Review

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The Spacetime Algebra and Basic General Relativity

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Review

The Spacetime Algebra and Basic General Relativity

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Abstract

This pedagogical paper presents General Relativity (GR) using the Spacetime Algebra (STA) in an approachable style allowing for a more accessible entry-point for students and professors. To maintain a focused and instructive tone, this paper concisely presents the STA and applies it in an equally concise and geometrically-intuitive crash-course to GR. This new, brief, yet sufficiently expansive pedagogical paper on the application of the STA to GR will serve students and professors well.

Keywords: geometric algebra; spacetime algebra; general relativity

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1. Foreword

This pedagogical paper was inspired by [1–3], which all outline notationally different yet equivalent methods of formulating General Relativity (GR) in the Spacetime Algebra (STA). However, while indeed more pedagogical than most papers and books (and still being excellent reads), a beginner would still find them to be difficult to parse. So the goal of this paper is to bridge the gap and present GR using STA in a *geometrically-intuitive style*, aimed for students and professors. It is the hope of the author that this paper can be a mostly self-contained crash-course to both the STA and using it for GR.[1] Before continuing, the author would like to reiterate that [1–3] are excellent resources and should be read after this if more depth is desired.

The scope and study of GR is quite far-reaching, and presenting it in its entirety would undermine the pedagogical nature of this writ. Therefore the derivation of the Einstein equation from a stationary action will be ignored in favor of simply introducing the equation *after* deriving the curvature tensors (multivectors). Likewise the discussion of the Christoffel symbols will be limited to the presentation in Equation (4.22) and Equation (4.23). For more information on both the stationary action derivation and Christoffel symbols within the Geometric Algebra formalism, [1] is recommended with fervor.

2. Conceptual Overview

Conceptually, GR is not so complicated. It states that mass curves spacetime, which in turn modifies the dynamics. Constructing GR in the STA will be very straightforward: First, one must posit that every coordinate x *infinitesimally close* to another coordinate x' is related via an infinitesimal *Lorentz transformation*. Second, one must create a derivative and gradient which transform *covariantly* as one moves throughout coordinate space (thereby keeping tally of curvature that might or might not be there). Using these covariant derivatives (which represent parallel-transporting a multivector field, and therefore represent the action of the spacetime curvature on said field), one then builds the appropriate curvature multivectors (tensors). This ends up being the *Einstein vector* (tensor) which expresses the "curvature flux" through the hyperplane geometrically represented by the vector. Finally, one obtains the *Einstein equation* by setting the Einstein vector equal to the *stress-energy vector* (tensor). The stress-energy vector expresses the "energy flux" through the hyperplane of the vector. Thus, the Einstein equation is a statement that the energy flux[2] results in a curvature flux.

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[1] A funny man would say this approach should be called STAGR, because of its staggering simplicity and geometric clarity.
[2] Recall that mass and energy are equivalent concepts from Special Relativity.

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3. The Spacetime Algebra

Both an algebraic and geometric interpretation are now to be given. Such a step was seen as necessary for the reader to fully grasp the approach to General Relativity (GR) in the Spacetime Algebra (STA). Moreover, including such an introduction will allow this document to serve as a mostly standalone text for learning the STA. This crash-course was written with reference to [1,4–6], with notation particularly inspired by [6].

3.1. Algebraic Essentials

The *Spacetime Algebra* is a *real* geometric (Clifford) algebra generated by the orthonormal vector basis

$$\gamma_0, \quad \gamma_1, \quad \gamma_2, \quad \gamma_3 \tag{3.1}$$

which satisfies the inner product

$$\gamma_\mu \cdot \gamma_\nu = \frac{1}{2}(\gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu) = \eta_{\mu\nu}, \tag{3.2}$$

where $\eta_{\mu\nu}$ is the (mostly-minus) Minkowski metric, and $\mu, \nu = 0, 1, 2, 3$. This implies that $\gamma_0^2 = +1$, while $\gamma_1^2 = \gamma_2^2 = \gamma_3^2 = -1$. As there is one *unipotent* basis vector with three *anti-unipotent* basis vectors, this algebra is denoted $\mathbb{G}_{1,3}$. Moreover, in the context of Minkowski geometric algebras, $\mathbb{G}_{1,n}$, positive squaring objects are referred to as *timelike*, negative squaring objects are referred to as *spacelike*, and nilpotent objects are referred to as *lightlike*. The basis in Equation (3.1) is called the *standard basis*, as opposed to the *reciprocal basis*

$$\gamma^0, \quad \gamma^1, \quad \gamma^2, \quad \gamma^3 \tag{3.3}$$

which satisfies the inner product

$$\gamma^\mu \cdot \gamma_\nu = \frac{1}{2}(\gamma^\mu\gamma_\nu + \gamma_\nu\gamma^\mu) = \delta^\mu_\nu, \tag{3.4}$$

where δ^μ_ν is the Kronecker delta. This implies $\gamma^j = -\gamma_j$ for $j = 1, 2, 3$. Any vector a can be written as a linear combination of the standard basis

$$a = a^\mu\gamma_\mu = (a^0\gamma_0 + a^1\gamma_1 + a^2\gamma_2 + a^3\gamma_3) \tag{3.5}$$

or of the reciprocal basis

$$a = a_\mu\gamma^\mu = (a_0\gamma^0 + a_1\gamma^1 + a_2\gamma^2 + a_3\gamma^3). \tag{3.6}$$

Both Equation (3.5) and Equation (3.6) are given for consistency and to make it easier to connect to traditional formalisms. However, in general it is possible to work coordinate-free. The (geometric) product of any two vectors a and b is

$$ab = a \cdot b + a \wedge b. \tag{3.7}$$

It is the sum of their *inner product*

$$a \cdot b = \frac{1}{2}(ab + ba) \tag{3.8}$$

and their *outer product*

$$a \wedge b = \frac{1}{2}(ab - ba). \tag{3.9}$$

In coordinate form their inner product is $a \cdot b = a^\mu b_\mu = a_\mu b^\mu$, and their outer product is $a \wedge b = a^\mu b^\nu \gamma_\mu\gamma_\nu(1 - \delta_{\mu\nu})$. The quantities $\gamma_\mu\gamma_\nu = \gamma_\mu \wedge \gamma_\nu = \gamma_{\mu\nu}$ are the orthonormal basis *bivectors*,

$$\gamma_{10}, \quad \gamma_{20}, \quad \gamma_{30}, \quad \gamma_{12}, \quad \gamma_{23}, \quad \gamma_{31} \tag{3.10}$$

where the γ_{j0} are timelike and the γ_{jk} are spacelike.

Before introducing the geometric product between bivectors, or between vectors and bivectors, a few concepts must be discussed. Firstly, geometric algebras are $\mathbb{Z}_2$-*graded*. Scalars are grade-0, vectors are grade-1, and bivectors are grade-2. For a d-dimensional algebra there are $d+1$ distinct grades, from grade-0 to grade-d, and in general a grade-k object is called a k-vector. Thus for the Spacetime Algebra, there are 5 grades, from grade-0 to grade-4. Scalars are grade-0, vectors are grade-1, and bivectors are grade-2. The orthonormal basis for *trivectors*, the grade-3 objects, is

$$\gamma_{123}, \quad \gamma_{012}, \quad \gamma_{023}, \quad \gamma_{031} \tag{3.11}$$

and the basis for *quadravectors*, the grade-4 objects, is

$$\gamma_{0123}. \tag{3.12}$$

Because there is only one basis element for grade-4, it is called the *pseudoscalar*[3] and is relabeled as $I = \gamma_{0123}$. The Spacetime Algebra can be written as the direct sum of its graded subspaces,

$$\mathbb{G}_{1,3} = \bigoplus_{j=0}^{4} \mathbb{G}_{1,3}^{j} = \mathbb{G}_{1,3}^{0\oplus\cdots\oplus 4}. \tag{3.13}$$

A general element of the Spacetime Algebra M is called a *multivector*, and is a linear combination of all bases, including 1, and the bases in Equation (3.1), Equation (3.10), Equation (3.11), and Equation (3.12). If J is the multivector index for this *complete algebraic basis* $\{1, \gamma_0, \ldots, \gamma_3, \gamma_{12}, \ldots, \gamma_{30}, \gamma_{123}, \cdots, \gamma_{031}, I\}$, then

$$M = M^{\mathrm{J}}\gamma_{\mathrm{J}}. \tag{3.14}$$

Alternatively, the *grade projection* can be defined as

$$\langle M \rangle_j \in \mathbb{G}_{1,3}^{j}, \tag{3.15}$$

satisfying

$$\langle M \rangle_{j\oplus k} = \langle M \rangle_j + \langle M \rangle_k. \tag{3.16}$$

It is often convenient to express multivectors as the sum of their graded subcomponents, as will be seen in the upcoming discussion of *conjugation*. Another useful concept is that of *blades*. A j-blade is a grade-j object that can be expressed as the outer product of j vectors. For example, γ_{12} is a 2-blade but $\gamma_{12} + \gamma_{30}$ is *not*. Combining the grade projector with the idea of blades enables a convenient definition of the geometric product between an arbitrary j-blade A and k-blade B,

$$AB = \langle AB \rangle_{|j-k|} + \langle AB \rangle_{|j-k|+2} + \cdots + \langle AB \rangle_{j+k}. \tag{3.17}$$

Then the inner product is always the *grade-lowering* product

$$A \cdot B = \langle AB \rangle_{|j-k|}, \tag{3.18}$$

and the outer product is always the *grade-raising* product

$$A \wedge B = \langle AB \rangle_{j+k}. \tag{3.19}$$

The vector-bivector, trivector-trivector, and vector-trivector products can be easily extrapolated from Equation (3.17), and are seen to only contain the sum of their respective inner and outer products. The

[3] Indeed the grade-d object is *always* the pseudoscalar.

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bivector-bivector product, however, includes an extra grade-preserving[4] commutator product. That is, for bivectors B and B',

$$BB' = B \cdot B' + [B, B'] + B \wedge B', \tag{3.20}$$

where the middle term is the *commutator product*,

$$[B, B'] = \frac{1}{2}(BB' - B'B). \tag{3.21}$$

Many geometric algebra sources denote the commutator product with the *times* operator, but this paper reserves that notation for the *cross product*,

$$B \times B' = -[B, B']I, \tag{3.22}$$

which is equivalent to the Gibbs-Heaviside vector cross product.

3.1.1. Conjugations

There are three "grade-selective" (involutory) conjugations inherent to every geometric algebra. For a general multivector M the first is *grade involution*,

$$\widehat{M} = \langle M\rangle_0 - \langle M\rangle_1 + \langle M\rangle_2 - \langle M\rangle_3 + \langle M\rangle_4, \tag{3.23}$$

the second is *reversion*,

$$\widetilde{M} = \langle M\rangle_0 + \langle M\rangle_1 - \langle M\rangle_2 - \langle M\rangle_3 + \langle M\rangle_4, \tag{3.24}$$

and the third is *Clifford conjugation*,

$$\overline{M} = \langle M\rangle_0 - \langle M\rangle_1 - \langle M\rangle_2 + \langle M\rangle_3 + \langle M\rangle_4. \tag{3.25}$$

Grade involution is so-named due to the fact it negates odd grades and ignores even grades. Reversion is so-named due to the fact it reverses all product orders. If N is also an arbitrary multivector, then $\widetilde{MN} = \widetilde{N}\widetilde{M}$. Clifford conjugation is the composition of grade involution and reversion, $\overline{M} = \widehat{\widetilde{M}} = \widetilde{\widehat{M}}$.

There are two *additional* conjugations which, unlike the first three, depend upon the use of a unit timelike vector. For convenience, the vector is chosen as γ_0. The first conjugation is *parity conjugation*,

$$M^- = \gamma_0 M \gamma_0, \tag{3.26}$$

and the second is *Hermitian conjugation*,

$$M^\dagger = \gamma_0 \widetilde{M} \gamma_0. \tag{3.27}$$

Both parity and Hermitian conjugation are well-known conjugations in physics. Geometrically, parity corresponds to the negative of a reflection in the γ_0 hyperplane defined in Section 3, while Hermitian conjugation corresponds to reversion followed by parity.

3.1.2. The Lorentz Group, Spin(1, 3)

At the heart of relativistic physics lies the *Lorentz group*, $\mathrm{Spin}(1,3) \approx \mathrm{SL}(2,\mathbb{C})$. This spin group exists naturally within the Spacetime Algebra,

$$\mathrm{Spin}(1,3) = \{\Lambda \in \mathbb{G}^+_{1,3} \,|\, \Lambda\widetilde{\Lambda} = \widetilde{\Lambda}\Lambda = 1\}, \tag{3.28}$$

[4] The commutator product between any multivector and a bivector preserves the multivector's grade.

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where $\mathbb{G}^{+}_{1,3} = \mathbb{G}^{0\oplus2\oplus4}_{1,3}$ is the *even subalgebra* of the Spacetime Algebra and Λ is called a *Lorentz rotor*. The existence of this group is thanks to the basis bivectors of Equation (3.10), which form the Lie algebra $\mathfrak{spin}(1,3) \approx \mathfrak{sl}(2,\mathbb{C})$. In general, a Lorentz rotor is of the form

$$\Lambda = RL = e^{-\frac{1}{2}B_{\text{sp}}}e^{\frac{1}{2}B_{\text{ti}}} = e^{\frac{1}{2}\Omega}, \tag{3.29}$$

where B_{ti} is a timelike bivector, B_{sp} is a spacelike bivector, and $\Omega = -B_{\text{sp}} + B_{\text{ti}}$ is the total[5] *spacetime rotation rate*. This means that R is a (Euclidean) rotation and L is a (Lorentz) boost. Note that in general, $LR \neq RL$. An arbitrary multivector is then Lorentz-transformed via a *sandwich product*,

$$M \mapsto \Lambda M \widetilde{\Lambda}. \tag{3.30}$$

3.1.3. Duality in $\mathbb{G}_{1,3}$

The pseudoscalar I functions as the generator of *duality*. This holds in arbitrary (non-degenerate) geometric algebras. For a multivector M in the Spacetime Algebra,

$$\star: \ M \mapsto MI^{-1} = -MI \tag{3.31}$$

is the *duality map*. In the Spacetime Algebra, this maps scalars to pseudoscalars, vectors to trivectors, timelike bivectors to spacelike bivectors, trivectors to vectors, and pseudoscalars to scalars. As the pseudoscalar is the generator of this duality, then maps of either the form (for $\theta \in \mathbb{R}$)

$$M \mapsto e^{\theta I} M \tag{3.32}$$

or

$$M \mapsto e^{\frac{1}{2}\theta I} M e^{-\frac{1}{2}\theta I} \tag{3.33}$$

are called *duality rotations*. Notice that Equation (3.33) leaves even-graded multivectors invariant.

3.2. The Mirror-Based View

Geometric algebras are traditionally described in the *point-based view* which does not holistically incorporate geometric objects like space itself, hyperplanes, hyperlines, lines, or points as being represented by graded objects. The *mirror-based view*, which employs a top-down approach to geometry does exactly this. As shown in Figure 1. For a $(p+q)$-dimensional[6] real geometric algebra, $d = p + q$ is the dimension of the (pseudo)Euclidean space. Vectors always represent $(d-1)$-dimensional hyperplanes, bivectors represent $(d-2)$-dimensional hyperlines. This pattern continues until $(d-1)$-vectors give 1-dimensional lines and d-vectors give 0-dimensional points.

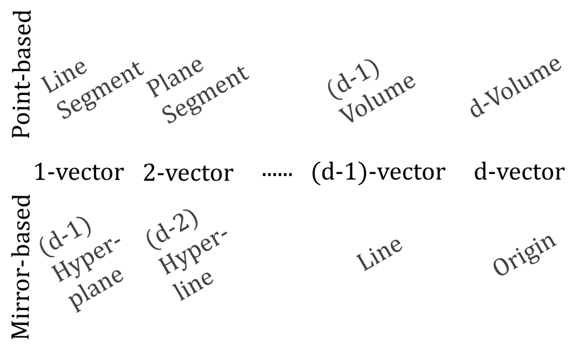

Figure 1. Image originally from [7]. List of geometric interpretation assigned to different grades in both the point-based view and mirror-based view.

[5] Note that inside the exponential's argument, $-B_{\text{sp}} + B_{\text{ti}} \neq B_{\text{ti}} - B_{\text{sp}}$ because the exponentials do not commute in general.
[6] Here p and q respectively denote the number of unipotent and anti-unipotent orthonormal basis vectors.

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If $A, B \in \mathbb{G}_{p,q}$ are respectively a j-blade and k-blade, then their outer product as specified by Equation (3.19) represents the subspace intersection of the two blades. Similarly, their outer product as specified by Equation (3.18) represents the subspace orthogonal to the higher graded blade which contains the subspace of the lower graded blade. If $\star$ represents the duality map like in Equation (3.31) then the *regressive product* is

$$A \vee B = \star^{-1}(\star A \wedge \star B) \tag{3.34}$$

and represents the join of the two blades' subspaces.

As all algebraic objects correspond to geometric objects within the mirror-based view, the geometric interpretation of real scalars is a natural question. The apparent answer to this question is that real scalars represent the *space itself*. Conceptually, this is a rather clever solution. Geometric objects in the space have orientations relative to the space, but the space cannot have an orientation with respect to itself that differs from plus-or-minus unity. Moreover, the scaling of any geometric object is equivalent to an outer product. That is, if α is a real scalar and M is some multivector (in any algebra), then

$$\alpha \wedge M = \alpha M. \tag{3.35}$$

This implies an interpretation of coefficients as intersections between geometric objects with scalar fields intrinsic to the space they inhabit. Furthermore, since the multivector is "preserved" on both sides of Equation (3.35), the intersection of a multivector with the space itself returns the multivector. That is, the operation $1 \wedge M = M$ acts as the *identity intersection*.

4. General Relativity

As stated in the conceptual overview of Section 2, constructing General Relativity (GR) in the Spacetime Algebra (STA) will be very straightforward: Starting with infinitesimal Lorentz transformations and building up to the Einstein equation. If the reader has not yet read about the algebraic and geometric essentials presented in Section 3, it is recommended reading.

4.1. Tetrad Bases

When working with curved manifolds, there are objects called *tangent spaces*. If $\mathcal{M}$ is the manifold, then $T_x\mathcal{M}$ is the tangent space at coordinate x. Since the approach of this paper is to work with the (geometric) Spacetime Algebra (STA), the notation from [3] will be adopted: Promoting the tangent space to a tangent geometric algebra is labeled by $GT_x\mathcal{M} = \mathbb{G}_{1,3}(x)$, and called the *geometric tangent space* (GTS).

Without reference to a specific GTS, the coordinate x is a vector in the STA,

$$x = x^\mu \gamma_\mu = x_\mu \gamma^\mu. \tag{4.1}$$

Recall, the standard basis $\{\gamma_\mu\}$ of the STA is given by Equation (3.1) and the reciprocal basis $\{\gamma^\mu\}$ of the STA in Equation (3.3). When moving to the GTS $GT_x\mathcal{M}$, one shifts from the standard basis to the *coordinate basis* $\{g_\alpha\}$ defined by

$$g_\alpha = \partial_\alpha x, \tag{4.2}$$

and from the reciprocal basis to the *reciprocal coordinate basis* $\{g^\mu\}$ defined by

$$g^\alpha \cdot g_\beta = \delta^\alpha_\beta. \tag{4.3}$$

Here, $\partial_\alpha = g_\alpha \cdot \partial$ is the spacetime derivative in the g_α direction. This derivative is dependent upon the definition of the *spacetime gradient*,

$$\partial = \gamma^\mu \partial_\mu = \gamma^\mu \frac{\partial}{\partial x^\mu}. \tag{4.4}$$

It might appear recursive to define the coordinate basis $\{g_\alpha\}$ in terms of $g_\alpha \cdot \partial$, and it is! But it is not a problem. In practice, ∂_α is defined first and then $\{g_\alpha\}$ follows. *An example is with polar coordinates*:

First the indices are defined as $\alpha = t, r, \theta, \phi$ and correspond with the scalar values t (time), r (radius), θ (polar angle), and ϕ (azimuthal angle). Then ∂_α is the derivative with respect to *each scalar value*. So if the coordinate vector is given in polar coordinates

$$x = t\gamma_0 + r\sin\theta(\cos\phi\gamma_1 + \sin\phi\gamma_2) + r\cos\theta\gamma_3,$$

then

$$\begin{aligned} g_t &= \gamma_0 \\ g_r &= \sin\theta(\cos\phi\gamma_1 + \sin\phi\gamma_2) + \cos\theta\gamma_3 \\ g_\theta &= r\cos\theta(\cos\phi\gamma_1 + \sin\phi\gamma_2) - r\sin\theta\gamma_3 \\ g_\phi &= r\sin\theta(-\sin\phi\gamma_1 + \cos\phi\gamma_2) \end{aligned} \tag{4.5}$$

is the coordinate basis. It is important to realize that the difference in notation will tell whether or not one is working with the coordinate basis or the standard basis: This paper's convention uses ∂_α *only* for $g_\alpha \cdot \partial$, and ∂_μ *only* for $\partial/\partial x^\mu$. A big selling point for Geometric Algebra is the fact that everything *can* be coordinate-free (as exemplified in $g^\alpha\partial_\alpha = \gamma^\mu\partial_\mu$), but it is not always easiest to learn a subject coordinate-free. For a beginner, whatever coordinate system is being used could be quite ambiguous, thus the difference between the coordinate indices α and standard indices μ is useful.

The inner product of Equation (3.8) defines the (coordinate) *metric tensor*,

$$g_{\alpha\beta} = g_\alpha \cdot g_\beta. \tag{4.6}$$

The metric tensor therefore has an elegant interpretation within the STA: The *mutual overlap* of the different coordinate basis vectors. Moreover, it is easy to see that the symmetric nature of the metric tensor comes from the fact that the inner product of two vectors is symmetric!

In general, $\{g_\alpha\}$ is non-orthonormal. But one can move into an orthonormal basis at coordinate x. This new basis is called the *tetrad* of x, $\{\underline{\gamma}_m\}$. The tetrad satisfies

$$\underline{\gamma}_m \cdot \underline{\gamma}_n = \eta_{mn}, \tag{4.7}$$

where η_{mn} is the usual Minkowski metric, like in Equation (3.2). No matter the coordinate indices $\alpha = \alpha_0, \alpha_1, \alpha_2, \alpha_3$, the tetrad indices are $m = 0, 1, 2, 3$, as is the case with the standard basis indices $\mu = 0, 1, 2, 3$. The tetrad thereby functions as a basis for a local STA at coordinate x, $\mathbb{G}_{1,3}(x)$, which is just the GTS $GT_x\mathcal{M}$. In order to notationally distinguish the tetrad from the standard basis, the reader will notice an underbar is given and Latin indices replace Greek indices. But how does one move into a tetrad $\{\underline{\gamma}_m\}$ from a coordinate basis $\{g_\alpha\}$? Via the *vierbein* $\{e^m_\alpha\}$ and $\{\underline{e}^\alpha_m\}$:

$$g_\alpha = e^m_\alpha\underline{\gamma}_m \quad \text{and} \quad g^\alpha = \underline{e}^\alpha_m\underline{\gamma}^m. \tag{4.8}$$

The vierbein are like the positive square roots of the respective metric tensors $g_{\alpha\beta}$ and $g^{\alpha\beta}$. Notice that Equation (4.8) is equivalent to

$$\underline{e}^\alpha_m g_\alpha = \underline{\gamma}_m \quad \text{and} \quad e^m_\alpha g^\alpha = \underline{\gamma}^m. \tag{4.9}$$

As a last note, all bases and their reciprocals satisfy the following,

$$g^\alpha \cdot g_\beta = \delta^\alpha_\beta \quad \text{and} \quad \underline{\gamma}^m \cdot \underline{\gamma}_n = \delta^m_n. \tag{4.10}$$

And

$$e^m_\alpha \underline{e}^\alpha_n = \delta^m_n \quad \text{and} \quad e^m_\alpha \underline{e}^\beta_m = \delta^\beta_\alpha \tag{4.11}$$

hold for the vierbein.

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To better understand these concepts, the metric tensor $g^{\alpha\beta}$, vierbein $\{e^m_\alpha\}$, and tetrad $\underline{\gamma}_m$ will now be calculated for Equation (4.5) (the coordinate basis in polar coordinates). Directly from the inner products $g^\alpha \cdot g^\beta$,

$$g^{\alpha\beta} \leftrightarrow \begin{Bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & \frac{1}{-r^2} & 0 \\ 0 & 0 & 0 & \frac{1}{-r^2\sin^2\theta} \end{Bmatrix}. \tag{4.12}$$

The vierbein (positive square root) is trivial since the metric tensor is diagonal:

$$\{\underline{e}^\alpha_m\} \leftrightarrow \begin{Bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{r} & 0 \\ 0 & 0 & 0 & \frac{1}{r\sin\theta} \end{Bmatrix}. \tag{4.13}$$

Then the tetrad must be

$$\begin{aligned} \underline{\gamma}_0 &= g_t = \gamma_0 \\ \underline{\gamma}_1 &= g_r = \sin\theta(\cos\phi\gamma_1 + \sin\phi\gamma_2) + \cos\theta\gamma_3 \\ \underline{\gamma}_2 &= \frac{1}{r}g_\theta = \cos\theta(\cos\phi\gamma_1 + \sin\phi\gamma_2) - \sin\theta\gamma_3 \\ \underline{\gamma}_3 &= \frac{1}{r\sin\theta}g_\phi = (-\sin\phi\gamma_1 + \cos\phi\gamma_2). \end{aligned} \tag{4.14}$$

4.2. The Covariant Derivative

Because GR deals with a curved manifold (presumably Spacetime), there must be a derivative operator which accounts for said curvature! It would be of the form

$$D_\alpha = \partial_\alpha + \Gamma. \tag{4.15}$$

where $\Gamma_\alpha = \Gamma(g_\alpha)$ is some as-of-yet undefined *connection* (the curvature tally). But heuristically determining the connection is easy! This is due to GR's curvature being bound by the following restriction: A coordinate x *infinitesimally close* to coordinate x' is related by an *infinitesimal Lorentz transformation*. That is, if Λ_ϵ is an infinitesimal Lorentz rotor, and the coordinates x and x' are respectively given the tetrads $\{\underline{\gamma}_m\}$ and $\{\underline{\gamma}'_m\}$, then

$$\underline{\gamma}'_m = \Lambda_\epsilon \underline{\gamma}_m \widetilde{\Lambda}_\epsilon. \tag{4.16}$$

Yet this form isn't much use as it stands. The goal is to heuristically determine the connection Γ by first observing the explicit form of an infinitesimal rotation (which bounds the curvature by the aforewritten restriction) and claiming that *it is* the connection! The explicit form of Λ_ϵ can be determined by inserting an infinitesimal scalar ϵ (such that $\epsilon^n \approx 0$ for $n > 1$) into Equation (3.29) and applying its Taylor expansion:

$$\Lambda_\epsilon = e^{\frac{1}{2}\epsilon\Omega} = 1 + \frac{1}{2}\epsilon\Omega. \tag{4.17}$$

Immediately plugging this into Equation (4.16),

$$\begin{aligned} &(1 + \frac{1}{2}\epsilon\Omega)\underline{\gamma}_m(1 - \frac{1}{2}\epsilon\Omega) \\ &\qquad = \underline{\gamma}_m + \frac{1}{2}\epsilon\Omega\underline{\gamma}_m - \frac{1}{2}\epsilon\underline{\gamma}_m\Omega - \frac{1}{4}\epsilon^2\Omega\underline{\gamma}_m\Omega \\ &\qquad = \underline{\gamma}_m + \epsilon[\Omega, \underline{\gamma}_m], \end{aligned} \tag{4.18}$$

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where the last step invoked the definition of the commutator product, Equation (3.21). Now the heuristic identification of the connection is

$$\Gamma_\alpha = [\Omega_\alpha, \cdot], \tag{4.19}$$

where $\Omega_\alpha = \Omega(g_\alpha)$ is the *connection bivector* and $(\cdot)$ represents the argument upon which the commutator acts. Therefore the *covariant derivative* is defined as

$$D_\alpha = \partial_\alpha + [\Omega_\alpha, \cdot], \tag{4.20}$$

and then

$$D = g^\alpha D_\alpha \tag{4.21}$$

is the *covariant gradient*. The connection has a nice interpretation as the generator of the Lorentz transformation between two coordinates, and geometrically represents the act of rotating about the hyperline described by the spacetime rotation rate Ω_α.

4.2.1. The Connection Bivector

An expression for directly calculating the connection bivector Ω_α is natural in the STA. However, discussion about the Christoffel symbols in the STA is warranted. Recall that the metric tensor $g_{\alpha\beta}$ is the inner product of g_α and g_β, and thereby measures their mutual overlap. The *Christoffel symbols* of GR are defined as

$$\Gamma^\delta_{\alpha\beta} = \frac{1}{2} g^{\delta\varepsilon}(\partial_\alpha g_{\beta\varepsilon} + \partial_\beta g_{\alpha\varepsilon} - \partial_\varepsilon g_{\beta\alpha}). \tag{4.22}$$

In the parentheses, the first term is the rate of change of the overlap $g_{\beta\varepsilon}$ in the g_α direction, the second term is the rate of change of the overlap $g_{\alpha\varepsilon}$ in the g_β direction, and the third term is the (negative) rate of change of the overlap $g_{\beta\alpha}$ in the g_ε direction. These rates are then summed over the (reciprocal) overlap $g^{\delta\varepsilon}$. Therefore the Christoffel symbols are a measure of how changes in overlap within different directions affect the overlap in another direction. Christoffel symbols play an important role through the *Christoffel identity*,

$$D_\beta g_\alpha = \Gamma^\delta_{\alpha\beta} g_\delta, \tag{4.23}$$

and greatly simplify the calculation of the connection bivector Ω_α. The Christoffel identity is quite elegant, saying that the covariant change of the coordinate basis $\{g_\alpha\}$ is simply the measure of how changes in overlap within different directions affect the overlap in another direction.

Calculating the connection bivector is straightforward: Consider the covariant derivative of the tetrad $\{\underline{\gamma}_m\}$, then isolate Ω_α. First, the covariant derivative of the tetrad is taken directly, giving

$$\begin{aligned} D_\beta \underline{\gamma}_m &= \partial_\beta \underline{\gamma}_m + [\Omega_\beta, \underline{\gamma}_m] \\ &= [\Omega_\beta, \underline{\gamma}_m] \end{aligned} \tag{4.24}$$

since $\partial_\beta \underline{\gamma}_m = 0$. Second, the covariant derivative of the tetrad expressed as the vierbein-coordinate product in Equation (4.9) is taken, giving

$$\begin{aligned} D_\beta \underline{\gamma}_m &= D_\beta(\underline{e}^\alpha_m g_\alpha) \\ &= D_\beta \underline{e}^\alpha_m g_\alpha + \underline{e}^\alpha_m D_\beta g_\alpha \\ &= \partial_\beta \underline{e}^\alpha_m g_\alpha + \underline{e}^\alpha_m \Gamma^\delta_{\alpha\beta} g_\delta. \end{aligned} \tag{4.25}$$

In the last line, the Christoffel identity of Equation (4.23) was used, as was the fact that the commutator product of a scalar is zero, $[\Omega_\beta, \underline{e}^\alpha_m] = 0$. Third, it must be realized that $[\Omega_\beta, \underline{\gamma}_m] = \Omega \cdot \underline{\gamma}_m$ by

Equation (3.18). Then, by an important property in Geometric Algebra for a k-vector A, $\gamma^\mu(\gamma_\mu \cdot A) = kA$, the connection bivector can be isolated,

$$\begin{aligned}\underline{\gamma}^m[\Omega_\beta, \underline{\gamma}_m] &= -\underline{\gamma}^m[\underline{\gamma}_m, \Omega_\beta] \\ &= -\underline{\gamma}^m\left(\underline{\gamma}_m \cdot \Omega_\beta\right) \\ &= -2\Omega_\beta.\end{aligned} \tag{4.26}$$

Multiplying the last line of Equation (4.25) gives

$$\begin{aligned}-2\Omega_\beta &= \underline{\gamma}^m \partial_\beta \underline{e}^\alpha_m g_\alpha + \underline{\gamma}^m \underline{e}^\alpha_m \Gamma^\delta_{\alpha\beta} g_\delta \\ &= \partial_\beta g^\alpha g_\alpha + \frac{1}{2} g^\alpha g^{\delta\varepsilon}\left(\partial_\alpha g_{\beta\varepsilon} + \partial_\beta g_{\alpha\varepsilon} - \partial_\varepsilon g_{\beta\alpha}\right) g_\delta \\ &= \partial_\beta g^\alpha g_\alpha + \frac{1}{2} g^\alpha g^\varepsilon\left(\partial_\alpha g_{\beta\varepsilon} + \partial_\beta g_{\alpha\varepsilon} - \partial_\varepsilon g_{\beta\alpha}\right).\end{aligned} \tag{4.27}$$

Now, the lefthandside is a bivector, which means that the righthandside must also be a bivector. Therefore, the grade-2 projection of Equation (3.15) can be applied, filtering out any non-bivector terms:

$$\begin{aligned}-2\Omega_\beta &= \left\langle \partial_\beta g^\alpha g_\alpha + \frac{1}{2} g^\alpha g^\varepsilon\left(\partial_\alpha g_{\beta\varepsilon} + \partial_\beta g_{\alpha\varepsilon} - \partial_\varepsilon g_{\beta\alpha}\right)\right\rangle_2 \\ &= \left(\partial_\beta g^\alpha\right) \wedge g_\alpha + \frac{1}{2} g^\alpha \wedge g^\varepsilon\left(\partial_\alpha g_{\beta\varepsilon} + \partial_\beta g_{\alpha\varepsilon} - \partial_\varepsilon g_{\beta\alpha}\right).\end{aligned} \tag{4.28}$$

The next simplification is a bit tricky, and involves examining the term $(g^\alpha \wedge g^\varepsilon)\partial_\beta g_{\alpha\varepsilon}$ and seeing that it vanishes. The first observation is that the outer product is antisymmetric while the metric tensor is symmetric. And since the term is independent (the ∂_β plays no part in the summation over α and ε), it is possible to relabel α to ε and ε to α. Mathematically,

$$\begin{aligned}(g^\alpha \wedge g^\varepsilon)\partial_\beta g_{\alpha\varepsilon} &= (g^\varepsilon \wedge g^\alpha)\partial_\beta g_{\varepsilon\alpha} \\ &= -(g^\alpha \wedge g^\varepsilon)\partial_\beta g_{\alpha\varepsilon}.\end{aligned}$$

So like any object equal to its negative, it must be zero! Therefore

$$-2\Omega_\beta = \left(\partial_\beta g^\alpha\right) \wedge g_\alpha + \frac{1}{2} g^\alpha \wedge g^\varepsilon\left(\partial_\alpha g_{\beta\varepsilon} - \partial_\varepsilon g_{\beta\alpha}\right). \tag{4.29}$$

Dividing both sides by -2,

$$\begin{aligned}\Omega_\beta &= -\frac{1}{2}\left[\left(\partial_\beta g^\alpha\right) \wedge g_\alpha + \frac{1}{2} g^\alpha \wedge g^\varepsilon\left(\partial_\alpha g_{\beta\varepsilon} - \partial_\varepsilon g_{\beta\alpha}\right)\right] \\ &= \frac{1}{2}\left[g_\alpha \wedge \left(\partial_\beta g^\alpha\right) + \frac{1}{2} g^\varepsilon \wedge g^\alpha\left(\partial_\alpha g_{\beta\varepsilon} - \partial_\varepsilon g_{\beta\alpha}\right)\right].\end{aligned} \tag{4.30}$$

The final simplification is for the last term:

$$\begin{aligned}\frac{1}{2} g^\varepsilon \wedge g^\alpha\left(\partial_\alpha g_{\beta\varepsilon} - \partial_\varepsilon g_{\beta\alpha}\right) &= \frac{1}{2}(g^\varepsilon \wedge g^\alpha)\partial_\alpha g_{\beta\varepsilon} - \frac{1}{2}(g^\varepsilon \wedge g^\alpha)\partial_\varepsilon g_{\beta\alpha} \\ &= \frac{1}{2}(g^\varepsilon \wedge g^\alpha)\partial_\alpha g_{\beta\varepsilon} + \frac{1}{2}(g^\alpha \wedge g^\varepsilon)\partial_\varepsilon g_{\beta\alpha} \\ &= \frac{1}{2} g^\varepsilon \wedge (g^\alpha \partial_\alpha) g_{\beta\varepsilon} + \frac{1}{2} g^\varepsilon \wedge (g^\alpha \partial_\alpha) g_{\beta\varepsilon} \\ &= g^\varepsilon \wedge \partial g_{\beta\varepsilon}.\end{aligned}$$

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In the penultimate line, the indices of the second term were relabeled between α and ε. Therefore, the explicit expression for the connection bivector $\Omega(g_\beta) = \Omega_\beta$ is

$$\Omega_\beta = \frac{1}{2}\left(g_\alpha \wedge (\partial_\beta g^\alpha) + g^\varepsilon \wedge \partial g_{\beta\varepsilon}\right). \tag{4.31}$$

The interpretation is again clear: The connection bivectors are the hyperlines about which the coordinate basis $\{g_\alpha\}$ changes (hyperaxes of rotation). This serves as a curvature tally, and the curvature tensors are to be expressed in terms of said tally. Notice the difference between a *tally* and a *measurement*: The connection bivector keeps a tally of the curvature, yet does not measure it directly. It is the curvature tensors that then use the connection bivectors' tallies to "calculate a measurement".

To demonstrate a calculation using Equation (4.31), the connection bivectors are shown below from the polar coordinate basis of EQ.4.5:

$$\begin{aligned}
\Omega_t &= 0 \\
\Omega_r &= 0 \\
\Omega_\theta &= r g^r \wedge g^\theta = \underline{\gamma}^{12} \\
\Omega_\phi &= r \sin^2\theta g^r \wedge g^\phi + r^2 \cos\theta \sin\theta g^\theta \wedge g^\phi = \sin\theta \underline{\gamma}^{13} + \cos\theta \underline{\gamma}^{23}.
\end{aligned} \tag{4.32}$$

While bivectors are presented as hyperlines in Section 3, it is also important to understand that traditionally $g^\alpha \wedge g^\beta$ represents the differential area between the two directions. Thus, in the coordinate basis, Equation (4.32) says that the hyperlines for the t-connection bivector and r-connection bivector are zero, that the hyperline for the θ-connection bivector is defined by the *radially-growing* differential area between g^r and g^θ, and that the hyperline for the ϕ-connection bivector is defined by the radially-growing and θ-*dependent* differential areas between g^r and g^θ respectively with g^ϕ.

4.3. The Einstein Equation

To discuss the Einstein equation, one must first derive the curvature multivectors (tensors). The first group of multivectors is the Riemann bivectors, the second is the Ricci vectors, and the last is the Ricci scalar. And to derive these multivectors, one must also understand what curvature is!

Curvature can be intuitively understood through the following question and answer. **Q**: Suppose somebody is in a field of hills and walks along path A followed by B and arrives at point P, and then somebody else takes path B followed by A and arrives at point Q; For what reason would the points P and Q be different? **A**: Each path is offset by the hills a different amount, so the order of paths matters! That is, computing the commutator of the paths recovers information about the curvature! To "walk the paths" in the math, one must use the covariant derivative/gradient to "tally the hills" of a multivector field. And to compare two "paths", the commutator of the covariant derivative/gradient naturally follows! This is the nature of what is called *parallel transport*.

The first step in obtaining the Riemann bivectors is by considering the commutator of two covariant gradients acting upon a multivector field M,

$$\begin{aligned}
[D, D]M &= [g^\alpha D_\alpha, g^\beta D_\beta]M \\
&= 2 g^\alpha \wedge g^\beta [D_\alpha, D_\beta]M,
\end{aligned} \tag{4.33}$$

where the factor of 2 comes from separating the coordinate basis vectors g^α and g^β from their respective coefficients (this is because the outer product $g^\alpha \wedge g^\beta$ has a canceling factor of $1/2$ as defined in

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Equation (3.9)). The Riemann bivectors arises from the $2[D_\alpha, D_\beta]M$ term, so that is what is next expanded:

$$\begin{aligned}2[D_\alpha, D_\beta]M &= \big(D_\alpha D_\beta - D_\beta D_\alpha\big)M \\ &= D_\alpha\big(D_\beta M\big) - D_\beta\big(D_\alpha M\big) \\ &= D_\alpha\big(\partial_\beta M + [\Omega_\beta, M]\big) - D_\beta\big(\partial_\alpha M + [\Omega_\alpha, M]\big) \\ &= \partial_\alpha\partial_\beta M + \partial_\alpha[\Omega_\beta, M] + [\Omega_\alpha, \partial_\beta M] + [\Omega_\alpha, [\Omega_\beta, M]] \\ &\quad - \partial_\beta\partial_\alpha M - \partial_\beta[\Omega_\alpha, M] - [\Omega_\beta, \partial_\alpha M] - [\Omega_\beta, [\Omega_\alpha, M]] \\ &= \partial_\alpha[\Omega_\beta, M] + [\Omega_\alpha, \partial_\beta M] - \partial_\beta[\Omega_\alpha, M] - [\Omega_\beta, \partial_\alpha M] \\ &\quad + [\Omega_\alpha, [\Omega_\beta, M]] - [\Omega_\beta, [\Omega_\alpha, M]],\end{aligned} \tag{4.34}$$

where the last line is obtained using $(\partial_\alpha\partial_\beta - \partial_\beta\partial_\alpha)M = 0$. Now, both $\partial_\alpha[\Omega_\beta, M] + [\Omega_\alpha, \partial_\beta M] - \partial_\beta[\Omega_\alpha, M] - [\Omega_\beta, \partial_\alpha M]$ and $[\Omega_\alpha, [\Omega_\beta, M]] - [\Omega_\beta, [\Omega_\alpha, M]]$ must be expanded separately. First,

$$\begin{aligned}&\partial_\alpha[\Omega_\beta, M] + [\Omega_\alpha, \partial_\beta M] - \partial_\beta[\Omega_\alpha, M] - [\Omega_\beta, \partial_\alpha M] \\ &\quad = [\partial_\alpha\Omega_\beta, M] + [\Omega_\beta, \partial_\alpha M] + [\Omega_\alpha, \partial_\beta M] \\ &\qquad - [\partial_\beta\Omega_\alpha, M] - [\Omega_\alpha, \partial_\beta M] - [\Omega_\beta, \partial_\alpha M] \\ &\quad = [\partial_\alpha\Omega_\beta, M] - [\partial_\beta\Omega_\alpha, M] \\ &\quad = [\partial_\alpha\Omega_\beta - \partial_\beta\Omega_\alpha, M].\end{aligned} \tag{4.35}$$

Second,

$$\begin{aligned}&[\Omega_\alpha, [\Omega_\beta, M]] - [\Omega_\beta, [\Omega_\alpha, M]] \\ &\quad = \frac{1}{4}\big(\Omega_\alpha\big(\Omega_\beta M - M\Omega_\beta\big) - \big(\Omega_\beta M - M\Omega_\beta\big)\Omega_\alpha\big) \\ &\qquad - \frac{1}{4}\big(\Omega_\beta(\Omega_\alpha M - M\Omega_\alpha) - (\Omega_\alpha M - M\Omega_\alpha)\Omega_\beta\big) \\ &\quad = \frac{1}{4}\big(\Omega_\alpha\Omega_\beta M - \Omega_\alpha M\Omega_\beta - \Omega_\beta M\Omega_\alpha + M\Omega_\beta\Omega_\alpha\big) \\ &\qquad - \frac{1}{4}\big(\Omega_\beta\Omega_\alpha M - \Omega_\beta M\Omega_\alpha - \Omega_\alpha M\Omega_\beta + M\Omega_\alpha\Omega_\beta\big) \\ &\quad = \frac{1}{4}\big((\Omega_\alpha\Omega_\beta - \Omega_\beta\Omega_\alpha)M - M(\Omega_\alpha\Omega_\beta - \Omega_\beta\Omega_\alpha)\big) \\ &\quad = [[\Omega_\alpha, \Omega_\beta], M].\end{aligned} \tag{4.36}$$

Therefore

$$\begin{aligned}&2[D_\alpha, D_\beta]M \\ &\qquad = [\partial_\alpha\Omega_\beta - \partial_\beta\Omega_\alpha, M] + [[\Omega_\alpha, \Omega_\beta], M] \\ &\qquad = [\partial_\alpha\Omega_\beta - \partial_\beta\Omega_\alpha + [\Omega_\alpha, \Omega_\beta], M] \\ &\qquad = [R_{\alpha\beta}, M],\end{aligned} \tag{4.37}$$

so the commutator of two covariant derivatives acting on a multivector field is equivalent to the commutator product of some bivectors with said multivector! These bivectors are

$$R(g_\alpha \wedge g_\beta) = R_{\alpha\beta} = \partial_\alpha\Omega_\beta - \partial_\beta\Omega_\alpha + [\Omega_\alpha, \Omega_\beta], \tag{4.38}$$

the *Riemann bivectors*. Geometrically, they are the hyperlines that generate the rotations experienced by a multivector field which is parallel transported in an infinitesimal closed loop within the plane between the two coordinate directions g_α and g_β.

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The next tensors are the Ricci vectors. They are obtained by considering how the Riemann bivectors rotate the coordinate bases, and this consideration itself comes from the parallel transport of coordinate bases:

$$\begin{aligned}[D_\alpha, D_\beta]g^\beta &= [R_{\alpha\beta}, g^\beta] \\ &= R_{\alpha\beta} \cdot g^\beta,\end{aligned} \tag{4.39}$$

and the last line comes from the definition of the inner product, Equation (3.18), and returns a vector. The resulting vectors are

$$R(g_\alpha) = R_\alpha = R_{\alpha\beta} \cdot g^\beta, \tag{4.40}$$

the *Ricci vectors*. Now, the Ricci vectors show how the coordinate basis is rotated via curvature, and geometrically vectors represent hyperplanes. So the Ricci vectors express the *curvature flux* through the hyperplane!

From the Ricci vector is obtained the *Ricci scalar*:

$$\mathcal{R} = g^\alpha \cdot R_\alpha. \tag{4.41}$$

It is the measure of overlap between the Ricci vectors R_α and the coordinate basis vectors g_α. This overlap quantifies the *average curvature* of the space at the coordinate x (e.g. $\mathcal{R} > 0$ says the space is overall positively curved). The Ricci scalar can also be rewritten in terms of the Riemann bivectors,

$$\begin{aligned}\mathcal{R} &= g^\alpha \cdot R_\alpha \\ &= g^\alpha \cdot \left(R_{\alpha\beta} \cdot g^\beta\right) \\ &= -g^\alpha \cdot \left(g^\beta \cdot R_{\alpha\beta}\right) \\ &= \left(g^\beta \wedge g^\alpha\right) \cdot R_{\alpha\beta},\end{aligned} \tag{4.42}$$

which expresses the same overlap but phrased using the Riemann bivectors. Then the final curvature tensors are the *Einstein vectors*,

$$G(g_\alpha) = G_\alpha = R_\alpha - \frac{1}{2}\mathcal{R}g_\alpha, \tag{4.43}$$

which simply represents the curvature flux minus half the overlap between said flux's direction and the coordinate bases. Put otherly, the Einstein vectors represent the curvature flux minus half the average curvature of the space. Finally, if $T(g_\alpha) = T_\alpha$ is the *stress-energy vector* in the g_α direction, then

$$G_\alpha = \kappa T_\alpha + \Lambda g_\alpha \tag{4.44}$$

is the *Einstein equation* in the g_α direction. The *Einstein constant* is $\kappa = 8\pi G_{\text{Newton}}/c^4$, and the *cosmological constant* is Λ. As stated in the paper's beginning, the interpretation of the Einstein equation is quite simple: The curvature flux (minus overlap) is equal to the energy flux (plus vacuum energy flux). It is worth noting that this is not the traditional way in which the Einstein equation is presented! To obtain this traditional presentation, one must simply take the inner product with another coordinate basis direction, g_β:

$$G_{\alpha\beta} = \kappa T_{\alpha\beta} + \Lambda g_{\alpha\beta}, \tag{4.45}$$

with $G_{\alpha\beta} = G_\alpha \cdot g_\beta$ and $T_{\alpha\beta} = T_\alpha \cdot g_\beta$. The interpretation is the same, but it is found by considering the overlap between the vectors in Equation (4.44) and the coordinate basis vectors $\{g_\alpha\}$. Thus Equation (4.45) is really a set of *scalar* equations in the STA, while Equation (4.44) is really a set of *vector* equations in the STA. The first looks at the geometry indirectly by considering overlap information, while the second looks at the geometry directly by considering objects (vectors) which represent oriented hyperplanes.

To further demonstrate the calculations in the STA, below are the Riemann bivectors (computed using the tetrad basis) obtained using Equation (4.38) and the connection bivectors found from polar coordinates in Equation (4.32):

$$R_{\alpha\beta} = 0 \tag{4.46}$$

for all indices. This should not be a surprise, as polar coordinates describe flat space! Therefore the curvature is zero.

Afterword

Having concluded this paper, it is now recommended that the reader peruse the list of citations. Each one contains valuable information that expands vastly upon this text's crash-course.

Perhaps the most expansive (and reasonably articulated) textbook in the applications of Geometric Algebra to modern physics is [1], and it is the author's foremost recommendation. Admittedly, the nature of its expansiveness can at times be daunting or cause steps to be omitted inside calculations which are not always clear to even experienced readers. Therefore the author's secondary and tertiary recommendations are [6] and [5], as they supplement [1] quite well. Moreover, [6] was crucial to the author's development within the field of Geometric Algebra, so it is recommended with a hint of academic love and respect.

While [1–3] all contributed to the General Relativity portion of this paper, the notation and terminology of [3] was most closely followed, with [2] thereafter. Again, all three are recommended for the reader if they wish to more completely study General Relativity within the Geometric Algebra formalism.

At the current time of writing-completion, the author is currently working on an accompanying lecture series on *YouTube*. For readers who wish to learn through audio-visual presentation, please visit General Relativity but Easy.

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