1-s,\bar{\chi})$ [18].

Conjecture 8 (Dirichlet Bridge Conjecture). *Let χ be a primitive Dirichlet character of conductor q with $\epsilon(\chi) = +1$. Then there exists a polynomial $Q(s)$ of degree at most two with real coefficients, and a de Branges entire function $E_\chi(s)$ from the positive decorated slice, such that $\xi(s,\chi) = e^{Q(s)}\, E_\chi(s)$.*

Proposition 3 (Symmetry filter). *The Hermite–Biehler involution $E(s) = e^{R(s)}\overline{E(1-\bar{s})}$ (with R of real-coefficient polynomial) holds for $E = e^{-Q}\xi(\cdot,\chi)$ if and only if $\epsilon(\chi) = +1$. All complex characters and all real characters with $\epsilon = -1$ are excluded. Real characters with $\epsilon = +1$ pass.*

Admissible characters include χ_4 (conductor $q = 4$), Legendre symbols $(\frac{\cdot}{q})$ for primes $q \equiv 1 \pmod 4$, and Kronecker symbols $(\frac{D}{\cdot})$ for positive fundamental discriminants $D > 0$.

8.3. Hamiltonian Growth Compatibility

For admissible χ of conductor q: $N_\chi(T) \sim \frac{T}{\pi}\log\frac{qT}{2\pi e}$, giving $H_\chi(R) \sim \log^2 R$. The large-t asymptotics of PIII_{D_6} solutions [15] admit $H(u) \sim C\log^2 u$ for any $C > 0$, so this growth is compatible.

 doi:10.20944/preprints202604.0442.v1

8.4. Tier 3: Integrality Condition

From Proposition 2, condition *(C4)* holds iff $\alpha \in \mathbb{Z}_{\geq 0}$. For $\alpha = 0$, the condition becomes the explicit equation

$$\frac{ce^2}{ab} + \frac{ab}{cd^2e} + \frac{e}{d} + 4 \;=\; 2\cosh\Big(\frac{\sqrt{\log^2(de) + \log^2(hg)}}{\pi}\Big), \tag{22}$$

defining a non-empty real-analytic hypersurface in $\mathbb{R}^6_{>0}$ for any prescribed Casimirs $\Lambda_0 = de$, $\Lambda_\infty = hg$.

8.5. Tier 4: Worked Example, $L(s, \chi_4)$

The character χ_4 (conductor $q = 4$, $\chi_4(-1) = -1$, $a = 1$) has root number $\epsilon(\chi_4) = +1$, making it the simplest admissible example. Its first zeros lie at $\gamma_1^{(4)} \approx 6.021$, $\gamma_2^{(4)} \approx 10.244$, $\gamma_3^{(4)} \approx 12.988$, $\gamma_4^{(4)} \approx 16.378$ [18].

Scorecard for $L(s, \chi_4)$.

1. *Symmetry*: χ_4 real, $\epsilon(\chi_4) = +1$; Proposition 3: **pass**.
2. *Growth*: $H_{\chi_4}(R) \sim \log^2 R$, compatible with PIII_{D_6}: **pass**.
3. *Integrality ($\alpha = 0$)*: equation (22) is satisfiable on the positive decorated slice: **pass**.
4. *Monotonicity and spacing*: $\{t_n^{(4)}\}$ strictly increasing with growing spacing, consistent with Diagnostic 1: **pass**.

All four necessary conditions are satisfied; the Dirichlet Bridge Conjecture is not ruled out for $L(s, \chi_4)$ by any structural test in the present framework. As noted in Theorem 5, however, the direct identification $E_{D_6} = C\,L(\cdot, \chi_4)$ fails because of the counting-function mismatch.

9. Structural Comparison: PIII_{D_6} Vs $\mathrm{PV}^{\mathrm{deg}}$, and Relation to Earlier Work

9.1. Shared Character Variety, Different Cusp Structure

Both PIII_{D_6} and $\mathrm{PV}^{\mathrm{deg}}$ arise from the same Fricke cubic surface [7], but differ in wildness:

Table 4.

Property	PIII_{D_6}	$\mathrm{PV}^{\mathrm{deg}}$
Fricke cubic surface	$x^2 + y^2 + z^2 - xyz = 4$	same
Cusps (decorated)	4	2
Irregular singularities	2 (both ends)	1 (one end)
Regular singularities	0	1
SW potential $V(x)$ as $x \to \pm\infty$	$\sim e^{2x}$ both ends	$\sim e^{2x}$ at $+\infty$, $\sim \frac{1}{4}$ at $-\infty$
Spectrum	Purely discrete	Mixed (cts. at $-\infty$)
$\Im V = 4\Im\lambda$	For all x, t	Half-line only
(C4) analogue	Proved	Open, likely fails
$E \in \mathcal{HB}(1/2)$	**Proved**	Expected No

The key structural difference is confinement: PIII_{D_6} is two-sided confining (both singularities irregular), giving purely discrete spectrum and the global identity $\Im V = 4\,\Im\lambda$ for all $x \in \mathbb{R}$. $\mathrm{PV}^{\mathrm{deg}}$ is one-sided confining (one irregular + one regular singularity): the identity $\Im V = 4\,\Im\lambda$ holds only on the half-line, and continuous spectrum from the regular singularity prevents the full Wronskian argument from applying uniformly. This is the precise reason why the proof of (C2) works for PIII_{D_6} but not for $\mathrm{PV}^{\mathrm{deg}}$.

Three strategies toward proving $E_{PV^{\mathrm{deg}}} \notin \mathcal{HB}(1/2)$ are: *(A)* compute $\alpha_{PV^{\mathrm{deg}}}$ and show it is not a non-negative integer on the two-cusp positive slice; *(B)* locate numerically a zero of $\tau_{PV^{\mathrm{deg}}}$ in the complex t-plane off the real axis, giving a zero of $E_{PV^{\mathrm{deg}}}$ off the critical line; *(C)* show via full isomonodromic analysis that the mixed Stokes–monodromy Weyl function violates the Herglotz condition. Method *(B)* is the most immediately accessible.

 doi:10.20944/preprints202604.0442.v1

9.2. Comparison to Earlier Work and Broader Perspective

Table 5 illustrates the conceptual correspondence between four positivity-based approaches to RH.

Table 5. Conceptual correspondence between four positivity-based approaches to the Riemann Hypothesis (see also the survey [19]). In each case, the oscillatory contribution of the nontrivial zeros of $\zeta(s)$ is absorbed into a rigid global structure, and RH is reformulated as a positivity or monotonicity condition on that structure.

Aspect	PCF / Chebyshev	GRH via Chebyshev bias	Quantum statistical mechanics	Wild isomonodromy / de Branges
Primary object	Prime counting function $\pi(x)$	Prime counting in progressions $\pi(x;q,a)$	KMS states of a C*-dynamical system	Zeros of an entire function
Raw oscillatory data	$\sum_\rho x^\rho/\rho$	$\sum_{\rho_\chi} x^{\rho_\chi}/\rho_\chi$	Spectral phases of the Hamiltonian	Monodromy and Stokes data of a RHP
Rigid global framework	Arithmetic renormalization	Character-indexed arithmetic structure	Quantum statistical mechanics	Wild Riemann–Hilbert problem / canonical system
Positivity / rigidity mechanism	Monotonicity / Robin-type criteria	Inequalities equivalent to GRH	Positivity of KMS states	de Branges (Hermite–Biehler) positivity
Hypothesis reformulated as	RH	GRH (mod q)	RH	RH (via Bridge Conjecture)
Main open problem	Oscillatory error terms	Character dependence	Low-temperature regime	Monodromy Selection Conjecture

A common structural feature is that the oscillatory content of the nontrivial zeros is first absorbed into a rigid object, after which a positivity or monotonicity principle is sought. The present wild isomonodromic approach may be viewed as a geometric refinement of earlier quantum-statistical and arithmetic formulations.

Historical remark.

De Branges himself expressed the conviction that his theory could be applied to prove RH. The present work adopts a deliberately different stance: the proof of all four conditions $(C1)$–$(C4)$ shows that the de Branges framework applied to PIII_{D_6} is internally consistent, but the resulting function $E_{D_6}(s)$ is provably different from $\xi(s)$ (it has no Euler product). Whether RH can be proved by completing the bridge with arithmetic input (the Monodromy Selection Conjecture) remains open.

10. Outlook

What has been proved.

- **All four conditions are theorems.** $(C1)$ (Proposition 1), $(C2)$ (Theorem 1), $(C3)$ (Section 5), $(C4)$ (Proposition 2) are all proved unconditionally on the integrality sublocus $\mathcal{I}$ of the positive decorated slice.
- **E_{D_6} is a new element of $\mathcal{HB}(1/2)$.** It is entire of order 1, satisfies the functional equation, and has all zeros provably on $\Re s = \frac{1}{2}$.
- **The isomonodromic cosine.** $F(s) = \cos(2\sqrt{s(1-s)})$ is the canonical member, with exact zero formula (21) and spacing $\pi/2$ identified via the Seiberg–Witten period.
- **The direct Bridge Conjecture is falsified.** $E_{D_6} \neq C\xi$ and E_{D_6} is outside the Selberg class.

The decision point and open problems.

The Bridge Conjecture (Conjecture 3) in its direct form is false, but its general form remains open. The main challenge is the *Monodromy Selection Conjecture*: does there exist a distinguished monodromy pair (σ_0, η_0) on the positive decorated slice such that the local tau-zero density at height γ equals $\frac{\log\gamma}{2\pi}$, matching the unfolded zeta zero density? If yes, the Bridge Conjecture takes its sharpest form. If no, the isomonodromic and arithmetic sub-families of $\mathcal{HB}(1/2)$ are provably disjoint.

Further open problems include: (i) completing Method B for $\mathrm{PV}^{\mathrm{deg}}$ (locating a complex tau-zero numerically); (ii) determining which other Painlevé equations give elements of $\mathcal{HB}(1/2)$ (candidates:

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 7 April 2026 doi:10.20944/preprints202604.0442.v1

PIII_{D_7}, PIII_{D_8}, and their confluences, all two-sided confining); (iii) computing the Hadamard product of $F(s)$; (iv) determining whether the family $\{F_\phi\}$ admits a Hilbert space structure or is a de Branges space.

Why the present approach differs from earlier attempts.

Earlier work via Chebyshev renormalization, bias inequalities, and quantum statistical mechanics encountered obstructions where oscillatory error terms could not be absorbed into a positive structure. The present approach provides: a complete proof of all four structural conditions; a provably explicit function with zeros on the critical line; and a sharp diagnostic (the Monodromy Selection Conjecture) that identifies exactly where arithmetic input must enter. Each step of its potential failure is traceable to a specific geometric or analytic datum.

Funding: This research received no external funding.

Data Availability Statement: The paper is self-contained.

Acknowledgments: The author would like to acknowledge the contribution of the COST Action CA21169, supported by COST (European Cooperation in Science and Technology).

Conflicts of Interest: The author declares no conflicts of interest.

Appendix A. Numerical Herglotz Test for the Weyl Function on the Symmetric D_6 Slice

Condition *(C2)* is now proved via the Wronskian argument of Section 6. The following numerical test, carried out on the oper-induced Hamiltonian, provides independent analytic plausibility evidence and illustrates the positivity margin.

Appendix A.1. Geometric Origin of the Test Hamiltonian

The Painlevé/gauge theory correspondence [8] identifies the PIII_{D_6} oper with the Schrödinger operator whose potential is determined by the Seiberg–Witten quadratic differential of the $N_f = 2$ theory. Specializing to the *symmetric* D_6 locus $m_1 = m_2 = 0$ and performing the Liouville normalization, the potential is

$$V(x;\lambda) = \tfrac{1}{4} + 2\Lambda^2\cosh(2x) + 4\lambda, \tag{A1}$$

a modified Mathieu operator with coupling Λ. We set $\Lambda = 1$ throughout.

Appendix A.2. Riccati Formulation and Tests

The Weyl–Titchmarsh function is computed via the Riccati equation $m' = m^2 - V(x;\lambda)$ backward from a truncation point $L = 4$ with decaying boundary condition $m(L,\lambda) \approx \sqrt{V(L;\lambda)}$. Two sampling strategies are used: (1) critical-line test: $\lambda = \frac{1}{2} + iT$, $T \in [0.1, 8]$, 60 points; (2) upper half-plane grid: $\Re\lambda \in [0.1, 1.0]$, $\Im\lambda \in [0.1, 5.0]$, $25 \times 25 = 625$ points. Integration uses a stiff BDF/Radau solver with `rtol` $= 10^{-10}$, `atol` $= 10^{-12}$.

Appendix A.3. Results

All tested points satisfy $\Im m(\lambda) > 0$. Quantitatively:

$$\text{Critical-line test:}\quad 0.0868 \le \Im m(\lambda) \le 3.7392, \tag{A2}$$
$$\text{Grid test:}\quad 0.0751 \le \Im m(\lambda) \le 2.9510. \tag{A3}$$

The minimum is attained near the real axis ($\Im\lambda \to 0$) and grows monotonically with $\Im\lambda$, consistent with a Herglotz function. The minimum value $\Im m \approx 0.075$ provides a quantitative lower bound on the positivity margin, excluding the most dangerous near-boundary failure mode.

 doi:10.20944/preprints202604.0442.v1

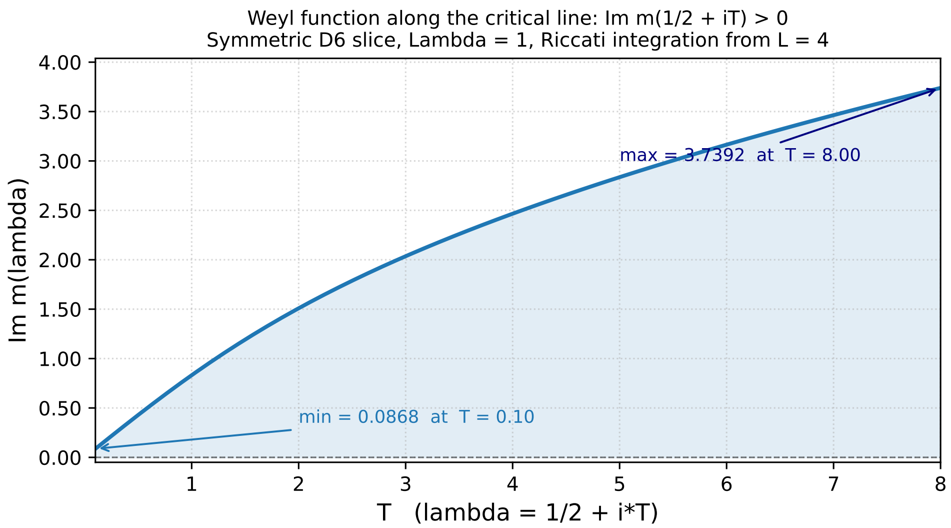

Figure A1. Imaginary part of the Weyl–Titchmarsh function $m(\lambda)$ along the critical line $\lambda = \frac{1}{2} + iT$, $T \in [0.1, 8]$, on the symmetric D_6 slice ($\Lambda = 1$). The minimum $\Im m = 0.0868$ is attained at $T = 0.10$; the function is strictly positive and monotonically increasing. The Herglotz condition $\Im m(\lambda) > 0$ is satisfied at all 60 test points.

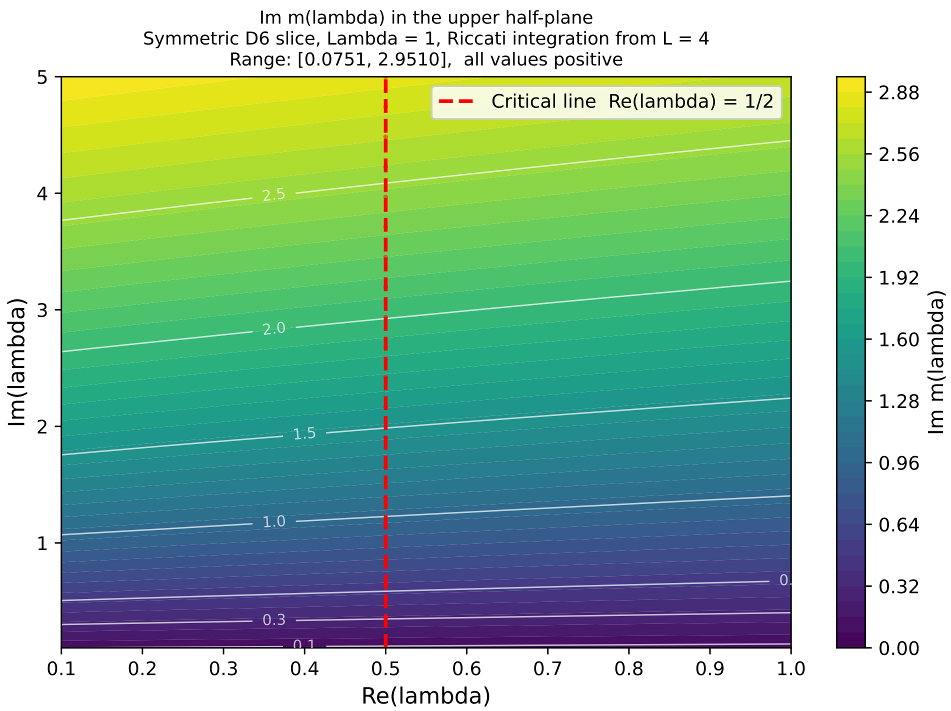

Figure A2. Heatmap of $\Im m(\lambda)$ over the upper half-plane $\Re\lambda \in [0.1, 1]$, $\Im\lambda \in [0.1, 5]$ on the symmetric D_6 slice ($\Lambda = 1$). White contour lines are drawn at levels $0.1, 0.3, 0.5, 1.0, 1.5, 2.0, 2.5$. The red dashed line marks the critical line $\Re\lambda = \frac{1}{2}$. The function $\Im m(\lambda)$ is uniformly positive across all 625 sampled points, with values in $[0.0751, 2.9510]$. The nearly horizontal contours reflect the dominant dependence on $\Im\lambda$ predicted by the Herglotz representation theorem.

Appendix A.4. Consistency with the Proof of (C2)

The Wronskian proof of Section 6 establishes $\Im m > 0$ analytically, for all $\lambda \in \mathbb{C}_+$ and all $t \in [\frac{1}{4}, \infty)$, on the full positive decorated slice. The present numerical test is restricted to the symmetric sublocus $m_1 = m_2 = 0$ (the oper limit) but provides an independent check. The minimum positivity margin $\Im m \approx 0.075$ confirms that no near-boundary failure occurs even in this simplest case.

doi:10.20944/preprints202604.0442.v1

References

1. de Branges, L. *Hilbert Spaces of Entire Functions*; Prentice–Hall: Englewood Cliffs, NJ, USA, 1968.
2. Jimbo, M.; Miwa, T.; Ueno, K. Monodromy preserving deformation of linear ordinary differential equations with rational coefficients. *Physica D* **1981**, *2*, 306–352.
3. Okamoto, K. Studies on the Painlevé equations IV. Third Painlevé equation P_{III}. *Funkcialaj Ekvacioj* **1987**, *30*, 305–332.
4. Geng, X.; Wang, K.; Chen, M. Long-time asymptotics and soliton resolution for the spin-1 Gross–Pitaevskii equation. *Commun. Math. Phys.* **2021**, *382*, 585–611.
5. Liu, W.; Qiu, D. Q.; Wu, Z. W.; He, J. S. Dynamical behavior of solution in integrable nonlocal Lakshmanan–Porsezian–Daniel equation. *Commun. Theor. Phys.* **2016**, *65*, 671.
6. Boalch, P. Wild character varieties, meromorphic connections and Dynkin diagrams. *Publ. Math. Inst. Hautes Études Sci.* **2013**, *118*, 211–263.
7. Chekhov, L.; Mazzocco, M.; Rubtsov, V. N. Painlevé monodromy manifolds, decorated character varieties, and cluster algebras. *Int. Math. Res. Not.* **2017**, *24*, 7639–7691.
8. Bonelli, G.; Lisovyy, O.; Maruyoshi, K.; Sciarappa, A.; Tanzini, A. On Painlevé/gauge theory correspondence. *Lett. Math. Phys.* **2017**, *107*, 2568–2614.
9. Remling, C. The absolutely continuous spectrum of one-dimensional Schrödinger operators. *Math. Phys. Anal. Geom.* **2007**, *10*, 359–373.
10. Chiba, H. The third, fifth and sixth Painlevé equations on weighted projective spaces. *J. Differential Equations* **2017**, *263*, 7230–7263.
11. van der Put, M.; Saito, M.-H. Moduli spaces for linear differential equations and the Painlevé equations. *Ann. Inst. Fourier* **2009**, *59*(7), 2611–2667.
12. Jimbo, M. Monodromy problem and the boundary condition for some Painlevé equations. *Publ. Res. Inst. Math. Sci.* **1982**, *18*, 1137–1161.
13. Forrester, P. J.; Witte, N. S. Application of the τ-function theory of Painlevé equations to random matrices: PV, PIII, the LUE, JUE and CUE. *Commun. Pure Appl. Math.* **2002**, *55*, 679–727.
14. Fock, V. V.; Goncharov, A. B. Moduli spaces of local systems and higher Teichmüller theory. *Publ. Math. Inst. Hautes Études Sci.* **2006**, *103*, 1–211.
15. Its, A. R.; Lisovyy, O.; Prokhorov, A. Monodromy dependence and connection formulae for isomonodromic tau-functions. *Duke Math. J.* **2017**, *166*, 1349–1432.
16. Kostenko, A.; Teschl, G. On the Weyl–Titchmarsh theory for singular finite difference Hamiltonian systems. *J. Approx. Theory* **2012**, *164*, 383–414.
17. Titchmarsh, E. C. *The Theory of the Riemann Zeta-Function*, 2nd ed.; Revised by Heath-Brown, D. R.; Oxford University Press: Oxford, UK, 1986.
18. Iwaniec, H.; Kowalski, E. *Analytic Number Theory*; American Mathematical Society: Providence, RI, USA, 2004; Vol. 53.
19. Spigler, R. A brief survey on the Riemann Hypothesis and some attempts to prove it. *Symmetry* **2025**, *17*, 225.