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Article

On Weak Probabilistic φ -Contractions in Menger Spaces

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Abstract: In this paper, we introduce two new concepts: $(0, \infty)$ -dense set and weak probabilistic φ -contraction. Then we prove a fixed point theorem for weak probabilistic φ -contraction in Menger spaces, which generalizes and unifies several results in existing literatures. Finally, we provide an example to demonstrate the validity of our results.

Keywords: $(0, \infty)$ -dense set; probabilistic φ -contraction; weak probabilistic φ -contraction; menger spaces

1. Introduction

Recall a mapping $T : X \rightarrow X$ is said a probabilistic k -contraction if it satisfies $F_{Tx, Ty}(kt) \geq F_{x, y}(t)$ for all $x, y \in X$ and $t > 0$, where $k \in (0, 1)$.

And a mapping $T : X \rightarrow X$ is called a probabilistic φ -contraction if it satisfies $F_{Tx, Ty}(\varphi(t)) \geq F_{x, y}(t)$ for all $x, y \in X$ and $t > 0$, where $\varphi : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is a gauge function satisfying certain conditions.

In 1972, Sehgal and Bharucha-Ried proved the following theorem which is the probabilistic version of the classical Banach Contraction Principle.

Theorem 1. ([9]) Let (X, F, Δ_M) be a complete Menger space with t-norm Δ_M defined by $\Delta_M = \min\{a, b\}$. If $T : X \rightarrow X$ is a probabilistic k -contraction, then T is a Picard mapping.

Since then, different authors studied probabilistic φ -contractions and got some fixed point theorems. But, they assumed that φ satisfied $\sum_{n=1}^{\infty} \varphi^n(t) < \infty$ for all $t > 0$ to get corresponding results. As [2] has pointed out, the condition is very strong and difficult for testing in practice. Therefore, a natural question arises whether this condition can be omitted or improved.

Ćirić [2] have made a positive attempt. The following theorem is a revised result of [2].

Theorem 2. ([2,7]) Let (X, F, Δ) be a complete Menger space with Δ of H -type and $\varphi : [0, \infty) \rightarrow [0, \infty)$ be a function satisfying the condition:

$$\varphi(0) = 0, \varphi(t) < t \text{ and } \lim_{r \rightarrow t^+} \sup \varphi(t) < t \text{ for all } t > 0.$$

If T is a probabilistic φ -contraction, then T is a Picard mapping.

Later, Jachymski in [7] obtained the following result as a generalization of Theorem 1.2.

Theorem 3. ([7]) Let (X, F, Δ) be a complete Menger space with Δ of H -type and $\varphi : [0, \infty) \rightarrow [0, \infty)$ be a function such that $0 < \varphi(t) < t$ and $\lim_{n \rightarrow \infty} \varphi^n(t) = 0$ for all $t > 0$. If T is a probabilistic φ -contraction, then T is a Picard mapping.

Fang [4] relaxed restrictions on the gauge functions and gave a more general class of functions denoted by Φ_ω , where Φ_ω is the class of gauge functions $\phi_\omega : [0, \infty) \rightarrow [0, \infty)$ satisfying the condition: for each $t > 0$ there exists $r \geq t$ such that $\lim_{n \rightarrow \infty} \phi_\omega^n(r) = 0$. And he obtained the following theorem as a generalization of Theorem 1.3.

Theorem 4. ([4]) Let (X, F, Δ) be a complete Menger space with Δ of H -type and $\phi_\omega \in \Phi_\omega$. If T is a probabilistic ϕ_ω -contraction, then T is a Picard mapping.

Mihet and Zaharia [8] investigated the existence of fixed points for several classes of probabilistic ϕ -contractions under Fang-type conditions. They also raised several open questions, one of which is whether Theorem 1.3 and Theorem 1.4 are equivalent?

Alegre and Romaguera [1], Gregori et al. [5] proved the equivalence of Theorem 1.3 and Theorem 1.4 using different methods. In their work, they used the set $A = \{t > 0 \mid \lim_{n \rightarrow \infty} \phi^n(t) = 0\}$ or $A' = \{t > 0 \mid \lim_{n \rightarrow \infty} \phi^n(t) \neq 0\}$, respectively, where A' is the complement of set A . It can be seen from their work that set A plays an important role.

Taking inspiration from set A , we propose a new concept in this paper, called $(0, \infty)$ -dense set, which plays a crucial role in the definition of weak probabilistic ϕ -contraction.

We notice that a probabilistic ϕ -contraction involves the following three aspects:

- (i) Two sets: X and R^+ ;
- (ii) Two mappings: $T : X \rightarrow X$ and $\phi : R^+ \rightarrow R^+$;
- (iii) The bridge to contact the above two sets and two mappings:
 $F_{Tx, Ty}(\phi(t)) \geq F_{x, y}(t)$ for all $x, y \in X$ and $t \in R^+ - \{0\}$.

In this paper, we introduce a "small" set which we call $(0, \infty)$ -dense set. The reason why we say it "small" set is because sometimes the elements contained in the set is relatively little, and sometimes its Lebesgue measure may be 0.

we use $(0, \infty)$ -dense set—the "small" set P to replace R^+ and then define a weak probabilistic ϕ -contraction which involves the following three aspects:

- (i)' Two sets: X and P ;
- (ii)' Two mappings: $T : X \rightarrow X$ and $\phi : P \rightarrow P$;
- (iii)' The bridge to contact the above two sets and two mappings:
 $F_{Tx, Ty}(\phi(t)) \geq F_{x, y}(t)$ for all $x, y \in X$ and $t \in P - \{0\}$.

Because P is relatively small, the mapping $\phi : P \rightarrow P$ is relatively simple. Thus, the relational formula $F_{Tx, Ty}(\phi(t)) \geq F_{x, y}(t)$ which is difficult to be verified and satisfied in the whole scope is relatively easy to be verified and satisfied. In fact, in this paper, the condition " $0 < \phi(t) < t$ and $\lim_{n \rightarrow \infty} \phi^n(t) = 0$ for all $t > 0$ " in Theorem 1.3 can be weakened to the condition " $\lim_{n \rightarrow \infty} \phi^n(p) = 0$ for $t \in P - \{0\}$, where $P \subset R^+$ is a $(0, \infty)$ -dense set."

2. Preliminaries

Throughout this paper, D_+ is the space of all mappings $F : R \rightarrow [0, 1]$, such that F is left-continuous, non-decreasing on R , $F(0) = 0$ and $F(+\infty) = 1$.

A triangular norm (for short t -norm) is a binary operation $\Delta : [0, 1] \times [0, 1] \rightarrow [0, 1]$ which is commutative, associative, monotone and has 1 as the unit element. Basic examples are the Łukasiewicz t -norm Δ_L , $\Delta_L(a, b) = \max(a + b - 1, 0)$ and the t -norms Δ_P, Δ_M , where $\Delta_P(a, b) = ab$, $\Delta_M(a, b) = \min\{a, b\}$. The minimum t -norm Δ_M is the strongest t -norm.

If Δ is a t -norm, then $\Delta^n(t)$ is defined for every $t \in [0, 1]$ and $n \in N$ as 1 if $n = 0$ and $\Delta^n(t) = \Delta(\Delta^{n-1}(t), t)$ if $n \geq 1$. A t -norm Δ is said to be of H -type if the family $\{\Delta^n(t) : n \in N\}$ is equicontinuous at $t = 1$. Obviously, the minimum Δ_M is a t -norm of H -type, in addition, there are a large number of t -norms of H -type.

Definition 2.1. ([6]) A Menger space is a triple (X, F, Δ) where X is a nonempty set, Δ is a t -norm and F is a mapping from $X \times X$ to D_+ , ($F(x, y)$ is denoted by $F_{x, y}$), satisfying the following conditions:

- (PM1) $F_{x, y}(t) = 1$ for all $t > 0$ iff $x = y$,
- (PM2) $F_{x, y}(t) = F_{y, x}(t)$ for all $x, y \in X$,
- (PM3) $F_{x, z}(t + s) \geq \Delta(F_{x, y}(t), F_{y, z}(s))$ for $x, y, z \in X, t, s \geq 0$.

Let (X, F, Δ) be a Menger space. The (ϵ, λ) –topology in X is introduced by the family of neighborhoods $\{U_{\epsilon, \lambda}(u) : u \in X, \epsilon > 0, \lambda > 0\}$, where $U_{\epsilon, \lambda}(u) = \{v \in X : F_{u,v}(\epsilon) > 1 - \lambda\}$. If the t-norm Δ is such that $\sup_{a < 1} T(a, a) = 1$, then in the (ϵ, λ) –topology X is a metrizable topological space.

A sequence $\{x_n\} \subset X$ converges to some point $x \in X$ in the (ϵ, λ) –topology if and only if for every $\epsilon > 0$ and $\lambda \in (0, 1)$ there exists $N_{\epsilon, \lambda} \in \mathbb{N}$ such that

$$F_{x_n, x}(\epsilon) > 1 - \lambda.$$

A sequence $\{x_n\}$ in X is a Cauchy sequence if and only if for every $\epsilon > 0$ and $\lambda \in (0, 1)$ there exists $N_{\epsilon, \lambda} \in \mathbb{N}$ such that

$$F_{x_n, x_m}(\epsilon) > 1 - \lambda \text{ for all } m, n > N_{\epsilon, \lambda}.$$

A Menger probabilistic metric space (X, F, Δ) is said to be complete if every Cauchy sequence in X is convergent.

Definition 2.2. ([3]) A function $\phi : [0, \infty) \rightarrow [0, \infty)$ is said to be a Φ -function if it satisfies the following conditions:

- (i) $\phi(t) = 0$ if and only if $t = 0$.
- (ii) ϕ is strictly increasing and $\phi(t) \rightarrow \infty$ as $t \rightarrow \infty$.
- (iii) ϕ is left-continuous in $(0, \infty)$.
- (iv) ϕ is continuous at 0.

The class of all functions ϕ satisfying (i)-(iv) will be denoted by Φ .

Based on Φ -functions, a type of altering distance functions in probabilistic metric spaces, Choudhury and Das introduced a new type of contraction mapping and obtained the following theorem [3].

Theorem 5. ([3]) Let (X, F, Δ_M) be a complete Menger probabilistic metric space with $\Delta_M = \min\{a, b\}$ and $T : X \rightarrow X$ satisfies the following conditions:

$$F_{Tx, Ty}(\phi(ct)) \geq F_{x, y}(\phi(t)) \text{ for } x, y \in X \text{ and } t > 0,$$

where ϕ is a Φ -function and $c \in (0, 1)$, then T has a unique fixed point.

3. Main Results

Definition 3.1. A set $P \subset \mathbb{R}^+$ is said a $(0, \infty)$ -dense set if for each $n \in \mathbb{N}$ there exists $p_1, p_2 \in P$ such that $p_1 \geq n$ and $p_2 \leq 1/n$.

Example 3.2. Let $P = \mathbb{R}^+$, then P is a $(0, \infty)$ -dense set.

Example 3.3. Let $P_1 = \mathbb{N} \cup \{0\}$, $P_2 = \mathbb{N} \cup \{\frac{1}{n} : n \in \mathbb{N}\}$, $P_3 = \{m + \frac{1}{n} : m, n \in \mathbb{N}\}$, then P_1, P_2, P_3 are $(0, \infty)$ -dense sets. It is obvious that P_1, P_2, P_3 are countable and $u(P_i) = 0$ for $i = 1, 2, 3$, where u is the Lebesgue measure .

Definition 3.4. A function $\varphi : P \rightarrow P$ is said a weak gauge function if $P \subset \mathbb{R}^+$ is a $(0, \infty)$ -dense set and $\lim_{n \rightarrow \infty} \varphi^n(p) = 0$ for each $p \in P$.

Example 3.5. Let $P = P_1 = \mathbb{N} \cup \{0\}$ and $\varphi : P \rightarrow P$ defined by

$$\varphi(p) = \begin{cases} p - 1, & \text{if } p > 1 \\ 0, & \text{if } p = 1 \end{cases},$$

then φ is a weak gauge function.

Example 3.6. Let $P = P_2 = N \cup \{\frac{1}{n} : n \in N\}$ and $\varphi : P \rightarrow P$ defined by

$$\varphi(p) = \begin{cases} n-1, & \text{if } p = n \geq 2 \\ \frac{1}{n+1}, & \text{if } p = \frac{1}{n} \end{cases},$$

then φ is a weak gauge function.

Denote $\Psi(P)$ the class of weak gauge functions for a $(0, \infty)$ -dense set P .

Definition 3.7. Let (X, F, Δ) be a Menger space. A mapping $T : X \rightarrow X$ is called a weak probabilistic φ -contraction if there exists a $(0, \infty)$ -dense set P and $\varphi \in \Psi(P)$ such that

$$F_{Tx, Ty}(\varphi(t)) \geq F_{x, y}(t) \quad (3.1)$$

for $x, y \in X$ and $t \in P - \{0\}$.

The following lemma plays a crucial role in the proof of our main results.

Lemma 3.8. Let (X, F, Δ) be a Menger space and $T : X \rightarrow X$ is a weak probabilistic φ -contraction for $\varphi \in \Phi(P)$, where P is a $(0, \infty)$ -dense set. Then for each $t > 0$ and $t_0 \in P - \{0\}$, there exists $n_0 \in N$ such that

- (i) $\varphi^{n_0+1}(t_0) < \varphi^{n_0}(t_0) < t$;
- (ii) $\varphi^n(t_0) < t$ as $n \geq n_0$.

Proof. Firstly, we show that for $t_0 \in P - \{0\}$ and $n \in N, \varphi^n(t_0) > 0$.

Conversely, suppose that there exists $n \in N$ such that $\varphi^n(t_0) = 0$, then

$0 = F_{T^n x, T^n x}(0) = F_{T^n x, T^n x}(\varphi^n(t_0)) \geq F_{T^{n-1}x, T^{n-1}x}(\varphi^{n-1}(t_0)) \geq \dots \geq F_{x, x}(t_0) = 1$, which is a contradiction.

Secondly, since $\lim_{n \rightarrow \infty} \varphi^n(t_0) = 0$, then for each $t > 0$, there exists $n_1 \in N$ such that when $n \geq n_1$,

$$\varphi^n(t_0) < t.$$

We claim there exists $n_0 \geq n_1$ such that $\varphi^{n_0+1}(t_0) < \varphi^{n_0}(t_0)$.

Suppose that for $n \geq n_1$, $\varphi^{n+1}(t_0) \geq \varphi^n(t_0)$. Then we have $\lim_{n \rightarrow \infty} \varphi^n(t_0) \geq \varphi^{n_1}(t_0) > 0$, which is a contradiction. Thus, for each $t > 0$ and $t_0 \in P - \{0\}$, there exists $n_0 \in N$ such that

- (i) $\varphi^{n_0+1}(t_0) < \varphi^{n_0}(t_0) < t$;
- (ii) $\varphi^n(t_0) < t$ as $n \geq n_0$. $\square$

Theorem 6. Let (X, F, Δ) be a complete Menger space with Δ of H -type and $T : X \rightarrow X$ is a weak probabilistic φ -contraction for some $(0, \infty)$ -dense set P with $\varphi \in \Phi(P)$. Then T is a Picard mapping.

Proof. Let $x_0 \in X$ be an arbitrary point, we define the sequence $\{x_n\}$ in X by $x_n = T^n x_0$ for all $n \in N$. If $x_{n+1} = x_n$ for some $n \in N$, then T has a fixed point. Therefore, in the following proof, we can suppose $x_{n+1} \neq x_n$ for each $n \in N$.

We complete the proof by the following five steps.

Step 1. We show that for $\forall t > 0$,

$$\lim_{n \rightarrow \infty} F_{x_n, x_{n+1}}(t) = 1. \quad (3.2)$$

Fix $t > 0$ and let $\epsilon \in (0, 1]$. Since $\lim_{t \rightarrow \infty} F_{x_0, x_1}(t) = 1$, there exists $t_0 \in P$ such that $F_{x_0, x_1}(t_0) > 1 - \epsilon$.

For $t > 0$ and $t_0 \in P$, by Lemma 3.8, there exists $n_0 \in N$ such that when $n \geq n_0$, $\varphi^n(t_0) < t$. Then

$F_{x_n, x_{n+1}}(t) \geq F_{x_n, x_{n+1}}(\varphi^n(t_0)) \geq \cdots \geq F_{x_0, x_1}(t_0) > 1 - \epsilon$,
thus, we have $\lim_{n \rightarrow \infty} F_{x_n, x_{n+1}}(t) = 1$.

$$\text{Step 2. } F_{x_n, x_{n+k}}(\varphi^{n_0}(t_0)) \geq \Delta^{k-1}(F_{x_n, x_{n+1}}(\varphi^{n_0}(t_0) - \varphi^{n_0+1}(t_0))) \quad (3.3)$$

for $k \in N$, where $t_0 \in P$, $\varphi^{n_0+1}(t_0) < \varphi^{n_0}(t_0)$.

It is obvious for $k = 1$, since

$$F_{x_n, x_{n+1}}(\varphi^{n_0}(t_0)) \geq F_{x_n, x_{n+1}}(\varphi^{n_0}(t_0) - \varphi^{n_0+1}(t_0)) = \Delta^0(F_{x_n, x_{n+1}}(\varphi^{n_0}(t_0) - \varphi^{n_0+1}(t_0))).$$

Assume that (3.3) holds for some k , then

$$\begin{aligned} F_{x_n, x_{n+k+1}}(\varphi^{n_0}(t_0)) &= F_{x_n, x_{n+k+1}}(\varphi^{n_0}(t_0) - \varphi^{n_0+1}(t_0) + \varphi^{n_0+1}(t_0)) \\ &\geq \Delta(F_{x_n, x_{n+1}}(\varphi^{n_0}(t_0) - \varphi^{n_0+1}(t_0)), F_{x_{n+1}, x_{n+k+1}}(\varphi^{n_0+1}(t_0))) \\ &\geq \Delta(F_{x_n, x_{n+1}}(\varphi^{n_0}(t_0) - \varphi^{n_0+1}(t_0)), F_{x_n, x_{n+k}}(\varphi^{n_0}(t_0))) \\ &\geq \Delta(F_{x_n, x_{n+1}}(\varphi^{n_0}(t_0) - \varphi^{n_0+1}(t_0)), \Delta^{k-1}(F_{x_n, x_{n+1}}(\varphi^{n_0}(t_0) - \varphi^{n_0+1}(t_0)))) \\ &= \Delta^k(F_{x_n, x_{n+1}}(\varphi^{n_0}(t_0) - \varphi^{n_0+1}(t_0))). \end{aligned}$$

Step 3. $\{x_n\}$ is a Cauchy sequence in X .

For $t > 0$, as proved in Lemma 3.8, we can find $t_0 \in P$ and n_0 such that,

$$\varphi^{n_0+1}(t_0) < \varphi^{n_0}(t_0) < t. \text{ Then}$$

$$F_{x_n, x_{n+k}}(t) \geq F_{x_n, x_{n+k}}(\varphi^{n_0}(t_0)) \geq \Delta^{k-1}(F_{x_n, x_{n+1}}(\varphi^{n_0}(t_0) - \varphi^{n_0+1}(t_0)))$$

Let $\epsilon \in (0, 1]$, since $\{\Delta^n(t)\}$ is equicontinuous at $t = 1$ and $\Delta^n(1) = 1$, so there exists $\delta > 0$ such that

$$\Delta^n(s) > 1 - \epsilon \text{ for all } s \in (1 - \delta, 1] \text{ and } n \in N. \quad (3.4)$$

By **step 1**, we have that $\lim_{n \rightarrow \infty} F_{x_n, x_{n+1}}(\varphi^{n_0}(t_0) - \varphi^{n_0+1}(t_0)) = 1$.

Thus there exists $n_1 > n_0$ such that $F_{x_n, x_{n+1}}(\varphi^{n_0}(t_0) - \varphi^{n_0+1}(t_0)) > 1 - \delta$ for all $n > n_1$.

It follows from (3.4) that

$$F_{x_n, x_{n+k}}(t) \geq \Delta^{k-1}(F_{x_n, x_{n+1}}(\varphi^{n_0}(t_0) - \varphi^{n_0+1}(t_0))) > 1 - \epsilon \text{ for all } n > n_1 \text{ and } k \in N.$$

Thus, $\{x_n\}$ is a Cauchy sequence in X .

Step 4. T has a fixed point.

Since X is complete and $\{x_n\}$ is a Cauchy sequence in X , there exists a $x^* \in X$ such that $\lim_{n \rightarrow \infty} F_{x_n, x^*}(t) = 1$ for $\forall t > 0$. For $t > 0$, as proved in Lemma 3.8, we can find $t_0 \in P$ and n_0 such that,

$$\varphi^{n_0+1}(t_0) < \varphi^{n_0}(t_0) < t.$$

Then,

$$\begin{aligned} F_{x^*, Tx^*}(t) &\geq F_{x^*, Tx^*}(\varphi^{n_0}(t_0)) \\ &\geq \Delta(F_{x^*, x_{n+1}}(\varphi^{n_0}(t_0) - \varphi^{n_0+1}(t_0)), F_{x_{n+1}, Tx^*}(\varphi^{n_0+1}(t_0))) \\ &\geq \Delta(F_{x^*, x_{n+1}}(\varphi^{n_0}(t_0) - \varphi^{n_0+1}(t_0)), F_{x_n, x^*}(\varphi^{n_0}(t_0))) \\ &\geq \Delta(b_n, b_n), \end{aligned}$$

where $b_n = \min\{F_{x^*, x_{n+1}}(\varphi^{n_0}(t_0) - \varphi^{n_0+1}(t_0)), F_{x_n, x^*}(\varphi^{n_0}(t_0))\}$.

Since $\lim_{n \rightarrow \infty} b_n = 1$ and Δ is continuous, passing $n \rightarrow \infty$, we get

$$F_{x^*, Tx^*}(t) = 1 \text{ for all } t > 0.$$

Thus, $Tx^* = x^*$

Step 5. T has at most one fixed point.

Suppose, on the contrary, that there exists another fixed point y^* of T such that

$$Tx^* = x^* \neq Ty^* = y^*.$$

Then, there exists $t_1 > 0$ such that $F_{x^*, y^*}(t_1) < 1$.

Since $\lim_{t \rightarrow \infty} F_{x^*, y^*}(t) = 1$, there exists $t_0 \in P$ and $t_0 > t_1$ such that

$$F_{x^*, y^*}(t_0) > F_{x^*, y^*}(t_1).$$

By Lemma 3.8, there exists $n_0 \in N$ such that $\varphi^{n_0}(t_0) < t_1$.

$$\text{Then we have } F_{x^*, y^*}(t_1) \geq F_{x^*, y^*}(\varphi^{n_0}(t_0)) \geq F_{x^*, y^*}(t_0) > F_{x^*, y^*}(t_1),$$

which is a contradiction. Therefore, the fixed point of T is unique. $\square$

Next, we will demonstrate the generality and validity of our results. Firstly, we are going to show that Theorem 1.4 can be obtained as an easy consequence of Theorem 3.9.

Proof of Theorem 1.4. Let (X, F, Δ) be a complete Menger space with Δ of H -type and $\phi_\omega \in \Phi_\omega$ and let T be a probabilistic ϕ_ω -contraction.

Let $P = \{r : \lim_{n \rightarrow \infty} \phi_\omega^n(r) = 0\}$, then P is a $(0, \infty)$ -dense set by the definition of Φ_ω .

Define $\varphi : P \rightarrow P$ by $\varphi(p) = \phi_\omega(p)$ for $p \in P$, then φ is well-defined and $\varphi \in \Psi(P)$. In fact, for $p \in P$, by the definition of P , $\lim_{n \rightarrow \infty} \phi_\omega^n(p) = 0$, then

$$\lim_{n \rightarrow \infty} \phi_\omega^n(\varphi(p)) = \lim_{n \rightarrow \infty} \phi_\omega^n(\phi_\omega(p)) = \lim_{n \rightarrow \infty} \phi_\omega^{n+1}(p) = 0.$$

Thus, $\varphi(p) \in P$.

$$\varphi^2(p) = \varphi(\varphi(p)) = \phi_\omega(\varphi(p)) = \phi_\omega(\phi_\omega(p)) = \phi_\omega^2(p).$$

We can get $\varphi^n(p) = \phi_\omega^n(p)$ for $n \in N$ in a similar way.

So, $\lim_{n \rightarrow \infty} \varphi^n(p) = \lim_{n \rightarrow \infty} \phi_\omega^n(p) = 0$. Thus, $\varphi \in \Psi(P)$.

Since, $F_{Tx, Ty}(\phi_\omega(t)) \geq F_{x, y}(t)$ for all $t > 0$. In particular, $F_{Tx, Ty}(\phi_\omega(t)) \geq F_{x, y}(t)$ for all $t \in P - \{0\}$.

That is, $F_{Tx, Ty}(\varphi(t)) \geq F_{x, y}(t)$ for all $t \in P - \{0\}$. So, T is a weak probabilistic φ -contraction. Therefore, we can apply Theorem 3.9 and thus T is a Picard mapping.

Secondly, we use Theorem 3.9 to prove a more general theorem than Theorem 2.3. As a consequence, we answer an open question raised by Choudhury and Das [3].

Theorem 7. Suppose (X, F, Δ) is a complete Menger space, Δ is a t -norm of H -type. Suppose that $T : X \rightarrow X$ satisfies:

$$F_{Tx, Ty}(\phi(ct)) \geq F_{x, y}(\phi(t))$$

for $x, y \in X, t > 0$, where $\phi \in \Phi, c \in (0, 1)$, then T is a Picard mapping.

Proof. Let $P = \{\phi(t) : t \in R^+\}$, then P is a $(0, \infty)$ -dense set by the definition of Φ .

Define $\varphi : P \rightarrow P$ by $\varphi(p) = \phi(c\phi^{-1}(p))$ for $p \in P$, then φ is well-defined and $\varphi \in \Psi(P)$. In fact, for $p \in P$, by the definition of P , $\varphi(p) \in P$.

$$\varphi^2(p) = \varphi(\varphi(p)) = \phi(c\phi^{-1}(\varphi(p))) = \phi(c\phi^{-1}(\phi(c\phi^{-1}(p)))) = \phi(c^2\phi^{-1}(p)).$$

We can get $\varphi^n(p) = \phi(c^n\phi^{-1}(p))$ for $n \in N$ in a similar way.

By the definition of Φ , $\lim_{n \rightarrow \infty} \phi(c^n\phi^{-1}(p)) = 0$. So, $\lim_{n \rightarrow \infty} \varphi^n(p) = \lim_{n \rightarrow \infty} \phi_\omega^n(p) = 0$. Thus, $\varphi \in \Psi(P)$.

Since, $F_{Tx, Ty}(\phi(ct)) \geq F_{x, y}(\phi(t))$ for all $t > 0$ is equivalent to

$$F_{Tx, Ty}(\phi(c\phi^{-1}(s))) \geq F_{x, y}(s) \text{ for } s = \phi(t) \in P.$$

That is, $F_{Tx, Ty}(\varphi(s)) \geq F_{x, y}(s)$ for all $s \in P - \{0\}$. So, T is a weak probabilistic φ -contraction. Therefore, we can apply Theorem 3.9 and thus T is a Picard mapping. $\square$

Finally, we provide an example to demonstrate the validity of our results.

Example 3.9. Let $X = [0, +\infty)$ with the metric $d(x, y) = |x - y|$, then (X, d) is a complete metric space.

Let $F_{x, y}(t) = H(t - d(x, y)) = \begin{cases} 0, & \text{if } t \leq d(x, y) \\ 1, & \text{if } t > d(x, y) \end{cases}$, then (X, F, Δ) is a complete Menger space, where Δ is a t -norm of H -type.

Now, define $T : X \rightarrow X$ by $Tx = \frac{x}{2}$ and $\varphi : X \rightarrow X$ by $\varphi(t) = \begin{cases} \frac{t}{2}, & \text{if } t \in P \\ \frac{t}{3}, & \text{if } t \in X - P \end{cases}$,

where P is the set of all rational numbers of X .

We notice that $\varphi(P) \subset P$, that is $\varphi|_P : P \rightarrow P$.

In the following, we show that T is a weak probabilistic φ -contraction while T is not a probabilistic φ -contraction. In fact, $\frac{t}{2} \leq d(\frac{x}{2}, \frac{y}{2}) \Leftrightarrow t \leq d(x, y)$ and $\frac{t}{2} > d(\frac{x}{2}, \frac{y}{2}) \Leftrightarrow t > d(x, y)$. So if $t \in P$, then $F_{Tx, Ty}(\varphi(t)) = F_{\frac{x}{2}, \frac{y}{2}}(\frac{t}{2}) = F_{x, y}(t)$.

But, T is not a probabilistic φ -contraction. To see it, let $x = 2, y = 4$ and $t = e$, where $e = \lim_{n \rightarrow \infty} (1 + \frac{1}{n})^n$, then $F_{x, y}(t) = F_{2, 4}(e) = 1$, while $F_{Tx, Ty}(\varphi(t)) = F_{1, 2}(\frac{e}{3}) = 0$.

4. Conclusions

In this paper, we introduce the concept of $(0, \infty)$ -dense set, and based on which, we give the concept of weak probabilistic φ -contraction. We have proved a new fixed point theorem for weak probabilistic contraction in Menger spaces with a t-norm of H -type, namely Theorem 3.9. It is worth noting that in the theorem, the contractive gauge function φ only needs to meet a quite weak condition. Therefore, this theorem improves, generalizes and unifies some important fixed point theorems, including the results of Ćirić [2], Jachymski [7], Fang [4] and Choudhury and Das [3]. The weak gauge function and weak probabilistic φ -contraction proposed in this paper can also be considered as an answer to the question raised in the Conclusion section of [4].

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References

1. C. Alegre, S. Romaguera, A note on φ -contractions in probabilistic and fuzzy metric Spaces, *Fuzzy Sets Syst.* 313 (2017) 119-121.
2. L. Ćirić, Solving the Banach fixed point principle for nonlinear contractions in probabilistic metric spaces, *Nonlinear Anal.* 72 (2010) 2009-2018.
3. B. S. Choudhury, K. Das, A new contraction principle in Menger spaces, *Acta Math. Sin. (Engl. Ser.)*, 24 (2008) 1379-1386.
4. J. X. Fang, On φ -contractions in probabilistic and fuzzy metric spaces, *Fuzzy Sets Syst.* 267 (2015) 86-99.
5. V. Gregori, J. J. Minana, S. Morillas, On probabilistic φ -contractions in Menger spaces, *Fuzzy Sets Syst.* 313 (2017) 114-118.
6. O. Hadžić, E. Pap, *Fixed Point Theory in Probabilistic Metric Spaces*, Kluwer Academic Publishers, 2001.
7. J. Jachymski, On probabilistic φ -contractions on Menger spaces, *Nonlinear Anal.* 73 (2010) 2199-2203.
8. D. Mihet, C. Zaharia, On some classes of nonlinear contractions in probabilistic metric spaces, *Fuzzy Sets Syst.* 300 (2016) 84-92.
9. V. M. Sehgal, A. T. Bharucha-Reid, Fixed points of contraction mappings on PM-spaces, *Math. Syst. Theory* 6 (1972) 97-102.

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