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Revisiting Probabilistic Metric Spaces

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Revisiting Probabilistic Metric Spaces

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Abstract

The field of probabilistic metric spaces has an intrinsic interest based on a blend of ideas drawn from metric space theory and probability theory. The goal of the present paper is to introduce and study new ideas in this field. In general terms, we investigate the following concepts: *linearly ordered* families of distances and associated *continuity properties*, *geometric properties* of distances, *finite range* weak probabilistic metric spaces, *generalized Menger* spaces, and a *categorical framework* for weak probabilistic metric spaces. Hopefully, the results will contribute to the foundations of the subject.

Keywords: weak probabilistic metric space; menger space; generalized menger space; distance-space; pseudometric; linearly ordered family; finite range; non-expansive; category; reflective; coreflective; normal sequence; modulus of continuity; regular écart

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1. Introduction

1.1. Preliminary Remarks

The original and primary source on probabilistic metric spaces is [1]. Current research in the field appears to fall into three basic areas: (*i*) *foundational topics* ([2–5]), (*ii*) *fixed point theory* ([6–8]), and (*iii*) *models* for applications ([9,10]) with the primary emphasis being on area (*ii*). The admittedly lofty aim of the present paper is to rejuvenate the foundations of the subject by reexamining the original concepts before most research was devoted to *Menger* spaces and triangle functions.

1.2. Organization

The main body of the paper consists of three sections. Section two presents background material on weak probabilistic metric spaces (*WPMSs*) and introduces the notions of *finite range WPMSs* and *linearly ordered* families of distances. Section three presents results on constructing *Menger* spaces and approximating them by *finite range Menger* spaces, as well as characterizing the *generalized Menger* spaces. Section four introduces a categorical framework for *WPMSs* and establishes several key properties of the category WP. It also shows that the *Menger* spaces are a *reflective* subcategory of WP. Section five discusses the potential contributions of the work. There is also an appendix devoted to an overview of distance-spaces that includes several results used in other sections of the paper.

1.3. Conventions

We use the symbol § to refer to sections in the paper and □ to denote the end of proofs. The symbol $\boldsymbol{N}$ (resp. $\boldsymbol{Z}$ or $\boldsymbol{R}$) denotes the natural numbers (resp. integers or real numbers), $\boldsymbol{N}^+ = \boldsymbol{N}\backslash\{0\}$, and $\boldsymbol{R}^+ = [0, +\infty)$. The symbol $\boldsymbol{I}$ denotes $[0, 1]$. For each $x \in \boldsymbol{R}$, $\lfloor x \rfloor$ (resp. $\lceil x \rceil$) denotes the largest integer $\leq x$ (resp. smallest integer $\geq x$). For a set S, $id_S : S \to S$ denotes the identity mapping, and for each $A \subseteq S$, $\chi_A : S \to \{0, 1\}$ (resp. $incl_A : A \to S$) denotes the characteristic function of A (resp. the inclusion mapping). The symbol $\mathrm{P}(S)$ denotes the power-set of S and $\mathrm{P_F}^+(S)$ denotes the family of non-empty finite subsets of S. Given sets S and T, $S \cup_d T$ denotes their disjoint union, $F(S, T)$ is the family of mappings $S \to T$, and for each $t \in T$, $const_t : S \to T$ is the *constant* mapping at t. A mapping $\varphi : \boldsymbol{R}^+ \to \boldsymbol{R}^+$ is a *modulus of continuity* if φ is *non-decreasing* and $\varphi(0+) = \varphi(0) = 0$.

2. Weak Probabilistic Metric Spaces

2.1. Background

Here we present the notion of a weak probabilistic metric space, the associated family of distances, and several examples. The reader familiar with these ideas can skip to **§2.2**.

Define the set of distribution functions

$\Delta = \{F : \mathbf{R} \to \mathbf{I} \mid F$ is left-continuous, non-decreasing, $\mathbf{inf}(F) = 0$, and $\mathbf{sup}(F) = 1\}$

and let $\leqq$ denote the point-wise order on Δ. The symbol $\boldsymbol{H}$ denotes the member of Δ defined by $\boldsymbol{H}(x) = 0$ if $x \leq 0$ and $\boldsymbol{H}(x) = 1$ if $x > 0$. Let $\Delta^+ = \{F \in \Delta \mid F(0) = 0\}$. If $F \in \Delta^+$, then $F(x) = 0$ for each $x < 0$, so we can assume that $\boldsymbol{R}^+$ is the domain of each member of Δ^+.

A *weak probabilistic metric space* (*WPMS*) consists of a set S and $\mathrm{F} = \{F_{pq} \mid p, q \in S\} \subseteq \Delta^+$ that satisfies the following properties:

(w_1) For each $p \in S$, $F_{pp} = \boldsymbol{H}$.

(w_2) For each $p, q \in S$, $F_{pq} = F_{qp}$.

(w_3) For each *distinct* $p, q, r \in S$ and $x, y \in \boldsymbol{R}^+$, $F_{pq}(x) = F_{qr}(y) = 1 \Rightarrow F_{pr}(x + y) = 1$.

The standard interpretation is that for $p, q \in S$ and $x \in \boldsymbol{R}^+$, $F_{pq}(x)$ is the probability that the distance from p to q is *less than* x. For example, $F_{pq}(0) = 0$ implies a zero probability that the distance between points p and q is *less than* 0. Property (w_3) postulates a probabilistic triangle inequality: if the probability that the distance from p to q is *less than* x and the probability that the distance from q to r is *less than* y, then the probability that the distance from p to r is *less than* $x + y$. We note that (w_3) also holds if p, q, and r are *not distinct*; for instance, if $p = q$, then $F_{pr}(y) = 1$, so since F_{pr} is non-decreasing, $F_{pr}(x + y) = 1$.

The property (w_1) is a weaker version of the following property:

(w_1') For each $p, q \in S$, $F_{pq} = \boldsymbol{H} \Leftrightarrow p = q$.

Otherwise, our definition of a *WPMS* agrees with the one used in ([**1**], 1.4.2).

➤ In the literature, many authors use the phrase "probabilistic metric space" to mean a *Menger* space under a *t-norm* (see **§3.1**).

In the sequel, a pair (S, F) denotes a *WPMS*. Frequently, we also use the alternate representation (S, $\boldsymbol{F}$), where $\boldsymbol{F} : S \times S \to \Delta^+$ is defined by $\boldsymbol{F}(p, q) = F_{pq}$.

We distinguish the following types of *WPMSs*. If (S, F) is a *WPMS* and (w_1') holds, then we say that (S, F) is a *WPMS*$^+$. We say that a *WPMS* (S, F) is *trivial* if for each $p, q \in S$, $F_{pq} = \boldsymbol{H}$. More generally, we say that a *WPMS* (S, F) is *special* if for each $p, q \in S$, $F_{pq}^{-1}(1) \neq \varnothing$.

Example 2.1.1. The *WPMS* (S, $\boldsymbol{G}$) *determined by* a *pseudometric* space (S, γ) ([**1**], 1.2.5).
For $p, q \in S$, define $G_{pq} : \boldsymbol{R}^+ \to \boldsymbol{I}$ by $G_{pq}(x) = \boldsymbol{H}(x - \gamma(p, q))$. Using the standard interpretation, the probability is zero that the distance from p to q is less than $\gamma(p, q)$.
Since the motivation for a *WPMS* is to model a probabilistic situation, it is natural to use a distribution function to define the members of F. The next example illustrates this idea.

Example 2.1.2. The 1-*simple WPMS* (S, $\boldsymbol{A}$, γ, Φ) based on a *pseudometric* space (S, γ) and a fixed distribution function $\Phi \in \Delta^+ \setminus \{\boldsymbol{H}\}$ ([**1**], 8.6.1). For $p, q \in S$, define $A_{pq} : \boldsymbol{R}^+ \to \boldsymbol{I}$ by

$$A_{pq}(x) = \begin{cases} \boldsymbol{H}(x) & \gamma(p, q) = 0 \\ \Phi(x/\gamma(p, q)) & \gamma(p, q) > 0. \end{cases}$$

If $\gamma(p, q) = 0$, then the probability is one that the distance from p to q is less than any positive number. If $\gamma(p, q) > 0$, then the probability that the distance from p to q is less than x is determined by the choice of Φ. In the same manner, we can define an α-*simple WPMS* (S, $\boldsymbol{A}$, γ, Φ, α) for each $0 < \alpha < 1$ by using $\Phi(x/\gamma(p, q)^{\alpha})$ in place of $\Phi(x/\gamma(p, q))$.

Example 2.1.3. For a distinct pair $p, q \in S = \boldsymbol{R}^+$, define $F_{pq} : \boldsymbol{R}^+ \to \boldsymbol{I}$ by

$$F_{pq}(x) = \begin{cases} 0 & 0 \le x < 1 \\ (x - 1)/a_{pq} & 1 \le x < 1 + a_{pq} \\ 1 & x \ge 1 + a_{pq} \end{cases}$$

where $a_{pq} = | p - q |$. Also, for each $p \in S$, let $F_{pp} = \boldsymbol{H}$. Then $(\boldsymbol{R}^+, \boldsymbol{F})$ is a special *WPMS*.

Example 2.1.4. For each $p, q \in \boldsymbol{R}$ satisfying $| p - q | < 1$, define $f_{pq} : \boldsymbol{R}^+ \to \boldsymbol{I}$ by

$$f_{pq}(x) = \begin{cases} 0 & x = 0 \\ 1 - | p - q | & 0 < x \le 1 \\ 1 & x > 1. \end{cases}$$

Also, for each $p, q \in \boldsymbol{R}$, define $F_{pq} : \boldsymbol{R}^+ \to \boldsymbol{I}$ by

$$F_{pq}(x) = \begin{cases} \boldsymbol{H}(x - 1) & | p - q | \ge 1 \\ f_{pq}(x) & | p - q | < 1. \end{cases}$$

Then $(\boldsymbol{R}, \boldsymbol{F})$ is a *WPMS*$^+$.

There is a natural way to associate distances with a *WPMS*. For each $F \in \Delta^+$, define the non-decreasing mapping $F^\wedge: (0, 1] \to [0, +\infty]$ by $F^\wedge(t) = \boldsymbol{sup}(X(t))$, where $X(t) = \{x \in \boldsymbol{R}^+ \mid F(x) < t\}$ ([**1**], section 4.4). Since $F(0) = 0$, $X(t) \ne \varnothing$ for each $t > 0$ and if $t < 1$, then $X(t)$ is also bounded since $\boldsymbol{sup}(F) = 1$, so $F^\wedge(t)$ is well-defined. If $F^{-1}(1) \ne \varnothing$, then $F^\wedge(1)$ is also well-defined.
Given a *WPMS* $(S, \boldsymbol{F})$ and $0 < t < 1$, define $\omega_t : S \times S \to \boldsymbol{R}^+$ by

$$\omega_t(p, q) = F_{pq}{}^\wedge(t) = \boldsymbol{sup}\{x \in \boldsymbol{R}^+ \mid F_{pq}(x) < t\}$$

and define the extended real-valued mapping $\omega_1 : S \times S \to [0, +\infty]$ by $\omega_1(p, q) = F_{pq}{}^\wedge(1)$.
The ω_t's correspond to the d_c's introduced in ([**1**], 8.2.3). For each $0 < t < 1$, ω_t is a *distance*, i.e., for each $p, q \in S$, $\omega_t(p, p) = 0$ and $\omega_t(p, q) = \omega_t(q, p)$ (**§A.1**). In general, ω_t is not a *semi-metric*, that is, $\omega_t(p, q) = 0$ can occur for distinct p and q.
Given a *WPMS* $(S, \boldsymbol{F})$, let

$$\Omega_F = \{\omega_t \mid 0 < t < 1\}$$

denote the set of well-defined distances. Notice that Ω_F is a *linearly ordered* family of distances: if $0 < s < t < 1$, then $\omega_s \le \omega_t$.

For an *α-simple space* $(S, \boldsymbol{A}, \gamma, \Phi, \alpha)$ (**Example 2.1.2**), $\omega_t = \Phi^\wedge(t)\gamma^\alpha$ for each $0 < t \le 1$ and in **Example 2.1.3**, $\omega_t(p, q) = 1 + t a_{pq}$ for distinct $p, q \in S$ and $0 < t < 1$. Notice that in both cases, each ω_t is a pseudometric. In the following example, the distances are not pseudometrics.

Example 2.1.5. For each $p, q \in R^+$, define $F_{pq} : \boldsymbol{R}^+ \to \boldsymbol{I}$ by

$$F_{pq}(x) = [\boldsymbol{H}(x - | p - q |) + \boldsymbol{H}(x - | p^2 - q^2 |)]/2.$$

Then $(\boldsymbol{R}^+, \boldsymbol{F})$ is a special *WPMS*$^+$. As an illustration, if $0 < a = | p - q | < b = | p^2 - q^2 |$, then

$$F_{pq}(x) = \begin{cases} 0 & 0 \le x \le a \\ 1/2 & a < x \le b \\ 1 & x > b. \end{cases}$$

The distances fall into two groups:

$$\omega_t(p, q) = \begin{cases} e_1(p, q) & 0 < t \le 1/2 \\ e_2(p, q) & 1/2 < t \le 1, \end{cases}$$

where e_1 and e_2 are defined by $e_1(p, q) = \boldsymbol{min}\{a, b\}$ and $e_2(p, q) = \boldsymbol{max}\{a, b\}$, respectively. The mapping e_2 is a metric, but e_1 is not a pseudometric since the triangle inequality doesn't hold. In fact, we can make

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a stronger statement. For $n \in \mathbf{N}^+$, let $x_n = n + 1/n$, $y_n = n$, and $z_n = -n + 1/n^2$. Then $e_1(x_n, y_n) + e_1(y_n, z_n) = 3/n - 1/n^4 \to 0$ as $n \to +\infty$, but $e_1(x_n, z_n) = 2 + 1/n^2 + 2/n - 1/n^4 > 2$ for each n. Therefore, $(\boldsymbol{R}^+, e_1)$ doesn't satisfy the condition (*wtc*) discussed in **§3.3**.

2.2. Preliminary Results

In this subsection, we present some basic results that provide further background for the reader as well as a basis for subsequent sections. In proofs, we often assume that $p, q, r \in S$ and $x, y \in \boldsymbol{R}^+$ without explicitly mentioning the sets.

The first result summarizes the relationships between the values of the distribution functions and distances. We use it repeatedly in the text. The proof is elementary, but we include part of it to familiarize the reader with the notation.

Lemma 2.2.1. The following statements hold for each *WPMS* $(S, \boldsymbol{F})$ and $0 < t \le 1$.

(a) $\omega_t(p, q) < x \Rightarrow F_{pq}(x) \ge t$.

(b) $F_{pq}(x) > t \Rightarrow \omega_t(p, q) < x$.

(c) $F_{pq}(x) \ge t \Rightarrow \omega_t(p, q) \le x$ and $\omega_u(p, q) < x$ for each $0 < u < t$.

Proof

(**b**): Suppose $y = \omega_t(p, q) \ge x$. By the left-continuity of F_{pq}, for each $\varepsilon > 0$, there is $y_\varepsilon < y$ such that $F_{pq}(y) - F_{pq}(y_\varepsilon) < \varepsilon$. Hence, $F_{pq}(y_\varepsilon) \le t$, so $F_{pq}(y) < t + \varepsilon$. Since the inequality holds for each $\varepsilon > 0$, $F_{pq}(y) \le t$. Therefore, $F_{pq}(x) \le t$ since F_{pq} is non-decreasing.

□

To check when a member of Ω_F is a pseudometric, it is useful to introduce the following predicate. Given a *WPMS* $(S, \boldsymbol{F})$ and $0 < t \le 1$, define

$$(\&_t)\text{: } \forall p, q, r \in S, \forall x, y \in \boldsymbol{R}^+ \bullet F_{pq}(x) \ge t \text{ and } F_{qr}(y) \ge t \Rightarrow F_{pr}(x + y) \ge t.$$

It follows from property (w_3) that ($\&_1$) always holds. The next result is implicit in the literature.

Prop 2.2.1. Suppose $(S, \boldsymbol{F})$ is a *WPMS* and $0 < t < 1$.

(a) If ($\&_t$) holds, then ω_t is a *pseudometric*.

(b) If ω_s is a *pseudometric* for each $0 < s < t$, then ($\&_t$) holds.

Proof

(**a**): Let $\varepsilon > 0$ and choose x, y such that $\omega_t(p, q) < x < \omega_t(p, q) + \varepsilon/2$ and $\omega_t(q, r) < y < \omega_t(q, r) + \varepsilon/2$. By **Lemma 2.2.1**(a), $F_{pq}(x) \ge t$ and $F_{qr}(y) \ge t$, so by ($\&_t$), $F_{pr}(x + y) \ge t$. Hence, by **Lemma 2.2.1**(c), $\omega_t(p, r) \le x + y$. Therefore, $\omega_t(p, r) < \omega_t(p, q) + \omega_t(q, r) + \varepsilon$ for each $\varepsilon > 0$, so ω_t satisfies the triangle inequality.

(**b**): If ($\&_t$) doesn't hold, then there exists p, q, r, x, and y such that $F_{pq}(x) \ge F_{qr}(y) \ge t > F_{pr}(x + y)$. Choose $F_{pr}(x + y) < s < t$. By **Lemma 2.2.1**(b), $\omega_s(p, q) < x$ and $\omega_s(q, r) < y$. Since $F_{pr}(x + y) < s$, by **Lemma 2.2.1**(a), $x + y \le \omega_s(p, r)$. Therefore, $\omega_s(p, q) + \omega_s(q, r) < x + y \le \omega_s(p, r)$ shows that ω_s doesn't satisfy the triangle inequality.

□

The reader can see from the previous proof that **Lemma 2.2.1** might need to be used repeatedly. Therefore, in future proofs, we only use it implicitly.

The following elementary result complements the preceding result.

Lemma 2.2.2. The following statements are equivalent for a *WPMS* $(S, \boldsymbol{F})$.

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(a) $(S, \boldsymbol{F})$ is *special*.

(b) ω_1 is a *real-valued* mapping.

(c) ω_1 is a *pseudometric*.

Proof

The implication (**a**) ⇒ (**b**) follows from the definition.

(**b**) ⇒ (**c**): Let $x = \omega_1(p, q)$ and $y = \omega_1(q, r)$. By definition, $F_{pq}(x + \varepsilon/2) = 1$ and $F_{qr}(y + \varepsilon/2) = 1$ for each $\varepsilon > 0$, so by ($\boldsymbol{w}_3$), $F_{pr}(x + y + \varepsilon) = 1$. Hence, $\omega_1(p, r) \le x + y + \varepsilon$ for each $\varepsilon > 0$, so ω_1 satisfies the triangle inequality.

(**c**) ⇒ (**a**): Since $\omega_1(p, q)$ is finite, $F_{pq}(\omega_1(p, q) + 1) = 1$. Hence, $(S, \boldsymbol{F})$ is special.

□

The next two results present conditions that are equivalent to Ω_F being a family of pseudometrics and metrics, respectively.

Prop 2.2.2. The following statements are equivalent for a *WPMS* $(S, \boldsymbol{F})$.

(a) For each $0 < t < 1$, ω_t is a *pseudometric*.

(b) For each $0 < t < 1$, ($\&_t$) holds.

(c) For each $p, q, r \in S$ and $x, y \in \boldsymbol{R}^+$, $F_{pr}(x + y) \ge \boldsymbol{min}\{F_{pq}(x), F_{qr}(y)\}$.

Proof

The implication (**c**) ⇒ (**a**) follows from **Prop 2.2.1(a)** since (**c**) implies that each ($\&_t$) holds. The implication (**a**) ⇒ (**b**) follows from **Prop 2.2.1(b)**.

(**b**) ⇒ (**c**): Let $s = F_{pq}(x)$ and $t = F_{qr}(y)$. Clearly, if $s = 0$ or $t = 0$, then the inequality in (**c**) holds. Otherwise, if $s \le t$, then by ($\&_s$), $F_{pr}(x + y) \ge s = \boldsymbol{min}\{s, t\}$. If $s > t$, then $F_{pr}(x + y) \ge t = \boldsymbol{min}\{s, t\}$ follows from ($\&_t$).

□

The following result is not used in the rest of the paper, but it's a natural companion to the previous proposition.

Prop 2.2.3. The following statements are equivalent for a *WPMS* $(S, \boldsymbol{F})$.

(a) For each $0 < t < 1$, ω_t is a *metric*.

(b) (*i*) For each $p \ne q$, F_{pq} is *right-continuous* at 0.

(*ii*) For each $p, q, r \in S$ and $x, y \in \boldsymbol{R}^+$, $F_{pr}(x + y) \ge \boldsymbol{min}\{F_{pq}(x), F_{qr}(y)\}$.

Proof

(**a**) ⇒ (**b**): Suppose $p \ne q$ and let $n \in \boldsymbol{N}\backslash\{0, 1\}$. Since $\omega_{1/n}(p, q) > 0$, there is $0 < \varepsilon_n < 1/n$ such that $\varepsilon_n < \omega_{1/n}(p, q)$. Hence, $F_{pq}(\varepsilon_n) \le 1/n$. Therefore, $F_{pq}(0+) = lim_{h\to 0+}F_{pq}(h) = 0$, which establishes (*i*). Condition (*ii*) follows from **Prop 2.2.2**.

(**b**) ⇒ (**a**): By **Prop 2.2.2** and (*ii*), each ω_t is a pseudometric. Suppose $p \ne q$ and $\omega_t(p, q) = 0$ for some $0 < t < 1$. Then each $F_{pq}(1/n) \ge t$, so F_{pq} is not right-continuous at 0, which contradicts (*i*).

□

We note that right continuity can be characterized by using the interplay between distances and distributions. Given a *WPMS* $(S, \boldsymbol{F})$, for each $x \in \boldsymbol{R}^+$ and $p, q \in S$, define the predicate

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$$(\#_{x,p,q}) \equiv 0 < t \le 1 \text{ and } \omega_t(p, q) = x \Rightarrow F_{pq}(x) = t.$$

Then $(S, \boldsymbol{F})$ *satisfies* $(\#_{x,p,q})$ if and only if F_{pq} is *right-continuous* at x. We leave this statement for the reader's consideration.

The condition stated in ([**5**], (IV)) is equivalent to $(\&_t)$ and the equivalence of parts (a) and (b) in **Prop 2.2.2** and **Prop 2.2.3** is proved in ([**5**], Lemma 1 and Theorem 1). Also, **Prop 2.2.2**(a) is proved in ([**1**], 8.2.3) and **Prop 2.2.3** is essentially established in ([**11**], Theorems 9 and 10).

2.3. Reconstruction and Representation

The first result shows that for any *WPMS* $(S, \boldsymbol{F})$, we can always reconstruct the family F = $\{F_{pq}\}$ from Ω_F.

PROP 2.3.1. For each *WPMS* $(S, \boldsymbol{F})$, $p, q \in S$, and $x \in \boldsymbol{R}^+$, $F_{pq}(x) = \boldsymbol{sup}\{0 < t < 1 \mid \omega_t(p, q) < x\}$, where $\Omega_F = \{\omega_t\}$.

<u>Proof</u>

For the proof, we suppress the pairs (p, q) and subscripts pq.

Let $x \in \boldsymbol{R}^+$ and $T = \{0 < t < 1 \mid \omega_t < x\}$. Define $v = 0$ if $T = \emptyset$ and $v = \boldsymbol{sup}(T)$ if $T \neq \emptyset$. If $T = \emptyset$, then $\omega_t \ge x$ for each $0 < t < 1$, so $F(x) \le t$ for each $0 < t < 1$. Hence, $F(x) = v$. If $T \neq \emptyset$, then for each $t \in T$, $F(x) \ge t$. Hence, $v \le F(x)$. If $v < F(x)$, choose $v < r < F(x)$. Then $\omega_r < x$, so $r \in T$. Hence, $v \ge r$, which is a contradiction. Therefore, $v = F(x)$.

□

We remark that the following definition of ω_t is used in ([**5**], (I)),: $\omega_t(p, q) = \boldsymbol{inf}\{x \mid F_{pq}(x) > t\}$. Then ([**5**], (III)) asserts that $F_{pq}(x) = \boldsymbol{sup}\{t \mid \omega_t(p, q) < x\}$.

Next, we present the first new idea in the paper. We say that a *WPMS* $(S, \boldsymbol{F})$ has *finite range* if $Rng(\boldsymbol{F}) = \cup\{Rng(F_{pq}) \mid p, q \in S\}$ is a *finite* set. For this type of *WPMS*, we can establish the following representation theorem. Also, we'll revisit these types of *WPMSs* in **§3.2**.

Theorem 2.3.1. Assume that the *WPMS* $(S, \boldsymbol{F})$ has the *finite range* $Rng(\boldsymbol{F}) = \{t_i \mid 0 \le i \le n\}$, where $t_0 = 0 < t_1 < \ldots < t_n = 1$ and $\Omega_F = \{\omega_t\}$. Then the following statements hold:

(a) $\Omega_F = \{\omega_{t(i)} \mid 1 \le i \le n\}$ and $\mid \Omega_F \mid = n$.

(b) For each $p, q \in S$ and $x \in \boldsymbol{R}^+$, $F_{pq}(x) = \Sigma\{(t_{i+1} - t_i)\boldsymbol{H}(x - \omega_{t(i+1)}(p, q)) \mid 0 \le i < n\}$.

<u>Proof</u>

In various places, we write $t(i)$ in place of t_i.

(**a**): Given $0 < t \le 1$, choose $1 \le i \le n$ such that $t_{i-1} < t \le t_i$. Then for $p, q \in S$,

$$\omega_t(p, q) = \boldsymbol{sup}\{x \mid F_{pq}(x) < t\} = \boldsymbol{sup}\{x \mid F_{pq}(x) < t_i\} = \omega_{t(i)}(p, q).$$

This establishes $\omega_t = \omega_{t(i)}$. Let $1 \le i < n$ and choose p, q, and y such that $F_{pq}(y) = t_i$. Then $z = \omega_{t(i+1)}(p, q) \ge y$. Since F_{pq} is left-continuous, there exists $0 < x < y$ such that $F_{pq}(x) = t_i$. If $\omega_{t(i)}(p, q) \ge z$, then $\omega_{t(i)}(p, q) > x$, so $F_{pq}(x) < t_i$, which gives a contradiction. Therefore, we have $\omega_{t(i)}(p, q) < z = \omega_{t(i+1)}(p, q)$. It follows that $\mid \Omega_F \mid = n$.

(**b**): For each $p, q \in S$, define G_{pq} by $G_{pq}(x) = \Sigma\{(t_{i+1} - t_i)\boldsymbol{H}(x - \omega_{t(i+1)}(p, q)) \mid 0 \le i < n\}$.

(1) $(S, \boldsymbol{G})$ is a *WPMS*.

[Each $G_{pq} \in \Delta^+$ since $\boldsymbol{H}$ is left-continuous and $G_{pq}(x) = 1$ for $x > \omega_1(p, q)$. It is easy to show that $(\boldsymbol{w}_1)$ and $(\boldsymbol{w}_2)$ hold. Suppose $G_{pq}(x) = G_{qr}(y) = 1$. Since $(S, \boldsymbol{F})$ is special, by **Lemma 2.2.2**, ω_1 is a pseudometric. Then by definition, $x > \omega_1(p, q)$ and $y > \omega_1(q, r)$, so $x + y > \omega_1(p, r)$. Therefore, $G_{pr}(x + y) = 1$, so $(\boldsymbol{w}_3)$ holds.]

Let $\Omega_G = \{\nu_t\}$.

(2) $\nu_t = \omega_t$ for each $0 < t \leq 1$.

[Let $p, q \in S$ and choose $1 \leq i \leq n$ such that $t_{i-1} < t \leq t_i$. Since $\{\omega_t\}$ is linearly ordered, by definition, $Rng(\boldsymbol{G}) \subseteq \{t_i \mid 0 \leq i \leq n\}$. Hence, $\nu_t(p, q) = \boldsymbol{sup}\{x \mid G_{pq}(x) < t\} = \boldsymbol{sup}\{x \mid G_{pq}(x) < t_i\} = \nu_{t(i)}(p, q)$. By definition, if $G_{pq}(x) \leq t_{i-1}$, then $x \leq \omega_{t(i)}(p, q)$, so $\nu_{t(i)}(p, q) = \boldsymbol{sup}\{x \mid G_{pq}(x) < t_i\} \leq \omega_{t(i)}(p, q)$. If $F_{pq}(x) < t_i$, then $x \leq \omega_{t(i)}(p, q)$, so $G_{pq}(x) < t_i$. Therefore,

$$\omega_{t(i)}(p, q) = \boldsymbol{sup}\{x \mid F_{pq}(x) < t_i\} \leq \boldsymbol{sup}\{x \mid G_{pq}(x) < t_i\} = \nu_{t(i)}(p, q).$$

This establishes $\omega_{t(i)} = \nu_{t(i)}$.]

Since $(S, \boldsymbol{F})$ is a *WPMS* and by (1), $(S, \boldsymbol{G})$ is a *WPMS*, it follows from (2) and **Prop 2.3.1** that $\boldsymbol{F} = \boldsymbol{G}$.

□

The *WPMS* $(S, \boldsymbol{F})$ in **Example 2.1.5** illustrates **Theorem 2.3.1** since

$$F_{pq}(x) = [\boldsymbol{H}(x - | p - q |) + \boldsymbol{H}(x - | p^2 - q^2 |)]/2 = [\boldsymbol{H}(x - e_1(a, b)) + \boldsymbol{H}(x - e_2(a, b))]/2.$$

In this case, $| \Omega_F | = \{e_1, e_2\}$ and $Rng(e_1) = Rng(e_2) = \boldsymbol{R}^+$. Therefore, even for a *WPMS* with a finite range, the distances may not have finite ranges. In fact, **Example 3.1.1** shows that the distances can have the values {0, 1} without the *WPMS* itself having a finite range.

The following result is an easy consequence of the previous theorem that fits nicely with one's intuition. We can view it as a "0-1 law": if the probability is 0 or 1 that the distance from p to q is less than some value, then the distance is described by a unique pseudometric.

Corollary 2.3.1. The following statements are equivalent for a *WPMS* $(S, \boldsymbol{F})$.

(a) $Rng(\boldsymbol{F}) = \{0, 1\}$.

(b) $(S, \boldsymbol{F})$ is *determined* by a *pseudometric*.

(c) $\Omega_F = \{\omega_1\}$ and ω_1 is a *pseudometric*.

(d) $| \Omega_F | = 1$.

Proof

(**a**) ⇒ (**b**): By **Theorem 2.3.1**, $F_{pq}(x) = \boldsymbol{H}(x - \omega_1(p, q))$ for each $p, q \in S$ and $x \in \boldsymbol{R}^+$. Since $(S, \boldsymbol{F})$ is special, by **Lemma 2.2.2**, ω_1 is a pseudometric.

(**b**) ⇒ (**c**): Suppose $F_{pq}(x) = \boldsymbol{H}(x - \omega(p, q))$ for a pseudometric ω and each $p, q \in S$ and $x \in \boldsymbol{R}^+$. Then for each $0 < t \leq 1$, $\omega_t(p, q) = \boldsymbol{sup}\{x \mid F_{pq}(x) < t\} = \boldsymbol{sup}\{x \mid x \leq \omega(p, q)\} = \omega(p, q)$.

Clearly, (**c**) ⇒ (**d**).

¬(**a**) ⇒ ¬(**d**): If $0 < F_{pq}(x) < 1$ for some $p, q \in S$ and $x \in \boldsymbol{R}^+$, choose $0 < s < F_{pq}(x) < t < 1$. Then $\omega_s(p, q) < x \leq \omega_t(p, q)$ shows that $\omega_s \neq \omega_t$. Hence, $| \Omega_F | > 1$.

□

2.4. Linearly Ordered Families

Here, we present the second new idea in the paper – the application of linearly ordered families of distances. For clarity, we are referring to a family $\Omega = \{\rho_t \mid 0 < t < 1\}$ of distances defined on a set S that satisfies $\rho_s \leq \rho_t$ for each $0 < s < t < 1$. We say that Ω satisfies the left-continuity property (lcp) (resp. right-continuity property (rcp)) if $\rho_t = \sup\{\rho_s \mid 0 < s < t\}$ (resp. $\rho_t = \inf\{\rho_s \mid t < s < 1\}$ for each $0 < t < 1$.

Prop 2.4.1. The following statements hold for a *WPMS* $(S, \boldsymbol{F})$.

(a) The family $\Omega_F = \{\omega_t \mid 0 < t < 1\}$ satisfies (***lcp***) and if $(S, \boldsymbol{F})$ is a *special WPMS*, then $\omega_1 = \boldsymbol{sup}\{\omega_t \mid 0 < t < 1\}$.

(b) The family Ω_F satisfies (***rcp***) if and only if (#): for each distinct pair $p, q \in S$, F_{pq} is *strictly increasing* on $F_{pq}^{-1}(0, 1)$.

Proof

(**a**): Let $0 < t < 1$. For $p, q \in S$ and $\varepsilon > 0$, there is $x \in \mathbf{R}^+$ such that $\omega_t(p, q) - \varepsilon < x$ and $F_{pq}(x) < t$. If $F_{pq}(x) < s < t$, then $\omega_s(p, q) \geq x > \omega_t(p, q) - \varepsilon$. Therefore, $\omega_t = \boldsymbol{sup}\{\omega_s \mid 0 < s < t\}$. If $(S, \boldsymbol{F})$ is special, then by **Lemma 2.2.2**, ω_1 is real-valued, so by a similar argument, we can establish that $\omega_1 = \boldsymbol{sup}\{\omega_t \mid 0 < t < 1\}$.

(**b**): Suppose Ω_F satisfies (***rcp***) and $p, q \in S$ is a distinct pair. Suppose $0 < F_{pq}(x) = F_{pq}(y) = t < 1$ for some $0 \leq x < y$. If $t < s < 1$, $\omega_s(p, q) \geq y$ and $\omega_t(p, q) \leq x$, so $\omega_t(p, q) < \boldsymbol{inf}\{\omega_s(p, q) \mid t < s < 1\}$, which is a contradiction. Therefore, (#) holds. Conversely, if (#) holds and (***rcp***) doesn't, then $a = \omega_t(p, q) < b = \boldsymbol{inf}\{\omega_s(p, q) \mid t < s < 1\}$ for some p and q. Then $F_{pq}(x) \geq t$ for each $a < x$ and for each $x < b$ and $t < s < 1$, $F_{pq}(x) < s$, so $F_{pq}(x) \leq t$. Therefore, $F_{pq}(x) = t$ for each $a < x < b$, which is a contradiction. Hence, (***rcp***) holds.

□

The following example describes linearly ordered families of pseudometrics that don't satisfy the (***lcp***) and (***rcp***) conditions, respectively.

Example 2.4.1.

(1) ¬(***lcp***)

For each $n \in \mathbf{N}$ and $0 \leq i < 2^n$, let

$$B_{i,n} = \begin{cases} [i2^{-n}, (i+1)2^{-n}) & 0 \leq i \leq 2^n - 2 \\ [1 - 2^{-n}, 1] & i = 2^n - 1. \end{cases}$$

Each $\pi_n = \{B_{i,n} \mid 0 \leq i < 2^n\}$ is a partition of $\boldsymbol{I}$ and $\pi_{n+1} < \pi_n$. Define the pseudometric d_n on $\boldsymbol{I}$ by $d_n(p, q) = 0$ if $p, q \in B_{i,n}$ for some i and $d_n(p, q) = 1$ otherwise. For each $0 < t < 1$, let $\rho_t = d_k$, where $k = \lfloor t/(1 - t) \rfloor$ and let $a_k = k/(k + 1)$ for $k \in \mathbf{N}$. Then $\rho_t = d_k$ if and only if $a_k \leq t < a_{k+1}$. Suppose $0 < s < t < 1$. If $s \in [a_m, a_{m+1})$ and $t \in [a_n, a_{n+1})$, where $m < n$, then $\rho_s = d_m$ and $\rho_t = d_n$. Since $\pi_n < \pi_m$, $\rho_s \leq \rho_t$, so $\Omega = \{\rho_t\}$ is a linearly ordered family. However, $\rho_{1/2} = d_1$ is not the supremum of the family $\{\rho_t \mid 0 < t < 1/2\} = \{d_0\}$, so (***lcp***) doesn't hold.

(2) ¬(***rcp***)

For each distinct pair $p, q \in \boldsymbol{I}$, define $F_{pq} : \mathbf{R}^+ \to \boldsymbol{I}$ by

$$F_{pq}(x) = \begin{cases} 0 & x = 0 \\ \boldsymbol{min}\{p, q\} & 0 < x \leq 1 \\ 1 & x > 1 \end{cases}$$

and let $F_{pp} = \boldsymbol{H}$. Then $(\boldsymbol{I}, \boldsymbol{F})$ is a *WPMS* and for each $0 < t \leq 1$,

$$\omega_t(p, q) = \begin{cases} 1 & p \neq q \text{ and } \boldsymbol{min}\{p, q\} < t. \\ 0 & \text{otherwise.} \end{cases}$$

If $p = 1/2$ and $q = 3/4$, then $F_{pq} = 1/2$ on $F_{pq}^{-1}(0, 1)$, so by **Prop 2.4.1(b)**, (***rcp***) doesn't hold. Alternately, $\omega_{1/4}(1/4, 3/4) = 0$, but $\omega_s(1/4, 3/4) = 1$ for each $1/4 < s < 1$, so (***rcp***) doesn't hold.

The characterization stated in **Prop 2.4.1**(b) suggests another unexplored property. We say that a *WPMS* $(S, \boldsymbol{F})$ satisfies the *collective increasing property* (***cip***) if for each $0 \leq x < y$, there exists $p, q \in S$ such that $F_{pq}(x) < F_{pq}(y)$. We can characterize this property by using the following set: given a *WPMS* $(S, \boldsymbol{F})$, the *spectrum* of $\boldsymbol{F}$ is $\boldsymbol{spec}(\boldsymbol{F}) = \cup\{\boldsymbol{spec}(\omega_t) \mid 0 < t < 1\}$, where $\Omega_F = \{\omega_t\}$ and $\boldsymbol{spec}(\omega_t) = \{\omega_t(p, q) \mid p, q \in S\}$. The next result shows that the property (***cip***) is equivalent to a density criterion.

Prop 2.4.2. A *WPMS* $(S, \boldsymbol{F})$ satisfies (***cip***) if and only if $\boldsymbol{spec}(\boldsymbol{F})$ is a *dense* subset of $\mathbf{R}^+$.

Proof

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Suppose $(S, \boldsymbol{F})$ satisfies (***cip***). Let $0 \le x < y$ and let $x < a < b < y$. By definition, there is $p, q \in S$ such that $F_{pq}(a) < F_{pq}(b)$. Choose $F_{pq}(a) < t < F_{pq}(b)$. Then $0 < t < 1$ and $x < a \le \omega_t(p, q) < b$, so ***spec***$(\boldsymbol{F})$ is dense in $\boldsymbol{R}^+$. Conversely, suppose ***spec***$(\boldsymbol{F})$ is dense in $\boldsymbol{R}^+$ and let $0 \le x < y$. By assumption, there is $0 < t < 1$ and $p, q \in S$ such that $x < \omega_t(p, q) < y$. Then $F_{pq}(x) < t \le F_{pq}(y)$, so (***cip***) holds.

□

Because of the similar definitions, we can ask if (***rcp***) implies (***cip***). However, this isn't the case. In **Example 2.1.3**, for distinct p and q, F_{pq} is strictly increasing on $F_{pq}^{-1}(0, 1) = (1, 1 + a_{pq})$, so by **Prop 2.4.1(b)**, (***rcp***) holds. Since each $\omega_t(p, q) = 1 + ta_{pq}$, ***spec***$(\boldsymbol{F}) \subseteq \{0\} \cup [1, +\infty)$, so by **Prop 2.4.2**, (***cip***) doesn't hold.

The next result shows that we can easily modify a linearly ordered family of pseudometrics so that (***lcp***) holds. We leave the proof to the reader.

Lemma 2.4.1. Let $\Omega = \{\rho_t \mid 0 < t < 1\}$ be a *linearly ordered* family of *distances* (resp. *pseudometrics*) on a set S and for each $0 < t < 1$, define $\rho'_t =$ ***sup***$\{\rho_s \mid 0 < s < t\}$. Then $\Omega' = \{\rho_t'\}$ is a *linearly ordered* family of *distances* (resp. *pseudometrics*) on S that satisfies (***lcp***) and $\rho_t' \le \rho_t$ for each t.

Other authors have shown that a *WPMS* can be reconstructed from its family of distances, but the following question doesn't seem to have been addressed. What conditions on a family of distances *guarantee* that it has the form Ω_F for some *WPMS* $(S, \boldsymbol{F})$? Based on **Prop 2.4.1(a)**, (***lcp***) is a necessary condition and as we noted earlier, Ω_F is a linearly ordered family. The next result shows how to construct a special *WPMS* from a linearly ordered family of distances. A companion result for *Menger* spaces is presented in **§3.2**.

Theorem 2.4.1. Let $\Omega = \{\rho_t \mid 0 < t \le 1\}$ be a *linearly ordered* family of *distances* on a set S that satisfies the following conditions:

(a) ρ_1 is a *pseudometric* and $\rho_1 =$ ***sup***$\{\rho_t \mid 0 < t < 1\}$.

(b) If $p, q \in S$ and $\rho_1(p, q) > 0$, then $\rho_t(p, q) < \rho_1(p, q)$ for each $0 < t < 1$.

Then there is a *special WPMS* $(S, \boldsymbol{F})$ such that $\omega_1 = \rho_1$ and $\omega_t \le \rho_t$ for each $0 < t < 1$, where $\Omega_F = \{\omega_t\}$. If Ω also satisfies (***lcp***), then $\omega_t = \rho_t$ for each $0 < t < 1$.

<u>Proof</u>

For $p, q \in S$ and $x \in \boldsymbol{R}^+$, let $T_{pq}(x) = \{0 < t < 1 \mid \rho_t(p, q) < x\}$ and define $F_{pq} : \boldsymbol{R}^+ \to \boldsymbol{I}$ by

$$F_{pq}(x) = \begin{cases} 0 & T_{pq}(x) = \varnothing \\ \boldsymbol{sup}(T_{pq}(x)) & T_{pq}(x) \neq \varnothing. \end{cases}$$

(1) Each F_{pq} belongs to Δ^+ and $F_{pq}^{-1}(1) \neq \varnothing$.

[Since $T_{pq}(0) = \varnothing$, $F_{pq}(0) = 0$. If $x > 0$, then $T_{pp}(x) = (0, 1)$, so $F_{pp}(x) = 1$. Therefore, $F_{pp} = \boldsymbol{H}$. Suppose $p, q \in S$ is a distinct pair and let $g = F_{pq}$. It is routine to show that $g(0) = 0$ and g is non-decreasing. Also, by definition, $g(\rho_1(p, q) + 1) = 1$, so ***sup***$(g) = 1$. Let $x > 0$ and let $\{x_n\}$ be a sequence in $(0, x)$ such that $x_n \to x$. Let $t = g(x)$. If $t = 0$, then $g(x_n) = 0$ since g is non-decreasing, so $\boldsymbol{lim}_{n\to+\infty} g(x_n) = g(x)$. Let $t > 0$ and choose $0 < \varepsilon < t$. Since $t =$ ***sup***$\{0 < s < 1 \mid \rho_s(p, q) < x\}$, there exists $t - \varepsilon < s < 1$ such that $\rho_s(p, q) < x$. Then $\rho_{t-\varepsilon}(p, q) \le \rho_s(p, q) < x_n$ for sufficiently large n, so $g(x_n) \ge t - \varepsilon$. Hence, $\boldsymbol{lim}_{n\to+\infty} g(x_n) = g(x)$. Therefore, g is left-continuous.]

(2) $(S, \boldsymbol{F})$ is a special *WPMS*.

[It's routine to show that (w_1) and (w_2) hold. Suppose $F_{pq}(x) = F_{qr}(y) = 1$ for distinct $p, q, r \in S$ and $x, y \in \boldsymbol{R}^+$. Then $x > 0$ and $y > 0$ and since Ω is linearly ordered, $T_{pq}(x) = T_{qr}(y) = (0, 1)$. By (**a**), $\rho_1(p, q) =$

sup$\{\rho_t(p, q)\} \le x$ and, similarly, $\rho_1(q, r) \le y$, so $\rho_1(p, r) \le x + y$. If $\rho_1(p, r) = 0$, then each $\rho_t(p, r) = 0$, so $T_{pr}(x + y) = (0, 1)$ since $x + y > 0$. If $\rho_1(p, r) > 0$, then it follows from (**b**) that $\rho_t(p, r) < \rho_1(p, r) \le x + y$ for each $0 < t < 1$, so $T_{pr}(x + y) = (0, 1)$. It follows that in either case, $F_{pr}(x + y) = 1$, so (w_3) holds. Then by (1), (S, ***F***) is a special *WPMS*.]

Assume that $\Omega_F = \{\omega_t\}$. For the rest of the proof, we suppress the pair (p, q) and subscript pq.

(3) For each $0 < t < 1$, $\omega_t \le \rho_t$ and $\omega_1 = \rho_1$.

[Given $\varepsilon > 0$, choose $x > \omega_t - \varepsilon$ such that $F(x) < t$. Then $t \notin T(x)$, so $\rho_t \ge x$. Therefore, $\omega_t \le \rho_t$. If $x > \omega_1$, then $F(x) = 1$, so $T(x) = (0, 1)$. Hence, $\rho_t < x$ for each $0 < t < 1$, so by (**a**), $\rho_1 \le x$. This establishes $\rho_1 \le \omega_1$. By (2) and **Prop 2.4.1(a)**, $\omega_1 =$ ***sup***$\{\omega_t\} \le$ ***sup***$\{\rho_t\} = \rho_1$, so we have $\omega_1 = \rho_1$.]

(4) If Ω satisfies (***lcp***), then $\omega_t = \rho_t$ for each $0 < t < 1$.

[If $\omega_t < \rho_t$ for some $0 < t < 1$, then since (S, ***F***) satisfies (***lcp***), there is $0 < u < t$ such that $\omega_t < \rho_u$. Hence, $t \le F(\rho_u)$, so $F(\rho_u) =$ ***sup***$\{0 < s < 1 \mid \rho_s < \rho_u\} > 0$. Since Ω is linearly ordered, if $\rho_s < \rho_u$, then $s < u$. Hence, $F(\rho_u) \le u$, which is a contradiction. Therefore, $\omega_t = \rho_t$.]

□

In some cases, we can replace condition (**b**) in the previous result without altering the conclusion. For example, if ρ_1 satisfies (**a**) and (**b′**): $\rho_1(p, r) < \rho_1(p, q) + \rho_1(q, r)$ for *distinct* $p, q, r \in S$, then in the proof of statement (2), $\rho_1(p, r) < \rho_1(p, q) + \rho_1(q, r) \le x + y$, so we obtain $\rho_1(p, r) < x + y$. Therefore, $F_{pr}(x + y) = 1$.

Theorem 2.4.1 is suggested by **Prop 2.3.1** and by the work on *metrically generated* spaces and *E-spaces* ([**1**], §1.7 and §9.2, [**12**]). The latter topic involves using a probability space and a family of metrics as a base space without referring to a linear order.

The following example illustrates the construction used in **Theorem 2.4.1.**

Example 2.4.2. For each $0 < t \le 1$, define the distance ρ_t on $(0, 1)$ by $\rho_t(p, q) = | p - q |^{2/(t+1)}$. Then $\Omega = \{\rho_t \mid 0 < t \le 1\}$ satisfies the conditions in **Theorem 2.4.1** and (***lcp***) holds, so $\omega_t = \rho_t$ for each $0 < t \le 1$. Now (S, ***F***) is a special *WPMS* with the following distributions: for $0 < p < q < 1$,

$$F_{pq}(x) = \begin{cases} 0 & x = 0 \\ 2ln(q - p)/ln(x) - 1 & 0 < x < q - p \\ 1 & x \ge q - p. \end{cases}$$

3. Menger Spaces

In this section, we discuss the families of *Menger* spaces and *generalized Menger* spaces.

3.1. Definitions and Examples

Various conditions can be imposed on a *WPMS* (S, ***F***) that reflect the properties found in standard examples. One well-known condition has the following form: there is a mapping $T : \boldsymbol{I} \times \boldsymbol{I} \to \boldsymbol{I}$ such that the following condition holds:

(**T**) For each $p, q, r \in S$ and $x, y \in \boldsymbol{R}^+$, $F_{pr}(x + y) \ge T(F_{pq}(x), F_{qr}(y))$.

The mapping T is called a *norm* and other requirements are usually imposed. For instance, T is a *t-norm* ([**1**], 5.6.1) if it is *commutative*, *associative*, and satisfies the following conditions: (*i*) for $a \in \boldsymbol{I}$, $T(a, 1) = a$ and (*ii*) for $a, b, c, d \in \boldsymbol{I}$, $a \le c$ and $b \le d \Rightarrow T(a, b) \le T(c, d)$. Standard examples of *t*-norms are $Min(a, b) =$ ***min***$\{a, b\}$, $\Pi(a, b) = ab$, and $T_m(a, b) =$ ***max***$\{a + b - 1, 0\}$ ([**1**], 11.1.1).

We say that a *WPMS* (S, ***F***) is a *Menger space under T* ([**1**], 8.1.4) if T is a *t-norm* satisfying (**T**). If (S, ***F***) is a *Menger* space under *Min*, then we simply refer to (S, ***F***) as a *Menger* space. This is a non-standard use of the term that's appropriate for our purposes. We note that if T is a *t-norm* satisfying (**T**), then $T \le Min$, so each *Menger* space is a *Menger* space under T.

Based on **Prop 2.2.2**, a *WPMS* $(S, \boldsymbol{F})$ is a *Menger* space if and only if Ω_F consists of pseudometrics; the sufficiency is noted [**1**], 8.2.3). The *WPMSs* in **Examples 2.1.1** and **2.1.2** are *Menger* spaces. **In Example 2.1.5**, some distances are not pseudometrics, so the *WPMS* is not a *Menger* space. Here is another example of a *Menger* space.

Example 3.1.1. For distinct $p, q \in \boldsymbol{I}$, define $F_{pq} : \boldsymbol{R}^+ \to \boldsymbol{I}$ by

$$F_{pq}(x) = \begin{cases} 0 & x = 0 \\ \boldsymbol{min}\{p, q\} & 0 < x \le 1 \\ 1 & x > 1. \end{cases}$$

Then $(S, \boldsymbol{F})$ is a *Menger* space and for each $0 < t \le 1$,

$$\omega_t(p, q) = \begin{cases} 1 & p \ne q \text{ and } \boldsymbol{min}\{p, q\} < t. \\ 0 & \text{otherwise.} \end{cases}$$

3.2. Main Results

The first result is the promised companion to **Theorem 2.4.1.**

Theorem 3.2.1. Let $\Omega = \{\rho_t \mid 0 < t < 1\}$ be a *linearly ordered* family of *pseudometrics* on a set S.

(a) There is a *Menger* space $(S, \boldsymbol{G})$ such that $\omega_t \le \rho_t$ for each $0 < t < 1$, where $\Omega_G = \{\omega_t\}$.

(b) $(S, \boldsymbol{G})$ is a *special WPMS* $\Leftrightarrow$ $\boldsymbol{sup}\{\rho_t(p, q) \mid 0 < t < 1\} < +\infty$ for each $p, q \in S$. In this case, ρ_1 is a *pseudometric* and $\omega_1 \le \rho_1 = \boldsymbol{sup}\{\rho_t\}$.

(c) The family Ω satisfies (***lcp***) $\Leftrightarrow$ $\omega_t = \rho_t$ for $0 < t < 1$. In this case, if $(S, \boldsymbol{G})$ is *special*, then $\omega_1 = \rho_1$.

Proof

(**a**): For each $p, q \in S$ and $x \in \boldsymbol{R}^+$, let $T_{pq}(x) = \{0 < t < 1 \mid \rho_t(p, q) < x\}$ and define $G_{pq} : \boldsymbol{R}^+ \to \boldsymbol{I}$ by

$$G_{pq}(x) = \begin{cases} 0 & T_{pq}(x) = \varnothing \\ \boldsymbol{sup}(T_{pq}(x)) & T_{pq}(x) \ne \varnothing. \end{cases}$$

(1) Each G_{pq} belongs to Δ^+.

[Since $G_{pq}(\rho_t(p, q) + 1) \ge t$ for each $0 < t < 1$, $\boldsymbol{sup}(G_{pq}) = 1$. Now the rest of the proof used to establish statement (1) in **Theorem 2.4.1** shows that G_{pq} belongs to Δ^+.]

(2) $(S, \boldsymbol{G})$ is a *Menger* space.

[It's routine to show that ($\boldsymbol{w_1}$) and ($\boldsymbol{w_2}$) hold. Suppose $0 < s = G_{pq}(x) \le t = G_{qr}(y)$ for $p, q, r \in S$ and $x, y \in \boldsymbol{R}^+$. Given $\varepsilon > 0$, choose $s - \varepsilon < s' < s$ such that $\rho_{s'}(p, q) < x$ and $t - \varepsilon < t' < t$ such that $\rho_{t'}(q, r) < y$. If $s' \le t'$, then $\rho_{s'}(p, r) \le \rho_{s'}(p, q) + \rho_{t'}(q, r) < x + y$, so $G_{pr}(x + y) \ge s' > s - \varepsilon$. Similarly, if $t' \le s'$, then $G_{pr}(x + y) \ge t' > t - \varepsilon \ge s - \varepsilon$. Hence, $G_{pr}(x + y) \ge s = \boldsymbol{min}\{G_{pq}(x), G_{qr}(y)\}$, so by (1), $(S, \boldsymbol{G})$ is a *Menger* space.]

(3) For each $0 < t < 1$, $\omega_t \le \rho_t$.

[The same argument used to prove (3) in **Theorem 2.4.1** establishes the statement.]

It follows from (2) and **Prop 2.2.2** that each member of $\Omega_G = \{\omega_t\}$ is a pseudometric.

(**b**): Let $p, q \in S$. If $(S, \boldsymbol{G})$ is special, then $G_{pq}(x) = 1$ for some $x \in \boldsymbol{R}^+$, so $\boldsymbol{sup}\{\rho_t(p, q)\} \le x$. Conversely, if $y = \boldsymbol{sup}\{\rho_t(p, q)\} < +\infty$, then $G_{pq}(y + 1) = 1$, so $(S, \boldsymbol{G})$ is special. In addition, if $\boldsymbol{sup}\{\rho_t(p, q)\} < +\infty$, then by **Prop 2.4.1(a)**, $\rho_1 = \boldsymbol{sup}\{\rho_t\}$ is a pseudometric on S.

If $G_{pq}(x) < 1$ for $p, q \in S$ and $x \in \boldsymbol{R}^+$, then $\rho_1(p, q) \ge \rho_t(p, q) \ge x$ for each $G_{pq}(x) < t < 1$, so $\omega_1 \le \rho_1$.

(**c**): If $\omega_t = \rho_t$ for each $0 < t < 1$, then by (2) and **Prop 2.4.1(a)**, Ω satisfies (***lcp***). Conversely, suppose Ω satisfies (***lcp***). If $\omega_t(p, q) < \rho_t(p, q)$ for some $0 < t < 1$ and $p, q \in S$, choose $u < t$ that satisfies $\omega_t(p, q) < x =$

$\rho_u(p, q)$. Then $t \leq G_{pq}(x)$. Since Ω is linearly ordered, if $\rho_s(p, q) < x$, then $s < u$, so $G_{pq}(x) \leq u$. This is a contradiction, so $\omega_t = \rho_t$ for each $0 < t < 1$.

If $(S, \boldsymbol{G})$ is special, then by **Prop 2.4.1(a)** and part (**b**), $\omega_1 = \boldsymbol{sup}\{\omega_t\} = \boldsymbol{sup}\{\rho_t\} = \rho_1$.

□

Based on a summary of [**13**], a result similar to **Theorem 3.2.1(b)** seems to be established, but I can't give an exact statement since I haven't seen the paper. The following examples illustrate the construction used in the previous result.

Example 3.2.1. In **Example 2.4.1(1)**, for distinct $p, q \in \boldsymbol{I}$, the distribution function in the corresponding *Menger* space is

$$G_{pq}(x) = \begin{cases} 0 & x = 0 \\ (n+1)/(n+2) & 0 < x \leq 1 \\ 1 & x > 1 \end{cases}$$

where n is the largest integer such that p and q belong to the same member of π_n. By **Theorem 3.2.1(a)**, $\omega_t \leq \rho_t$ for each $0 < t < 1$, where $\Omega_G = \{\omega_t\}$. Since $\Omega = \{\rho_t\}$ doesn't satisfy (***lcp***) and Ω_G does satisfy (***lcp***), $\omega_t < \rho_t$ for some t. For example, $\omega_{1/2}(0, 1/2) = 0$ and $\rho_{1/2}(0, 1/2) = 1$.

Example 3.2.2. For each $0 < t < 1$, define the distance ρ_t on $\boldsymbol{I}$ by $\rho_t(p, q) = |\, p - q \,|^{1-t}$. Then $\Omega = \{\rho_t\}$ is a linearly ordered family of pseudometrics satisfying (***lcp***). If $0 < a = |\, p - q \,| < 1$, then

$$G_{pq}(x) = \begin{cases} 0 & 0 \leq x \leq a \\ 1 - ln(x)/ln(a) & a < x < 1 \\ 1 & x \geq 1 \end{cases}$$

and

$$G_{01}(x) = \begin{cases} 0 & 0 \leq x \leq 1 \\ 1 & x > 1. \end{cases}$$

The *WPMS* $(\boldsymbol{I}, \boldsymbol{G})$ is the *Menger* space referred to in **Theorem 3.2.1** that satisfies $\Omega = \Omega_G$.

In **Theorem 2.3.1**, we presented a representation result for *WPMSs* with finite range. Here, we show that the *Menger* spaces satisfying this condition can be used to approximate any *Menger* space. We begin with the following preliminary result.

Lemma 3.2.1. Let $(S, \boldsymbol{F})$ be a *WPMS* and $V = \{v_0, \ldots, v_n\}$, where $v_0 = 0 < v_1 < \ldots < v_n = 1$. For each $p, q \in S$, define $F^V_{pq} : \boldsymbol{R}^+ \to \boldsymbol{I}$ by $F^V_{pq}(x) = \boldsymbol{min}\{v \in V \mid F_{pq}(x) \leq v\}$. Then the following statements hold:

(a) For $p, q \in S$, $F^V_{pq} \in \Delta^+$ and $F^V{}_{pq}{}^{-1}(1) \neq \varnothing$.

(b) $(S, \boldsymbol{F}^V)$ satisfies $(\boldsymbol{w}_1)$ and $(\boldsymbol{w}_2)$.

(c) For $p, q \in S$, $F_{pq} \leq F^V_{pq}$ and $\| F^V_{pq} - F_{pq} \|_\infty \leq \boldsymbol{max}\{v_{k+1} - v_k \mid 0 \leq k < n\}$.

(d) For $0 < t \leq 1$, define $\omega^V_t : S \times S \to \boldsymbol{R}^+$ by

$$\omega^V_t(p, q) = \boldsymbol{sup}\{x \in \boldsymbol{R}^+ \mid F^V_{pq}(x) < t\}.$$

Then ω^V_t is a *distance* and ω^V_1 is a *pseudometric*.

Proof

Let $p, q \in S$ and let $G = F^V_{pq}$. For $x \in \boldsymbol{R}^+$, let $A(p, q, x) = \{v \in V \mid F_{pq}(x) \leq v\}$.

(**a**): Since $1 \in V$, G is well-defined. Since $F_{pq}(0) = 0$, $G(0) = 0$. Since $\boldsymbol{sup}(F_{pq}) = 1$, there is x such that $F_{pq}(x) > v_{n-1}$, so $G(x) = 1$. Since F_{pq} is non-decreasing, if $x \leq y$, then $A(p, q, y) \subseteq A(p, q, x)$, so $G(x) \leq G(y)$. Let $x >$

0 and suppose $G(x) = v_k$. If $k = 0$, then $G(x) = 0$, so $G(p) = 0$ for $0 < p < x$. Hence, G is left-continuous at x. If $k > 0$, then $v_{k-1} < F_{pq}(x) \le v_k$. Since F_{pq} is left-continuous, there exists $0 < h < x$ such that $v_{k-1} < F_{pq}(x - h)$. Then $G(p) = v_k$ for each $x - h < p < x$, so G is left-continuous at x. This shows that $G \in \Delta^+$.
(**b**): If $p = q$ and $x > 0$, then by (w_1) for (S, $\boldsymbol{F}$), $G(x) = \boldsymbol{min}(A(p, p, x)) = \boldsymbol{min}\{v \in V \mid \boldsymbol{H}(x) \le v\} = 1$. Hence, since $G(0) = 0$, $G = \boldsymbol{H}$, so (w_1) holds. Since (S, $\boldsymbol{F}$) satisfies (w_2), for $p, q \in S$ and $x \in \boldsymbol{R}^+$, $A(p, q, x) = A(q, p, x)$, so $F^V_{pq} = F^V_{qp}$. Hence, (w_2) holds.
(**c**): If $G(x) = v_k$, then $F_{pq}(x) \le v_k$, so $F_{pq} \le G$. If $k = 0$, then $G(x) = 0 = F_{pq}(x)$. Otherwise, $k > 0$, so $v_{k-1} < F_{pq}(x)$ and $G(x) - F_{pq}(x) < v_k - v_{k-1}$. Hence, $\| F^V_{pq} - F_{pq} \|_\infty \le \boldsymbol{max}\{v_{k+1} - v_k\}$.
(**d**): It follows from part (**b**) that each ω^V_t is a distance. Also, by part (**a**), ω^V_1 is well-defined, so by **Lemma 2.2.2**, it is a pseudometric.

□

We say that a *WPMS* (S, $\boldsymbol{F}$) can be *approximated* by a family W of *WPMSs* if for each $\varepsilon > 0$, there exists (S, $\boldsymbol{G}$) ∈ W such $\| F_{pq} - G_{pq} \|_\infty < \varepsilon$ for each $p, q \in S$. Now we can establish the following result.

Theorem 3.2.2. Each *Menger* space can be *approximated* by the family of *special Menger* spaces with a *finite range*.

<u>Proof</u>
Let (S, $\boldsymbol{F}$) be a *Menger* space and suppose $V = \{v_0, \ldots, v_n\}$ with $v_0 = 0 < v_1 < \ldots < v_n = 1$.

(1) (S, $\boldsymbol{F}^V$) is a special *Menger* space with a finite range.

[Let $p, q, r \in S$, $x, y \in \boldsymbol{R}^+$, $G = F^V_{pq}$, and $H = F^V_{qr}$. Suppose $G(x) = v_k$ and $H(y) = v_m$ with $k \le m$. If $k = 0$, then $\boldsymbol{min}\{G(x), H(y)\} = 0$. If $k > 0$, then $G(x) > v_{k-1}$ and $H(y) > v_{m-1}$, so $F_{pq}(x + y) > v_{k-1}$. Hence, $F^V_{pq}(x + y) \ge v_k = \boldsymbol{min}\{G(x), H(y)\}$. A similar argument works if $m < k$. Therefore, (S, $\boldsymbol{F}^V$) satisfies condition (**T**) for $T = Min$, so by **Lemma 3.2.1(a)(b)**, it is a special *Menger* space.
Given $\varepsilon > 0$, let $V = \{k/n \mid 0 \le k \le n\}$, where $n \in \boldsymbol{N}^+$ satisfies $1/n < \varepsilon$. Therefore, for each $p, q \in S$, $\| F^V_{pq} - F_{pq} \|_\infty \le 1/n$ by **Lemma 3.2.1(c)**.

□

To establish **Theorem 3.2.2**, it appears that (S, $\boldsymbol{F}$) must be a *Menger* space. Otherwise, there exists $p, q, r \in S$ and $x, y \in \boldsymbol{R}^+$ such that $a = F_{pr}(x + y) < b = F_{pq}(x) \le F_{qr}(y)$. Let $V = \{0, (a + b)/2, 1\}$. Then $F^V_{pq}(x) = F^V_{qr}(y) = 1$, but $F^V_{pr}(x + y) \le (a + b)/2$, so ($S$, $\boldsymbol{F}^V$) doesn't satisfy (w_3).

3.3. Geometric Conditions

The *Menger* spaces are characterized among *WPMSs* by the fact that each associated distance is a pseudometric. This is the gold standard. However, before the definition of a metric space was codefied, other properties of distances were studied ([**14**] – [**17**]). In particular one largely forgotten property of a distance-space (S, d) is the following *weak triangle condition*:

$$(\boldsymbol{wtc}) \equiv \forall \varepsilon > 0\ \exists \delta > 0\ \forall x, y, z \in M \bullet d(x, z) < \delta \text{ and } d(z, y) < \delta \Rightarrow d(x, y) < \varepsilon.$$

Clearly, each pseudometric satisfies (***wtc***), but in **Example 2.1.5**, some distances do not satisfy it. Also, in **Example 4.2.1(1)**, ($\boldsymbol{R}$, $\boldsymbol{F}$) is a *WPMS*⁺ and for each $0 < t \le 1$,

$$\omega_t(p, q) = \begin{cases} 0 & \mid p - q \mid \le 1 - t \\ 1 & \mid p - q \mid > 1 - t. \end{cases}$$

Given $0 < t < 1$, let $p = 0$, $q = 1 - t$, and $r = 2(1 - t)$. Then $\omega_t(p, q) = \omega_t(q, r) = 0$, but $\omega_t(p, r) = 1$. Therefore, no member of Ω_F satisfies (***wtc***). On the other hand, in **Example 2.4.2**, the (***wtc***) property holds for each distance.

The importance of the (***wtc***) property is demonstrated by **Theorem A.3.1**. If a distance-space (S, d) satisfies (***wtc***), then d is *uniformly equivalent* to a *pseudometric* on S. More generally, other special

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properties of distances may be useful for the classification of *WPMSs*. For purposes of comparison, we've included a second (***wpc***) property in **§A.3**.

3.4. Generalized Menger Spaces

Now we present the third new idea in the paper. Much study has been devoted to the study of *Menger* spaces, but it seems that a natural generalizations has not received much attention. Consider a *WPMS* that satisfies the following weaker version of (**T**), where $T = Min$:

(**M′**) $\forall \varepsilon > 0\ \exists \delta > 0\ \forall p, q, r \in S \bullet F_{pr}(\varepsilon) \geq \boldsymbol{min}\{F_{pq}(\delta), F_{qr}(\delta)\}$.

In this case, we say that $(S, \boldsymbol{F})$ is a *generalized Menger space*. If $(S, \boldsymbol{F})$ is a *Menger* space, then (**M′**) holds since $F_{pr}(\varepsilon) \geq \boldsymbol{min}\{F_{pq}(\varepsilon/2), F_{qr}(\varepsilon/2)\}$ for each $\varepsilon > 0$.

The next result gives several characterizations of generalized Menger spaces.

Theorem 3.4.1. The following conditions are equivalent for a *WPMS* $(S, \boldsymbol{F})$.

(a) (**M′**) holds.

(b) There is a *right-continuous modulus of continuity* φ such that for $p, q, r \in S$ and $0 < t < 1$,

$$\omega_t(p, r) < \varphi(\boldsymbol{max}\{\omega_t(p, q), \omega_t(q, r)\}).$$

(c) There is a *right-continuous modulus of continuity* φ such that $F_{pr} \circ \varphi \geq \boldsymbol{min}\{F_{pq}, F_{qr}\}$ for $p, q, r \in S$.

(d) Ω_F satisfies (***equi-wtc***): for each $\varepsilon > 0$, there exists $\delta > 0$ such that for each $0 < t < 1$ and $p, q, r \in S$, $\omega_t(p, r) < \delta$ and $\omega_t(q, r) < \delta \Rightarrow \omega_t(p, q) < \varepsilon$.

Proof

(**a**) ⇒ (**b**): In the definition of (**M′**), δ depends only on ε, so the proof of (**a**) ⇒ (**b**) in **Theorem A.3.1** shows that there is a non-decreasing mapping τ: $(0, +\infty) \to (0, +\infty)$ that satisfies the following conditions:

(***ec***$_1$) For each $p, q, r \in S$ and $0 < t < 1$, if $\omega_t(p, q) < x$ and $\omega_t(q, r) < x$, then $\omega_t(p, r) < \tau(x)$.

(***ec***$_2$) $\tau(0+) = 0$.

Define the mapping φ: $\boldsymbol{R}^+ \to \boldsymbol{R}^+$ by $\varphi(x) = \tau(x+)$. Since τ is non-decreasing, φ is non-decreasing and an elementary argument shows that φ is right-continuous. Hence, by (***ec***$_2$), $\varphi(0+) = \varphi(0) = 0$, so φ is a modulus of continuity that satisfies (***ec***$_1$). Let $p, q, r \in S$, $0 < t < 1$, and assume that $\omega_t(p, q) \leq \omega_t(q, r)$. By (***ec***$_1$), if $\omega_t(q, r) < x$, then $\omega_t(p, r) < \varphi(x)$. Hence, by the right-continuity of φ, $\omega_t(p, r) \leq \varphi(\omega_t(q, r))$.

(**b**) ⇒ (**c**): Let $x \in \boldsymbol{R}^+$ and assume that $s = F_{pq}(x) \leq t = F_{qr}(x)$. Then $\omega_s(q, r) \leq \omega_t(q, r) \leq x$ and $\omega_s(p, q) \leq x$, so $\omega_s(p, r) < \varphi(\boldsymbol{max}\{\omega_s(p, q), \omega_s(q, r)\}) \leq \varphi(x)$. Hence, $F_{pr}(\varphi(x)) \geq s$.

(**c**) ⇒ (**d**): Since $\varphi(0+) = \varphi(0) = 0$, given $\varepsilon > 0$, there is $\delta > 0$ such that $\varphi(\delta) < \varepsilon/2$. Suppose $\omega_t(p, q) < \delta$ and $\omega_t(q, r) < \delta$ for some $0 < t < 1$. Then $F_{pq}(\delta) \geq t$ and $F_{qr}(\delta) \geq t$, so by (**c**), we obtain $F_{pr}(\varepsilon/2) \geq F_{pr}(\varphi(\delta)) \geq t$. Hence, $\omega_t(p, r) \leq \varepsilon/2 < \varepsilon$. Therefore, Ω_F satisfies (***equi-wtc***).

(**d**) ⇒ (**a**): Let $\varepsilon > 0$ and choose $\delta > 0$ such that the conclusion in (**d**) holds. Let $p, q, r \in S$ and suppose $\alpha = F_{pq}(\delta/2) \leq \beta = F_{qr}(\delta/2)$. If $\alpha = 0$, then $F_{pr}(\varepsilon) \geq \boldsymbol{min}\{\alpha, \beta\}$. If $\alpha = 1$, then $\beta = 1$. Let $0 < t < 1$. Then $F_{pq}(\delta/2) = F_{qr}(\delta/2) > t$, so $\omega_t(q, r) < \delta$ and $\omega_t(q, r) < \delta$. Hence, $\omega_t(p, r) < \varepsilon$, so $F_{pr}(\varepsilon) \geq t$. Since t is arbitrary, $F_{pr}(\varepsilon) = 1$, so $F_{pr}(\varepsilon) \geq \boldsymbol{min}\{\alpha, \beta\}$. If $0 < \alpha < 1$, then $\omega_\alpha(p, q) < \delta$ and $\omega_\alpha(q, r) < \delta$, so $\omega_\alpha(p, r) < \varepsilon$. Hence, $F_{pr}(\varepsilon) \geq \alpha = \boldsymbol{min}\{\alpha, \beta\}$. This establishes (**M′**).

□

In **Example 2.1.5**, ω_t doesn't satisfy (***wtc***) for $0 < t \leq 1/2$, so by the previous result, the *WPMS* is not a generalized *Menger* space. The following example shows that a generalized Menger space may not be a *Menger* space.

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Example 3.4.1. We reuse **Example 2.4.2.** For $0 < t \le 1$, the distance ρ_t on (0, 1) is defined by $\rho_t(p, q) = | p - q |^{2/(t+1)}$ and (*S*, ***F***) is a special *WPMS*.

We claim that $\Omega = \{\rho_t\}$ is an (***equi-wtc***) family. For $0 < \varepsilon < 1$, let $\delta = \varepsilon/4$ and let *p*, *q*, and *r* be distinct points in (0, 1). Let $a = | p - q |$, $b = | q - r |$, and $c = | p - r |$. For $0 < t < 1$, let $y = 2/(t + 1)$, and suppose $a^y < \delta$ and $b^y < \delta$. If $c = a + b$, then $c^y < (2\delta^{1/y})^y = 2^y\delta = 2^{y-2}\varepsilon < \varepsilon$. If $c = a - b$, then $c^y < a^y < \varepsilon$ and, similarly, if $c = b - a$, then $c^y < b^y < \varepsilon$. By **Theorem 3.4.1**, (*S*, ***F***) satisfies (**M′**). Choose $0 < t < 1$ and let $p = 1/4$, $q = 1/2$, $r = 3/4$, and $y = 2/(t + 1)$. Then $\rho_t(p, q) = \rho_t(q, r) = (1/4)^y$, so $\rho_t(p, q) + \rho_t(q, r) = 2(1/4)^y = 2^{1-2y}$ and $\rho_t(p, r) = 2^{-y}$. Since $y > 1$, $2^{-y} > 2^{1-2y}$, so ρ_t is not a pseudometric. Hence, by **Prop 2.2.2**, (*S*, ***F***) is not a *Menger* space.

The following result shows that each generalized *Menger* space has a closely associated *Menger* space.

Theorem 3.4.2. If a *WPMS* (*S*, ***F***) satisfies (**M′**), then there is a *Menger* space (*S*, ***G***) and *right-continuous moduli of continuity* *f* and *g* such that the following statements hold for $p, q \in S$ and $0 < t < 1$, where $\Omega_F = \{\omega_t\}$ and $\Omega_G = \{\sigma_t\}$:

(a) $\sigma_t(p, q) < \delta < 1/8 \Rightarrow \omega_t(p, q) < f(\delta)$.

(b) $\omega_t(p, q) < \delta < 1 \Rightarrow \sigma_t(p, q) < g(\delta)$.

In terms of distribution functions, for $0 < \delta < 1/8$, $G_{pq}(\delta) \le F_{pq}(f(\delta))$ and $F_{pq}(\delta) \le G_{pq}(g(\delta))$.

Proof

The proof is essentially the same as the proof of (**b**) ⇒ (**c**) in **Theorem A.3.1** except that we use the regular écart φ from **Theorem 3.4.1**. In statements (1) and (2), ω denotes one of the mappings ω_t. Their proofs follow the proof of **Theorem A.3.1**. Recall from **§A.1** that if *d* is a distance on *S*, then $S(d, r) = S_d(r) = \{S_d(x, r) \mid x \in S\}$ for each $r > 0$, where $S_d(x, r) = \{y \in S \mid d(x, y) < r\}$.

(1) For each $x \in M$ and $r > 0$, $St(x, S_\omega(r)) \subseteq S_\omega(x, \varphi(r))$.

(2) There is a decreasing sequence $\{\varepsilon_k\} \subseteq (0, 1]$ converging to 0 with $\varepsilon_0 = \varepsilon_1 = 1$ such that $\varphi(\varepsilon_{k+1}) < \varepsilon_k$ and $S_\omega(\varepsilon_{k+1}) <^* S_\omega(\varepsilon_k)$ for each $k > 0$.

For each $0 < t < 1$, let $A_{0,t} = \{S\}$ and $A_{k,t} = S(\omega_t, \varepsilon_k)$ for $k > 0$. By (2), each $\{A_{k,t}\}$ is a normal sequence. Define $\mu_t : S \times S \to \boldsymbol{R}$ by $\mu_t(p, q) = \boldsymbol{inf}\{2^{-k} \mid q \in St(p, A_{k,t})\}$ and let ρ_t be the corresponding path-metric based on **Prop A.2.2(i)**. If $0 < s < t < 1$, then $A_{k,t} < A_{k,s}$ for each *k* since $\omega_s \le \omega_t$. Therefore, $\mu_s \le \mu_t$, so $\rho_s \le \rho_t$.

Define $f : \boldsymbol{R}^+ \to \boldsymbol{R}^+$ by

$$f(x) = \begin{cases} 0 & x = 0 \\ \varphi(\varepsilon_k) & 0 < x < 1/4 \text{ and } k = \lceil -lg(4x) \rceil \\ \varphi(1) & x \ge 1/4 \end{cases}$$

and define $h : \boldsymbol{R}^+ \to \boldsymbol{N}^+$ by

$$h(x) = \begin{cases} \boldsymbol{max}\{k \mid x < \varepsilon_k\} & 0 < x < 1 \\ 1 & x \ge 1. \end{cases}$$

Define $g : \boldsymbol{R}^+ \to \boldsymbol{R}^+$ by $g(0) = 0$ and $g(x) = x + 2^{-h(x)}$ if $x > 0$. One can verify that *f* and *g* are right-continuous moduli of continuity.

(3) (i) $\rho_t(p, q) < \delta < 1/4 \Rightarrow \omega_t(p, q) < f(\delta)$.

(ii) $\omega_t(p, q) < \delta < 1 \Rightarrow \rho_t(p, q) < g(\delta)$.

[(i): By **Prop A.2.2(ii)**, $\rho_t \le \mu_t \le 2\rho_t$, so we obtain $\mu_t(p, q) < 2\delta < 1$. Let $k = \lceil -lg(2\delta) \rceil$. Then $2^{-k-1} \le 2\delta < 2^{-k}$, so $\mu_t(p, q) \le 2^{-k}$. Since $q \in St(p, A_{k,t}) = St(p, S_{\omega(t)}(\varepsilon_k))$, it follows from (1) that $\omega_t(p, q) < \varphi(\varepsilon_k) = f(\delta)$.

(ii): Let $k = h(\delta)$. Since $\delta < \varepsilon_k$, it follows that $\omega_t(p, q) < \varepsilon_k$, so we have $q \in St(p, A_{k,t})$. Therefore, $\rho_t(p, q) \le \mu_t(p, q) \le 2^{-k} < g(\delta)$.]

For each $0 < t < 1$, let $\mu'_t = \boldsymbol{sup}\{\mu_s \mid 0 < s < t\}$ and $\rho'_t = \boldsymbol{sup}\{\rho_s \mid 0 < s < t\}$. Then $\mu'_t \leq \mu_t$, $\rho'_t \leq \rho_t$, and by **Lemma 2.4.1**, $\{\rho'_t\}$ is a linearly ordered family of pseudometrics satisfying (***lcp***). Therefore, by **Theorem 3.2.1(a)(c)**, there is a *Menger* space $(S, \boldsymbol{G})$ with $\sigma_t = \rho'_t$ for each t, where $\Omega_G = \{\sigma_t\}$.

(4) For each $0 < t < 1$, $\mu_t \leq 2\mu'_t$ and $\rho_t \leq 2\rho'_t$.

[Assume that $\mu'_t(p, q) = 2^{-k}$ and choose $0 < s' < t$ such that $\mu_{s'}(p, q) > 2^{-k-1}$. Let $s' < s < t$. Then $\mu_s(p, q) = 2^{-k}$, so $q \in St(p, A_{k,s})$. Hence, by (1), $\omega_s(p, q) \leq \varphi(\varepsilon_k)$. By **Prop 2.4.1(a)**, the family $\{\omega_t\}$ satisfies (***lcp***), so by (2), $\omega_t(p, q) \leq \varphi(\varepsilon_k) < \varepsilon_{k-1}$. Therefore, $\mu_t(p, q) \leq 2^{1-k} = 2\mu'_t(p, q)$. This establishes the first statement. The second statement follows from the first one and the definition of a path-metric.]

(5) (i) $\sigma_t(p, q) < \delta < 1/8 \Rightarrow \omega_t(p, q) < f(\delta)$

(ii) $\omega_t(p, q) < \delta < 1 \Rightarrow \sigma_t(p, q) < g(\delta)$.

[(i): By (4), $\rho_t(p, q) \leq 2\sigma_t(p, q) < 2\delta < 1/4$, so by (3)(i), $\omega_t(p, q) < f(\delta)$.

(ii): By (3)(ii), $\sigma_t(p, q) \leq \rho_t(p, q) < g(\delta)$.]

To verify the final statement, assume that $G_{pq}(\delta) > F_{pq}(f(\delta))$ and choose $G_{pq}(\delta) > t > F_{pq}(f(\delta))$. Then $\sigma_t(p, q) < \delta$, so by (5)(i), $\omega_t(p, q) < f(\delta)$. Hence, $F_{pq}(f(\delta)) \geq t$, which is a contradiction. Therefore, $G_{pq}(\delta) \leq F_{pq}(f(\delta))$. The other statement is established similarly.]

□

4. Operations on WPMSs

4.1. The Category WP

In this section, we introduce the fourth new idea in the paper – a *categorical framework* for *WPMSs* and accompanying operations. The idea has been touched on in [**4**] and [**18**], where a generalized notion of probabilistic metric spaces is introduced in the context of quantales.

The reader is referred to [**19**] for general categorical notions. Let | WP | denote the family of *WPMSs*. For $(S, \boldsymbol{F})$ and $(T, \boldsymbol{G})$ in | WP |, a *morphism* $[f]: (S, \boldsymbol{F}) \to (T, \boldsymbol{G})$ is a mapping $f: S \to T$ such that $F_{pq} \leqq G_{f(p)f(q)}$ for each $p, q \in S$, where $\leqq$ denotes the point-wise order on Δ^+. The symbol WP($(S, \boldsymbol{F})$, $(T, \boldsymbol{G})$) denotes the family of morphisms from $(S, \boldsymbol{F})$ to $(T, \boldsymbol{G})$. The composition of morphisms $[f]: (S, \boldsymbol{F}) \to (T, \boldsymbol{G})$ and $[g]: (T, \boldsymbol{G}) \to (V, \boldsymbol{K})$ is the morphism $[g \circ f]: (S, \boldsymbol{F}) \to (V, \boldsymbol{K})$. Also, $[id_S]$ is the identity morphism for each *WPMS* $(S, \boldsymbol{F})$, so WP is a category.

The first result characterizes the morphisms in WP in terms of the families of distances.

Lemma 4.1.1. The following statements are equivalent for *WPMSs* $(S, \boldsymbol{F})$ and $(T, \boldsymbol{G})$ and a mapping $f: S \to T$, where $\Omega_F = \{\omega_t\}$ and $\Omega_G = \{\rho_t\}$.

(a) $[f]: (S, \boldsymbol{F}) \to (T, \boldsymbol{G})$ is a WP-*morphism* $[f]: (S, \boldsymbol{F}) \to (T, \boldsymbol{G})$.

(b) For each $0 < t < 1$, $f: (S, \omega_t) \to (T, \rho_t)$ is a *non-expansive* mapping.

Proof

$\neg$(**b**) $\Rightarrow \neg$(**a**): Choose t, p, and q such that $\omega_t(p, q) < \rho_t(f(p), f(q))$. Let $\omega_t(p, q) < x < \rho_t(f(p), f(q))$. Therefore, $G_{f(p)f(q)}(x) < t \leq F_{pq}(x)$, so $\neg$(**a**) holds.

$\neg$(**a**) $\Rightarrow \neg$(**b**): Choose p, q, and x such that $G_{f(p)f(q)}(x) < F_{pq}(x)$ and let $G_{f(p)f(q)}(x) < t < F_{pq}(x)$. Then $\omega_t(p, q) < x \leq \rho_t(f(p), f(q))$. Hence, $\omega_t(p, q) < \rho_t(f(p), f(q))$, so $\neg$(**b**) holds.

□

Corollary 4.1.1. The following statements are equivalent for *WPMSs* $(S, \boldsymbol{F})$ and $(T, \boldsymbol{G})$ and a mapping $h: S \to T$, where $\Omega_F = \{\omega_t\}$ and $\Omega_G = \{\rho_t\}$.

(a) $[h]: (S, \boldsymbol{F}) \to (T, \boldsymbol{G})$ is a WP-*isomorphism*.

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(b) h is a *bijection* and $F_{pq} = G_{h(p)h(q)}$ for each $p, q \in S$.

(c) For each $0 < t < 1$, $h : (S, \omega_t) \to (T, \rho_t)$ is a *surjective isometry*.

Proof

The equivalence of (**a**) and (**b**) follows from the definition of a WP-isomorphism. The equivalence of (**a**) and (**c**) follows from **Lemma 4.1.1**.

□

We note that the notion of *isomorphic WPMS*'s is the same as the idea of *isometric WPMS*'s found in ([**1**], 8.1.2).

The next result lists some elementary facts about the category WP. The proof is left to the reader.

Lemma 4.1.2. The following statements hold for *WPMSs* $(S, \boldsymbol{F})$ and $(T, \boldsymbol{G})$.

(a) The *monomorphisms* (resp. *epimorphisms*) in WP are exactly the *morphisms* that are *injections* (resp. *surjections*) on the base sets.

(b) For each $t \in T$, $const_t : S \to T$ defines a WP-morphism $[const_t] : (S, \boldsymbol{F}) \to (T, \boldsymbol{G})$.

(c) Let $H \subseteq S$ and $\boldsymbol{F}_H = \boldsymbol{F}_{|H \times H}$. Then $(H, \boldsymbol{F}_H)$ is a *WPMS* and $incl_H : H \to S$ defines a WP-morphism $[incl_H] : (H, \boldsymbol{F}_H) \to (S, \boldsymbol{F})$.

In the literature, both countable products and arbitrary products of *WPMSs* have been studied. In our setting, we can characterize when products exist.

For a set W = $\{(S_i, \boldsymbol{F}_i) \mid i \in I\} \subseteq$ | WP |, let $S = \Pi S_i$ and define the mapping $\boldsymbol{F} : S \times S \to F(\boldsymbol{R}^+, \boldsymbol{I})$ by $\boldsymbol{F}(p, q) = \boldsymbol{inf}\{\boldsymbol{F}_i(p_i, q_i) \mid i \in I\}$. Then the following statements hold:

(1) For each $p, q \in S$, $F_{pq}(0) = 0$, $Rng(F_{pq}) \subseteq \boldsymbol{I}$, and F_{pq} is non-decreasing. (This follows since each $\boldsymbol{F}_i(p_i, q_i) \in \rho^+$.)

(2) $(S, \boldsymbol{F})$ satisfies $(w_1) - (w_3)$. (Clearly, (w_1) and (w_2) hold. If $F_{pq}(x) = F_{qr}(y) = 1$ for some $p, q, r \in S$ and $x, y \in \boldsymbol{R}^+$, then each $\boldsymbol{F}_i(p_i, q_i)(x) = \boldsymbol{F}_i(q_i, r_i)(y) = 1$, so $\boldsymbol{F}_i(p_i, r_i)(x + y) = 1$. Hence, $F_{pq}(x + y) = 1$, so (w_3) holds.)

(3) For each $0 < t < 1$, $\omega_t = \boldsymbol{sup}\{\omega_{i,t}\}$, where $\Omega_F = \{\omega_t\}$ and each $\Omega_{F(i)} = \{\omega_{i,t}\}$.

It may not be the case that $\boldsymbol{sup}(F_{pq}) = 1$ for each $p, q \in S$ (**Example 4.1.1**), so F_{pq} may not belong to ρ^+. To remedy this problem, we introduce the following property.

We say that W satisfies the *uniform limit property* (***ulp***) if for each $\varepsilon > 0$ and each $\{p_i, q_i\} \subseteq S_i$, $i \in I$, there is $x \in \boldsymbol{R}^+$ such that $\boldsymbol{inf}\{\boldsymbol{F}_i(p_i, q_i)(x) \mid i \in I\} > 1 - \varepsilon$. Clearly, (***ulp***) holds if W is a finite family. Also, if W satisfies (***ulp***), then each $F_{pq} \in \rho^+$, so by (1) and (2), $(S, \boldsymbol{F})$ is a *WPMS*.

Prop 4.1.1. The following statements hold for a set W = $\{(S_i, \boldsymbol{F}_i) \mid i \in I\} \subseteq$ | WP |.

(a) If W satisfies (***ulp***), then $(S, \boldsymbol{F})$ is the *product* in WP of the members of W. Hence, each *finite* family of *WPMSs* has a *product* in WP.

(b) If W satisfies (***ulp***) and W consists of *Menger* spaces, then $(S, \boldsymbol{F})$ is a *Menger* space.

(c) If W is an *infinite* set and its *product* in WP *exists*, then W satisfies (***ulp***).

Proof

(**a**): Based on our earlier discussion, $(S, \boldsymbol{F})$ is a *WPMS*, where $S = \Pi S_i$ and $\boldsymbol{F} : S \times S \to \Delta^+$ is defined by $\boldsymbol{F}(p, q) = \boldsymbol{inf}\{\boldsymbol{F}_i(p_i, q_i) \mid i \in I\}$. For each $i \in I$, let $\pi_i : S \to S_i$ be the standard projection mapping. Since $\boldsymbol{F} \le \boldsymbol{F}_i$, $[\pi_i] : (S, \boldsymbol{F}) \to (S_i, \boldsymbol{F}_i)$ is a morphism. Let $(T, \boldsymbol{G})$ be a *WPMS* and assume that each $[f_i] : (T, \boldsymbol{G}) \to (S_i, \boldsymbol{F}_i)$ is a morphism. Let $f : T \to S$ be the mapping that satisfies $\pi_i \circ f = f_i$ for each i. Then $\boldsymbol{G}(a, b) \le \boldsymbol{F}_i(f_i(a),$

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$f_i(b)$) for each $a, b \in T$ and i, so $\boldsymbol{G}(a, b) \leq \boldsymbol{F}(f(a), f(b))$. Therefore, $[f] : (T, \boldsymbol{G}) \rightarrow (S, \boldsymbol{F})$ is a morphism. If the morphism $[g] : (T, \boldsymbol{G}) \rightarrow (S, \boldsymbol{F})$ satisfies $\pi_i \circ g = f_i$ for each i, then clearly, $f = g$. Hence, $(S, \boldsymbol{F})$ is the product.

(**b**): We refer to (2) above. Suppose $\boldsymbol{F}(p, r)(x + y) < b = \boldsymbol{F}(p, q)(x) \leq c = \boldsymbol{F}(q, r)(y)$ for $p, q, r \in S$ and $x, y \in \boldsymbol{R}^+$. Choose $i \in I$ with $\boldsymbol{F}_i(p_i, r_i)(x + y) < b$. Since $b \leq \boldsymbol{F}_i(p_i, q_i)(x)$ and $c \leq \boldsymbol{F}_i(q_i, r_i)(y)$, $\boldsymbol{F}_i(p_i, r_i)(x + y) < \boldsymbol{min}\{\boldsymbol{F}_i(p_i, q_i)(x), \boldsymbol{F}_i(q_i, r_i)(y)\}$. This is a contradiction, so $(S, \boldsymbol{F})$ is a Menger space.

(**c**): Assume that $(T, \boldsymbol{G})$ is the product of the family W based on the projection mappings $\{[\sigma_i] : (T, \boldsymbol{G}) \rightarrow (S_i, \boldsymbol{F}_i)\}$. If W doesn't satisfy (***ulp***), then there is $0 < \varepsilon < 1$ and $\{p_i, q_i\} \subseteq S_i$, $i \in I$, such that $\boldsymbol{inf}\{\boldsymbol{F}_i(p_i, q_i)(x) \mid i \in I\} \leq \delta = 1 - \varepsilon$ for each $x \in \boldsymbol{R}^+$.

Let $(\{*\}, \boldsymbol{H})$ be the trivial *WPMS* and for each i, define the morphism $[f_i] : (\{*\}, \boldsymbol{H}) \rightarrow (S_i, \boldsymbol{F}_i)$ by $f_i(*) = p_i$. By assumption, there is a morphism $[f] : (\{*\}, \boldsymbol{H}) \rightarrow (T, \boldsymbol{G})$ such that $\sigma_i \circ f = f_i$ for each i.

Similarly, for each i, define the morphism $[g_i] : (\{*\}, \boldsymbol{H}) \rightarrow (S_i, \boldsymbol{F}_i)$ by $g_i(*) = q_i$. Then there is a morphism $[g] : (\{*\}, \boldsymbol{H}) \rightarrow (T, \boldsymbol{G})$ such that $\sigma_i \circ g = g_i$ for each i. Let $s = f(*)$ and $t = g(*)$. Then for each $i \in I$ and $x \in \boldsymbol{R}^+$, $G_{st}(x) \leq \boldsymbol{F}_i(f_i(*), g_i(*))(x) = \boldsymbol{F}_i(p_i, q_i)(x)$, so $\boldsymbol{sup}(G_{st}) \leq \boldsymbol{inf}\{\boldsymbol{F}_i(p_i, q_i)(x)\} \leq \delta$, which contradicts the fact that $G_{st} \in \Delta^+$. Therefore, W satisfies (***ulp***).

□

Many authors have studied products of *WPMSs* independently of any categorical framework. For example, ([**20**], Theorem 1) establishes the finite case of **Prop 4.1.1(a)** and ([**21**], Proposition 3) "purports" to establish a general product theorem. Here are some additional details about finite products based on the previous construction:

- If $(S, \boldsymbol{F})$ is the product of a *finite* family $\{(S_i, \boldsymbol{F}(i))\}$ of *WPMS*'s, then $\omega_t = \boldsymbol{max}\{\omega_{i,t}\}$ for each t, where $\Omega_{\boldsymbol{F}} = \{\omega_t\}$ and $\Omega_{\boldsymbol{F}(i)} = \{\omega_{i,t}\}$ for each i.
- The product of a *finite* family of *Menger* spaces is a *Menger* space. (By **Prop 2.2.2**, each coordinate distance is a pseudometric, so by the previous remark, each distance in the product is a pseudometric. Hence, by **Prop 2.2.2**, the product is a *Menger* space.)
- The product $(S, \boldsymbol{F})$ of a *finite* family $\{(S_i, \boldsymbol{F}_i)\}$ of *special WPMS*'s is a *special WPMS*. (For each $p, q \in S$ and i, choose x_i such that $\boldsymbol{F}_i(p_i, q_i)(x_i) = 1$. Then $\boldsymbol{F}(p, q)(\boldsymbol{max}\{x_i\}) = 1$.)

The following example shows that the (***ulp***) property doesn't hold in general.

Example 4.1.1. For each $n \in \boldsymbol{N}^+$, define $\boldsymbol{F}_n : [0, n] \times [0, n] \rightarrow \Delta^+$ by $\boldsymbol{F}_n(p, q)(x) = \boldsymbol{H}(x - \mid p - q \mid)$.
Each $([0, n], \boldsymbol{F}_n) \in \mid \text{WP}_{\text{M}} \mid$, but $W = \{([0, n], \boldsymbol{F}_n) \mid n \in \boldsymbol{N}^+\}$ doesn't satisfy (***ulp***). To see this, note that for each $x \in \boldsymbol{R}^+$, $\boldsymbol{inf}\{\boldsymbol{F}_n(0, n)(x)\} = \boldsymbol{inf}\{\boldsymbol{H}(x - n)\} = 0$.
By way of contrast with **Prop 4.1.1**, we have the following rather dramatic result.

Prop 4.1.2. *No pair* of *WPMSs* has a *coproduct* in WP.

Proof

Let $(S, \boldsymbol{F})$ and $(T, \boldsymbol{G})$ be *WPMSs* and assume that $(C, \boldsymbol{H})$ is their coproduct in WP based on the morphisms $[i] : (S, \boldsymbol{F}) \rightarrow (C, \boldsymbol{H})$ and $[j] : (T, \boldsymbol{G}) \rightarrow (C, \boldsymbol{H})$. Suppose $\Omega_{\boldsymbol{H}} = \{\rho_t\}$. Choose $p \in S$ and $q \in T$ and let $a = i(p)$ and $b = j(q)$. Let $r = \rho_{1/2}(a, b) + 1$. Let $D = \{0, r\}$ and let $K_{r0} = K_{0r} : \boldsymbol{R}^+ \rightarrow \boldsymbol{I}$ be the characteristic function of $(r, +\infty)$. Also, let $K_{00} = K_{rr} = \boldsymbol{H}$. Then $(D, \boldsymbol{K})$ is a *WPMS* and $\Omega_{\boldsymbol{K}} = \{\omega_t\}$, where each ω_t satisfies $\omega_t(0, r) = r$. By **Lemma 4.1.2(b)**, each constant mapping defines a morphism, so there exists a morphism $[h] : (C, \boldsymbol{H}) \rightarrow (D, \boldsymbol{K})$ such that $h \circ i = const_0$ and $h \circ j = const_r$. By **Lemma 4.1.1**, $h : (C, \rho_{1/2}) \rightarrow (D, \omega_{1/2})$ is non-expansive, so

$$\omega_{1/2}(h(a), h(b)) = \omega_{1/2}(0, r) = r \leq \rho_{1/2}(a, b) = r - 1,$$

which is a contradiction. Hence, the coproduct doesn't exist.

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□

A result related to **Prop 4.1.2** is found in ([**22**], Proposition 1) stating that the category Met (**§4.2**) has no coproducts. Next, we consider limits in WP.

Prop 4.1.3. Suppose W = $\{(S_i, \boldsymbol{F}_i) \mid i \in D\} \subseteq \mid$ WP $\mid$, where $(D, \leq)$ is a *directed set*, and $(\{(S_i, \boldsymbol{F}_i) \mid i \in I\}, \{[f_{ij}] : (S_j, \boldsymbol{F}_j) \to (S_i, \boldsymbol{F}_i) \mid i, j \in D \text{ and } i \leq j\})$ is an *inverse system* in **WP**. If W satisfies (***ulp***), then the *inverse system* has an *inverse limit* in WP.

Proof

Let $P = \Pi S_i$ and define $\boldsymbol{F} : P \times P \to \Delta^+$ by $\boldsymbol{F}(p, q) = \boldsymbol{inf}\{\boldsymbol{F}_i(p_i, q_i) \mid i \in I\}$. By **Prop 4.1.1(a)**, $(P, \boldsymbol{F})$ is the product of $\{(S_i, \boldsymbol{F}_i)\}$. Let $S = \{s \in P \mid i, j \in D \text{ and } i \leq j \Rightarrow f_{ij}(s_j) = s_i\}$. By **Lemma 4.1.2(c)**, $(S, \boldsymbol{F}_S)$ is also a *WPMS*. We claim that $((S, \boldsymbol{F}_S), \{[\pi_i] : (S, \boldsymbol{F}_S) \to (S_i, \boldsymbol{F}_i) \mid i \in D\})$ is the inverse limit, where each π_i is a projection mapping. If $((T, \boldsymbol{G}), \{[\varphi_i] : (T, \boldsymbol{G}) \to (S_i, \boldsymbol{F}_i) \mid i \in I\})$ is a source ($f_{ij} \circ \varphi_j = \varphi_i$ for $i, j \in D$ satisfying $i \leq j$), then $\varphi : T \to S$ defined by $\varphi(t) = (\varphi_i(t))$ is well-defined and $\pi_i \circ \varphi = \varphi_i$ for each i. By assumption, for $u, v \in T$ and $i \in D$, $G_{uv} \leq \boldsymbol{F}_i(\varphi_i(u), \varphi_i(v))$, so $G_{uv} \leq \boldsymbol{F}_S(\varphi(u), \varphi(v))$. Therefore, $[\varphi] : (T, \boldsymbol{G}) \to (S, \boldsymbol{F}_S)$ is a WP-morphism.

□

Prop 4.1.4. Each *directed system* in WP has a *direct limit*.

Proof

Let $(\{(S_i, \boldsymbol{F}_i) \mid i \in I\}, \{f_{ij} : (S_i, \boldsymbol{F}_i) \to (S_j, \boldsymbol{F}_j) \mid i, j \in I \text{ and } i \leq j\})$ be a directed system in WP, where $(I, \leq)$ is a directed set. For each $i, j \in I$, let $I(i, j) = \{k \in I \mid i \leq k \text{ and } j \leq k\}$. For each $i \in I$, let $S^\wedge_i = S_i \times \{i\}$, and let $S^\wedge = \cup\{S^\wedge_i \mid i \in I\}$. For $(p, i) \in S^\wedge_i$ and $(q, j) \in S^\wedge_j$, define $(p, i) \sim (q, j)$ if there is $k \in I(i, j)$ such that $f_{ik}(p) = f_{jk}(q)$. Then $\sim$ is an equivalence relation on $S^\wedge$.

Let

$$S = \{[(p, i)] : i \in I \text{ and } p \in S_i\}$$

denote the set of equivalence classes for $\sim$. Let $a = [(p, i)]$, $b = [(q, j)] \in S$. If $a = b$, let $\boldsymbol{F}_{ab} = \boldsymbol{H}$. If $a \neq b$, let

$$\boldsymbol{F}_{ab} = \boldsymbol{sup}\{\boldsymbol{F}_k(f_{ik}(p), f_{jk}(q)) \mid k \in I(i, j)\}.$$

It is easy to verify that $(S, \boldsymbol{F})$ is a *WPMS*. For each $i \in I$, define $\psi_i : S_i \to S$ by $\psi_i(p) = [(p, i)]$.

(1) Each $[\psi_i] : (S_i, \boldsymbol{F}_i) \to (S, \boldsymbol{F})$ is a morphism and $\{[\psi_i]\}$ is a sink for the directed system.

[Let $a = [(p, i)]$, $b = [(q, i)] \in S$, and $x \in \boldsymbol{R}^+$. If $a = b$, then $\boldsymbol{F}_i(p, q)(x) \leq 1 = \boldsymbol{H}(x) = \boldsymbol{F}_{ab}(x)$. If $a \neq b$, then $\boldsymbol{F}_i(p, q) \leq \boldsymbol{sup}\{\boldsymbol{F}_k(f_{ik}(p), f_{ik}(q)) \mid i \leq k\} = \boldsymbol{F}_{ab}(x)$. If $i, j \in I$ and $i \leq j$, then $f_{ij}(p) = f_{jj}(f_{ij}(p))$ for each $p \in S_i$, so $(p, i) \sim (f_{ij}(p), j)$. Hence, for each $p \in S_i$, $\psi_j(f_{ij}(p)) = [(f_{ij}(p), j)] = [(p, i)] = \psi_i(p)$. Therefore, $\psi_j \circ f_{ij} = \psi_i$ for each $i, j \in I$.]

(2) $((S, \boldsymbol{F}), \{[\psi_i] : (S_i, \boldsymbol{F}_i) \to (S, \boldsymbol{F}) \mid i \in I\})$ is the direct limit of the directed system.

[Let $((T, \boldsymbol{G}), \{[\varphi_i] : (S_i, \boldsymbol{F}_i) \to (T, \boldsymbol{G}) \mid i \in I\})$ be a sink for the directed system and define the mapping $h : S \to T$ by $h([(p, i)]) = \varphi_i(p)$. If $(p, i) \sim (q, j)$, then there exists $k \in I(i, j)$ such that $f_{ik}(p) = f_{jk}(q)$, so $\varphi_i(p) = \varphi_k(f_{ik}(p)) = \varphi_k(f_{jk}(q)) = \varphi_j(q)$. Hence, h is well-defined and $h \circ \psi_i = \varphi_i$ for each $i \in I$.]

We claim that $[h] : (S, \boldsymbol{F}) \to (T, \boldsymbol{G})$ is a morphism. Let $a = [(p, i)]$, $b = [(q, j)] \in S$. If $a = b$, then $\boldsymbol{F}_{ab} = \boldsymbol{H} = \boldsymbol{G}_{h(a)h(b)}$. Suppose $a \neq b$. If $k \in I(i, j)$, then since $[\varphi_k]$ is a morphism,

$$\boldsymbol{F}_k(f_{ik}(p), f_{jk}(q)) \leq \boldsymbol{G}(\varphi_k(f_{ik}(p)), \varphi_k(f_{jk}(q))) = \boldsymbol{G}(\varphi_i(p), \varphi_j(q)) = \boldsymbol{G}(h(a), h(b)).$$

Therefore, $\boldsymbol{F}_{ab} \leq \boldsymbol{G}(h(a), h(b))$.

□

Here are two useful examples that illustrate the previous results.

Example 4.1.2. Each *WPMS* is the *direct limit* of the family of *finite subspace WPMSs*.

Let $(S, \boldsymbol{F})$ be a *WPMS* and assume that $\mathrm{I} \subseteq \mathrm{P_F^+}(S)$ satisfies $S = \cup\mathrm{I}$ and $(\mathrm{I}, \leq)$ is a directed set, where $\leq$ is defined by $A \leq B$ if $A \subseteq B$. Let $\mathrm{W} = \{(A, \boldsymbol{F}_A) \mid A \in \mathrm{I}\}$ be the family of subspace *WPMSs* (**Lemma 4.1.2(c)**) and let $\mathrm{F} = \{[f_{AB}] : (A, \boldsymbol{F}_A) \to (B, \boldsymbol{F}_B) \mid A, B \in \mathrm{I}$ and $A \leq B\}$, where each f_{AB} is the inclusion mapping. Since $\{\mathrm{W}, \mathrm{F}\}$ is a directed system, by **Prop 4.1.4**, $\{(T, \boldsymbol{G}), \{[\psi_A] : A \in \mathrm{I}\}$ is the direct limit, where $T = \{[(a, A)] : A \in \mathrm{I}$ and $a \in A\}$ is the set of equivalence classes of ~ and each $\psi_A : S \to T$ is defined by $\psi_A(a) = [(a, A)]$. Since $(a, A) \sim (b, B)$ if there exists $C \in \mathrm{I}(A, B)$ such that $f_{AC}(a) = f_{BC}(b)$, we have $[(a, A)] = \{(a, B) : a \in B \in \mathrm{I}\}$.
By **Lemma 4.1.2(c)**, $\{(S, \boldsymbol{F}), \{[incl_A] : (S, \boldsymbol{F}_A) \to (S, \boldsymbol{F}) \mid A \in \mathrm{I}\}$ is a sink for the directed system, so there exists a morphism $[h] : (T, \boldsymbol{G}) \to (S, \boldsymbol{F})$ such that $h \circ \psi_A = incl_A$ for each $A \in \mathrm{I}$. It is easy to verify that h is a bijection since $S = \cup\mathrm{I}$. Also, $\boldsymbol{G}_{tt'} = \boldsymbol{F}_{h(a)h(b)}$ for each $t = [(a, A)]$ and $t' = [(b, B)]$, so by **Corollary 4.1.1**, $[h]$ is an isomorphism. Therefore, $(S, \boldsymbol{F})$ is the direct limit.

Example 4.1.3. Each *Menger* space $(S, \boldsymbol{F})$ is the *inverse limit* of *special Menger* spaces with a *finite range*. Let $\mathrm{V} = \{V \in \mathrm{P_F^+}(S) \mid \{0, 1\} \subseteq V\}$ and for $V, W \in \mathrm{V}$, define $V \leq W$ if $V \subseteq W$. Then $(\mathrm{V}, \leq)$ is an down-directed set. Statement (1) in **Theorem 3.2.2** shows that for each $V \in \mathrm{V}$, $(S, \boldsymbol{F}^V)$ is a *Menger* space with a finite range. Also, by **Lemma 3.2.1(c)**, each $\pi_V = [id_S] : (S, \boldsymbol{F}) \to (S, \boldsymbol{F}^V)$ is a morphism, so $\mathrm{W} = \{(S, \boldsymbol{F}^V) \mid V \in \mathrm{V}\}$ satisfies (***ulp***). If $V, W \in \mathrm{V}$ and $V \leq W$, then $F^W_{pq}(x) \leq F^V_{pq}(x)$ for each $p, q \in S$ and $x \in \boldsymbol{R}^+$. Therefore, each $f_{VW} = [id_S] : (S, \boldsymbol{F}^W) \to (S, \boldsymbol{F}^V)$ is a morphism, so $(\mathrm{W}, \{f_{VW} \mid V, W \in \mathrm{V}$ and $V \leq W\})$ is an inverse system in **WP**. Based on the construction in **Prop 4.1.3**, the inverse limit is $(\rho, \boldsymbol{G})$, where $\Delta = \{\delta \in S^{\mathrm{V}} \mid V, W \in \mathrm{V}$ and $V \leq W \Rightarrow f_{VW}(\delta_W) = \delta_V\}$ and $\boldsymbol{G}(\delta, \xi) = \boldsymbol{inf}\{\boldsymbol{F}^V(\delta_V, \xi_V) \mid V \in \mathrm{V}\}$ for $\delta, \xi \in \Delta$. Let $\delta \in \Delta$ and let $A, B \in \mathrm{V}$. Let $V = A \cap B$. Then $V \in \mathrm{V}$ and $\delta_A = f_{VA}(\delta_A) = \delta_V = f_{VB}(\delta_B) = \delta_B$, so Δ is the diagonal in S^{V}, that is,

$$\Delta = \{\delta \in S^{\mathrm{V}} \mid \delta_V = \delta_W \text{ for each } V, W \in \mathrm{V}\}.$$

Since $\{(S, \boldsymbol{F}), \{\pi_V \mid V \in \mathrm{V}\})$ is a source for the inverse system, the bijection $\varphi: S \to \Delta$ defined by $\varphi(s)_V = s$ for each $V \in \mathrm{V}$ defines a morphism $[\varphi] : (S, \boldsymbol{F}) \to (\Delta, \boldsymbol{G})$. Let $p, q \in S$ and $x \in \boldsymbol{R}^+$. Then $y = F_{pq}(x) \leq z = G_{\varphi(p)\varphi(q)}(x) = \boldsymbol{inf}\{\boldsymbol{F}^V_{pq}(x) \mid V \in \mathrm{V}\}$. Choose $A \in \mathrm{V}$ that contains y. Then $z \leq \boldsymbol{F}^A_{pq}(x) = y$, so $F_{pq} = G_{\varphi(p)\varphi(q)}$. Hence, by **Corollary 4.1.1**, $[\varphi]$ is an isomorphism.
Notice that the same conclusion holds by using the family $\mathrm{V} = \{\{0, 1\}\} \cup \{\{0, r, 1\} : 0 < r < 1\}$.

4.2. Subcategories of ***WP***

In this section, we introduce and compare several special subcategories of ***WP***. Here are the categories that we'll be using:

Dist – objects are *distance-spaces* and morphisms are *non-expansive* mappings.

PMet (resp. ***Met***) – objects are *pseudometric* (resp. *metric*) spaces and morphisms are *non-expansive* mappings.

$\boldsymbol{WP}_D$ – full subcategory of ***WP*** based on *WPMSs determined by pseudometric* spaces.

$\boldsymbol{WP}_M$ – full subcategory of ***WP*** based on *Menger* spaces.

$\boldsymbol{WP}_S$ – full subcategory of ***WP*** based on *special WPMSs*.

The following result describes some relationships between the various categories.

Theorem 4.2.1.

(a) ***PMet*** is a *reflective* subcategory of ***Dist***.

(b) ***PMet*** and $\boldsymbol{WP}_{\mathcal{D}}$ are *isomorphic categories*.

(c) $\boldsymbol{WP}_{\mathcal{D}}$ is a *coreflective* subcategory of $\boldsymbol{WP}_{\mathcal{S}}$ and the *coreflection* of a *special WPMS* $(S, \boldsymbol{F})$ is the *WPMS determined by the pseudometric* space (S, ω_1).

(d) $\boldsymbol{WP}_{\mathcal{M}}$ is a *reflective* subcategory of ***WP***.

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Proof

(**a**): Let (M, ω) be a distance-space. By **Prop A.2.1**, there is a largest pseudometric d_ω such that $d_\omega \le \omega$, so $r = id_M : (M, \omega) \to (M, d_\omega)$ is a non-expansive mapping. Let $f : (M, \omega) \to (N, \sigma)$ be a non-expansive mapping to a pseudometric space and define τ by $\tau(x, y) = \sigma(f(x), f(y))$. Since σ is a pseudometric, τ is a pseudometric on M satisfying $\tau \le \omega$, so $\tau \le d_\omega$. Hence, $f : (M, d_\omega) \to (N, \sigma)$ is a morphism and $f \circ r = f$. Clearly, f is the unique mapping with this property.

(**b**): For each $(S, \gamma) \in | \mathcal{PMet} |$, let $\mathcal{E}((S, \gamma)) = (S, \mathbf{G}_\gamma)$ (**Example 2.1.1**) and for each non-expansive mapping $g : (S, \gamma) \to (S', \gamma')$, where $(S', \gamma') \in | \mathcal{PMet} |$, let $\mathcal{E}(g) = g$. By definition, for $p, q \in S$, $G_{pq}(x) = \boldsymbol{H}(x - \gamma(p, q))$. Hence, if $G_{pq}(x) = 1$, then $x > \gamma(p, q)) \ge \gamma'(g(p), g(q))$, so we obtain $G'_{g(p)g(q)}(x) = \boldsymbol{H}(x - \gamma'(g(p), g(q))) = 1$. Therefore, $G_{pq} \le G'_{g(p)g(q)}$, so $\mathcal{E}(g)$ is a morphism. This defines a functor $\mathcal{E} : \mathcal{PMet} \to \mathcal{WP}_D$. For each $(S, \mathbf{G}_\gamma) \in | \mathcal{WP}_D |$, let $\mathcal{F}((S, \mathbf{G}_\gamma)) = (S, \gamma)$ and for each morphism $[h] : (S, \mathbf{G}_\gamma) \to (S', \mathbf{G}_{\gamma'})$, where $(S', \gamma') \in | \mathcal{WP}_D |$, let $\mathcal{F}(h) = h$. Given $p, q \in S$ and $\varepsilon > 0$, choose $\gamma(p, q) < x < \gamma(p, q) + \varepsilon$. Then $G_{pq}(x) = 1$, so $G'_{h(p)h(q)}(x) = 1$ since $[h]$ is a morphism. Therefore, $x > \gamma'(h(p), h(q))$. Hence, $\gamma(p, q) + \varepsilon > \gamma'(h(p), h(q))$ for each $\varepsilon > 0$, so $h : (S, \gamma) \to (S', \gamma')$ is a non-expansive mapping. This defines a functor $\mathcal{F} : \mathcal{WP}_D \to \mathcal{PMet}$. Based on the definitions, $\mathcal{E} \circ \mathcal{F}$ (resp. $\mathcal{F} \circ \mathcal{E}$) is the identity functor on $\mathcal{WP}_D$ (resp. $\mathcal{PMet}$), so $\mathcal{E}$ is an isomorphism.

(**c**): If $(S, \boldsymbol{F})$ is a special *WPMS*, then by **Lemma 2.2.2**, $\gamma = \omega_1$ is a pseudometric. Let $(S, \mathbf{G}_\gamma)$ be the *WPMS* determined by (S, γ) which belongs to $| \mathcal{WP}_D |$.

(1) $[id_S] : (S, \mathbf{G}_\gamma) \to (S, \boldsymbol{F})$ is a morphism.

[Let $p, q \in S$. Since $G_{pq}(x) = \boldsymbol{H}(x - \omega_1(p, q))$, if $x \le \omega_1(p, q)$, then $G_{pq}(x) = 0 \le F_{pq}(x)$. On the other hand, if $x > \omega_1(p, q)$, then $F_{pq}(x) = 1$, so $G_{pq}(x) \le F_{pq}(x)$.]

(2) $(S, \mathbf{G}_\gamma)$ is the coreflection of $(S, \boldsymbol{F})$ in $\mathcal{WP}_D$.

[Suppose $[g] : (M, \mathbf{G}_{\gamma'} = \{G'_{ab}\}) \to (S, \boldsymbol{F})$ is a $\mathcal{WP}$-morphism where the domain *WPMS* is determined by the pseudometric space (M, γ'). Let $a, b \in M$. Then $G'_{ab} \le F_{pq}$, where $p = g(a)$ and $q = g(b)$. If $\gamma'(a, b) < \gamma(p, q)$, choose $\gamma'(a, b) < x < \gamma(p, q)$. Since $x < \gamma(p, q) = \omega_1(p, q)$, $F_{pq}(x) < 1$, but $G'_{ab}(x) = \boldsymbol{H}(x - \gamma'(a, b)) = 1$, so $F_{pq}(x) = 1$, which is a contradiction. Hence, $\gamma(p, q) \le \gamma'(a, b)$, so g is a non-expansive mapping. Therefore, by **Lemma 4.1.1**, $[g] : (M, \mathbf{G}_{\gamma'}) \to (S, \mathbf{G}_\gamma)$ is a morphism, which establishes (2).]

(**d**): Let $(S, \boldsymbol{F})$ be a *WPMS* and $\Omega_F = \{\omega_t\}$. By **Prop A.2.1**, for each $0 < t < 1$, the path-metric ρ_t based on ω_t is the largest pseudometric on S satisfying $\rho_t \le \omega_t$. Since Ω_F is linearly ordered, $\Omega = \{\rho_t\}$ is also linearly ordered. For each $0 < t < 1$, let $\rho_t' = \boldsymbol{sup}\{\rho_s \mid 0 < s < t\}$. By **Lemma 2.4.1**, $\{\rho_t'\}$ is a linearly ordered family of pseudometrics that satisfies (***lcp***), so by **Theorem 3.2.1(a)(c)**, there is a *Menger* space $(S, \boldsymbol{G})$ such that $\Omega_G = \{\rho_t'\}$. Since each $\rho_t' \le \omega_t$, by **Lemma 4.1.1**, $[id_S] : (S, \boldsymbol{F}) \to (S, \boldsymbol{G})$ is a morphism.

(3) $(S, \boldsymbol{G})$ is the reflection of $(S, \boldsymbol{F})$ in $\mathcal{WP}_M$.

[Let $[f] : (S, \boldsymbol{F}) \to (S^*, \boldsymbol{F}^*)$ be a $\mathcal{WP}$-morphism to a Menger space, where $\Omega_{F^*} = \{\omega_t^*\}$. By **Lemma 4.1.1**, for each t, $f : (S, \omega_t) \to (S^*, \omega_t^*)$ is non-expansive, so $\omega_t^*(f(p), f(q)) \le \omega_t(p, q)$ for $p, q \in S$. Since $(S^*, \boldsymbol{F}^*)$ is a *Menger* space, by **Prop 2.2.2**, Ω_{F^*} consists of pseudometrics, so for each t, the equation $\sigma_t(p, q) = \omega_t^*(f(p), f(q))$ defines a pseudometric σ_t on S satisfying $\sigma_t \le \omega_t$. Hence, by **Prop A.2.1**, each $\sigma_t \le \rho_t$. By **Prop 2.4.1(a)**, $\sigma_t = \boldsymbol{sup}\{\sigma_s \mid 0 < s < t\} = \boldsymbol{sup}\{\rho_s \mid 0 < s < t\} = \rho_t'$. Hence, by **Lemma 4.1.1**, $[f] : (S, \boldsymbol{G}) \to (S^*, \boldsymbol{F}^*)$ is a morphism. Therefore, $[id_S] : (S, \boldsymbol{F}) \to (S, \boldsymbol{G})$ is the reflection mapping associated with $\mathcal{WP}_M$.]

□

The following example illustrates the reflection into *Menger* spaces.

Example 4.2.1.

(1) In **Example 2.1.4**, $(\boldsymbol{R}, \boldsymbol{F})$ is *not* a *Menger* space and the distances have the following form: for each $0 < t \le 1$ and $p, q \in \boldsymbol{R}$,

$$\omega_t(p, q) = \begin{cases} 0 & |p - q| \le 1 - t. \\ 1 & \text{otherwise.} \end{cases}$$

Let $0 < t < 1$ and $\varepsilon = 1 - t$, and let ρ be a pseudometric satisfying $\rho \leq \omega_t$. If $|p - q| \leq \varepsilon$, then $\rho(p, q) \leq \omega_t(p, q) = 0$, so $\rho(p, q) = 0$. If $|p - q| > \varepsilon$, choose n satisfying $1/n < \varepsilon/|p - q|$ and let $x_k = (1 - k/n)p + (k/n)q$ for $0 \leq k \leq n$. For each k, $|x_k - x_{k+1}| = |p - q|/n < \varepsilon$, so $\omega_t(x_k, x_{k+1}) = 0$. Hence, $\rho(p, q) \leq \Sigma\rho(x_k, x_{k+1}) \leq \Sigma\omega_t(x_k, x_{k+1}) = 0$. Therefore, ρ is the zero pseudometric.

The construction used in the proof of **Theorem 4.2.1(d)** shows that the Menger space reflection of $(\boldsymbol{R}, \boldsymbol{F})$ is the trivial *WPMS* $(\boldsymbol{R}, \{\boldsymbol{H}\})$.

(1) In **Example 2.1.5**, $(\boldsymbol{R}^+, \boldsymbol{F})$ is *not* a Menger space and the distances have the following form: for each $0 < t \leq 1$ and $p, q \in \boldsymbol{R}^+$,

$$\omega_t(p, q) = \begin{cases} e_1(p, q) & 0 < t \leq 1/2 \\ e_2(p, q) & 1/2 < t \leq 1. \end{cases}$$

By definition, $e_1(p, q) = \boldsymbol{min}\{a, b\}$ and $e_2(p, q) = \boldsymbol{max}\{a, b\}$, where $a = |p - q|$ and $b = |p^2 - q^2|$. In addition, e_2 is a pseudometric. Let $c = 1/2$, $A = [0, c]$, and $B = (c, +\infty)$. An analysis of cases shows that the largest pseudometric $d_1 \leq e_1$ is defined by

$$d_1(p, q) = \begin{cases} e_1(p, q) & p, q \in A \text{ or } p, q \in B \\ q - p^2 - c^2 & p \in A \text{ and } q \in B \\ p - q^2 - c^2 & q \in A \text{ and } p \in B. \end{cases}$$

The construction used in the proof of **Theorem 4.2.1(d)** shows that the Menger space reflection $(\boldsymbol{R}^+, \boldsymbol{G})$ of $(\boldsymbol{R}^+, \boldsymbol{F})$ has the pseudometrics defined by

$$\rho_t' = \begin{cases} d_1 & 0 < t \leq 1/2 \\ e_2 & 1/2 < t \leq 1. \end{cases}$$

The construction is the one used in the proof of **Theorem 3.2.1**. One can show that if $p, q \in A$ or $p, q \in B$, then $G_{pq} = F_{pq}$. However, if $p \in A$ and $q \in B$, then

$$G_{pq}(x) = \begin{cases} 0 & 0 \leq x \leq d_1(p, q) \\ 1/2 & d_1(p, q) < x \leq e_2(p, q) \\ 1 & x > e_2(p, q). \end{cases}$$

For example, if $p = 1/4$ and $q = 1$, then $a = 3/4$ and $b = 15/16$. Therefore, $F_{pq}(x) = 0$ for $0 \leq x \leq 3/4$ and $G_{pq}(x) = 1/2$ for $11/16 < x < 3/4$, so $F_{pq} \neq G_{pq}$.

5. Discussion

In some sense, the paper represents a step backwards instead of forwards since it addresses the fundamental issues rather than more contemporary work. However, the results speak for themselves. The most significant ones are (*i*) the construction of *WPMSs* from *linearly ordered* families (**Theorems 2.4.1** and **Theorem 3.2.1**), (*ii*) the introduction of *finite range WPMSs* and related results (**Theorems 2.3.1** and **3.2.2**, and **Example 4.1.3**), (*iii*) the definition and characterization of *generalized Menger* spaces (**Theorem 3.4.1**), and (*iv*) the presentation of a *categorical framework* where the *Menger* spaces are a *reflective* subcategory (**Theorem 4.2.1**).

In the text, I've alluded to the general problem of classifying *WPMSs*. The originator of the subject was aware of this issue and proposed some interesting ideas in [**23**] that haven't gained much attention. The geometric properties of distances that we've mentioned may play a role. In my opinion, the classification problem is the most interesting foundational issue. A potential sequel to the present paper may address it in some detail.

Appendix A. Distance Spaces

In this appendix, we discuss distance-spaces and present some accompanying geometric conditions that can be imposed on them.

A.1. Distances

We say that a mapping $d : M \times M \to \mathbf{R}^+$ is a *distance* on a set M if $d(x, x) = 0$ for each $x \in M$ and $d(x, y) = d(y, x)$ for each $x, y \in M$. If $d(x, y) > 0$ for each distinct pair $x, y \in M$, then d is called a *semi-metric*. We say that (M, d) is a *distance-space* (resp. *semi-metric space*) if d is a *distance* (resp. *semi-metric*) on M. Let $f : (M, d) \to (N, e)$ be a mapping between distance-spaces. We say that f is *uniformly continuous* if for each $\varepsilon > 0$, there exists $\delta > 0$ such that for each $x, y \in M$, $d(x, y) < \delta \Rightarrow e(f(x), f(y)) < \varepsilon$; *non-expansive* if for each $x, y \in M$, $e(f(x), f(y)) \le d(x, y)$, and a *uniform equivalence* if f is a *bijection* and f and f^{-1} are *uniformly continuous.*

A.2. Covers and Pseudometrics

Let $\mathcal{C}_X$ denote the family of covers on a set X. For $\mathcal{U}, \mathcal{V} \in \mathcal{C}_X$, we write $\mathcal{U} < \mathcal{V}$ if for each $U \in \mathcal{U}$, there is $V \in \mathcal{V}$ such that $U \subseteq V$. For $\mathcal{U} \in \mathcal{C}_X$,

$\mathcal{U}^\delta = \{St(x, \mathcal{U}) \mid x \in X\}$, where $St(x, \mathcal{U}) = \cup\{U \in \mathcal{U} \mid x \in U\}$

$\mathcal{U}^* = \{St(V, \mathcal{U}) \mid V \in \mathcal{U}\}$, where $St(V, \mathcal{U}) = \cup\{U \in \mathcal{U} \mid U \cap V \neq \varnothing\}$.

We write $\mathcal{U} <^\delta \mathcal{V}$ if $\mathcal{U}^\delta < \mathcal{V}$ and $\mathcal{U} <^* \mathcal{V}$ if $\mathcal{U}^* < \mathcal{V}$. If $\mathcal{U}, \mathcal{V}, \mathcal{W} \in \mathcal{C}_X$, $\mathcal{U} <^\delta \mathcal{V}$, and $\mathcal{V} <^\delta \mathcal{W}$, then $\mathcal{U} <^* \mathcal{W}$. A family $\{\mathcal{U}_k \mid k \in \mathbf{N}\} \subseteq \mathcal{C}_X$ is a *normal sequence* if $\mathcal{U}_{k+1} <^* \mathcal{U}_k$ for each k.

For a distance space (X, d) and $r > 0$, $\mathcal{S}(d, r) = \mathcal{S}_d(r) = \{S_d(x, r) \mid x \in X\}$ is the cover by open spheres and $\mathcal{B}_d(r) = \{B_d(x, r) \mid x \in X\}$ is the cover by closed spheres, where $S_d(x, r) = \{y \in X \mid d(x, y) < r\}$ and $B_d(x, r) = \{y \in X \mid d(x, y) \le r\}$.

For an undirected graph $G = (V, E)$ and mapping $\omega : E \to \mathbf{R}^+$, we say that $P = (x_0, \ldots, x_n)$ is a *path* in E from x to y if $x = x_0$, $y = x_n$, and each $(x_k, x_{k+1}) \in E$. Then the path-length ω_P is defined by

$$\omega_P(x, y) = \Sigma\{\omega(x_k, x_{k+1}) \mid 0 \le k < n\}.$$

The *path-metric* d_ω on V is defined by

$$d_\omega(x, y) = \mathbf{inf}\{\omega_P \mid P \text{ is a path in } E \text{ from } x \text{ to } y\}$$

with the convention that $d_\omega(x, x) = 0$ for each $x \in V$.

Prop A.2.1. If $G = (V, E)$ is a *connected undirected* graph and $\omega : E \to \mathbf{R}^+$ is a mapping, then d_ω is the *largest pseudometric* on V such that $d_\omega(x, y) \le \omega(x, y)$ for each $(x, y) \in E$.

The proof is left to the reader. If (M, ω) is a distance-space and G is the complete graph on M, then d_ω is the largest pseudometric on M satisfying $d_\omega \le \omega$.

The next result states the well-known connection between pseudometrics and normal sequences of covers.

Prop A.2.2. Let $\{\mathcal{U}_k\}$ be a *normal sequence* of covers on a set X with $\mathcal{U}_0 = \{X\}$ and let $r_k = 2^{-k}$ for each $k \ge 0$. For each $x, y \in X$, define $\omega(x, y) = \mathbf{inf}\{r_k \mid y \in St(x, \mathcal{U}_k)\}$. Then (*i*) $\mathcal{U}_k < \mathcal{B}_d(2^{-k})$ and $\mathcal{S}_d(2^{-k}) < \mathcal{U}_k^\delta$ for each $k > 0$ and (*ii*) $d \le \omega \le 2d$, where $d = d_\omega$.

<u>Proof</u>

The result (*i*) is found in ([**24**], I.14). To prove (*ii*), we use a statement similar to the one on ([**24**], page 8, line 22): $(\#) \equiv d(x, y) < r_k \Rightarrow y \in St(x, \mathcal{U}_{k+1}^*)$. By (#), if $d(x, y) = 0$, then $y \in St(x, \mathcal{U}_k)$ for each k, so $\omega(x, y) = 0$. If $d(x, y) > 0$, then either (**a**) $d(x, y) = r_k$ or (**b**) $r_{k+1} < d(x, y) < r_k$ for some k. (**a**): If $k = 0$, then $\omega(x, y) \le r_0 = d(x, y)$; otherwise, by (#), $d(x, y) < r_{k-1} \Rightarrow y \in St(x, \mathcal{U}_{k-1})$. Hence, $\omega(x, y) \le r_{k-1} = 2r_k = 2d(x, y)$. (**b**): By (#), $y \in St(x, \mathcal{U}_k)$, so $\omega(x, y) \le r_k = 2r_{k+1} < 2d(x, y)$.

□

A.3. Geometric Conditions

Here are two examples of geometric conditions that can be imposed on a distance-space (M, ω):

- weak polygonal condition

$$(\textbf{\textit{wpc}}) \equiv \forall\varepsilon > 0\ \exists\delta > 0\ \forall \text{paths } P \text{ between } x, y \in M \bullet \omega_P < \delta \Rightarrow \omega(x, y) < \varepsilon$$

- weak triangle condition

$$(\textbf{\textit{wtc}}) \equiv \forall\varepsilon > 0\ \exists\delta > 0\ \forall x, y, z \in M \bullet \omega(x, z) < \delta \text{ and } \omega(z, y) < \delta \Rightarrow \omega(x, y) < \varepsilon.$$

Clearly, (***wpc***) implies (***wtc***), but the converse is false. Corresponding to each condition, there is a theorem that states a uniform equivalence with a pseudometric space. First, we establish the following result.

Prop A.3.1. A *distance-space* (M, ω) satisfies (***wpc***) if and only if $id_M : (M, \omega) \to (M, d_\omega)$ is a *uniform equivalence*.

Proof

Suppose (M, ω) satisfies (***wpc***). Given $\varepsilon > 0$, there is $\delta > 0$ such that for any path P from x to y, $\omega_P < r \Rightarrow \omega(x, y) < \varepsilon$. If $d_\omega(x, y) < \delta$, then there is a path P from x to y with $\omega_P < \delta$. Therefore, $\omega(x, y) < \varepsilon$, so $id_M : (M, d_\omega) \to (M, \omega)$ is uniformly continuous. Also, since $d_\omega \le \omega$, the mapping $id_M : (M, \omega) \to (M, d_\omega)$ is non-expansive, so id_M is a uniform equivalence. Conversely, suppose $id_M : (M, d_\omega) \to (M, \omega)$ is uniformly continuous. Given $\varepsilon > 0$, there is $\delta > 0$ such that for $x, y \in M$, $d_\omega(x, y) < \delta \Rightarrow \omega(x, y) < \varepsilon$. If P is a path between x and y such that $\omega_P < \delta$, then $d_\omega(x, y) \le \omega_P < \delta$. Hence, $\omega(x, y) < \varepsilon$, so ω satisfies (***wpc***).

□

For the next result, we need to introduce the following notion. We say that a distance-space (M, ω) is a *regular écart* if there is a *non-decreasing* mapping $\varphi : [0, +\infty) \to (0, +\infty)$ that satisfies

(***ec***1) For each $x, y, z \in M$, $\omega(x, z) < r$ and $\omega(y, z) < r \Rightarrow \omega(x, y) < \varphi(r)$

(***ec***2) $\varphi(0+) = \varphi(0) = 0$.

Theorem A.3.1. The following statements are equivalent for a *distance-space* (M, ω).

(a) (M, ω) satisfies (***wtc***).

(b) (M, ω) is a regular écart.

(c) There is a *path-metric* d on M such that $id_M : (M, \omega) \to (M, d)$ is a *uniform equivalence*.

(d) There is a *pseudometric* d on M such that $id_M : (M, \omega) \to (M, d)$ is a *uniform equivalence*.

Proof

Given $\varepsilon > 0$ and $r > 0$, define the predicate

$$P_\omega(\varepsilon, r) \equiv \forall x, y \in M \bullet \omega(x, y) \ge \varepsilon \Rightarrow \forall z \in M \bullet \omega(x, z) + \omega(y, z) \ge r.$$

Clearly, (**c**) ⇒ (**d**).

(**d**) ⇒ (**a**): Let $\varepsilon > 0$ and choose $\delta > 0$ such that $d(x, y) < \delta \Rightarrow \omega(x, y) < \varepsilon$. Also, choose $r > 0$ such that $\omega(x, y) < r \Rightarrow d(x, y) < \delta/2$. Suppose $\omega(x, y) \ge \varepsilon$ and let $z \in M$. Then $d(x, y) \ge \delta$, so without loss of generality, we can assume that $d(x, z) \ge \delta/2$. Therefore, $\omega(x, z) + \omega(y, z) \ge \omega(x, z) \ge r$. Hence, $P_\omega(\varepsilon, r)$ holds, so (M, ω) satisfies (***wtc***).

To make the remaining proof less cluttered, we put the proofs of statements (1) – (4) at the end.

(**b**) ⇒ (**c**): By assumption, there is a non-decreasing mapping $\varphi : (0, +\infty) \to (0, +\infty)$ that satisfies (***ec***1) and (***ec***2), so we can assume that $\varphi(0) = 0$. Then the following statements hold.

(1) For each $x \in M$ and $r > 0$, $St(x, \boldsymbol{S}_\omega(r)) \subseteq S_\omega(x, \varphi(r))$.

(2) There is a decreasing sequence $\{\varepsilon_k\} \subseteq (0, 1]$ converging to 0 with $\varepsilon_0 = \varepsilon_1 = 1$ such that $\varphi(\varepsilon_{k+1}) < \varepsilon_k$ and $\boldsymbol{S}_\omega(\varepsilon_{k+1}) <^* \boldsymbol{S}_\omega(\varepsilon_k)$ for each $k > 0$.

Let $\{\varepsilon_k\}$ be a sequence that satisfies the conditions in (2). Let $\boldsymbol{\mathcal{U}}_0 = \{M\}$ and $\boldsymbol{\mathcal{U}}_k = \boldsymbol{S}_\omega(\varepsilon_k)$ for $k > 0$. By (2), $\{\boldsymbol{\mathcal{U}}_k\}$ is a normal sequence. Define the distance σ on M by $\sigma(x, y) = \boldsymbol{inf}\{2^{-k} \mid y \in St(x, \boldsymbol{\mathcal{U}}_k)\}$ and let d_σ be the induced path-metric. Then by **Prop A.2.2(ii)**, $d_\sigma \le \sigma \le 2d_\sigma$, so σ and d_σ are uniformly equivalent.

(3) ω and σ are uniformly equivalent, so d_σ and ω are uniformly equivalent. Hence, part (**c**) holds.

(a) ⇒ (b): Without loss of generality, assume that $\Delta = \mathrm{diam}_\omega(M) > 0$.

Case 1: $\Delta = +\infty$ or $\Delta < +\infty$ and $Rng(\omega) \subseteq [0, \Delta)$.

(4) (*i*): If $\Delta = +\infty$, then there is a non-decreasing $g : (0, +\infty) \to (0, +\infty)$ such that $P_\omega(\varepsilon, g(\varepsilon))$ holds for each $\varepsilon > 0$. (*ii*) If $\Delta < +\infty$ and $Rng(\omega) \subseteq [0, \Delta)$, then there is a non-decreasing $g : (0, \Delta) \to (0, \Delta]$ such that $P_\omega(\varepsilon, g(\varepsilon))$ holds for each $0 < \varepsilon < \Delta$.

Subcase 1: $L = lim_{\varepsilon \to \Delta\text{-}}\, g(\varepsilon) = +\infty$.

For each $r > 0$, $B_r = \{\varepsilon > 0 \mid g(\varepsilon) \geq r\} \neq \emptyset$. Define $\tau : (0, +\infty) \to [0, +\infty)$ by $\tau(r) = \boldsymbol{inf}(B_r)$. If $0 < r < r'$, then $B_{r'} \subseteq B_r$, so $\tau(r) \leq \tau(r')$. Hence, τ is non-decreasing. Define $\varphi : (0, +\infty) \to (0, +\infty)$ by $\varphi(r) = \tau(2r) + r$. Then φ is also non-decreasing.

Suppose $x, y \in M$ and $\omega(x, y) \geq \varphi(r)$ for some $r > 0$. Let $z \in M$ and let $s = \omega(x, z) + \omega(y, z)$. Since $\omega(x, y) > \tau(2r) = \boldsymbol{inf}(B_{2r})$, there is $\varepsilon \in B_{2r}$ such that $\omega(x, y) > \varepsilon$. Then by (4), $s \geq g(\varepsilon) \geq 2r$, so either $\omega(x, z) \geq r$ or $\omega(y, z) \geq r$. Hence, φ satisfies (***ec***$_1$).

Subcase 2: $L = lim_{\varepsilon \to \Delta^-} g(\varepsilon) < +\infty$.

For each $0 < r < L$, $B_r = \{0 < \varepsilon < \Delta \mid g(\varepsilon) \geq r\} \neq \emptyset$. Define $\tau : (0, L) \to [0, \Delta)$ by $\tau(r) = \boldsymbol{inf}(B_r)$. As in Subcase 1, τ is non-decreasing, so $K = lim_{r \to L^-}\tau(r)$ exists and $K \leq \Delta$. Now define τ on $(0, +\infty)$ by assigning $\tau(r) = K$ for each $r \geq L$. Then $\tau : (0, +\infty) \to (0, \Delta]$ is non-decreasing. Define $\varphi : (0, +\infty) \to (0, +\infty)$ by $\varphi(r) = \tau(2r) + 4r\Delta/L$. Then φ is also non-decreasing.

Suppose $x, y \in M$ and $\omega(x, y) \geq \varphi(r)$ for some $r > 0$. Let $z \in M$ and let $s = \omega(x, z) + \omega(y, z)$. If $r \geq L/2$, then $\omega(x, y) \geq 4r\Delta/L \geq 2\Delta$, which is impossible. Since $r < L/2$, $\omega(x, y) > \tau(2r) = \boldsymbol{inf}(B_{2r})$, so there is $\varepsilon \in B_{2r}$ such that $\omega(x, y) > \varepsilon$. As in Subcase 1, by (4), either $\omega(x, z) \geq r$ or $\omega(y, z) \geq r$, so φ satisfies (***ec***$_1$).

If $a = lim_{n \to +\infty}\tau(1/n) > 0$, choose $0 < x < a$. If $L < +\infty$, then for each $n > 1/L$, $x < \tau(1/n) = \boldsymbol{inf}(B_{1/n})$, so $g(x) < 1/n$. Hence, $g(x) = 0$, which is a contradiction. Hence, $a = 0$, so $\varphi(0+) = 0$. If $L = +\infty$, the same proof works, so in either case, φ satisfies (***ec***$_2$).

Case 2: $\Delta < +\infty$ and $\Delta \in Rng(\omega)$.

Let $M' = M \cup_d B$, where $B = (0, \Delta + 1)$ and define $\omega' : M' \times M' \to \mathbf{R}^+$ by $\omega'(x, y) = \omega(x, y)$ if $x, y \in M$, $\omega'(x, y) = | x - y |$ if $x, y \in B$, and $\omega'(x, y) = \omega'(y, x) = 1$ if $x \in M$ and $y \in B$. Evidently, $\boldsymbol{diam}_{\omega'}(M') = \Delta + 1$ and $Rng(\omega') \subseteq [0, \Delta + 1)$. By part (**a**), given $\varepsilon > 0$, choose $r > 0$ such that $P_\omega(\varepsilon, r)$ holds. Then $P_{\omega'}(\varepsilon, \boldsymbol{min}\{r, \varepsilon, 1\})$ holds, so (M', ω') satisfies (***wtc***). Hence, by Case 1, (M', ω') is a regular écart, so (M, ω) is also a regular écart.

<u>Proofs</u> (intermediate statements)

(1) For each $x \in M$ and $r > 0$, $St(x, \boldsymbol{S}_\omega(r)) \subseteq S_\omega(x, \varphi(r))$.

Let $x \in M$. If $x \in S_\omega(p, r)$ for some $p \in M$, then $\omega(x, p) < r$ and $\omega(p, y) < r$ for each $y \in S_\omega(p, r)$, so by (***ec***$_1$), $\omega(x, y) < \varphi(r)$. Hence, $St(x, \boldsymbol{S}_\omega(r)) \subseteq S_\omega(x, \varphi(r))$.

(2) There is a decreasing sequence $\{\varepsilon_k\} \subseteq (0, 1]$ converging to 0 with $\varepsilon_0 = \varepsilon_1 = 1$ such that $\varphi(\varepsilon_{k+1}) < \varepsilon_k$ and $\boldsymbol{S}_\omega(\varepsilon_{k+1}) <^* \boldsymbol{S}_\omega(\varepsilon_k)$ for each $k > 0$.

Let $\varepsilon_0 = \varepsilon_1 = 1$ and suppose $\varepsilon_1, \ldots, \varepsilon_k$ have been defined for $k > 0$ such that the two conditions hold. By (***ec***$_2$), there is $r > 0$ such that $r < \varepsilon_k$ and $\varphi(r) < \varepsilon_k$ and there is $0 < \varepsilon_{k+1} < \varepsilon_k$ such that $\varphi(\varepsilon_{k+1}) < r$. Then $\varphi(\varepsilon_{k+1}) < \varepsilon_k$ and by (1), $\boldsymbol{S}_\omega(\varepsilon_{k+1}) <^\delta \boldsymbol{S}_\omega(\varphi(\varepsilon_{k+1}))$ and $\boldsymbol{S}_\omega(\varphi(\varepsilon_{k+1})) < \boldsymbol{S}_\omega(r) <^\delta \boldsymbol{S}_\omega(\varphi(r)) < \boldsymbol{S}_\omega(\varepsilon_k)$. Hence, by our earlier remarks, $\boldsymbol{S}_\omega(\varepsilon_{k+1}) <^* \boldsymbol{S}_\omega(\varepsilon_k)$.

(3) ω and σ are uniformly equivalent, so d_σ and ω are uniformly equivalent.

Let $x, y \in M$. If $\sigma(x, y) \leq 2^{-(k+1)}$ for some $k \geq 0$, then $y \in St(x, \boldsymbol{a}_{k+1})$, so there is $z \in M$ such that $x, y \in S_\omega(z, \varepsilon_{k+1})$. Hence, by (***ec***$_1$) and (2), $\omega(x, y) < \varphi(\varepsilon_{k+1}) < \varepsilon_k$. Conversely, if $\omega(x, y) < \varepsilon_k$ for some $k > 1$, then $y \in St(x, \boldsymbol{a}_k)$, so $\sigma(x, y) \leq 2^{-k}$.

(4) (*i*): If $\Delta = +\infty$, then there is a non-decreasing $g : (0, +\infty) \to (0, +\infty)$ such that $P_\omega(\varepsilon, g(\varepsilon))$ holds for each $\varepsilon > 0$. (*ii*) If $\Delta < +\infty$ and $Rng(\omega) \subseteq [0, \Delta)$, then there is a non-decreasing $g : (0, \Delta) \to (0, \Delta]$ such that $P_\omega(\varepsilon, g(\varepsilon))$ holds for each $0 < \varepsilon < \Delta$.

By part (**a**), for each $\varepsilon > 0$, $S_\varepsilon = \{r > 0 \mid P_\omega(\varepsilon, r)\} \neq \emptyset$.

(*i*): Let $\varepsilon > 0$ and choose $x, y \in M$ such that $\omega(x, y) > \varepsilon$. If $r \in S_\varepsilon$, then $\omega(x, y) + \omega(y, y) \geq r$, so $S_\varepsilon \subseteq (0, \omega(x, y)]$. Define $g : (0, +\infty) \to (0, +\infty)$ by $g(\varepsilon) = \boldsymbol{sup}(S_\varepsilon)$. If $0 < \varepsilon < \varepsilon'$, then $S_\varepsilon \subseteq S_{\varepsilon'}$, so $g(\varepsilon) \leq g(\varepsilon')$. Hence, g is non-decreasing. If $x, y \in M$ satisfies $\omega(x, y) \geq \varepsilon > 0$, then for each $r \in S_\varepsilon$ and $z \in M$, $\omega(x, z) + \omega(y, z) \geq r$. Hence, $\omega(x, z) + \omega(y, z) \geq g(\varepsilon)$, so $P_\omega(\varepsilon, g(\varepsilon))$ holds.

(*ii*): Let $0 < \varepsilon < \Delta$. The argument used in (*i*) shows that $S_\varepsilon \subseteq (0, \Delta]$. Define $g : (0, \Delta) \to (0, \Delta]$ by $g(\varepsilon) = \boldsymbol{sup}(S_\varepsilon)$. Then the previous argument shows that each $P_\omega(\varepsilon, g(\varepsilon))$ holds.

□

In [**17**], the (wtc) property is called Axiom V and the definition of a regular écart is attributed to Fréchet. A regular écart is also referred to as property (V) in [**14**] and [**15**]. We can call **Theorem A.3.1** the Chittenden-Wilson Theorem since the equivalence of (**a**) and (**b**) is established in ([**17**], Theorem I), (**b**) ⇒ (**d**) is proved in [**15**], and (**d**) ⇒ (**b**) is proved in ([**17**], Theorem II). Our proof that (**a**) ⇒ (**b**) is a modified and more detailed version of the original proof and our proof that (**b**) ⇒ (**c**) differs significantly from the original proof.

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