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Ji Peng *

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Article

Symmetry in Formal Calculation

Peng Ji

Department of Electronic Information, Nanjing University, Nanjing, China. mcfroo@sina.com

Abstract: Formal Calculation uses an auxiliary form to calculate various nested sums and provides results in three forms. It is also a powerful tool for analysis. This article studies the symmetry of the coefficients in Formal Calculation. Three types of extended numbers were introduced, and many formulas for symmetric functions were obtained.

Keywords: formal calculation; symmetry functions; stirling number; eulerian number

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1. Introduction

The notion of formal computation was introduced in [1].

Definition 1. *The definition of* ∇^p *is recursive.* $p \in \mathbb{Z}$,
$\nabla^0 f(n) = f(n),\ \sum\limits_{n=0}^{N-1} \nabla^1 f(n+1) = f(N),\ \sum\limits_{n=0}^{N-1} f(n+1) = \nabla^{-1} f(N), \nabla^1 = \nabla.$

Definition 2. *The definition of* $SUM(N) = SUM(N, PS, PT)$ *is recursive.* $K_i, D_i \in \mathbb{C}; T_i \in \mathbb{N}$.
$SUM(N, [K_1 : D_1], [T_1 = 1]) = \sum\limits_{n=0}^{N-1} (K_1 + nD_1).$
$SUM(N, [K_1 : D_1, K_2 : D_2], [T_1, T_2 = T_1 + 2 - p]) = \sum\limits_{n=0}^{N-1} (K_2 + nD_2) \nabla^p SUM(n+1, [K_1 : D_1], [T_1]).$

$[K_1 : D, K_2 : D ... K_M : D] = [K_1, K_2 ... K_M] : D$, $[K_1, K_2 ... K_M] : 1 = [K_1, K_2 ... K_M]$
Use $\mathbb{K}$ to represent the set $\{K_1 ... K_M\}$, $\mathbb{T}$ to represent the set $\{T_1 ... T_M\}$.
Use the auxiliary form: $(K_1 + T_1)(K_2 + T_2) ... (K_M + T_M) = \sum \prod_{i=1}^{M} X_i, X_i = T_i \text{ or } K_i$.

Definition 3. $X(T)$*=Number of* $\{X_1, X_2 ... X_M\} \in \mathbb{T}$.

Definition 4. X_{T-1}*=Number of* $\{X_1, X_2 ... X_{i-1}\} \in \mathbb{T}$, X_{K-1}*=Number of* $\{X_1, X_2 ... X_{i-1}\} \in \mathbb{K}$.

Obviously:$X_{T-1} + X_{K-1} = i - 1$.
Use the auxiliary form and each X_i cannot be exchanged, [1] draws conclusions:
$SUM(N, PS, PT) =$
$Form_1 \rightarrow \sum_{g=0}^{M} H_1(g) \binom{N+T_M-M}{N-1-g} = \sum_{g=0}^{M} H_1(g) \binom{N+T_M-M}{T_M-M+1+g},\ B_i = \begin{cases} (T_i - X_{K-1}) D_i,\ X_i = T_i \\ K_i + X_{T-1} D_i,\ X_i = K_i \end{cases},$
$Form_2 \rightarrow \sum_{g=0}^{M} H_2(g) \binom{N+T_M-M+g}{N-1} = \sum_{g=0}^{M} H_2(g) \binom{N+T_M-M+g}{T_M-M+1+g},\ B_i = \begin{cases} (T_i - X_{K-1}) D_i,\ X_i = T_i \\ K_i + (X_{K-1} - T_i) D_i,\ X_i = K_i \end{cases},$
$Form_3 \rightarrow \sum_{g=0}^{M} H_3(g) \binom{N+T_M-g}{N-1-g} = \sum_{g=0}^{M} H_3(g) \binom{N+T_M-g}{T_M+1},\ B_i = \begin{cases} -K_i + (T_i - X_{T-1}) D_i,\ X_i = T_i \\ K_i + X_{T-1} D_i,\ X_i = K_i \end{cases}.$
$H_i(g) = H_i(g, PS, PT) = H_i(g, M)$, is defined as $\sum_{X(T)=g} \prod_{i=1}^{M} B_i$.

For example:
$SUM(N, PS, [1, 2, 3 ... M]) = \sum_{n=0}^{N-1} \prod_{i=1}^{M} (K_i + nD_i)$.
$SUM(N, PS, [1, 3 ... 2M-1]) = \sum_{n_M=0}^{N-1} (K_M + n_M D_M) ... \sum_{n_2=0}^{n_3} (K_2 + n_2 D_2) \sum_{n_1=0}^{n_2} (K_1 + n_1 D_1)$.
$SUM(N, PS, [1, 2, 4]) = \sum_{n_3=0}^{N-1} (K_3 + n_3 D_3) \sum_{n=0}^{n_3} (K_1 + nD_1)(K_2 + nD_2)$.
$SUM(N, PS, [1, 3, 4]) = \sum_{n_3=0}^{N-1} (K_3 + n_3 D_3)(K_2 + n_3 D_2) \sum_{n=0}^{n_3} (K_1 + nD_1)$.

$PS = [K_1 : D_1, K_2 : D_2 ... K_M : D_M], PT = [T_1, T_2 ... T_M]$. There are recursive relationships:

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- $H_1(g,M) = (A_M + gD_M)H_1(g,M-1) + (B_M + (g-1)D_M)H_1(g-1,M-1)$,
  $A_M = K_M, B_M = D_M(T_M - ((i-1)-(g-1))) - (g-1)D_M = T_M D_M - (i-1)D_M$.
- $H_2(g,M) = (A_M - gD_M)H_2(g,M-1) + (B_M + (g-1)D_M)H_2(g-1,M-1)$,
  $A_M = K_M + (i-1-g-T_M)D_M + gD_M = K_M + (i-1-T_M)D_M, B_M = T_M D_M - (i-1)D_M$.
- $H_3(g,M) = (A_M + gD_M)H_3(g,M-1) + (B_M - (g-1)D_M)H_3(g-1,M-1)$,
  $A_M = K_M, B_M = -K_M + (T_M - (g-1))D_M + (g-1)D_M = -K_M + T_M D_M$.

Consider the general two-dimensional second-order linear recursive equations:
$R(M,g) = (A_M + gD_M)R(M-1,g) + (B_M + (g-1)E_M)R(M-1,g-1)$.
It can also be calculated in a similar way to $H(g)$, which is easier to understand. $H(g)$ itself requires $|D_i| = |E_i|$ and can't change the sign, so that three forms exist. Many conclusions have been drawn [1]:

1. $H_1(g) = \sum_{k=g}^{M} H_2(k)\binom{k}{g} = \sum_{k=0}^{g} H_3(k)\binom{M-k}{M-g}$.
2. $H_2(g) = \sum_{k=g}^{M} (-1)^{k+g} H_1(k)\binom{k}{g}$. $H_3(g) = \sum_{k=0}^{g} (-1)^{k+g} H_1(k)\binom{M-k}{M-g}$.
3. $\sum_{g=0}^{M} H_1(g)q^g(1-q)^{M-g} = \sum_{g=0}^{M} H_2(g)(1-q)^{M-g} = \sum_{k=0}^{M} H_3(g)q^g$
4. $\sum_{g=0}^{M} H_1(g) = \sum_{g=0}^{M} H_2(g)2^g = \sum_{g=0}^{M} H_3(g)2^{M-g}$
5. $\sum_{g=0}^{M} H_1(g)\binom{X}{Y-g} = \sum_{g=0}^{M} H_2(g)\binom{X+g}{Y} = \sum_{g=0}^{M} H_3(g)\binom{X+M-g}{Y-g}, Y \in \mathbb{N}, X \in \mathbb{C}$.
6. $SUM(N,[L_1,L_2...L_q,PS],[L_1,L_2...L_q,PT]) = \prod_{i=1}^{q} L_i \times SUM(N,PS,PT)$.

Recurrence relations $\to$ 1), inversion $\to$ 2). 1) $\to$ 3)4)5). 5) is the basis of $SUM(N)$.
6) show that T_1 can be great than 1. Regardless of the practical implications, we can make the definition domain of PT extend to $\mathbb{C}$.

When $D_i \neq 0, K_i + D_i n = D_i(\frac{K_i}{D_i} + 1)$, so only the case $D_i = 1$ needs to be dealt with. In this paper, if not specified, the default is $D_i = 1$. $PS = [K_1, K_2...K_M], PT = [T_1, T_2...T_M]$.

Definition 5. $F_g^{\mathbb{K}} = \sum\limits_{1 \le \lambda_1 < \lambda_2 < ... < \lambda_g \le M} K_{\lambda_1}K_{\lambda_2}...K_{\lambda_g}, F_0^{\mathbb{K}} = 1, F_g^N = F_g^{\{1,2...N\}}$.

Definition 6. $E_g^{\mathbb{K}} = \sum\limits_{1 \le \lambda_1 \le \lambda_2 \le ... \le \lambda_g \le M} K_{\lambda_1}K_{\lambda_2}...K_{\lambda_g}, E_0^{\mathbb{K}} = 1, E_g^N = E_g^{\{1,2...N\}}$.

$F_M^{N+M-1} = S_1(N+M,N)$, S_1 is unsigned stirling number of the first kind.
$E_M^N = S_2(N+M,N)$, S_2 is stirling number of the second kind.

2. Properties of H(g)

In the calculation of $H(g), \prod X_i = (\prod X_i, X_i \in \mathbb{T})(\prod X_i, X_i \in \mathbb{K})$.

Definition 7. $H(g,T) = H(g,T,PS,PT) = \prod_{X_i \in \mathbb{T}} B_i$, $H(g,\sum T) = \sum \prod_{X_i \in \mathbb{T}} B_i$.
Also define $H(g,K), H(g,\sum K)$.

Theorem 1.

1. $H_1(g,\sum K) = H_3(g,\sum K) = F_{M-g}^{\mathbb{K}}E_0^g + F_{M-g-1}^{\mathbb{K}}E_1^g + ... + F_0^{\mathbb{K}}E_{M-g}^g$.
2. $H_1(g,\sum T) = H_2(g,\sum T) = F_g^{\mathbb{T}}E_0^{M-g} - F_{g-1}^{\mathbb{T}}E_1^{M-g} + ... + (-1)^g F_0^{\mathbb{T}}E_g^{M-g}$.
3. $H_2(g,\sum K) = (-1)^{M-g}(F_{M-g}^{\mathbb{S}}E_0^g + ... + F_0^{\mathbb{S}}E_{M-g}^g), \mathbb{S} = \{T_i - K_i - i + 1\}$.
4. $H_3(g,\sum T) = (-1)^g F_g^{\mathbb{S}}E_0^{M-g} + (-1)^{g-1}F_{g-1}^{\mathbb{S}}E_1^{M-g} ... + F_0^{\mathbb{S}}E_g^{M-g}, \mathbb{S} = \{-(T_i - K_i - i + 1)\}$.
5. $H_1(g,\sum K) = \sum \prod_{i=1}^{M-g}(K_{i+\lambda_1} + \lambda_i), 0 \le \lambda_1 \le \lambda_2 \le ... \le \lambda_{M-g} \le g$.
6. $H_1(g,\sum T) = \sum \prod_{i=1}^{g}(T_{i+\lambda_1} - \lambda_i), 0 \le \lambda_1 \le \lambda_2 \le ... \le \lambda_g \le M-g$.

Proof.
$PS1 = [PS, K_{M+1}], PT1 = [PT, T_{M+1}]$. Using induction to prove 2).
$H_1(g,\sum T, PS1, PT1) = H_1(g,\sum T) + H_1(g-1,\sum T)(T_{M+1} - (M-g+1))$.
$F_{g-x}^{\{PT1\}}$ in $H_1(g,\sum T, PS1, PT1)$ has three sources.

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$= (-1)^x F^{\mathbb{T}}_{g-x} E_x^{M-g} + (-1)^x F^{\mathbb{T}}_{(g-1)-x} E_x^{M-(g-1)} T_{M+1}$
$+(-1)^{x-1} F^{\mathbb{T}}_{(g-1)-(x-1)} E_{x-1}^{M-(g-1)}(-(M-g+1))$
$= (-1)^x F^{\mathbb{T}}_{g-x}(E_x^{M-g} + E_{x-1}^{M+1-g}(M+1-g)) + (-1)^x F^{\mathbb{T}}_{g-x-1} E_x^{M+1-g} T_{M+1}$ (*)
$= (-1)^x F^{\mathbb{T}}_{g-x} E_x^{M+1-g} + (-1)^x F^{\mathbb{T}}_{g-x-1} E_x^{M+1-g} T_{M+1} = (-1)^x E_x^{M+1-g} F_{g-x}^{\{PT1\}}$.
$E_x^{M-g} + E_{x-1}^{M+1-g}(M+1-g)$
$= S_2(M-g+x, M-g) + (M+1-g) S_2(M-g+x, M+1-g)$
$= S_2(M-g+x+1, M+1-g) = E_x^{M+1-g} \rightarrow$ (*).
5) and 6) are definitions.
□

Theorem 2. $PS = [K_i : D_i], D_i \neq 0,$
$H_1(g) = (-1)^{M-g} H_2(g, [-K_i + T_i - (i-1) : D_i], PT) = (-1)^g H_3(g, PS, [\frac{K_i}{D_i} - T_i + i - 1]).$
$H_2(g) = (-1)^{M-g} H_1(g, [-K_i + T_i - (i-1) : D_i], PT), H_3(g) = (-1)^g H_1(g, PS, [\frac{K_i}{D_i} - T_i + i - 1]).$
$H_2(g) = (-1)^M H_3(g, [-T_i : D_i], [\frac{K_i}{D_i} - T_i + i - 1]).$
$H_3(g) = (-1)^M H_2(g, [-K_i + T_i - (i-1) : D_i], [-\frac{K_i}{D_i}]).$

3. Symmetric Expressions in H(g)

$PT = [T, T+1 \ldots T+M-1]$. It can be inferred from the definition of $SUM(N)$ that K_i can exchange orders. $H_1(g) = [T+g-1]_g H_1(g, \sum K)$. It is clearly a symmetric function of $\mathbb{K}$, we also reached the same conclusion. $H_2(g), H_3(g)$ are also symmetric functions, and there's no K_i^2 factors.

Definition 8. $F_T(N+M, N) = F_M^{\{T,T+1 \ldots T+N+M-1\}}, E_T(N+M, N) = E_M^{\{T,T+1 \ldots T+N-1\}}$.

Obviously:

- $F_T(0,0) = E_T(0,0) = 1, F_T(1,0) = E_T(1,0) = 0, F_T(1,1) = E_T(1,1) = 1.$
- $F_0(N+M, N) = S_1(N+M, N), E_1(N+M, N) = S_2(N+M, N).$
- $F_T(M+1, g+1) = F_{M-g}^{\{T,T+1 \ldots T+M\}} = (T+M) F_T(M, g+1) + F_T(M, g).$
- $E_T(M+1, g+1) = E_{M-g}^{\{T,T+1 \ldots T+g\}} = (T+g) E_T(M, g+1) + E_T(M, g).$
- $F_T(N+M, N) = SUM(N, [T, T+1 \ldots T+M-1], [1, 3 \ldots 2M-1]).$
- $E_T(N+M, N) = SUM(N, [T, T \ldots T], [1, 3 \ldots 2M-1]).$

Theorem 3. $PT = [T \ldots T+M-1], H_2(g, \sum K) = \sum_{j=0}^{M-g} (-1)^{M-g-j} F_j^{\mathbb{K}} E_{M-g-j}^{\{T,T+1 \ldots T+g\}}$.

Proof.
$H_2(0,1) = K_1 - T, H_2(1,1) = T$. It's holds when $M = 1$.
$H_2(g, M) = (K_M - T - g) H_2(g, M-1) + (T+g-1) H_2(g-1, M-1).$
$H_2(g)$ is a symmetric function $= \sum_{j=0}^{M} A_M^g(j) F_j^{\mathbb{K}}$

$A_1^0(0) = -T, A_1^1(0) = T.$
$A_M^g(0) = -(T+g) A_{M-1}^g(0) + (T+g-1) A_{M-1}^{g-1}(0) \rightarrow$
$A_M^g(0) = (-1)^{M-g} [T+g-1]_g E_T(M+1, g+1) = (-1)^{M-g} [T+g-1]_g E_{M-g}^{\{T,T+1 \ldots T+g\}}$.

The rest only need to consider terms that multiply with K_M.
$A_M^g(j) = A_{M-1}^g(j-1) = A_{M-j}^g(0) = (-1)^{M-g-j} [T+g-1]_g E_T(M+1-j, g+1).$
$H_2(g) = [T+g-1]_g \sum_{j=0}^{M-g} (-1)^{M-g-j} F_j^{\mathbb{K}} E_{M-g-j}^{\{T,T+1 \ldots T+g\}}$.
$H_2(g, T) = [T+g-1]_g \rightarrow$ the conclusion. □

Definition 9. $\left\langle {n \atop j} \right\rangle_T = (T-1+n-j) \left\langle {n-1 \atop j-1} \right\rangle_T + (j+1) \left\langle {n-1 \atop j} \right\rangle_T, \left\langle {1 \atop 0} \right\rangle_T = T, \left\langle {1 \atop 1} \right\rangle_T = 0, \left\langle {n \atop j>n, j<0} \right\rangle_T = 0.$

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Obviously: $\langle{n \atop 0}\rangle_1 = T, \langle{n \atop n}\rangle_T = 0.\ n > 0, \left\langle{n \atop j}\right\rangle_1 = \left\langle{n \atop j}\right\rangle$ is Eulerian number.

Definition 10. $\left\langle{M \atop g}\right\rangle_T^j = \sum_{i=0}^{j} (-1)^i \left\langle{M-j \atop g-1-i}\right\rangle_T \binom{j}{i}, 0 \le j < g, 0 < g < M.$

Theorem 4. $PT = [T, T+1 ... T+M-1], H_3(0) = \prod_{i=1}^{M} K_i, H_3(M) = \prod_{i=1}^{M} (T - K_i),$
$H_3(0 < g < M) = \sum_{j=0}^{M-1} \left\langle{M \atop g}\right\rangle_T^j F_j^{\mathbb{K}} + (-1)^g \binom{M}{g} F_M^{\mathbb{K}}.$

Proof.
The coefficient before $F_M^{\mathbb{K}}$ is obvious, so $H_3(g)$ can be written in that form.
$H_3(0 < g < M) = \sum_{j=0}^{M-1} A_M^g(j) F_j^{\mathbb{K}} + (-1)^g \binom{M}{g} F_M^{\mathbb{K}}.$
$H_3(0,1) = K_1, H_3(1,1) = T - K_1 \to A_1^0(0) = 0, A_1^1(0) = T.$
$H_3(g, M) = (K_M + g) H_3(g, M-1) + (T + M - g - K_M) H_3(g-1, M-1) \to$
$A_M^g(0) = g A_{M-1}^g(0) + (T + M - g) A_{M-1}^{g-1}(0) \to A_M^g(0) = \left\langle{M \atop g-1}\right\rangle_T.$
$A_M^g(j) = A_{M-1}^g(j-1) - A_{M-1}^{g-1}(j-1) \to A_M^g(j) = \left\langle{M \atop g}\right\rangle_T^j.$
□

Similarly, $PS = [K, K-1 ... K-M+1]$, then $H(g)$ are symmetric functions of $\mathbb{T}$.

Theorem 5. $PS = [K ... K-M+1], H_3(g, \sum T) = \sum_{j=0}^{g} (-1)^{g-j} F_j^{\mathbb{T}} E_{g-j}^{\{K, K-1 ... K-M+g\}}.$

Theorem 6. $PS = [K, K-1 ... K-M+1], H_2(M) = \prod_{i=1}^{M} T_i, H_2(0) = \prod_{i=1}^{M} (K - T_i),$
$H_2(0 < g < M) = \sum_{j=0}^{M-1} -\left\langle{M \atop M-g}\right\rangle_{-K}^j F_j^{\mathbb{T}} + (-1)^{M-g} \binom{M}{g} F_M^{\mathbb{T}}.$

Proof.
$H_2(0 < g < M) = \sum_{j=0}^{M-1} A_M^g(j) F_j^{\mathbb{T}} + (-1)^{M-g} \binom{M}{g} F_M^{\mathbb{T}}.$
$H_2(0,1) = K - T_1, H_2(1,1) = T_1 \to A_1^0(0) = K, A_1^1(0) = 0.$
$H_2(g, M) = (K - g - T_M) H_2(g, M-1) + (T_M - M + g) H_2(g-1, M-1) \to$
$A_M^g(0) = (K - g) A_{M-1}^g(0) - (M - g) A_{M-1}^{g-1}(0) \to A_M^g(0) = -\left\langle{M \atop M-g-1}\right\rangle_{-K}.$
$A_M^g(j) = -A_{M-1}^g(j-1) + A_{M-1}^{g-1}(j-1) \to A_M^g(j) = -\left\langle{M \atop M-g}\right\rangle_{-K}^j.$
□

4. Properties of Extended Numbers

Theorem 7.
$(x+T)(x+T+1)...(x+T+M-1) = \sum_{k=0}^{M} F_T(M,k) x^k.$
$(x+T)^M = \sum_{k=0}^{M} E_T(M+1, k+1) [x]_k.$

Proof.
$\nabla SUM(N, [T, T ... T], [1, 2 ... M]) = (T+n)^M = \sum_{g=0}^{M} H_1(g) \binom{n}{g}$
$= \sum_{g=0}^{M} g! E_{M-g}^{\{T, T+1 ... T+g\}} \binom{n}{g} = \sum_{g=0}^{M} g! E_T(M+1, g+1) \binom{n}{g} \to$ 2).
□

$x^M = \sum_{k=0}^{M} S_2(M,k) x^k$ because $E_0(M+1, k+1) = S_2(M,k)$.

Theorem 8. $E_T(M+1, g+1)$
$= \frac{1}{g!} \sum_{k=0}^{g} (-1)^{g+k} \binom{g}{k} (T+k)^M = \sum_{k=0}^{M-g} T^k \binom{M}{k} E_{M-k}^g.$

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Proof.
$H_1(g,[T,T...T],[1,2...M]) = g!E_T(M+1,g+1)$.
Based on Crammer's law, $H_1(g) = \sum_{k=1}^{g+1}(-1)^{g+1+k}\binom{T_M - M + g}{T_M - M + k - 1}\nabla SUM(k)$. [1]
$E_T(M+1,g+1) = \frac{1}{g!}\sum_{k=1}^{g+1}(-1)^{g+1+k}\binom{g}{k-1}(T+k-1)^M \to 1), 1 \to 2)$.
□

Theorem 9. $\left\langle {n \atop j} \right\rangle_{-1} = (-1)^{j-1}\binom{n-1}{j}$. $\left\langle {n \atop j} \right\rangle_{-2} = (-2)^{j-1}(\binom{n-2}{j} + 2^{n-1}\binom{n-2}{j})$.

Proof.
1) is to prove: $\binom{n-1}{j} = -(n-j-2)\binom{n-2}{j-1} + (j+1)\binom{n-2}{j}$.
$Right = -(n-1)\binom{n-2}{j-1} + (j+1)\binom{n-1}{j} = -j\binom{n-1}{j} + (j+1)\binom{n-1}{j} = Left$.
Prove 2) in a similar way.
□

By recurrence relation:

Theorem 10. $\left\langle {M \atop g} \right\rangle_T = H_3(g,[T,1...1],[T...T+M-1]) = T \times H_3(g,[1...1],[T+1...T+M-1])$.

Theorem 11. $T \in \mathbb{N}, \left\langle {M \atop g} \right\rangle_T = \frac{1}{(T-1)!}\sum_{k=0}^{g}(-1)^{g+k}\binom{T+M}{g-k}[T+k]_T(k+1)^{M-1}$
$= T \times (\sum_{k=0}^{M-2}\left\langle {M-1 \atop g} \right\rangle_{T+1}^{k}\binom{M-1}{k} + (-1)^g\binom{M-1}{g}), 0 < g < M-1.$

Proof.
Based on Crammer's law, $H_3(g) = \sum_{n=1}^{g+1}(-1)^{g+n+1}\binom{T_M+1}{g+1-n}\nabla SUM(n)$. [1]
$PS = [1...1], PT = [T+1...T+M-1], PS1 = [1,2...T,1...1], PT1 = [1,2...T,T+1...T+M-1]$.
$\left\langle {M \atop g} \right\rangle_T = T \times H_3(g) = \frac{1}{(T-1)!}H_3(g,PS1,PT1)$.
$= \frac{1}{(T-1)!}\sum_{n=1}^{g+1}(-1)^{g+n+1}\binom{T+M}{g+1-n}[T+n-1]_T \times n^{M-1} \to 1), 4 \to 2)$.
□

$\langle {M \atop 1} \rangle_T = T(T+1)2^{M-1} - T(T+M)$.

Theorem 12. $\sum_{g=0}^{M}(-1)^g g!E_T(M+1,g+1) = (T-1)^M$. $\sum_{g=0}^{M}\left\langle {M \atop g} \right\rangle_T = [T]^M$.

Proof.
$H_2(0) = (T-1)^M = \sum_{k=0}^{M}(-1)^k H_1(k) = \sum_{k=0}^{M}(-1)^k k!E_{M-k}^{\{T...T+k\}} \to 1)$.
$PS = [1...1], PT = [T+1...T+M-1], T \times H_1(M) = T \times \sum_{k=0}^{M}H_3(k) \to 2)$.
□

K_1 is a constant, $K_1K_2...K_M, K_{i+1} - K_i = 1$ or 2, there are M-1 intervals between factors. $K_{i+1} - K_i = 1$ is defined as continuity, $K_{i+1} - K_i = 2$ is defined as discontinuity.

Definition 11. $MIN_g^K(M) = K\sum K_2...K_M, K = K_1, MIN_g^1(M) = MIN_g(M)$, *count of discontinuities=g.*

Obviously, $0 \le g \le M-1$, $MIN_g^K(M)$ have $\binom{M-1}{g}$ items.
$PS = [T+1,T+2...T+M-1], PT = [T+2,T+4...T+2M-2], T \times H_1(g) = MIN_g^T(M)$
$PS = [T,T+1...T+M-1], PT = [T,T+2...T+2M-2], H_1(g) = MIN_g^T(M) + MIN_{g-1}^T(M)$
By definition:

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Theorem 13. $\lambda_1+\lambda_2+...+\lambda_{g+1}=M-g, \lambda_i\geq 0$,
$\left\langle {M \atop g} \right\rangle_T = T\sum 1^{\lambda_1}2^{\lambda_2}...(g+1)^{\lambda_{g+1}}(T+\lambda_1)(T+\lambda_1+\lambda_2)...(T+\lambda_1+...+\lambda_g)$.
$PS=[T,T...T], PT=[1,3...2M-1]$.
$H_1(g)=\sum T^{\lambda_1}(T+1)^{\lambda_2}...(T+g)^{\lambda_{g+1}}(1+\lambda_1)(3+\lambda_1+\lambda_2)...(2g-1+\lambda_1+...+\lambda_g)$.
$H_2(g)$=*Changed* $MIN_{g-1}(M)+MIN_g(M)$. *Select M-g from M factors, change i to (T-i).*
$PS=[T,T+1...T+M-1], PT=[1,3...2M-1]$.
$H_1(g)$=*Changed* $MIN_{g-1}(M)+MIN_g(M)$. *Select M-g from M factors, change i to (T+i-1).*
$H_2(g)=\sum(T-1)^{\lambda_1}(T-2)^{\lambda_2}...(T-g-1)^{\lambda_{g+1}}(1+\lambda_1)(3+\lambda_1+\lambda_2)...(2g-1+\lambda_1+...+\lambda_g)$.

For example, express the product in terms of ():
$MIN_0(3)+MIN_1(3)=(123)+(124)+(134)$,
$H_2(1)=(T-1,T-2,3)+(T-1,2,T-4)+(1,T-3,T-4)$.
$MIN_1(3)+MIN_2(3)=(124)+(134)+(135)$,
$H_1(2)=(T,2,4)+(1,T+2,3)+(1,3,T+4)$.

5. The Formulas for Symmetric Functions

If $1+K_i=K_{i+1}$ and $1+T_i=T_{i+1}$, $H_1(g)=\binom{M}{g}\prod_{i=1}^{g}T_i\prod_{i=g+1}^{M}K_i$. 1 $\rightarrow$ [1]:
$\sum_{j=0}^{M-g}F_j^{\{K,K+1...K+M-1\}}E_{M-g-j}^{g}=\prod_{i=g+1}^{M}(K+i-1)\binom{M}{g}$.
$\sum_{j=0}^{g}(-1)^jF_{g-j}^{\{T,T+1...T+M-1\}}E_j^{M-g}=\prod_{i=1}^{g}(T+i-1)\binom{M}{g}$.

1 $\rightarrow H_2(g,\sum K)=(-1)^{M-g}\sum_{j=0}^{M-g}F_j^{\{T-K_i\}}E_{M-g-j}^{g}$, 3 $\rightarrow$

Theorem 14. $\sum_{j=0}^{M-g}F_j^{\{T-K_i\}}E_{M-g-j}^{g}=\sum_{j=0}^{M-g}(-1)^jF_j^{\mathbb{K}}E_{M-g-j}^{\{T,T+1...T+g\}}$.

$K_i=T\rightarrow S_2(M,g)=\sum_{j=0}^{M-g}(-T)^j\binom{M}{j}E_T(M+1-j,g+1)$.
$K_i=T=1\rightarrow S_2(M,g)=\sum_{j=0}^{M-g}(-1)^j\binom{M}{j}S_2(M+1-j,g+1)$.
$K_i=-q\rightarrow \sum_{j=0}^{M-g}(T+q)^j\binom{M}{j}S_2(M-j,g)=\sum_{j=0}^{M-g}q^j\binom{M}{j}E_T(M+1-j,g+1)$.
$K_i=q, T=1\rightarrow \sum_{j=0}^{M-g}(1-q)^j\binom{M}{j}S_2(M-j,g)=\sum_{j=0}^{M-g}(-q)^j\binom{M}{j}S_2(M+1-j,g+1)$.

1 $\rightarrow PS=[K...K-M+1], H_3(g,\sum T)=\sum_{j=0}^{g}F_j^{\{T_i-K\}}E_{g-j}^{M-g}$, 5 $\rightarrow$

Theorem 15. $\sum_{j=0}^{M-g}F_j^{\{K_i-T\}}E_{M-g-j}^{g}=(-1)^{M-g}\sum_{j=0}^{M-g}(-1)^jF_j^{\mathbb{K}}E_{M-g-j}^{\{T,T-1...T-g\}}$.

Compared to 14, this is a little different.

Theorem 16. $PS1=[0,-1...-(p-1),K_{p+1}...K_M], PT1=[L_1,L_2...L_p,T_{p+1}...T_M]$,
$SUM(N,PS1,PT1)=\prod L_i\times SUM(N-p,[K_{p+1}+p...K_M+p],[T_{p+1}...T_M])$.
$\sum_{j=0}^{M-g}F_j^{\{0,-1...-(p-1),K_{p+1}...K_M\}}E_{M-g-j}^{g}=0, g<p$.
$\sum_{j=0}^{M-g-p}F_j^{\{0...-(p-1),K_{p+1}...K_M\}}E_{M-g-p-j}^{g+p}=\sum_{j=0}^{M-p-g}F_j^{\{K_{p+1}+p...K_M+p\}}E_{M-p-g-j}^{g}, 0\leq g\leq M-p$.

Proof.
By the definition of $H_1(g)$, there clearly is:
$H_1(g+p,PS1,PT1)=\prod L_i\times H_1(g,[K_{p+1}+p...K_M+p],[T_{p+1}...T_M])\rightarrow SUM(N)$.
Let $PT=[1,2...M]$, 1 $\rightarrow$ the rest.
□

Special:
$H_1(0,\sum K,[K_i+p],PT) \to \prod_{i=1}^{M}(p+K_i) = \sum_{j=0}^{M} F_j^{\{0,-1...-(p-1),K_1...K_M\}} E_{M-j}^p$.
$p^M = \sum_{j=0}^{p-1}(-1)^j F_j^{p-1} E_{M-j}^p, p>1$.

From $\prod_{i=1}^{M}(x+K_i) = \sum_{j=0}^{M} F_j^{\mathbb{K}} x^{M-j}$, it's easy to get:
$\sum_{j=0}^{M} k^{M-j} F_j^M = \prod_{i=1}^{M}(k+i)$. $\sum_{j=0}^{M}(-1)^j k^{M-j} F_j^M = 0, M \geq k \geq 1$.
The extension can be obtained with $H_3(g)$.
$PS = [0,-1...-(M-1)], PT = [T,T+1...T+M-1], H_3(g<M) = 0$,
$PS1 = PT1 = [T,T+1...T+M-1], H_3(g>0) = 0 \to$

Theorem 17.
$\sum_{j=0}^{M-1}(-1)^j \left\langle {M \atop g} \right\rangle_T^j F_j^{M-1} = 0, 0<g<M$.
$\sum_{j=0}^{M-1} \left\langle {M \atop g} \right\rangle_T^j F_j^{\mathbb{K}} + (-1)^g \binom{M}{g} F_M^{\mathbb{K}} = 0, 0<g<M, \mathbb{K} = \{T,T+1...T+M-1\}$.
$\left\langle {M \atop g} \right\rangle_T^j = \frac{1}{(T-1)!}\sum_{k=0}^{g}(-1)^{g+k}\binom{T+M}{g-k}[T+k]_T(k+1)^{M-1-j}, T \in \mathbb{N}$.

Proof.
Proof of the third equation. From $\sum_{j=0}^{M}(-1)^j k^{M-j} F_j^M = 0$ and the first equation, it can be seen that $\left\langle {M \atop g} \right\rangle_T^j$ and $\left\langle {M \atop g} \right\rangle_T^0 = \left\langle {M \atop g-1} \right\rangle_T$ have the same thing:
11 $\to \left\langle {M \atop g} \right\rangle_T^0 = a_1 1^x + a_2 2^y + ... \to \left\langle {M \atop g} \right\rangle_T^j = a_1 1^{x-j} + a_2 2^{y-j} + ... \to$ the expression.
The same conclusion can be obtained by combining definition of $\left\langle {M \atop g} \right\rangle_T^j$ and 11.
□

$PS = [T,T+1...T+p-1,K_{p+1}...K_M], PT = [T,T+1...T+M-1]$,
$H_3(g) = \prod_{i=1}^{p}(T+i-1)H_3(g,[K_{p+1}...K_M],[T+p...T+M-1]) \to$

Theorem 18. $\mathbb{K} = \{T,T+1...T+p-1,K_{p+1}...K_M\}, \mathbb{K}_{\not{K}} = \{K_{p+1}...K_M\}$,
$\sum_{j=0}^{M-1} \left\langle {M \atop g} \right\rangle_T^j F_j^{\mathbb{K}} + (-1)^g \binom{M}{g} F_M^{\mathbb{K}} = 0, M>g>M-p$.
$\sum_{j=0}^{M-1} \left\langle {M \atop g} \right\rangle_T^j F_j^{\mathbb{K}} + (-1)^g \binom{M}{g} F_M^{\mathbb{K}} =$
$\prod_{i=1}^{p}(T+i-1)(\sum_{j=0}^{M-p-1} \left\langle {M-p \atop g} \right\rangle_{T+p}^j F_j^{\mathbb{K}_{\not{K}}} + (-1)^g \binom{M-p}{g} F_{M-p}^{\mathbb{K}_{\not{K}}}), 0<g<M-p$.

$PS = [K,K-1..K-M+1], PT = [0,1...M-1], H_2(g>0) = 0$,
$PS1 = PT1 = [T,T+1...T+M-1], H_2(g<M) = 0 \to$

Theorem 19.
$\sum_{j=0}^{M-1} - \left\langle {M \atop M-g} \right\rangle_{-K}^j F_j^{M-1} = 0, 0<g<M$.
$\sum_{j=0}^{M-1} - \left\langle {M \atop M-g} \right\rangle_{1-M-T}^j F_j^{\mathbb{K}} + (-1)^{M-g}\binom{M}{g} F_M^{\mathbb{K}} = 0, 0<g<M, \mathbb{K} = \{T,T+1...T+M-1\}$.

$PS = [K,K-1...K-M+1], PT = [K,K-1...K-p+1,T_{p+1}...T_M]$,
$H_2(g+p) = \prod_{i=1}^{p}(K-i+1)H_2(g,[K-p...K-M+1],[T_{p+1}...T_M]) \to$

Theorem 20. $\mathbb{T} = \{K,K-1...K-p+1,T_{p+1}...T_M\}, \mathbb{T}_{\not{K}} = \{T_{p+1}...T_M\}$,
$\sum_{j=0}^{M-1} - \left\langle {M \atop M-g} \right\rangle_{-K}^j F_j^{\mathbb{T}} + (-1)^{M-g}\binom{M}{g} F_M^{\mathbb{T}} = 0, 0<g<p$.

$\sum_{j=0}^{M-1} -\left\langle {M \atop M-p-g} \right\rangle_{-K}^{j} F_j^{\mathbb{T}} + (-1)^{M-p-g} \binom{M}{g+p} F_M^{\mathbb{T}} =$
$\prod_{i=1}^{p} (K-i+1)(\sum_{j=0}^{M-p-1} -\left\langle {M-p \atop M-p-g} \right\rangle_{p-K}^{j} F_j^{\mathbb{T}_{\not K}} + (-1)^{g} \binom{M-p}{g} F_{M-p}^{\mathbb{T}_{\not K}}), 0 < g < M-p.$

Theorem 21. *$T \in \mathbb{Z}, p \in \mathbb{N}, 0 \le g \le M, p \le M,$*
$\sum_{j=0}^{M-g} F_j^{\{T,T+1...T+p-1,K_{p+1}...K_M\}} E_{M-g-j}^{g} =$
$\sum_{i=0}^{p} \binom{p}{i} (T+p+g-i-1)_{p-i} \sum_{j=0}^{M-p-g+i} F_j^{\{K_{p+1}...K_M\}} E_{M-p-g+i-j}^{g-i}.$

Proof.
$PS1 = [T, T+1...T+(p-1), K_{p+1}...K_M], PT1 = [T, T+1...T+M-1],$
$PS2 = [K_{p+1}...K_M], PT2 = [T+p-1...T+M-1].$
$H_1(g, [T, PS], [T, PT]) = T \times H_1(g) + T \times H_1(g-1) \rightarrow$
$H_1(g, PS1, PT1) = [T+g-1]_g H_1(g, \sum K, PS1, PT1)$
$= [T+p-1]_p \sum_{i=0}^{p} \binom{p}{i} H_1(g-i, PS2, PT2).$
$= [T+p-1]_p \sum_{i=0}^{p} \binom{p}{i} [T+p-1+g-i]_{g-i} H_1(g-i, \sum K, PS2, PT2).$
□

$K_i = 0, g = M-p \rightarrow \sum_{j=0}^{p} F_j^{\{T...T+p-1\}} E_{p-j}^{M-p} = \sum_{i=0}^{p} \binom{p}{i} [T-1+M-i]_{p-i} E_i^{M-p-i}.$
$g = 1 \rightarrow \sum_{j=0}^{M-1} F_j^{\{T...T+p-1,K_{p+1}...K_M\}} = \sum_{i=0}^{p} \binom{p}{i} [T-1+p-i]_{p-i} \sum_{j=0}^{M-p-1+i} F_j^{\{K_{p+1}...K_M\}}.$
$p = M-1 \rightarrow \sum_{j=0}^{p} F_j^{\{T,T+1...T+p-1\}} = [T+p]_p.$

6. PT=PS and Its Promotion

$H_1(g, PT, PT) = \prod_{i=1}^{M} T_i \binom{M}{g}$, promoting it:

Theorem 22. *$PS = [T_1, T_2...T_{M-p}, 0, -1... -(p-1)]$, then*
$H_1(g) = \prod_{i=1}^{M} T_i \binom{M-p}{g-p}, H_2(g) = (-1)^{M-g} \prod_{i=1}^{M} T_i \binom{p}{M-g},$
$H_3(p) = \prod_{i=1}^{M} T_i, H_3(g \neq p) = 0, SUM(N) = \prod_{i=1}^{M} T_i \binom{N+T_M-p}{T_M+1}.$

Proof.
$H_1(g, M) = (K_M + g) H_1(g, M-1) + (T_M - M + g) H_1(g-1, M-1)$
Using induction and the recurrence relationship one can obtain $H_1(g)$.
$PT1 = [T_{M-p+1}...T_M]$, 2 $\rightarrow$
$H_2(g, [0... -(p-1)], PT1) = (-1)^{M-g} H_1(g, PT1, PT1) = (-1)^{M-g} \prod_{i=M-p+1}^{M} T_i \binom{p}{g}.$
$H_2(g) = \prod_{j=1}^{M-p} T_i \times H_2(g-(M-p), [0... -(p-1)], PT1) \rightarrow H_2(g).$
$H_3(g)$ can be obtained from the definition. $H_3(g) \rightarrow SUM(N).$
□

Theorem 23. *$PS = [0, -1... -(M-p-1), T_{M-p+1} - (M-p)...T_M - (M-p)]$, then*
$H_2(g) = (-1)^{M-g} \prod_{i=1}^{M} T_i \binom{M-p}{g-p}, H_1(g) = \prod_{i=1}^{M} T_i \binom{p}{M-g},$
$H_3(M-p) = \prod_{i=1}^{M} T_i, H_3(g \neq M-p) = 0, SUM(N) = \prod_{i=1}^{M} T_i \binom{N+T_M-M+p}{T_M+1}.$

Proof.
2 & 22 $\rightarrow H_2(g).$
$p = 0, H_1(M) = \prod_{i=1}^{M} T_i$. Recurrence relation & induction on p $\rightarrow H_1(g).$

$H_3(g)$ can be obtained from the definition. $H_3(g) \to SUM(N)$.
□

Using similar methods:

Theorem 24. $PS = [T_1, T_2 ... T_{M-p}, 0, -1 ... -(p-1)]$,
$PT = [0, 1 ... (M-p-1), (M-p) - T_{M-p+1} ... (M-p) - T_M]$, *then*
$H_3(g) = (-1)^g \prod_{i=1}^{M} T_i \binom{M-p}{g-p}$, $H_1(p) = (-1)^p \prod_{i=1}^{M} T_i$, $H_1(g \neq p) = 0$,
$H_2(g) = (-1)^g \prod_{i=1}^{M} T_i \binom{M-p}{g}$,
$SUM(N) = (-1)^p \prod_{i=1}^{M} T_i \binom{N-T_M-p}{1-T_M}$, $p > 0$; $SUM(N) = \prod_{i=1}^{M} T_i$, $p = 0$.

7. Transformation of SUM(N)

$\prod_{i=1}^{M} (x + K_i) = \sum_{g=0}^{M} a_g x^{M-g} = \sum_{g=0}^{M} F_g^{\mathbb{K}} x^{M-g} = \nabla SUM(x+1, PS, [1, 2...M])$.
No need to know the value of K_i. Any polynomial can be converted to $\nabla SUM(N, PS, [1, 2...])$.

By choosing c and $K_1', K_2'...$ appropriately, $SUM(N)$ can be converted to $c \times \nabla^q SUM(N, [K_1', K_2'...], PT1)$. However, it is generally necessary to solve higher-order equations to solve for $\mathbb{K}'$. But specifies that every nested sum can be flattened, converting to $c \times \nabla^q SUM(N, PS, [1, 2...])$.

$(1) = \sum_{g=0}^{M} b_g \binom{N+Y}{g+X+1}$,$(2) = \sum_{g=0}^{M} c_g \binom{N+Y+g}{g+X+1}$, $(3) = \sum_{g=0}^{M} d_g \binom{N+Y+M+X-g}{M+1+X}$, can be converted to $c \times \nabla^{-X} SUM(N+Y-X, PS, [1, 2...M])$. $c = \frac{b_M}{M!} = \frac{c_M}{M!} = \frac{d_M}{\prod(1-K_i)}$. It is only necessary to solve the system of linear equations to find $F_j^{\mathbb{K}}$. so them can also be converted to $\sum_{g=0}^{M} a_g x^{M-g}$.
Special:
$b_0 = b_1 = ... = b_q = 0 \to K_1 = 0, K_2 = -1 ... K_q = -(q-1)$.
$c_0 = c_1 = ... = c_q = 0 \to K_1 = 1, K_2 = 2 ... K_q = q$.
$d_0 = d_1 = ... = d_q = 0 \to K_1 = 0, K_2 = -1 ... K_q = -(q-1)$.

$(1), (2), (3)$ can also be converted to $c \times \nabla SUM(N+Y-X, PS, [X+1, X+2 ... X+M])$.
$c = \frac{b_M}{(X+1)...(X+M)} = \frac{c_M}{(X+1)...(X+M)} = \frac{d_M}{\prod(X+1-K_i)}$.
Special: $c_0 = c_1 = ... = c_q = 0 \to K_1 = X+1, K_2 = X+2 ... K_q = X+q$.

References

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