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Enhancing Stability Criteria for Linear Systems with Interval Time-Varying Delays via Augmented Lyapunov-Krasovskii Functional

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Abstract: This work investigates the stability conditions for linear systems with time-varying delays via augmented Lyapunov-Krasovskii functional(LKF). Two types of augmented LKFs with cross terms in integral are suggested to improve the stability conditions for interval time-varying linear systems. Through this work, the compositions of the LKFs are considered to enhance the feasible region of stability criterion for linear systems. Mathematical tools such as Wirtinger-based integral inequality(WBII), zero equalities, reciprocally convex approach, and Finsler's lemma are utilized to solve the problem of stability criteria. Two sufficient conditions are derived to guarantee the asymptotic stability of the systems using linear matrix inequality(LMI). First, asymptotic stability criteria are induced by constructing the new augmented LKFs in Theorem 1. And then simplified LKFs in Corollary 1 are proposed to show the effectiveness of Theorem 1. Second, asymmetric LKFs are shown to reduce the conservatism and the number of decision variables in Theorem 2. Finally, the advantages of the proposed criteria are verified by comparing maximum delay bounds in two examples. Two numerical examples show that the proposed Theorem 1 and 2 obtained less conservative results than existing outcomes. Also, Example 2 shows that the asymmetric LKF methods of Theorem 2 can provide larger delay bounds and fewer decision variables than Theorem 1's in some specific systems.

Keywords: augmented approaches; linear system; stability analysis; time delay

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1. Introduction

It is well-known that time delays in system operation create unexpected dynamic situations such as quality degradation, vibration, and instability [1,2]. That is why time-delays have received lots of attention in many fields, such as aircraft, biological systems, chemical processes, networked control systems, neural networks, fuzzy systems, and so on. One of the crucial concerns in studying time-delay systems is the development of stability conditions while increasing the upper bounds of time-delay compared with others. On the other hand, stability analysis of time delay systems can be classified into two broad categories: delay-independent and delay-dependent. In the general case, the delay-dependent case is known to be less conservative than the delay-independent one when the size of the time delay is small [3]. Therefore, research on delay-dependent systems has been more actively conducted.

For the past decades, lots of stability criteria based on the Lyapunov stability theorem have been suggested [4]-[33]. Here, there are two methods available for deriving enhanced stability conditions for linear systems with interval time-varying delays. One approach involves determining the LKFs to derive less conservative stability conditions for the system [4]. Another method utilizes mathematical tools to handle issues such as quadratic terms, non-LMI forms, and so on [5,6]. Indicators evaluating these methods include the maximum delay upper bound and the number of decision variables [7].

It is well recognized that finding new LKFs is also an important and significant job [8]. The conservatism of the stability results is determined by how the LKFs are constructed. A notable trend is to design them as complex integrals, including double and triple integrals and even augmented forms, within LKFs to include more information on system dynamics and delay properties. In [9] and [10], the integral term was integrated into the $x^T(t)Px(t)$ form in the design of LKFs, with [10] further enhancing this approach by incorporating delay terms. Moreover, [11] and [12] expanded LKFs by proposing augmented quadratic forms for single and double integral terms, resulting in more conservative results. [3,13] improved LKFs by introducing an additional integral term for interval time-varying delays. And some researchers obtained conservative results by proposing various triple integral terms[14].

When extracting stability conditions based on LKFs, the quadratic terms within the integral term derived from the derivative of the double integral were of interest. To solve the quadratic terms within the integral, numerous integral inequalities offering the lower bound of integral terms have been proposed for decades. Many researchers have suggested various skills to reduce the conservatism of the LMI and computational burdens. The Jensen integral inequality was first used by Gu [15] to improve the stability conditions of time-delay systems. The Wirtinger-based integral inequality (WBII) [16] was introduced to obtain a tighter lower bound for the quadratic form in the integral term than that provided by Jensen's inequality [17]. Park [18] presented the Wirtinger-based double integral inequality for the quadratic double integral form in the stability of time-delay systems. Moreover, Generalized versions of the WBII were introduced using an auxiliary function in [19], while recent works in [20–22] have presented further generalized integral inequalities. Another mathematical technique is the reciprocally convex approach, which provides the lower bound of integral terms [23]. Since then, some improved versions have been introduced in [24–26]. Zhang et al.[24] proposed delay-dependent-matrix-based reciprocally convex inequality and estimation approaches for stability analysis with time-varying delays of linear systems. In addition to the methods mentioned so far, other techniques have been worked to improve the stability conditions of systems. By adding a cross term to the time derivative LKF term, [27] suggested several zero equations and ultimately yielded improved results. Generalized zero equality techniques that exploit the relationships among the integral terms have been introduced for linear systems with interval time-varying delays [28]. The work [29] applied the augmented zero equality method derived from Finsler Lemma and zero equalities to obtain improved results and proposed an approach to mitigate computational complexity. And numerous achievements in [30–32] adopted various functions to reduce the conservatism of stability conditions for time-delay systems. On the other hand, [33] proposed asymmetric LKF in which the involved matrix variables do not require to be all positive definite.

In this paper, stability criteria for linear systems with interval time-varying delays are studied. Two types of augmented LKFs with cross terms in integral are suggested to improve the stability conditions for interval time-varying linear systems. To solve the problem of stability criteria, WBII, zero equalities, reciprocally convex approach and Finsler's lemma are utilized. In theorem 1, stability criteria were derived by constructing augmented LKFs, with the process employing zero equalities and WBII. In Theorem 2, asymmetric LKFs were constructed to reduce the number of decision variables. Finally, two numerical examples show that the proposed Theorem 1 and 2 obtained less conservative results compared to existing outcomes.

Notations. $\mathbb{R}^n$, $\mathbb{R}^{m\times n}$, and $\mathbb{S}^n$ ($\mathbb{S}^n_+$) denote, respectively, n-vectors, $m\times n$ matrices, and $n\times n$ symmetric (positive) matrix. I_n denotes an $n\times n$ identity matrix. $0_{n\cdot m}$ denotes a $n\times m$ zero matrix. $\mathtt{Sym}\{X\}$ and $\mathtt{col}\{\cdots\}$ denote $X+X^T$ and the column vector. $*$ is used to represent symmetric terms as needed. $\odot$ denotes the quadratic form of the matrix. $X_\perp$ denotes a basis for the null-space of X. $X_{[\alpha]}$ means that $X_{[\alpha]}$ is the matrix with respect to α, i.e., $X_{[\alpha_0]} = X_{[\alpha=\alpha_0]}$.

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2. Problem Statements

In this paper, consider the following linear systems with time-varying delays as follows

$$\begin{aligned} \dot{x}(t) &= Ax(t) + A_d x(t-\kappa(t)) \\ x(s) &= \phi(s) \end{aligned} \tag{1}$$

where $x(t) \in \mathbb{R}^n$ represents the state vector, $\phi(s)^{\forall} s \in [-\kappa, 0]$ is the initial condition, $A, A_d \in \mathbb{R}^{n\times n}$ are the system's constant matrices. And $\kappa(t)$ is a time-varying delay satisfying $0 \le \kappa_L \le \kappa_U$ and $\dot{\kappa}(t) \le \kappa_D$ where κ_L and κ_U are known positive scalars and κ_D is any constant one.

The purpose of this paper is to derive stability criteria for linear systems (1) with interval time varying delays. On the other hand, the following lemmas are used to derive main results.

Lemma 1. *[5] For scalars $a < b$, an vector $x : [a,b] \to \mathbb{R}^n$, and a matrix $Q \in \mathbb{S}^n_+$, the following inequality holds*

$$\int_a^b \alpha^T(s)Q\alpha(s)ds \ge \frac{1}{b-a}\left(\int_a^b \mathbb{N}_N(s,b-a)\otimes\alpha(s)ds\right)^T \left(\mathbb{Q}_N^{-1}(b-a)\otimes Q\right)(\odot) \tag{2}$$

where $\mathbb{N}_N(s,\gamma) = \mathtt{col}\{I_n, (s-\gamma)I_n, \cdots, \frac{(s-\gamma)^{N-1}}{(N-1)!}I_n\}$ and

$$\mathbb{Q}_N^{-1}(h) = \begin{bmatrix} 1 & \frac{h}{2} & \cdots & \frac{h^{N-1}}{N!} \\ \frac{h}{2} & \frac{h^2}{3} & \cdots & \frac{h^N}{(N+1)\times(N-1)!} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{h^{N-1}}{N!} & \frac{h^N}{(N+1)\times(N-1)!} & \cdots & \frac{h^{2N-2}}{(2N-1)\times(N-1)!\times(N-1)!} \end{bmatrix}^{-1}.$$

Lemma 2. *[23] For a scalar $\alpha \in (0,1)$, vectors $x, y \in \mathbb{R}^n$, matrices M_1, $M_2 \in \mathbb{S}^n$, and $F \in \mathbb{R}^{n\times n}$, the following inequality holds $\frac{1}{\alpha}x^T M_1 x + \frac{1}{1-\alpha}y^T M_2 y \ge x^T M_1 x + y^T M_2 y + 2x^T F y$, with $M_1 - F M_2^{-1} F^T > 0$.*

Lemma 3. *[34] Let $v \in \mathbb{R}^n$, $F \in \mathbb{S}^n$, $G \in \mathbb{R}^{m\times n}$. The following formulas are equivalent: (i) $v^T F v < 0$, ${}^{\forall}Gv = 0$, $v \ne 0$, (ii) ${}^{\exists}E \in \mathbb{R}^{n\times m} : F + \mathtt{Sym}\{EG\} < 0$, (iii) $G_\perp^T F G_\perp < 0$.*

3. Main Results

This section presents the stability criteria for the system (1). At first, a stability criterion is obtained by utilizing an augmented LKF and Lemmas 1 - 3. Following this, stability criteria are improved through the an asymmetric LKF. To simplify the expression of LMIs, an augmented vector is defined as:

$$\zeta(t) = \left\{ \begin{bmatrix} x(t) \\ x(t-\kappa_L) \\ x(t-\kappa(t)) \\ x(t-\kappa_U) \\ \dot{x}(t) \\ \dot{x}(t-\kappa_L) \\ \dot{x}(t-\kappa_U) \end{bmatrix}, \begin{bmatrix} \varphi_{10}(x(s),\kappa_L,0) \\ \varphi_{10}(x(s),\kappa(t),\kappa_L) \\ \varphi_{10}(x(s),\kappa_U,\kappa(t)) \\ \varphi_{21}(x(u),\kappa_L,0) \\ \varphi_{21}(x(u),\kappa(t),\kappa_L) \\ \varphi_{21}(x(u),\kappa_U,\kappa(t)) \end{bmatrix}, \begin{bmatrix} \varphi_{11}(x(s),\kappa(t),\kappa_L) \\ \varphi_{11}(x(s),\kappa_U,\kappa(t)) \\ \varphi_{20}(x(u),\kappa(t),\kappa_L) \\ \varphi_{20}(x(u),\kappa_U,\kappa(t)) \end{bmatrix} \right\},$$

where

$$\varphi_{ij}(f(\cdot),\nu_a,\nu_b) = \frac{1}{(\nu_a-\nu_b)^j}\underbrace{\int_{t-\nu_a}^{t-\nu_b}\cdots\int_u^{t-\nu_b}}_{i-th\text{ integral}} f(\cdot)dv\cdots ds.$$

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And the block entry matrices $\varepsilon_i = [0_{n\cdot(i-1)n}, I_n, 0_{n\cdot(17-i)n}]^T \in \mathbb{R}^{17n\times n}$ $(i = 1, 2, \cdots, 17)$ are utilized, e.g., $\varepsilon_0 = 0_{17n\cdot n}$ and $\varepsilon_2^T\zeta(t) = x(t-\kappa_L)$. Also, the following notations are defined as:

$$
\begin{aligned}
\kappa_S &= \kappa_U - \kappa_L,\\
Q_{2i} &= Q_2 + \begin{bmatrix} 0_{n\cdot n} & S_i \\ S_i^T & 0_{n\cdot n} \end{bmatrix} (i = 1, 2),\\
\Omega_1 &= \mathtt{diag}\{Q_{21}, 3Q_{21}\}, \quad \Omega_2 \;=\; \mathtt{diag}\{Q_{22}, 3Q_{22}\}, \quad \Omega_3 = \begin{bmatrix} \Omega_1 & F_1 \\ * & \Omega_2 \end{bmatrix},\\
\Pi_{1,1[\kappa(t)]} &= [\varepsilon_1, \varepsilon_2, \varepsilon_4, \varepsilon_8, \kappa_L\varepsilon_{11}, \kappa_L(\varepsilon_8 - \varepsilon_{11}), \varepsilon_9 + \varepsilon_{10}, \varepsilon_{16} + \varepsilon_{17} + (\kappa_U - \kappa(t))\varepsilon_9,\\
&\quad (\kappa(t) - \kappa_L)\varepsilon_9 + \kappa_S\varepsilon_{10} - \varepsilon_{16} - \varepsilon_{17}],\\
\Pi_{1,2} &= [\varepsilon_5, \varepsilon_6, \varepsilon_7, \varepsilon_1 - \varepsilon_2, -\varepsilon_8 + \kappa_L\varepsilon_1, \varepsilon_8 - \kappa_L\varepsilon_2, \varepsilon_2 - \varepsilon_4,\\
&\quad -\varepsilon_9 - \varepsilon_{10} + \kappa_S\varepsilon_2, \varepsilon_9 + \varepsilon_{10} - \kappa_S\varepsilon_4],\\
\Pi_{2,1} &= [\varepsilon_5, \varepsilon_1, \varepsilon_0, \varepsilon_0, \varepsilon_1 - \varepsilon_2, \varepsilon_8],\\
\Pi_{2,2} &= [\varepsilon_6, \varepsilon_2.\varepsilon_1 - \varepsilon_2, \varepsilon_8, \varepsilon_0, \varepsilon_0],\\
\Pi_{2,3} &= [\varepsilon_6, \varepsilon_2, \varepsilon_0, \varepsilon_0, \varepsilon_2 - \varepsilon_4, \varepsilon_9 + \varepsilon_{10}],\\
\Pi_{2,4} &= [\varepsilon_7, \varepsilon_4, \varepsilon_2 - \varepsilon_4, \varepsilon_9 + \varepsilon_{10}, \varepsilon_0, \varepsilon_0],\\
\Pi_{2,5} &= [\varepsilon_1 - \varepsilon_2, \varepsilon_8, \kappa_L\varepsilon_1 - \varepsilon_8, \kappa_L\varepsilon_{11}, \varepsilon_8 - \kappa_L\varepsilon_2, \kappa_L(\varepsilon_8 - \varepsilon_{11})],\\
\Pi_{2,6} &= [\varepsilon_0, \varepsilon_0, \varepsilon_5, \varepsilon_1, -\varepsilon_6, -\varepsilon_2],\\
\Pi_{2,7[\kappa(t)]} &= [\varepsilon_2 - \varepsilon_4, \varepsilon_9 + \varepsilon_{10}, \kappa_S\varepsilon_2 - \varepsilon_9 - \varepsilon_{10}, \varepsilon_{16} + \varepsilon_{17} + (\kappa_U - \kappa(t))\varepsilon_9, \varepsilon_9 + \varepsilon_{10} - \kappa_S\varepsilon_4,\\
&\quad (\kappa(t) - \kappa_L)\varepsilon_9 + \kappa_S\varepsilon_{10} - \varepsilon_{16} - \varepsilon_{17}],\\
\Pi_{2,8} &= [\varepsilon_0, \varepsilon_0, \varepsilon_6, \varepsilon_2, -\varepsilon_7, -\varepsilon_4],\\
\Pi_{3,1} &= [\varepsilon_2, \varepsilon_0, \varepsilon_0, \varepsilon_2 - \varepsilon_4, \varepsilon_9 + \varepsilon_{10}],\\
\Pi_{3,2} &= [\varepsilon_3, \varepsilon_2 - \varepsilon_3, \varepsilon_9, \varepsilon_3 - \varepsilon_4, \varepsilon_{10}],\\
\Pi_{3,3[\kappa(t)]} &= [\varepsilon_9, (\kappa(t) - \kappa_L)\varepsilon_2 - \varepsilon_9, \varepsilon_{16}, \varepsilon_9 - (\kappa(t) - \kappa_L)\varepsilon_4, (\kappa(t) - \kappa_L)(\varepsilon_9 + \varepsilon_{10}) - \varepsilon_{16}],\\
\Pi_{3,4} &= [\varepsilon_0, \varepsilon_6, \varepsilon_2, -\varepsilon_7, -\varepsilon_4],\\
\Lambda_1 &= [\varepsilon_2 - \varepsilon_3, \varepsilon_9, -\varepsilon_2 - \varepsilon_3 + 2\varepsilon_{14}, \varepsilon_9 - 2\varepsilon_{12}],\\
\Lambda_2 &= [\varepsilon_3 - \varepsilon_4, \varepsilon_{10}, -\varepsilon_3 - \varepsilon_4 + 2\varepsilon_{15}, \varepsilon_{10} - 2\varepsilon_{13}],\\
\Lambda &= [\Lambda_1, \Lambda_2],\\
\Psi_{1[\kappa_t]} &= \mathtt{Sym}\left\{\Pi_{1,1[\kappa(t)]} R\, \Pi_{1,2}^T\right\},\\
\Psi_{2[\kappa(t)]} &= \Pi_{2,1} G_1 \Pi_{2,1}^T - \Pi_{2,2} G_1 \Pi_{2,2}^T + \Pi_{2,3} G_2 \Pi_{2,3}^T - \Pi_{2,4} G_2 \Pi_{2,4}^T\\
&\quad + \mathtt{Sym}\left\{\Pi_{2,5} G_1 \Pi_{2,6}^T + \Pi_{2,7[\kappa(t)]} G_2 \Pi_{2,8}^T\right\},\\
\Psi_{3[\kappa_t]} &= \Pi_{3,1} G_3 \Pi_{3,1}^T - (1 - \kappa_D)\Pi_{3,2} G_3 \Pi_{3,2}^T + \mathtt{Sym}\left\{\Pi_{3,3[\kappa(t)]} G_3 \Pi_{3,4}^T\right\},\\
\Psi_{41} &= \kappa_L^2[\varepsilon_5, \varepsilon_1] Q_1 [\varepsilon_5, \varepsilon_1]^T + \kappa_S^2[\varepsilon_6, \varepsilon_2] Q_2 [\varepsilon_6, \varepsilon_2]^T,\\
\Psi_{42} &= \kappa_S\left(\varepsilon_2 S_1 \varepsilon_2^T - \varepsilon_3(S_1 - S_2)\varepsilon_3^T - \varepsilon_4 S_2 \varepsilon_4^T\right),\\
\Psi_{43} &= [-\varepsilon_1 + \varepsilon_2, -\varepsilon_8] Q_1 [-\varepsilon_1 + \varepsilon_2, -\varepsilon_8]^T\\
&\quad + 3[\varepsilon_1 + \varepsilon_2 - \frac{2}{\kappa_L}\varepsilon_8, -\varepsilon_8 + 2\varepsilon_{11}] Q_1 [\varepsilon_1 + \varepsilon_2 - \frac{2}{\kappa_L}\varepsilon_8, -\varepsilon_8 + 2\varepsilon_{11}]^T,
\end{aligned}
$$

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$$\begin{aligned}
\Psi_{44} &= \Lambda_1\Omega_1\Lambda_1^T, \quad \Psi_{45} = \Lambda_2\Omega_2\Lambda_2^T, \quad \Psi_{46} = -\Lambda\Omega_3\Lambda^T, \\
\Psi_4 &= \Psi_{41} + \Psi_{42} + \Psi_{43} + \Psi_{46}, \\
\Psi_5 &= \kappa_U(\varepsilon_5 W\varepsilon_5^T - \varepsilon_7 W\varepsilon_7^T) - \mathtt{Sym}\{(\varepsilon_1 - \varepsilon_4)W(\varepsilon_5 - \varepsilon_7)^T\} \\
&\quad - 3\mathtt{Sym}\left\{\left(-\varepsilon_1 - \varepsilon_4 + \frac{2}{\kappa_U}(\varepsilon_8 + \varepsilon_9 + \varepsilon_{10})\right)W\left(-\varepsilon_5 - \varepsilon_7 + \frac{2}{\kappa_U}(\varepsilon_1 - \varepsilon_4)\right)^T\right\}, \\
\Psi_{[\kappa(t)]} &= \Psi_{1[\kappa(t)]} + \Psi_{2[\kappa(t)]} + \Psi_{3[\kappa(t)]} + \Psi_4 + \Psi_5, \\
\Phi_{[\kappa(t)]} &= \mathtt{Sym}\{\Pi((\kappa(t) - \kappa_L)[\varepsilon_{14}, \varepsilon_0, \varepsilon_{12}, \varepsilon_0] \\
&\quad + (\kappa_U - \kappa(t))[\varepsilon_0, \varepsilon_{15}, \varepsilon_0, \varepsilon_{13}] \\
&\quad - [\varepsilon_9, \varepsilon_{10}, \varepsilon_{16}, \varepsilon_{17}])^T\}, \\
\Gamma &= A\varepsilon_1^T + A_d\varepsilon_3^T - I_n\varepsilon_5^T.
\end{aligned} \tag{3}$$

Theorem 1. *For given scalars $\kappa_L, \kappa_U, \kappa_D$ satisfying (2), the systems (1) is asymptotically stable, if there exist matrices $R \in \mathbb{S}_+^{9n}, G_i \in \mathbb{S}_+^{6n}, G_3 \in \mathbb{S}_+^{5n}, Q_i \in \mathbb{S}_+^{2n}, W \in \mathbb{S}_+^{n}, S_i \in \mathbb{S}^n (i = 1,2), F_1 \in \mathbb{R}^{4n\times 4n}$ and $\Pi \in \mathbb{R}^{4n\times 17n}$ satisfying the following LMIs:*

$$\Gamma_\perp^T(\Psi_{[\kappa_L]} + \Phi_{[\kappa_L]})\Gamma_\perp < 0, \tag{4}$$

$$\Gamma_\perp^T(\Psi_{[\kappa_U]} + \Phi_{[\kappa_U]})\Gamma_\perp < 0, \tag{5}$$

$$\Omega_3 > 0. \tag{6}$$

Proof of Theorem 1. Consider the LKF candidate given by

$$V(t) = \sum_{i=1}^{5} V_i, \tag{7}$$

where

$$\begin{aligned}
V_1(t) &= \begin{bmatrix} x(t) \\ x(t-\kappa_L) \\ x(t-\kappa_U) \\ \varphi_{10}(x(s),\kappa_L,0) \\ \varphi_{20}(x(u),\kappa_L,0) \\ \int_{t-\kappa_L}^{t}\int_{t-\kappa_L}^{s} x(u)duds \\ \varphi_{10}(x(s),\kappa_U,\kappa_L) \\ \varphi_{20}(x(u),\kappa_U,\kappa_L) \\ \int_{t-\kappa_U}^{t-\kappa_L}\int_{t-\kappa_U}^{s} x(u)duds \end{bmatrix}^T R(\odot), \\
V_2(t) &= \int_{t-\kappa_L}^{t} \begin{bmatrix} \psi(s) \\ \hline \int_s^t \psi(u)du \\ \hline \int_{t-\kappa_L}^{s} \psi(u)du \end{bmatrix}^T G_1(\odot)ds + \int_{t-\kappa_U}^{t-\kappa_L} \begin{bmatrix} \psi(s) \\ \hline \int_s^{t-\kappa_L} \psi(u)du \\ \hline \int_{t-\kappa_U}^{s} \psi(u)du \end{bmatrix}^T G_2(\odot)ds, \\
V_3(t) &= \int_{t-\kappa(t)}^{t-\kappa_L} \begin{bmatrix} x(s) \\ \hline \int_s^{t-\kappa_L} \psi(u)du \\ \hline \int_{t-\kappa_U}^{s} \psi(u)du \end{bmatrix}^T G_3(\odot)ds, \\
V_4(t) &= \kappa_L\int_{t-\kappa_L}^{t}\int_s^t \psi^T(u)Q_1(\odot)duds + \kappa_S\int_{t-\kappa_U}^{t-\kappa_L}\int_s^{t-\kappa_L} \psi^T(u)Q_2(\odot)duds,
\end{aligned}$$

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$$
\begin{aligned}
V_5(t) &= \kappa_U \int_{t-\kappa_U}^{t} \dot{x}^T(s) W \dot{x}(s) ds - \left(\int_{t-\kappa_U}^{t} \dot{x}(s) ds \right)^T W(\odot) \\
&\quad -3 \left(\int_{t-\kappa_U}^{t} \dot{x}(s) ds - \frac{2}{\kappa_U} \int_{t-\kappa_U}^{t} \int_{s}^{t} \dot{x}(u) du ds \right)^T W(\odot),
\end{aligned}
$$

where $\psi(t) = \mathtt{col}\{\dot{x}(t), x(t)\}$.

The time derivative of Lyapunov Krasovskii functionals

$$
\begin{aligned}
\dot{V}_1 &= 2 \begin{bmatrix} x(t) \\ x(t-\kappa_L) \\ x(t-\kappa_U) \\ \varphi_{10}(x(s),\kappa_L,0) \\ \varphi_{20}(x(u),\kappa_L,0) \\ \int_{t-\kappa_L}^{t}\int_{t-\kappa_L}^{s} x(u) du ds \\ \varphi_{10}(x(s),\kappa_U,\kappa_L) \\ \varphi_{20}(x(u),\kappa_U,\kappa_L) \\ \int_{t-\kappa_U}^{t-\kappa_L}\int_{t-\kappa_U}^{s} x(u) du ds \end{bmatrix}^T R \begin{bmatrix} \dot{x}(t) \\ \dot{x}(t-\kappa_L) \\ \dot{x}(t-\kappa_U) \\ x(t)-x(t-\kappa_L) \\ \kappa_L x(t) - \varphi_{10}(x(s),\kappa_L,0) \\ \varphi_{10}(x(s),\kappa_L,0) - \kappa_L x(t-\kappa_L) \\ x(t-\kappa_L) - x(t-\kappa_U) \\ \kappa_S x(t-\kappa_L) - \varphi_{10}(x(s),\kappa_U,\kappa_L) \\ \varphi_{10}(x(s),\kappa_U,\kappa_L) - \kappa_S x(t-\kappa_U) \end{bmatrix} \\
&= \zeta^T(t) \Psi_{1[\kappa(t)]} \zeta(t), \qquad (8) \\
\dot{V}_2 &= \begin{bmatrix} \psi(t) \\ \hdashline 0_{2n\cdot 1} \\ \hdashline x(t)-x(t-\kappa_L) \\ \int_{t-\kappa_L}^{t} x(s) ds \end{bmatrix}^T G_1(\odot) - \begin{bmatrix} \psi(t-\kappa_L) \\ \hdashline x(t)-x(t-\kappa_L) \\ \int_{t-\kappa_L}^{t} x(s) ds \\ \hdashline 0_{2n\cdot 1} \end{bmatrix} G_1(\odot) \\
&\quad + \begin{bmatrix} \psi(t-\kappa_L) \\ \hdashline 0_{2n\cdot 1} \\ \hdashline \int_{t-\kappa_U}^{t-\kappa_L} \psi(s) ds \end{bmatrix}^T G_2(\odot) - \begin{bmatrix} \psi(t-\kappa_U) \\ \hdashline \int_{t-\kappa_U}^{t-\kappa_L} \psi(s) ds \\ \hdashline 0_{2n\cdot 1} \end{bmatrix}^T G_2(\odot) \\
&\quad + 2\int_{t-\kappa_L}^{t} \begin{bmatrix} \psi(s) \\ \hdashline \int_{s}^{t} \psi(s) ds \\ \hdashline \int_{t-\kappa_L}^{s} \psi(s) ds \end{bmatrix}^T G_1 \begin{bmatrix} 0_{2n\cdot 1} \\ \hdashline \psi(t) \\ \hdashline -\psi(t-\kappa_L) \end{bmatrix} ds \\
&\quad + 2\int_{t-\kappa_U}^{t-\kappa_L} \begin{bmatrix} \psi(s) \\ \hdashline \int_{s}^{t-\kappa_L} \psi(s) ds \\ \hdashline \int_{t-\kappa_U}^{s} \psi(s) ds \end{bmatrix}^T G_2 \begin{bmatrix} 0_{2n\cdot 1} \\ \hdashline \psi(t-\kappa_L) \\ \hdashline -\psi(t-\kappa_U) \end{bmatrix} ds \\
&= \zeta^T(t) \Psi_{2[\kappa(t)]} \zeta(t), \qquad (9) \\
\dot{V}_3 &= \begin{bmatrix} x(t-\kappa_L) \\ \hdashline 0_{2n\cdot 1} \\ \hdashline x(t-\kappa_L)-x(t-\kappa_U) \\ \int_{t-\kappa_U}^{t-\kappa_L} x(s) ds \end{bmatrix} G_3(\odot) - (1-\dot{\kappa}(t)) \begin{bmatrix} x(t-\kappa(t)) \\ \hdashline \int_{t-\kappa(t)}^{t-\kappa_L} \psi(s) ds \\ \hdashline \int_{t-\kappa_U}^{t-\kappa(t)} \psi(s) ds \end{bmatrix} G_3(\odot) \\
&\quad + 2\int_{t-\kappa(t)}^{t-\kappa_L} \begin{bmatrix} x(s) \\ \hdashline \int_{s}^{t-\kappa_L} \psi(s) ds \\ \hdashline \int_{t-\kappa_U}^{s} \psi(s) ds \end{bmatrix}^T G_3 \begin{bmatrix} 0_{n\cdot 1} \\ \hdashline \psi(t-\kappa_L) \\ \hdashline -\psi(t-\kappa_U) \end{bmatrix} ds \\
&\leq \zeta^T(t) \Psi_{3[\kappa(t)]} \zeta(t), \qquad (10)
\end{aligned}
$$

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$$
\begin{aligned}
\dot{V}_4 &= \kappa_L^2 \psi^T(t) Q_1(\odot) - \kappa_L \int_{t-\kappa_L}^{t} \psi^T(s) Q_1(\odot) ds \\
&\quad + \kappa_S^2 \psi^T(t-\kappa_L) Q_2(\odot) - \kappa_S \int_{t-\kappa_U}^{t-\kappa_L} \psi^T(s) Q_2(\odot) ds \\
&= \zeta^T(t) \Psi_{4[\kappa(t)]} \zeta(t) - \kappa_L \int_{t-\kappa_L}^{t} \psi^T(s) Q_1(\odot) ds - \kappa_S \int_{t-\kappa_U}^{t-\kappa_L} \psi^T(s) Q_2(\odot) ds. \qquad (11)
\end{aligned}
$$

Let introduce zero equalities for symmetric matrices S_1, S_2 inspired by the work [27].

$$
Z_1 = \kappa_S \Big(x^T(t-\kappa_L) S_1 x(t-\kappa_L) - x^T(t-\kappa(t)) S_1 x(t-\kappa(t)) - 2\int_{t-\kappa(t)}^{t-\kappa_L} \dot{x}^T(s) S_1 x(s) ds \Big), \qquad (12)
$$

$$
Z_2 = \kappa_S \Big(x^T(t-\kappa(t)) S_2 x(t-\kappa(t)) - x^T(t-\kappa_U) S_2 x(t-\kappa_U) - 2\int_{t-\kappa_U}^{t-\kappa(t)} \dot{x}^T(s) S_2 x(s) ds \Big). \qquad (13)
$$

And, adding (12) - (13) to (11), the following equality can be obtained as

$$
\begin{aligned}
\dot{V}_4(t) + Z_1 + Z_2 &= \zeta^T(t) \Psi_{41} \zeta(t) + \kappa_S \Big(x^T(t-\kappa_L) S_1 x(t-\kappa_L) \\
&\quad - x^T(t-\kappa(t))(S_1 - S_2) x(t-\kappa(t)) - x^T(t-\kappa_U) S_2 x(t-\kappa_U) \Big) \\
&\quad - \kappa_L \int_{t-\kappa_L}^{t} \psi^T(s) Q_1(\odot) ds - \kappa_S \int_{t-\kappa(t)}^{t-\kappa_L} \psi^T(s) Q_{21}(\odot) ds \\
&\quad - \kappa_S \int_{t-\kappa_U}^{t-\kappa(t)} \psi^T(s) Q_{22}(\odot) ds \\
&= \zeta^T(t)(\psi_{41} + \psi_{42}) \zeta - \kappa_L \int_{t-\kappa_L}^{t} \psi^T(s) Q_1(\odot) ds \\
&\quad - \kappa_S \int_{t-\kappa(t)}^{t-\kappa_L} \psi^T(s) Q_{21}(\odot) ds - \kappa_S \int_{t-\kappa_U}^{t-\kappa(t)} \psi^T(s) Q_{22}(\odot) ds. \qquad (14)
\end{aligned}
$$

Moreover, by employing Lemma 1 (N=1), upper bounds of integral terms in (14) can be obtained as follows:

$$
\begin{aligned}
-\kappa_L \int_{t-\kappa_L}^{t} \psi^T(s) Q_1(\odot) ds &\leq -\varphi_{10}^T(\psi(s), \kappa_L, 0) Q_1(\odot) \\
&\quad -3(\varphi_{10}(\psi(s), \kappa_L, 0) - 2\varphi_{21}(\psi(u), \kappa_L, 0))^T Q_1(\odot) \\
&= \zeta^T(t) \psi_{43} \zeta(t). \qquad (15) \\
-\kappa_S \int_{t-\kappa(t)}^{t-\kappa_L} \psi^T(s) Q_{21}(\odot) ds &\leq -\frac{\kappa_S}{\kappa(t)-\kappa_L} \varphi_{10}^T(\psi(s), \kappa(t), \kappa_L) Q_{21}(\odot) \\
&\quad -3\frac{\kappa_S}{\kappa(t)-\kappa_L} \Big(\varphi_{10}^T(\psi(s), \kappa(t), \kappa_L) - 2\varphi_{21}^T(\psi(u), \kappa(t), \kappa_L) \Big)^T \\
&\quad \times Q_{21}(\odot) \\
&= -\frac{\kappa_S}{\kappa(t)-\kappa_L} \zeta^T(t) \psi_{44} \zeta(t). \qquad (16) \\
-\kappa_S \int_{t-\kappa_U}^{t-\kappa(t)} \psi^T(s) Q_{22}(\odot) ds &\leq -\frac{\kappa_S}{\kappa_U-\kappa(t)} \varphi_{10}^T(\psi(s), \kappa_U, \kappa(t)) Q_{22}(\odot) \\
&\quad -3\frac{\kappa_S}{\kappa_U-\kappa(t)} (\varphi_{10}(\psi(s), \kappa_U, \kappa(t)) - 2\varphi_{21}(\psi(u), \kappa_U, \kappa(t)))^T \\
&\quad \times Q_{22}(\odot) \\
&= -\frac{\kappa_S}{\kappa_U-\kappa(t)} \zeta^T(t) \psi_{45} \zeta(t). \qquad (17)
\end{aligned}
$$

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By using Lemma 2,

$$-\frac{\kappa_S}{\kappa(t)-\kappa_L}\zeta^T(t)\psi_{44}\zeta(t)-\frac{\kappa_S}{\kappa_U-\kappa(t)}\zeta^T(t)\psi_{45}\zeta(t)\le\zeta^T(t)\psi_{46}\zeta(t). \tag{18}$$

Hence, from (11) $\sim$ (18), a novel upper bound for $\dot{V}_4(t)$ is obtained as:

$$\dot{V}_4(t)\le\zeta^T(t)\psi_4\zeta(t), \tag{19}$$

where $\psi_4=\psi_{41}+\psi_{42}+\psi_{43}+\psi_{46}$.

Based on the relationships between the elements of the augmented vector, the following equality is derived as

$$\begin{aligned} 0 &= \zeta^T(t)Sym\{\Pi((\kappa(t)-\kappa_L)[\varepsilon_{14},\varepsilon_0,\varepsilon_{12},\varepsilon_0]+(\kappa_U-\kappa(t))[\varepsilon_0,\varepsilon_{15},\varepsilon_0,\varepsilon_{13}] \\ &\quad -[\varepsilon_9,\varepsilon_{10},\varepsilon_{16},\varepsilon_{17}])\}\zeta(t) \\ &= \zeta^T(t)\Phi_{[h(t)]}\zeta(t). \end{aligned} \tag{20}$$

Also, by Lemma 1 (N=1), the Lyapunov functional is induced

$$\begin{aligned} V_5 &= \kappa_U\int_{t-\kappa_U}^{t}\dot{x}^T(s)W\dot{x}(s)ds-\left(\int_{t-\kappa_U}^{t}\dot{x}(s)ds\right)^T W(\odot) \\ &\quad -3\left(\int_{t-\kappa_U}^{t}\dot{x}(s)ds-\frac{2}{\kappa_U}\int_{t-\kappa_U}^{t}\int_{s}^{t}\dot{x}(u)duds\right)^T W(\odot). \end{aligned} \tag{21}$$

Thus, $\dot{V}_5(t)$ is

$$\begin{aligned} \dot{V}_5 &= \kappa_U\dot{x}^T(t)W_1\dot{x}(t)-\kappa_U\dot{x}^T(t-\kappa_U)W_1\dot{x}(t-\kappa_U)-2(x(t)-x(t-\kappa_U))^T W_1(\odot) \\ &\quad -6(-x(t)-x(t-\kappa_U)+2\varphi_{11}(x(s),\kappa_U,0))^T W_1(\odot) \\ &= \zeta^T(t)\Psi_5\zeta(t). \end{aligned} \tag{22}$$

From (11) $\sim$ (22), $\dot{V}(t)$ can be estimated as

$$\dot{V}(t)\le\zeta^T(t)\Big(\Psi_{1[\kappa(t)]}+\Psi_{2[\kappa(t)]}+\Psi_{3[\kappa(t)]}+\Psi_4+\Psi_5+\Phi_{[\kappa(t)]}\Big)\zeta(t). \tag{23}$$

Regarding the system (1), a stability condition can be expressed as

$$\zeta^T(t)\Big(\Psi_{\kappa(t)}+\Phi_{[\kappa(t)]}\Big)\zeta(t)<0 \text{ subject to } \Gamma\zeta(t)=0. \tag{24}$$

Considering parts i) and iii) of Lemma 3, the condition (24) is equivalent to

$$\Gamma_\perp^T\Big(\Psi_{[\kappa(t)]}+\Phi_{[\kappa(t)]}\Big)\Gamma_\perp<0. \tag{25}$$

And then, if the LMIs (4) and (5) are satisfied for the convexity of $\kappa(t)\in[\kappa_L,\kappa_U]$, then inequality (25) is satisfied, which means, the system (1) is asymptotically stable. This completes out proof. □

Remark 1. *In [3], the augmented vector for the single integral term included only $x(t)$, $\dot{x}(t)$, and $\int_s^t\dot{x}(s)ds$. However, in V_1 of Theorem 1, double integral terms were considered such as $\int_{t-\kappa_L}^{t}\int_{t-\kappa_L}^{s}x(u)$, and in V_2 and V_3, additional single integral term, such as $\int_{t-\kappa_L}^{s}\psi(u)du$, were considered. And accordingly, the augmented vector was further considered as $\zeta(t)$, and stability conditions were derived using the zero equality (20). In next section, it will be shown that Theorem 1 can provide larger delay bounds than those of [3].*

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In order to show the effectiveness of LKFs (7), let us consider LKFs as $\hat{V}(t) = \hat{V}_1(t) + \hat{V}_2(t) + \hat{V}_3(t) + \hat{V}_4(t) + \hat{V}_5(t)$. The LKFs are simplified by removing double integral terms in LKFs of Theorem 1. An augmented vector for Corollary 1 is defined follows:

$$\zeta_c(t) = \left\{ \begin{bmatrix} x(t) \\ x(t-\kappa_L) \\ x(t-\kappa(t)) \\ x(t-\kappa_U) \\ \dot{x}(t) \\ \dot{x}(t-\kappa_L) \\ \dot{x}(t-\kappa_U) \end{bmatrix}, \begin{bmatrix} \varphi_{10}(x(s),\kappa_L,0) \\ \varphi_{10}(x(s),\kappa(t),\kappa_L) \\ \varphi_{10}(x(s),\kappa_U,\kappa(t)) \end{bmatrix}, \begin{bmatrix} \varphi_{11}(x(s),\kappa(t),\kappa_L) \\ \varphi_{11}(x(s),\kappa_U,\kappa(t)) \end{bmatrix} \right\}.$$

And the block entry matrices $\hat{\varepsilon}_i = [0_{n\cdot(i-1)n}, I_n, 0_{n\cdot(12-i)n}]^T \in \mathbb{R}^{12n\times n}$ $(i = 1, 2, \cdots, 12)$ are utilized. Also, the following notations are defined as:

$$\begin{aligned}
\hat{\Omega}_1 &= \mathtt{diag}\{\hat{Q}_1, 3\hat{Q}_1\}, \quad \hat{\Omega}_2 = \mathtt{diag}\{\hat{Q}_2, 3\hat{Q}_2\}, \\
\hat{\Omega}_3 &= \begin{bmatrix} \hat{\Omega}_2 & F_2 \\ * & \hat{\Omega}_2 \end{bmatrix}, \\
\hat{\Lambda}_1 &= \left[\hat{\varepsilon}_1 - \hat{\varepsilon}_2, -\hat{\varepsilon}_1 - \hat{\varepsilon}_2 + \frac{2}{\kappa_L}\hat{\varepsilon}_8\right], \\
\hat{\Lambda}_2 &= [\hat{\varepsilon}_2 - \hat{\varepsilon}_3, -\hat{\varepsilon}_2 - \hat{\varepsilon}_3 + 2\hat{\varepsilon}_{11}], \\
\hat{\Lambda}_3 &= [\hat{\varepsilon}_3 - \hat{\varepsilon}_4, -\hat{\varepsilon}_3 - \hat{\varepsilon}_4 + 2\hat{\varepsilon}_{12}], \\
\hat{\Upsilon}_{1[\kappa(t)]} &= \mathtt{Sym}\left\{[\hat{\varepsilon}_1, \hat{\varepsilon}_2, \hat{\varepsilon}_4, \hat{\varepsilon}_8, \hat{\varepsilon}_9 + \hat{\varepsilon}_{10}]\hat{R}[\hat{\varepsilon}_5, \hat{\varepsilon}_6, \hat{\varepsilon}_7, \hat{\varepsilon}_1 - \hat{\varepsilon}_2, \hat{\varepsilon}_2 - \hat{\varepsilon}_4]^T\right\}, \\
\hat{\Upsilon}_{2[\kappa(t)]} &= [\hat{\varepsilon}_5, \hat{\varepsilon}_1, \hat{\varepsilon}_0, \hat{\varepsilon}_1 - \hat{\varepsilon}_2]\hat{G}_1[\hat{\varepsilon}_5, \hat{\varepsilon}_1, \hat{\varepsilon}_0, \hat{\varepsilon}_1 - \hat{\varepsilon}_2]^T - [\hat{\varepsilon}_6, \hat{\varepsilon}_2, \hat{\varepsilon}_1 - \hat{\varepsilon}_2, \hat{\varepsilon}_0]\hat{G}_1[\hat{\varepsilon}_6, \hat{\varepsilon}_2, \hat{\varepsilon}_1 - \hat{\varepsilon}_2, \hat{\varepsilon}_0]^T \\
&\quad + \mathtt{Sym}\left\{[\hat{\varepsilon}_1 - \hat{\varepsilon}_2, \hat{\varepsilon}_8, \kappa_L\hat{\varepsilon}_1 - \hat{\varepsilon}_8, \hat{\varepsilon}_8 - \kappa_L\hat{\varepsilon}_2]\hat{G}_1[\hat{\varepsilon}_0, \hat{\varepsilon}_0, \hat{\varepsilon}_5, -\hat{\varepsilon}_6]^T\right\} \\
&\quad + [\hat{\varepsilon}_6, \hat{\varepsilon}_2, \hat{\varepsilon}_0, \hat{\varepsilon}_2 - \hat{\varepsilon}_4]\hat{G}_2[\hat{\varepsilon}_6, \hat{\varepsilon}_2, \hat{\varepsilon}_0, \hat{\varepsilon}_2 - \hat{\varepsilon}_4]^T - [\hat{\varepsilon}_7, \hat{\varepsilon}_4, \hat{\varepsilon}_2 - \hat{\varepsilon}_4, \hat{\varepsilon}_0]\hat{G}_2[\hat{\varepsilon}_7, \hat{\varepsilon}_4, \hat{\varepsilon}_2 - \hat{\varepsilon}_4, \hat{\varepsilon}_0]^T \\
&\quad + \mathtt{Sym}\left\{[\hat{\varepsilon}_2 - \hat{\varepsilon}_4, \hat{\varepsilon}_9 + \hat{\varepsilon}_{10}, \kappa_S\hat{\varepsilon}_2 - \hat{\varepsilon}_9 - \hat{\varepsilon}_{10}, \hat{\varepsilon}_9 + \hat{\varepsilon}_{10} - \kappa_S\hat{\varepsilon}_4]\hat{G}_2[\hat{\varepsilon}_0, \hat{\varepsilon}_0, \hat{\varepsilon}_6, -\hat{\varepsilon}_7]^T\right\}, \\
\hat{\Upsilon}_{3[\kappa_t]} &= [\hat{\varepsilon}_2, \hat{\varepsilon}_0, \hat{\varepsilon}_2 - \hat{\varepsilon}_4]\hat{G}_3[\hat{\varepsilon}_2, \hat{\varepsilon}_0, \hat{\varepsilon}_2 - \hat{\varepsilon}_4]^T \\
&\quad - (1 - \kappa_D)[\hat{\varepsilon}_3, \hat{\varepsilon}_2 - \hat{\varepsilon}_3, \hat{\varepsilon}_3 - \hat{\varepsilon}_4]\hat{G}_3[\hat{\varepsilon}_3, \hat{\varepsilon}_2 - \hat{\varepsilon}_3, \hat{\varepsilon}_3 - \hat{\varepsilon}_4]^T \\
&\quad + \mathtt{Sym}\left\{[\hat{\varepsilon}_9, (\kappa(t) - \kappa_L)\hat{\varepsilon}_2 - \hat{\varepsilon}_9, \hat{\varepsilon}_9 - (\kappa(t) - \kappa_L)\hat{\varepsilon}_4]\hat{G}_3[\hat{\varepsilon}_0, \hat{\varepsilon}_6, -\hat{\varepsilon}_7]^T\right\}, \\
\hat{\Upsilon}_4 &= \kappa_L^2\hat{\varepsilon}_5\hat{Q}_1\hat{\varepsilon}_5^T + \kappa_S^2\hat{\varepsilon}_6\hat{Q}_2\hat{\varepsilon}_6^T - \hat{\Lambda}_1\hat{\Omega}_1\hat{\Lambda}_1^T - [\Lambda_2, \Lambda_3]\hat{\Omega}_3[\Lambda_2, \Lambda_3]^T, \\
\hat{\Upsilon}_5 &= \kappa_U(\hat{\varepsilon}_5\hat{W}\hat{\varepsilon}_5^T - \hat{\varepsilon}_7\hat{W}\hat{\varepsilon}_7^T) - \mathtt{Sym}\{(\hat{\varepsilon}_1 - \hat{\varepsilon}_4)\hat{W}(\hat{\varepsilon}_5 - \hat{\varepsilon}_7)^T\} \\
&\quad - 3\mathtt{Sym}\left\{\left(-\hat{\varepsilon}_1 - \hat{\varepsilon}_4 + \frac{2}{\kappa_U}(\hat{\varepsilon}_8 + \hat{\varepsilon}_9 + \hat{\varepsilon}_{10})\right)\hat{W}\left(-\hat{\varepsilon}_5 - \hat{\varepsilon}_7 + \frac{2}{\kappa_U}(\hat{\varepsilon}_1 - \hat{\varepsilon}_4)\right)^T\right\}, \\
\hat{\Upsilon}_{[\kappa(t)]} &= \hat{\Upsilon}_{1[\kappa(t)]} + \hat{\Upsilon}_{2[\kappa(t)]} + \hat{\Upsilon}_{3[\kappa(t)]} + \hat{\Upsilon}_4 + \hat{\Upsilon}_5, \\
\hat{\Phi}_{[\kappa(t)]} &= \mathtt{Sym}\{\hat{\Pi}((\kappa(t) - \kappa_L)[\hat{\varepsilon}_{11}, \hat{\varepsilon}_0] + (\kappa_U - \kappa(t))[\hat{\varepsilon}_0, \hat{\varepsilon}_{12}] - [\hat{\varepsilon}_9, \hat{\varepsilon}_{10}])\}, \\
\hat{\Gamma} &= A\hat{\varepsilon}_1^T + A_d\hat{\varepsilon}_3^T - I_n\hat{\varepsilon}_5^T.
\end{aligned}$$

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Corollary 1. *For given scalars $\kappa_L, \kappa_U, \kappa_D$ satisfying (2), the systems (1) is asymptotically stable, if there exist matrices $\hat{R} \in \mathbb{S}_+^{5n}, \hat{G}_i \in \mathbb{S}_+^{4n}, \hat{G}_3 \in \mathbb{S}_+^{3n}, \hat{Q}_i, \hat{W} \in \mathbb{S}_+^{n} (i = 1, 2), F_2 \in \mathbb{R}^{2n\times 2n}$ and $\hat{\Pi} \in \mathbb{R}^{2n\times 12n}$ satisfying the following LMIs:*

$$\hat{\Gamma}_\perp^T(\hat{\Psi}_{[\kappa_L]} + \hat{\Phi}_{[\kappa_L]})\hat{\Gamma}_\perp < 0, \tag{26}$$

$$\hat{\Gamma}_\perp^T(\hat{\Psi}_{[\kappa_U]} + \hat{\Phi}_{[\kappa_U]})\hat{\Gamma}_\perp < 0, \tag{27}$$

$$\hat{\Omega}_3 > 0. \tag{28}$$

Proof of Corollary 1. Consider the LKF candidate given by

$$\hat{V}(t) = \sum_{i=1}^{5} \hat{V}_i, \tag{29}$$

where

$$\begin{aligned}
\hat{V}_1(t) &= \begin{bmatrix} x(t) \\ x(t-\kappa_L) \\ x(t-\kappa_U) \\ \varphi_{10}(x(s),\kappa_L,0) \\ \varphi_{10}(x(s),\kappa_U,\kappa_L) \end{bmatrix}^T \hat{R}(\odot), \\
\hat{V}_2(t) &= \int_{t-\kappa_L}^{t} \begin{bmatrix} \psi(s) \\ \hdashline \int_s^t \dot{x}(u)du \\ \int_{t-\kappa_L}^s \dot{x}(u)du \end{bmatrix}^T \hat{G}_1(\odot)ds + \int_{t-\kappa_U}^{t-\kappa_L} \begin{bmatrix} \psi(s) \\ \hdashline \int_s^{t-\kappa_L} \dot{x}(u)du \\ \int_{t-\kappa_U}^s \dot{x}(u)du \end{bmatrix}^T \hat{G}_2(\odot)ds, \\
\hat{V}_3(t) &= \int_{t-\kappa(t)}^{t-\kappa_L} \begin{bmatrix} x(s) \\ \int_s^{t-\kappa_L} \dot{x}(u)du \\ \int_{t-\kappa_U}^s \dot{x}(u)du \end{bmatrix}^T \hat{G}_3(\odot)ds, \\
\hat{V}_4(t) &= \kappa_L \int_{t-\kappa_L}^{t}\int_s^t \dot{x}^T(u)\hat{Q}_1(\odot)duds + \kappa_S \int_{t-\kappa_U}^{t-\kappa_L}\int_s^{t-\kappa_L} \dot{x}^T(u)\hat{Q}_2(\odot)duds, \\
\hat{V}_5(t) &= \kappa_U \int_{t-\kappa_U}^{t} \dot{x}^T(s)\hat{W}\dot{x}(s)ds - \left(\int_{t-\kappa_U}^{t} \dot{x}(s)ds\right)^T \hat{W}(\odot) \\
&\quad -3\left(\int_{t-\kappa_U}^{t} \dot{x}(s)ds - \frac{2}{\kappa_U}\int_{t-\kappa_U}^{t}\int_s^t \dot{x}(u)duds\right)^T \hat{W}(\odot).
\end{aligned}$$

(29) has some augmented components removed from (7), and the method used can be seen in the proof of Theorem 1. Thus, it is omitted. □

In above results, LKFs are typically expected to be symmetric. However, the LKF in Theorem 2 is asymmetric. Then, the following theorem is introduced as the result with the asymmetric functionals. The augmented vector of Theorem 2 is presented as follows:

$$\xi(t) = \left\{ \begin{bmatrix} x(t) \\ x(t-\kappa_L) \\ x(t-\kappa_U) \end{bmatrix}, \begin{bmatrix} \varphi_{10}(x(s),\kappa_L,0) \\ \varphi_{10}(x(s),\kappa_U,\kappa_L) \end{bmatrix}, \begin{bmatrix} \varphi_{20}(x(u),\kappa_L,0) \\ \varphi_{20}(x(u),\kappa_U,\kappa_L) \end{bmatrix} \right\}$$

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And the block entry matrices $\epsilon_i = [0_{n\cdot(i-1)n}, I_n, 0_{n\cdot(7-i)n}]^T \in \mathbb{R}^{7n\times n}$ $(i = 1,2,\cdots,7)$ are utilized. Also, the following notations are defined as:

$$\check{G}_i = \begin{bmatrix} \check{G}_{i,11} & \check{G}_{i,12} & \check{G}_{i,13} \\ \check{G}_{i,21} & \check{G}_{i,22} & \check{G}_{i,23} \end{bmatrix}, \check{G}_{augi} = \begin{bmatrix} \check{G}_{i,11} & \check{G}_{i,12}/2 & (\check{G}_{i,13} + \check{G}_{i,21}^T)/2 \\ * & Q_i & \check{G}_{i,22}^T/2 \\ * & * & \check{G}_{i,23} \end{bmatrix}, (i = 1,2),$$

$$\begin{aligned} \Xi_1 &= [\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4, \epsilon_6, \kappa_L\epsilon_4 - \epsilon_6, \epsilon_5, \epsilon_7, \kappa_S\epsilon_5 - \epsilon_7]\check{R} \\ &\quad \times [\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4, \epsilon_6, \kappa_L\epsilon_4 - \epsilon_6, \epsilon_5, \epsilon_7, \kappa_S\epsilon_5 - \epsilon_7]^T, \\ \Xi_2 &= \frac{1}{\kappa_L}[\epsilon_1 - \epsilon_2, \epsilon_4, \kappa_L\epsilon_1 - \epsilon_4, \epsilon_6, \epsilon_4 - \kappa_L\epsilon_2, \kappa_L\epsilon_4 - \epsilon_6]\check{G}_{aug1} \\ &\quad \times [\epsilon_1 - \epsilon_2, \epsilon_4, \kappa_L\epsilon_1 - \epsilon_4, \epsilon_6, \epsilon_4 - \kappa_L\epsilon_2, \kappa_L\epsilon_4 - \epsilon_6]^T \\ &\quad + \frac{1}{\kappa_S}[\epsilon_2 - \epsilon_3, \epsilon_5, \kappa_S\epsilon_2 - \epsilon_5, \epsilon_7, \epsilon_5 - \kappa_S\epsilon_3, \kappa_S\epsilon_5 - \epsilon_7]\check{G}_{aug2} \\ &\quad \times [\epsilon_2 - \epsilon_3, \epsilon_5, \kappa_S\epsilon_2 - \epsilon_5, \epsilon_7, \epsilon_5 - \kappa_S\epsilon_3, \kappa_S\epsilon_5 - \epsilon_7]^T, \end{aligned}$$

$$\begin{aligned} \check{\Psi}_{1[\kappa_t]} &= \mathtt{Sym}\Big\{\Pi_{1,1[\kappa(t)]}\check{R}\,\Pi_{1,2}^T\Big\}, \\ \check{\Psi}_{2[\kappa(t)]} &= \frac{1}{2}sym\Big\{[\varepsilon_5, \varepsilon_1, \varepsilon_1 - \varepsilon_2, \varepsilon_8]\check{G}_1\Pi_{2,1}^T - [\varepsilon_6, \varepsilon_2, \varepsilon_0, \varepsilon_0]\check{G}_1\Pi_{2,2}^T, \\ &\quad + [\varepsilon_6, \varepsilon_2, \varepsilon_2 - \varepsilon_4, \varepsilon_9 + \varepsilon_{10}]\check{G}_2\Pi_{2,3}^T - [\varepsilon_7, \varepsilon_4, \varepsilon_0, \varepsilon_0]\check{G}_2\Pi_{2,4}^T, \\ &\quad + [\varepsilon_0, \varepsilon_0, -\varepsilon_6, -\varepsilon_2]\check{G}_1\Pi_{2,5}^T + [\varepsilon_1 - \varepsilon_2, \varepsilon_8, \varepsilon_8 - \kappa_L\varepsilon_2, \kappa_L(\varepsilon_8 - \varepsilon_{11})]\check{G}_1\Pi_{2,6}^T, \\ &\quad + [\varepsilon_0, \varepsilon_0, -\varepsilon_7, -\varepsilon_4]\check{G}_2\Pi_{2,7[\kappa(t)]}^T \\ &\quad + [\varepsilon_2 - \varepsilon_4, \varepsilon_9 + \varepsilon_{10}, \varepsilon_9 + \varepsilon_{10} - \kappa_S\varepsilon_4, \kappa_S\varepsilon_{10} + (\kappa(t) - \kappa_L)\varepsilon_9 - \varepsilon_{16} - \varepsilon_{17}]\check{G}_2\Pi_{2,8}^T\Big\}, \\ \check{\Psi}_{[\kappa(t)]} &= \check{\Psi}_{1[\kappa(t)]} + \check{\Psi}_{2[\kappa(t)]} + \Psi_{3[\kappa(t)]} + \Psi_4 + \Psi_5. \end{aligned}$$

Theorem 2. *For given scalars $\kappa_L, \kappa_U, \kappa_D$ satisfying (2), the systems (1) is asymptotically stable, if there exist matrices $\check{R} \in \mathbb{S}^{9n}, \check{G}_i \in \mathbb{R}^{4n\times 6n}$ with $\check{G}_{i,11} = \check{G}_{i,11}^T$ and $\check{G}_{i,23} = \check{G}_{i,23}^T$, $G_3 \in \mathbb{S}_+^{3n}, Q_i \in \mathbb{S}_+^{2n}, S_i, W \in \mathbb{S}^n (i = 1,2), F_1 \in \mathbb{R}^{4n\times 4n}$ and $\Pi \in \mathbb{R}^{4n\times 17n}$ satisfying the following LMIs:*

$$\check{G}_{aug1} > 0, \tag{30}$$
$$\check{G}_{aug2} > 0, \tag{31}$$
$$\Xi_1 + \Xi_2 > 0, \tag{32}$$
$$\Gamma_\perp^T(\check{\Psi}_{[\kappa_L]} + \Phi_{[\kappa_L]})\Gamma_\perp < 0, \tag{33}$$
$$\Gamma_\perp^T(\check{\Psi}_{[\kappa_U]} + \Phi_{[\kappa_U]})\Gamma_\perp < 0, \tag{34}$$
$$\Omega_3 > 0. \tag{35}$$

Proof of Theorem 2. Consider the LKF candidate given by

$$V(t) = \check{V}_1 + \check{V}_2 + \sum_{i=3}^{5} V_i, \tag{36}$$

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where

$$\begin{aligned}
\check{V}_1(t) &= \begin{bmatrix} x(t) \\ x(t-\kappa_L) \\ x(t-\kappa_U) \\ \varphi_{10}(x(s),\kappa_L,0) \\ \varphi_{20}(x(u),\kappa_L,0) \\ \int_{t-\kappa_L}^{t}\int_{t-\kappa_L}^{s} x(u)duds \\ \varphi_{10}(x(s),\kappa_U,\kappa_L) \\ \varphi_{20}(x(u),\kappa_U,\kappa_L) \\ \int_{t-\kappa_U}^{t-\kappa_L}\int_{t-\kappa_U}^{s} x(u)duds \end{bmatrix}^T \check{R}(\odot) \\
&= \xi^T(t)\Xi_1\xi(t), \\
\check{V}_2(t) &= \int_{t-\kappa_L}^{t} \begin{bmatrix} \psi(s) \\ \hdashline \int_{t-\kappa_L}^{s}\psi(u)du \end{bmatrix}^T G_1 \begin{bmatrix} \psi(s) \\ \hdashline \int_{s}^{t}\psi(u)du \\ \hdashline \int_{t-\kappa_L}^{s}\psi(u)du \end{bmatrix} ds \\
&\quad + \int_{t-\kappa_U}^{t-\kappa_L} \begin{bmatrix} \psi(s) \\ \hdashline \int_{t-\kappa_U}^{s}\psi(u)du \end{bmatrix}^T G_2 \begin{bmatrix} \psi(s) \\ \hdashline \int_{s}^{t-\kappa_L}\psi(u)du \\ \hdashline \int_{t-\kappa_U}^{s}\psi(u)du \end{bmatrix} ds.
\end{aligned}$$

Since the sum of LKF must be positive in asymmetric LKF method, it must be guaranteed that $\check{V}_1 + \check{V}_2 + V_4$ is positive if G_3 of V_3 and W of V_5 are positive.

By using Lemma 1 (N=0), V_4 can be bounded as

$$V_4 \geq \int_{t-\kappa_L}^{t}\left(\int_{s}^{t}\psi(u)du\right)^T Q_1(\odot)ds + \int_{t-\kappa_U}^{t-\kappa_L}\left(\int_{s}^{t-\kappa_L}\psi(u)du\right)^T Q_2(\odot)ds. \quad (37)$$

By adding the lower bound of V_4 to V_2, the following can be obtained.

$$\begin{aligned}
\check{V}_2 + V_4 &\geq \int_{t-\kappa_L}^{t} \begin{bmatrix} \psi(s) \\ \hdashline \int_{s}^{t}\psi(u)du \\ \hdashline \int_{t-\kappa_L}^{s}\psi(u)du \end{bmatrix}^T \check{G}_{aug1}(\odot)ds \\
&\quad + \int_{t-\kappa_U}^{t-\kappa_L} \begin{bmatrix} \psi(s) \\ \hdashline \int_{s}^{t-\kappa_L}\psi(u)du \\ \hdashline \int_{t-\kappa_U}^{s}\psi(u)du \end{bmatrix}^T \check{G}_{aug2}(\odot)ds.
\end{aligned} \quad (38)$$

And then, the lower bound of (38) can be bounded as follow by using Lemma 1(N=0), since $\check{G}_{aug1}$, $\check{G}_{aug2} > 0$,

$$\begin{aligned}
\check{V}_2 + V_4 &\geq \frac{1}{\kappa_L}\left(\int_{t-\kappa_L}^{t} \begin{bmatrix} \psi(s) \\ \hdashline \int_{s}^{t}\psi(u)du \\ \hdashline \int_{t-\kappa_L}^{s}\psi(u)du \end{bmatrix} ds\right)^T \check{G}_{aug1}(\odot) \\
&\quad + \frac{1}{\kappa_S}\left(\int_{t-\kappa_U}^{t-\kappa_L} \begin{bmatrix} \psi(s) \\ \hdashline \int_{s}^{t-\kappa_L}\psi(u)du \\ \hdashline \int_{t-\kappa_U}^{s}\psi(u)du \end{bmatrix} ds\right)^T \check{G}_{aug2}(\odot) \\
&= \xi^T(t)\Xi_2\xi(t).
\end{aligned} \quad (39)$$

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As a result, it is proven that $\check{V}_1 + \check{V}_2 + V_4$ is positive and (32) is obtained.

$$\check{V}_1 + \check{V}_2 + V_4 \geq \xi^T(t)(\Xi_1 + \Xi_2)\xi(t) > 0. \tag{40}$$

The time derivative of V_2 is induced as follow

$$\begin{aligned}
\dot{\check{V}}_2 = & \begin{bmatrix} \psi(t) \\ \int_{t-\kappa_L}^{t} \psi(s)ds \end{bmatrix}^T G_1 \begin{bmatrix} \psi(t) \\ 0_{2n\cdot 1} \\ \int_{t-\kappa_L}^{t} \psi(s)ds \end{bmatrix} - \begin{bmatrix} \psi(t-\kappa_L) \\ 0_{2n\cdot 1} \end{bmatrix}^T G_1 \begin{bmatrix} \psi(t-\kappa_L) \\ \int_{t-\kappa_L}^{t} \psi(s)ds \\ 0_{2n\cdot 1} \end{bmatrix} \\
& + \begin{bmatrix} \psi(t-\kappa_L) \\ 0_{2n\cdot 1} \end{bmatrix}^T G_2 \begin{bmatrix} \psi(t-\kappa_L) \\ \int_{t-\kappa_L}^{t} \psi(s)ds \\ 0_{2n\cdot 1} \end{bmatrix} - \begin{bmatrix} \psi(t-\kappa_U) \\ 0_{2n\cdot 1} \end{bmatrix}^T G_2 \begin{bmatrix} \psi(t-\kappa_U) \\ \int_{t-\kappa_U}^{t-\kappa_L} \psi(s)ds \\ 0_{2n\cdot 1} \end{bmatrix} \\
& + \int_{t-\kappa_L}^{t} \begin{bmatrix} 0_{2n\cdot 1} \\ -\psi(t-\kappa_L) \end{bmatrix}^T G_1 \begin{bmatrix} \psi(s) \\ \int_s^t \psi(s)ds \\ \int_{t-\kappa_L}^{s} \psi(s)ds \end{bmatrix} ds \\
& + \int_{t-\kappa_L}^{t} \begin{bmatrix} \psi(s) \\ \int_{t-\kappa_L}^{s} \psi(u)du \end{bmatrix}^T G_1 \begin{bmatrix} 0_{2n\cdot 1} \\ \psi(t) \\ -\psi(t-\kappa_L) \end{bmatrix} ds \\
& + \int_{t-\kappa_U}^{t-\kappa_L} \begin{bmatrix} 0_{2n\cdot 1} \\ -\psi(t-\kappa_U) \end{bmatrix}^T G_2 \begin{bmatrix} \psi(s) \\ \int_s^{t-\kappa_L} \psi(s)ds \\ \int_{t-\kappa_U}^{s} \psi(s)ds \end{bmatrix} ds \\
& + \int_{t-\kappa_U}^{t-\kappa_L} \begin{bmatrix} \psi(s) \\ \int_{t-\kappa_U}^{s} \psi(s)ds \end{bmatrix}^T G_2 \begin{bmatrix} 0_{2n\cdot 1} \\ \psi(t-\kappa_L) \\ -\psi(t-\kappa_U) \end{bmatrix} ds \\
= & \zeta^T(t)\check{\Psi}_{2[\kappa(t)]}\zeta(t).
\end{aligned} \tag{41}$$

Since the subsequent process is similar to Theorem 1, the proof of Theorem 2 is complete. □

Remark 2. *Asymmetric LKFs were proposed in [33], and asymmetric LKFs were also used in Theorem 2, but in [33] it was applied to part $\int_s^t x(u)du$, whereas in Theorem 2 it was applied to part $\int_s^t \psi(u)du$ because there was $V_4(t)$ to $\psi(u)$. In next section, Theorem 2 can provide larger delay bounds than [3,10,35] by comparing maximum delay bounds. Furthermore, Theorem 2 shows slightly larger delay bounds than Theorem 1 by Example 2, while the number of decision variables in Theorem 2 is used $20n^2 + 2n$ less than those of Theorem 1.*

4. Examples

In this section, the effectiveness and superiority of the proposed stability criteria are demonstrated through two examples.

Example 1 Consider the system (1) :

$$A = \begin{bmatrix} -2 & 0 \\ 0 & -0.9 \end{bmatrix}, A_d = \begin{bmatrix} -1 & 0 \\ -1 & -1 \end{bmatrix}. \tag{42}$$

Tables 1 and 2 list the maximum delay bounds for various κ_L, as determined by Theorem 1, 2, and Corollary 1, as well as those of [3,10,35], when κ_D is 0.1 and 0.5, respectively. It is listed in Table 1 that Theorem 1 and 2 provide larger delay bounds than those of [3,10,35]. Additionally, Theorem 1 is yields slightly larger delay bounds compared to Corollary 1. Table 2 also shows the superiority of Theorem

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1 and 2 when $\kappa_D = 0.5$. As a result, it can be confirmed that improved results can be obtained from suggested Theorem 1 and Theorem 2. And Figure 1 shows that the state responses of the system (1) converge to zero as time goes to infinity.

Table 1. Maximum delay bound κ_U for various κ_L with $\kappa_D = 0.1$ (Example 1).

κ_L	1	2	3	4	5
[10]	4.1935	4.4932	4.3979	4.1978	5.0275
[35]	4.4045	4.5729	4.5406	4.2367	5.0440
[3]	4.7561	4.7746	4.7931	4.7567	5.1372
Corollary 1	4.7577	4.7715	4.7634	4.7273	5.1373
Theorem 1	4.7952	4.8132	4.8110	4.7850	5.1511
Theorem 2	4.7951	4.8132	4.8109	4.7849	5.1500

Table 2. Maximum delay bound κ_U for various κ_L with $\kappa_D = 0.5$ (Example 1).

κ_L	1	2	3	4	5
[10]	2.3058	2.5663	3.3408	4.1690	5.0275
[35]	2.3513	2.6987	3.4186	4.2097	5.0440
[3]	2.4904	2.7994	3.4977	4.2939	5.1372
Corollary 1	2.4752	2.8111	3.4997	4.2946	5.1373
Theorem 1	2.5739	2.9247	3.5561	4.3134	5.1412
Theorem 2	2.5739	2.9247	3.5593	4.3133	5.1406

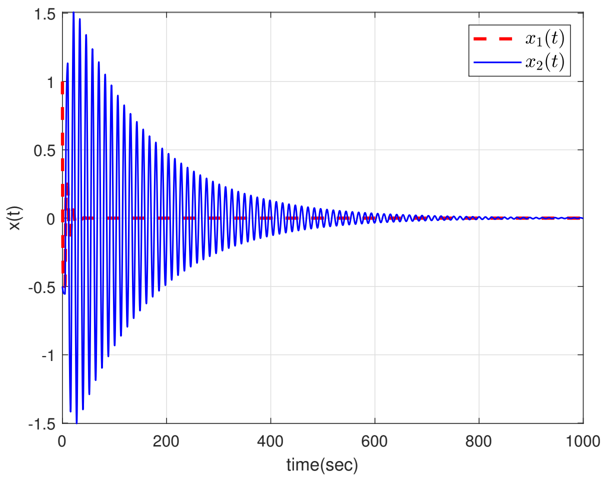

Figure 1. The state responses of the system (1) (Example 1).

Example 2 Considering the water pollution control problem based on [36], the dynamical behavior is as follows:

$$\frac{dz(t)}{dt} = -h_1 z(t) + \frac{Q_E m + Qz(t-\kappa) - (Q+Q_E)z(t)}{v}, \tag{43}$$

$$\frac{dq(t)}{dt} = -h_3 z(t) + h_2(q^s - q(t)) + \frac{Qq(t-\kappa) - (Q+Q_E)q(t)}{v}, \tag{44}$$

where $z(t)$ and $q(t)$ denote states, which represent the concentrations of biochemical oxygen demand (BOD) and dissolved oxygen (DO) in the reach at time t, respectively. And the definitions of parameter

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in (43) - (44) are shown in [36]. In the absence of disturbances and uncertainties, (43) - (44) can be described as the system (1), and the matrices of the system are as follows:

$$\begin{aligned} \bar{A} &= \begin{bmatrix} -1.32 & 0 \\ -0.32 & -1.2 \end{bmatrix}, A_d = \begin{bmatrix} 0.9 & 0 \\ 0 & 0.9 \end{bmatrix}, \\ B &= \begin{bmatrix} 0.1 & 0 \\ 0 & 1 \end{bmatrix}, K = \begin{bmatrix} -5 & -1 \\ 0.01 & -1 \end{bmatrix}, \\ A &= \bar{A} + BK. \end{aligned} \tag{45}$$

Table 3 lists the maximum delay bounds for various κ_L at $\kappa_D = 0.9$, comparing the results of Theorem 1, 2, and Corollary 1, and those of [3]. And it can also be shown that Theorem 2 provides larger delay bounds than Theorem 1. Table 4 also shows the effectiveness and superiority of Theorem 2 because the number of decision variables used is smaller than those of Theorem 1. Finally, it can be seen that the state responses of the system (1) in Figure 2 converge to zero.

Remark 3. *By Example 1, Theorem 1 and Theorem 2 show the effectiveness of the proposed criteria by comparing with [3,10,35] based on maximum delay bound. By Example 2, the asymmetric LKFs method of Theorem 2 can provide larger delay bounds than Theorem 1 and reduce decision variables in some specific systems. This supports the fact that the considered LKFs in Theorem 2 is effective in enhancing the feasible region by comparing Theorem 1.*

Table 3. Maximum delay bound κ_U for various κ_L with $\kappa_D = 0.9$ (Example 2).

κ_L	1	2	3	4	5
[3]	5.1893	6.0899	7.0461	8.0461	9.0461
Corollary 1	3.8906	4.8426	5.8413	6.8413	7.8413
Theorem 1	5.4731	6.2440	7.1456	8.0755	9.0564
Theorem 2	5.6896	6.3537	7.1932	8.0908	9.0578

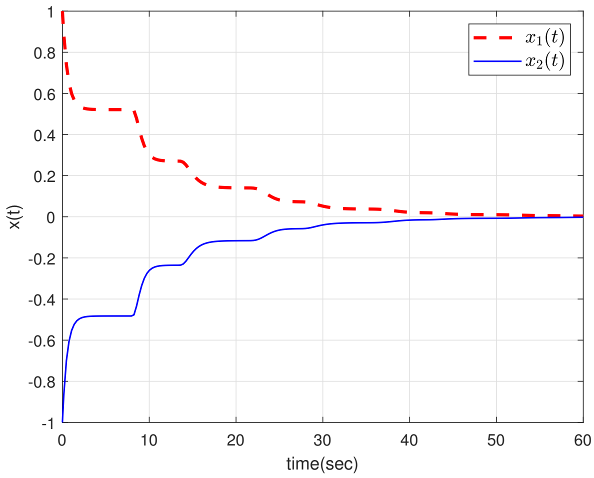

Figure 2. The state responses of the system (1) (Example 2).

Table 4. Number of decision variables.

Methods	NoDVs
[10]	$18n^2 + 8n$
[35]	$22.5n^2 + 9.5n$
[3]	$137.5n^2 + 9.5n$
Corollary 1	$62.5n^2 + 9.5n$
Theorem 1	$178.5n^2 + 16.5n$
Theorem 2	$158.5n^2 + 14.5n$

5. Conclusions

The stability problem for linear systems with interval time-varying delays was studied In this paper. The compositions of the LKFs were considered to improve the feasible region of stability criterion for linear system through this work. In Theorem 1, the asymptotic stability criteria of the systems have been derived by constructing the augmented LKFs. And then, simplified LKFs in Corollary 1 are proposed to show the effectiveness of Theorem 1. In Theorem 2, asymmetric LKFs are presented to reduce the conservatism and the number of decision variables. Two numerical examples have been given to show the effectiveness and superiority of the proposed criteria.

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Abbreviations

The following abbreviations are used in this manuscript:

LKF	Lyapunov-Krasovskii functional
WBII	Wirtinger-based integral inequality
LMI	Linear matrix inequality
BOD	biochemical oxygen demand
DO	dissolved oxygen

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