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Ismail's Ratio Conquers New Horizons the Non-stationary M/D/1 Queue's State Variable Closed Form Expression

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Posted Date: 12 February 2024

doi: 10.20944/preprints202402.0548.v2

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Ismail's Ratio Conquers New Horizons the Non-Stationary M/D/1 Queue's State Variable Closed Form Expression

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Abstract: This paper investigates the search for an exact analytic solution to a temporal first-order differential equation that represents the number of customers in a non-stationary or time-varying $M/D/1$ queueing system. Currently, the only known solution to this problem is through simulation. However, a study proposes a constant ratio β (Ismail's ratio) that relates the time-dependent mean arrival and mean service rates, offering an exact analytical solution. The stability dynamics of the time-varying $M/D/1$ queueing system are then examined numerically in relation to time, β, and the queueing parameters.

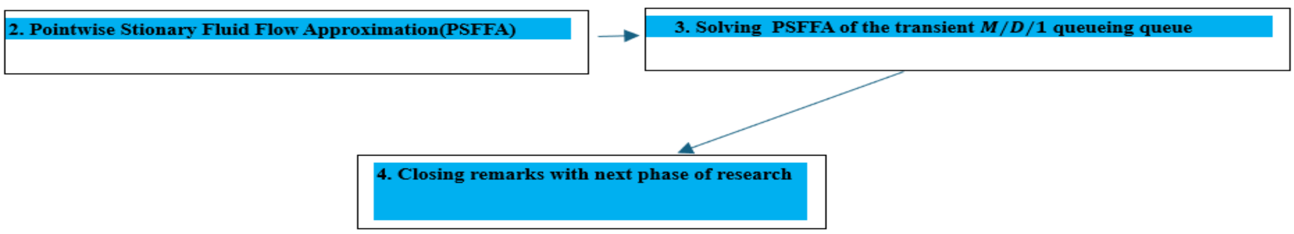
Keywords: time varying $M/D/1$ queueing system; number of customers; mean arrival rate; time; the pointwise stionary fluid flow approximation(PSFFA); time varying queueing systems(TVQSs)

1. Introduction

The field of transient/non-stationary analysis has limited literature, which can be categorized into simulation, transient analysis, analysis, and applications techniques. These categories encompass various approaches to studying systems that change over time, including simulations, analysing transient behaviour, and exploring non-stationary phenomena. In certain cases, mathematical transformations can be used to obtain a closed form expression for analysing non-stationary queueing systems. However, evaluating these expressions can be computationally complex. As a result, there has been a focus on numerically determining the transient behaviour of such systems instead of deriving closed form expressions.

The current exposition contributes to solving for first time ever, the longstanding unsolved problem of obtaining the state variable of the time varying $M/D/1$ queueing system.

The following flowchart shows how this paper is organized.



2. PSFFA

Let $f_{in}(t)$ and $f_{out}(t)$ serve as the temporal flow in, and fow out, respectively. Therefore,

$$\frac{dx(t)}{dt} = x^{\cdot}(t) = -f_{out}(t) + f_{in}(t), x(t) \text{ as the state variable} \tag{2.1}$$

$f_{out}(t)$ links server utilization, $\rho(t)$ and the time-dependent mean service rate, $\mu(t)$ by:

$$f_{out}(t) = \mu(t)\rho(t) \tag{2.2}$$

For an infite queue waiting space:

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$$f_{in}(t) = \text{Mean arrival rate} = \lambda(t) \quad (2.3)$$

Thus (2.1) rewrites to:

$$x^{\cdot}(t) = -\mu(t)\rho(t) + \lambda(t), \; 1 > \rho(t) = \frac{\lambda(t)}{\mu(t)} > 0 \quad (2.4)$$

The stable subcase of (2.4) (i.e., $x^{\cdot}(t) = 0$), implies:

$$x = G_1(\rho) \quad (2.5)$$

The numerical invertibility of $G_1(\rho)$, yields

$$\rho = G_1^{-1}(x) \quad (2.6)$$

Hence,

$$x^{\cdot}(t) = -\mu(t)(G_1^{-1}(x(t))) + \lambda(t) \quad (2.7)$$

The $M/D/1$ queueing system is made of Poisson arrival, one exponential (Poisson) server, FIFO (First-In-First-Out). Thus, $M/D/1$ queueing system's -G_1 (c.f., [1]) reads:

$$G_1(x) = \frac{x}{x+1} \quad (2.8)$$

Therefore, the time varying $M/D/1$ queueing system's -PSFFA model reads:

$$x^{\cdot} = -\mu\left((x+1) - \sqrt{(x^2+1)}\right) + \lambda \quad (2.9)$$

TVQSs' life example [6] is depicted by Figure1.

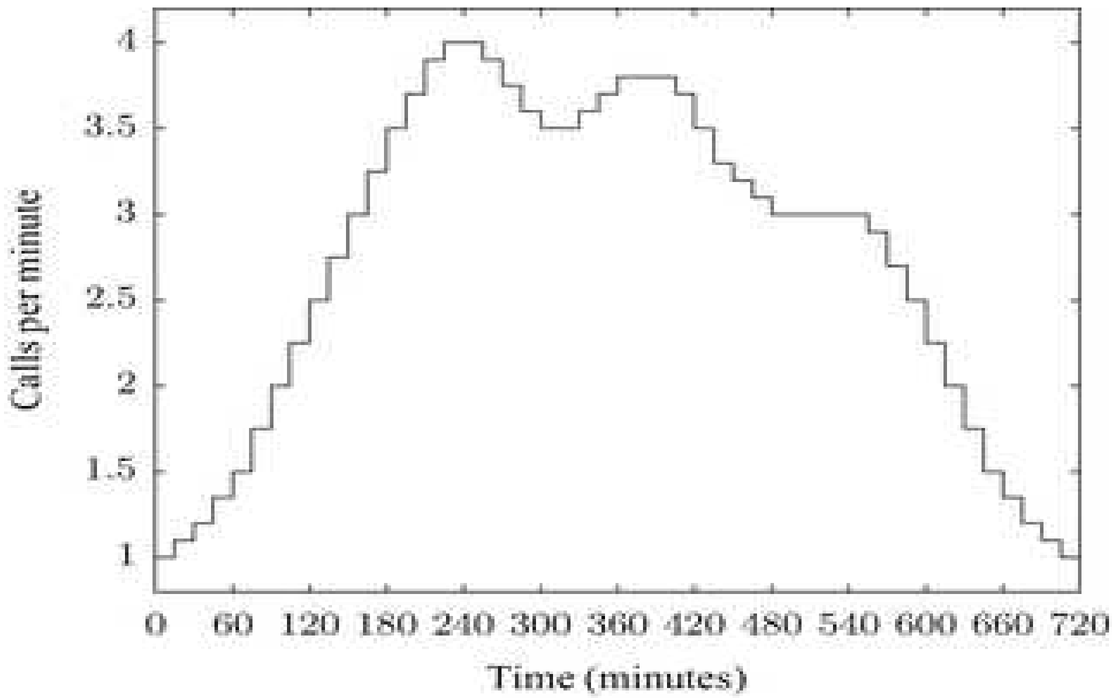

Figure 1.

3. Solving the Non-Stationary $M/D/1$ queueing System's PSFFA (c,f., (2.9))

Theorem 3.1 **Ismail's ration,** β solves (2.9), with a closed form expression to read as:

$$\left(x - \beta(x+1)\right)^{\frac{1}{(1-\beta)^2}} = \gamma e^{\left[\frac{(x+1)}{(1-\beta)} - \int \mu(t)dt\right]}, \beta = \frac{\lambda(t)}{\mu(t)} \quad (3.1)$$

Proof

We have

$$x^{\cdot} = -\mu\left((x+1) - \sqrt{(x^2+1)}\right) + \lambda \quad \text{(c.f., (2.9))}$$

Let, then $x^{\cdot} = -y^{\cdot}\, cothy\, cschy$. Setting, $\beta = \frac{\lambda(t)}{\mu(t)}$. Thus, we have

$$-y^{\cdot}\, cothy\, cschy = -\mu((1 + cschy) - cothy) + \lambda = -\mu[(1 + cschy) - cothy) - \beta] \quad (3.2)$$

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Therefore, we have

$$\frac{-cothy\ cschydy}{[(1+cschy)-cothy)-\beta]}=-\mu dt$$

$$\frac{coshydy}{[-(1+sinhy-coshy)+\beta sinhy]sinhy}=-\mu dt \tag{3.3}$$

$$\frac{\frac{2}{\beta}(e^{3y}+e^{y})dy}{\left[e^{2y}-\frac{e^{y}}{\beta}+(\frac{2}{\beta}-1)\right](e^{2y}-1)}=-\mu dt$$

$$e^{2y}-\frac{e^{y}}{\beta}+\left(\frac{2}{\beta}-1\right)=0\Longrightarrow e^{y}=\frac{\left(\frac{1}{\beta}\pm\sqrt{(\frac{1}{\beta^{2}}-4\left(\frac{2}{\beta}-1\right)}\right)}{2}=\frac{\left(\frac{1}{\beta}\pm\sqrt{(\frac{1}{\beta^{2}}-\frac{8}{\beta}+4}\right)}{2}=a,b$$

where

$$a=\frac{\left(\frac{1}{\beta}+\sqrt{(\frac{1}{\beta^{2}}-\frac{8}{\beta}+4)}\right)}{2},b=\frac{\left(\frac{1}{\beta}-\sqrt{(\frac{1}{\beta^{2}}-\frac{8}{\beta}+4)}\right)}{2}$$

Let

$$\frac{\frac{2}{\beta}(e^{3y}+e^{y})}{\left[e^{2y}-\frac{e^{y}}{\beta}+(\frac{2}{\beta}-1)\right](e^{2y}-1)}=\frac{A}{(e^{y}-a)}+\frac{B}{(e^{y}-b)}+\frac{C}{(e^{y}-1)}+\frac{D}{(e^{y}+1)} \tag{3.4}$$

Hence, it is implied that:

$$A+B+C+D=\frac{2}{\beta}\Longrightarrow A=\frac{2}{\beta}-(B+C+D)$$

$$\therefore B=\frac{(1-a)(C-D)}{\left(\frac{2}{\beta}+a\right)} \tag{3.5}$$

Thus,

$$C=\frac{\frac{4}{\beta}\left(\frac{2}{\beta}+a\right)-\left[\left(1+\frac{2}{\beta}+2a\right)-\left(ab-(a+b)\right)\left(\frac{2}{\beta}+a\right)\right]D}{\left(\frac{2}{\beta}\right)+a+[ab-(a+b)]\left(\frac{2}{\beta}+a\right)}$$

$$bA+aB+abC-abD=0$$

$$b\left[\frac{2}{\beta}-\frac{\left(1+\frac{2}{\beta}\right)C}{\left(\frac{2}{\beta}+a\right)}-\frac{\left(1+\frac{2}{\beta}+2a\right)D}{\left(\frac{2}{\beta}+a\right)}\right]+a\left[\frac{(1-a)(C-D)}{\left(\frac{2}{\beta}+a\right)}\right]+abC-abD=0$$

Therefore, we have

$$D=\frac{\frac{2b}{\beta}+a\left(ab-\frac{b\left(1+\frac{2}{\beta}\right)}{\left(\frac{2}{\beta}+a\right)}+a\left[\frac{(1-a)}{\left(\frac{2}{\beta}+a\right)}\right]\right)[\frac{\frac{4}{\beta}\left(\frac{2}{\beta}+a\right)}{\left(\frac{2}{\beta}\right)+a+[ab-(a+b)]\left(\frac{2}{\beta}+a\right)}]}{(ab+b\left[\frac{\left(1+\frac{2}{\beta}+2a\right)}{\left(\frac{2}{\beta}+a\right)}\right]+a\left[\frac{(1-a)}{\left(\frac{2}{\beta}+a\right)}\right])+a\left(ab-\frac{b\left(1+\frac{2}{\beta}\right)}{\left(\frac{2}{\beta}+a\right)}+a\left[\frac{(1-a)}{\left(\frac{2}{\beta}+a\right)}\right]\right)[\frac{\left[\left(1+\frac{2}{\beta}+2a\right)-\left(ab-(a+b)\right)\left(\frac{2}{\beta}+a\right)\right]}{\left(\frac{2}{\beta}\right)+a+[ab-(a+b)]\left(\frac{2}{\beta}+a\right)}]} \tag{3.6}$$

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$$C = \frac{\frac{4}{\beta}\left(\frac{2}{\beta}+a\right) - \left[\left(1+\frac{2}{\beta}+2a\right)-\left(ab-(a+b)\right)\left(\frac{2}{\beta}+a\right)\right]\frac{\frac{2b}{\beta}\left[\left(1+\frac{2}{\beta}+2a\right)-\left(ab-(a+b)\right)\left(\frac{2}{\beta}+a\right)\right]+\frac{4a}{\beta}\left(\frac{2}{\beta}+a\right)]\left(ab-\frac{b\left(1+\frac{2}{\beta}\right)}{\left(\frac{2}{\beta}+a\right)}+a\left[\frac{(1-a)}{\left(\frac{2}{\beta}+a\right)}\right]\right)}{(ab+b\left[\frac{b\left(1+\frac{2}{\beta}+2a\right)+a(1-a)}{\left(\frac{2}{\beta}+a\right)}\right])\frac{2}{\beta}+a+[ab-(a+b)]\left(\frac{2}{\beta}+a\right)+a\left(ab-\frac{b\left(1+\frac{2}{\beta}\right)}{\left(\frac{2}{\beta}+a\right)}+a\left[\frac{(1-a)}{\left(\frac{2}{\beta}+a\right)}\right]\right)\left[\left(1+\frac{2}{\beta}+2a\right)-\left(ab-(a+b)\right)\left(\frac{2}{\beta}+a\right)\right]})}{\left(\frac{2}{\beta}\right)+a+[ab-(a+b)]\left(\frac{2}{\beta}+a\right)}$$

(3.7)

$$A = \frac{2}{\beta} - \frac{\left(1+\frac{2}{\beta}\right)C}{\left(\frac{2}{\beta}+a\right)} - \frac{\left(1+\frac{2}{\beta}+2a\right)D}{\left(\frac{2}{\beta}+a\right)},\ B = \frac{(1-a)(C-D)}{\left(\frac{2}{\beta}+a\right)} \tag{3.8}$$

This finally solves the complicated mathematical computations to obtain:

$$\left[\frac{\boldsymbol{A}e^{-y}}{(1-ae^{-y})} + \frac{\boldsymbol{B}e^{-y}}{(1-be^{-y})} + \frac{Ce^{-y}}{(1-e^{-y})} + \frac{De^{-y}}{(1+e^{-y})}\right] dy = -\mu dt$$

Integrating both sides:

$$[\boldsymbol{A}ln\ |(1-ae^{-y})| + \boldsymbol{B}ln\ |(1-be^{-y})| + Cln|(1-e^{-y})| - Dln(1+e^{-y})] = -\int \mu dt + ln\eta$$

for some non − negative real constant parameter η

or

$$\frac{|(1-ae^{-y})|^{A}|(1-be^{-y})|^{\boldsymbol{B}}|(1-e^{-y})|^{C}}{(1+e^{-y})^{D}} = \eta e^{-\int \mu dt} \tag{3.9}$$

This transforms to the final required closed form solution:

$$\frac{|\left(1-ae^{-csch^{-1}(x)}\right)|^{A}|\left(1-be^{-csch^{-1}(x)}\right)|^{\boldsymbol{B}}|\left(1-e^{-csch^{-1}(x)}\right)|^{C}}{\left(1+e^{-csch^{-1}(x)}\right)^{D}} = \eta e^{-\int \mu dt} \tag{3.10}$$

By mathematical analysis, it is well known that:

$$csch^{-1}(x) = l\,n\left(\frac{1+\sqrt{1+x^2}}{x}\right) \tag{3.11}$$

with the domain of real line with zero removed

Thus, one gets

$$\frac{(|\left(1-\frac{ax}{1+\sqrt{1+x^2}}\right)|)^{A}\left(|\left(1-\frac{bx}{1+\sqrt{1+x^2}}\right)|\right)^{B}(|\left(1-\frac{x}{1+\sqrt{(1+x^2)}}\right)|)^{C}}{\left(1+\frac{x}{1+\sqrt{(1+x^2)}}\right)^{D}} = \eta e^{-\int \mu dt} \tag{3.12}$$

Where

$$a = \frac{\left(\frac{1}{\beta}+\sqrt{\left(\frac{1}{\beta^2}-\frac{8}{\beta}+4\right)}\right)}{2},\ b = \frac{\left(\frac{1}{\beta}-\sqrt{\left(\frac{1}{\beta^2}-\frac{8}{\beta}+4\right)}\right)}{2}$$

$$A = \frac{2}{\beta} - \frac{\left(1+\frac{2}{\beta}\right)C}{\left(\frac{2}{\beta}+a\right)} - \frac{\left(1+\frac{2}{\beta}+2a\right)D}{\left(\frac{2}{\beta}+a\right)}$$

$$B = \frac{(1-a)(C-D)}{\left(\frac{2}{\beta}+a\right)}$$

$$C = \frac{\left[\left(1+\frac{2}{\beta}+2a\right)-\left(ab-(a+b)\right)\left(\frac{2}{\beta}+a\right)\right]\dfrac{\frac{4}{\beta}\left(\frac{2}{\beta}+a\right)-\frac{2b}{\beta}\left[\left(1+\frac{2}{\beta}+2a\right)-\left(ab-(a+b)\right)\left(\frac{2}{\beta}+a\right)\right]+\frac{4a}{\beta}\left(\frac{2}{\beta}+a\right)]\left(ab-\frac{b\left(1+\frac{2}{\beta}\right)}{\left(\frac{2}{\beta}+a\right)}+a\left[\frac{(1-a)}{\left(\frac{2}{\beta}+a\right)}\right]\right)}{(ab+b\left[\frac{b\left(1+\frac{2}{\beta}+2a\right)+a(1-a)}{\left(\frac{2}{\beta}+a\right)}\right])\frac{2}{\beta}+a+[ab-(a+b)]\left(\frac{2}{\beta}+a\right)+a\left(ab-\frac{b\left(1+\frac{2}{\beta}\right)}{\left(\frac{2}{\beta}+a\right)}+a\left[\frac{(1-a)}{\left(\frac{2}{\beta}+a\right)}\right]\right)\left[\left(1+\frac{2}{\beta}+2a\right)-\left(ab-(a+b)\right)\left(\frac{2}{\beta}+a\right)\right]})}{\left(\frac{2}{\beta}\right)+a+[ab-(a+b)]\left(\frac{2}{\beta}+a\right)}$$

$$D = \frac{\frac{2b}{\beta}+a\left(ab-\frac{b\left(1+\frac{2}{\beta}\right)}{\left(\frac{2}{\beta}+a\right)}+a\left[\frac{(1-a)}{\left(\frac{2}{\beta}+a\right)}\right]\right)[\frac{\frac{4}{\beta}\left(\frac{2}{\beta}+a\right)}{\left(\frac{2}{\beta}\right)+a+[ab-(a+b)]\left(\frac{2}{\beta}+a\right)}]}{(ab+b\left[\frac{\left(1+\frac{2}{\beta}+2a\right)}{\left(\frac{2}{\beta}+a\right)}\right]+a\left[\frac{(1-a)}{\left(\frac{2}{\beta}+a\right)}\right])+a\left(ab-\frac{b\left(1+\frac{2}{\beta}\right)}{\left(\frac{2}{\beta}+a\right)}+a\left[\frac{(1-a)}{\left(\frac{2}{\beta}+a\right)}\right]\right)[\frac{\left[\left(1+\frac{2}{\beta}+2a\right)-\left(ab-(a+b)\right)\left(\frac{2}{\beta}+a\right)\right]}{\left(\frac{2}{\beta}\right)+a+[ab-(a+b)]\left(\frac{2}{\beta}+a\right)}}$$

Corollary 2:

As $x(t) \to 0$, we have

$$\eta e^{-\int \mu dt} = lim_{x(t)\to 0}\frac{\left(|1-\frac{ax}{1+\sqrt{1+x^2}}|\right)^{\boldsymbol{A}}\left(|\left(1-\frac{bx}{1+\sqrt{(1+x^2)}}\right)|\right)^{\boldsymbol{B}}\left(|1-\frac{x}{1+\sqrt{1+x^2}}|\right)^{C}}{\left(1+\frac{x}{1+\sqrt{(1+x^2)}}\right)^{D}} = 1$$

Implying

$$\int \mu dt = ln\gamma \qquad (3.13)$$

Corollary 3

As $x(t) \to \infty$, we have

$$\eta e^{-\int \mu dt} = lim_{x(t)\to \infty}\frac{\left(|1-\frac{ax}{1+\sqrt{1+x^2}}|\right)^{\boldsymbol{A}}\left(|\left(1-\frac{bx}{1+\sqrt{(1+x^2)}}\right)|\right)^{\boldsymbol{B}}\left(|1-\frac{x}{1+\sqrt{1+x^2}}|\right)^{C}}{\left(1+\frac{x}{1+\sqrt{(1+x^2)}}\right)^{D}}$$

$$= lim_{x(t)\to\infty} \frac{\left(|1-\frac{a}{\frac{1}{x}+\sqrt{1+\left(\frac{1}{x}\right)^2}}|\right)^{\boldsymbol{A}} \left(|\left(1-\frac{b}{\frac{1}{x}+\sqrt{(1+(\frac{1}{x})^2)}}\right)|\right)^{\boldsymbol{B}} \left(|1-\frac{1}{\frac{1}{x}+\sqrt{1+\left(\frac{1}{x}\right)^2}}|\right)^{C}}{\left(1+\frac{1}{\frac{1}{x}+\sqrt{(1+(\frac{1}{x})^2)}}\right)^{D}}$$

$$= \frac{(|1-a|)^{\boldsymbol{A}}(|(1-b)|)^{\boldsymbol{B}}\left(|1-\frac{1}{\frac{1}{x}+\sqrt{1+\left(\frac{1}{x}\right)^2}}|\right)^{C}}{\left(1+\frac{1}{\frac{1}{x}+\sqrt{(1+(\frac{1}{x})^2)}}\right)^{D}} = 0 \to \int \mu dt \to \infty \qquad (3.14)$$

Numerical experiments

Numerical experiment One

Let $\beta = 2$, then $a = 0.5, b = 0, \eta = 1, \mu(t) = t$, $A = -7.923076923, B = 1.846153846, C = 6, D = 0.4615384615$

We have

$$\sqrt{2}[\frac{(|\left(1-\frac{0.5x}{1+\sqrt{1+x^2}}\right)|)^{-7.923076923}(|\left(1-\frac{x}{1+\sqrt{(1+x^2)}}\right)|)^{6}}{\left(1+\frac{x}{1+\sqrt{(1+x^2)}}\right)^{0.4615384615}}] = t$$

Figure 2 shows a new phenomenon to queuing theorist. The possibility that time will converge to a certain value for sufficiently large number in the time varying $M/D/1$ queuing system. This is for an increasing temporal mean service rate.

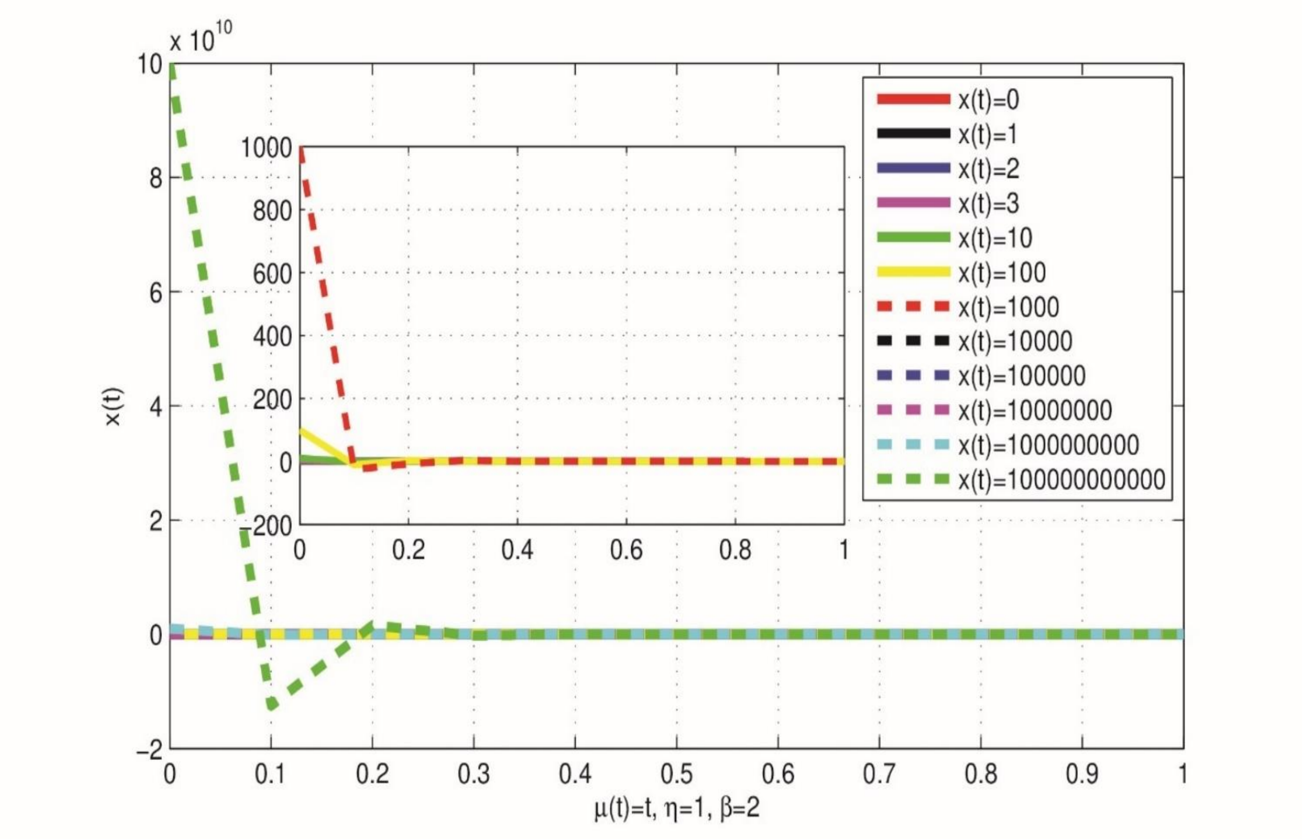

Figure 2.

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This shows that as the time varying $M/D/1$ queuing system's state variable becomes sufficiently large, time vanishes.

Numerical experiment Two

Let $\beta = 2$, then $a = 0.5$, $b = 0$, $\eta = 1, \mu(t) = \frac{1}{t}$

$$A = -7.923076923, B = 1.846153846, C = 6, D = 0.4615384615$$

We have

$$[\frac{(|\left(1-\frac{0.5x}{1+\sqrt{1+x^2}}\right)|)^{7.923076923}\left(\left|\left(1-\frac{x}{1+\sqrt{1+x^2}}\right)\right|\right)^{-6}}{\left(1+\frac{x}{1+\sqrt{(1+x^2)}}\right)^{-0.4615384615}}] = t$$

Figure 3 visualizes a new phenomenon to queuing theorist. The possibility that time will converge to a certain value for sufficiently large number in the time varying $M/D/1$ queuing system. This is for a decreasing temporal mean service rate.

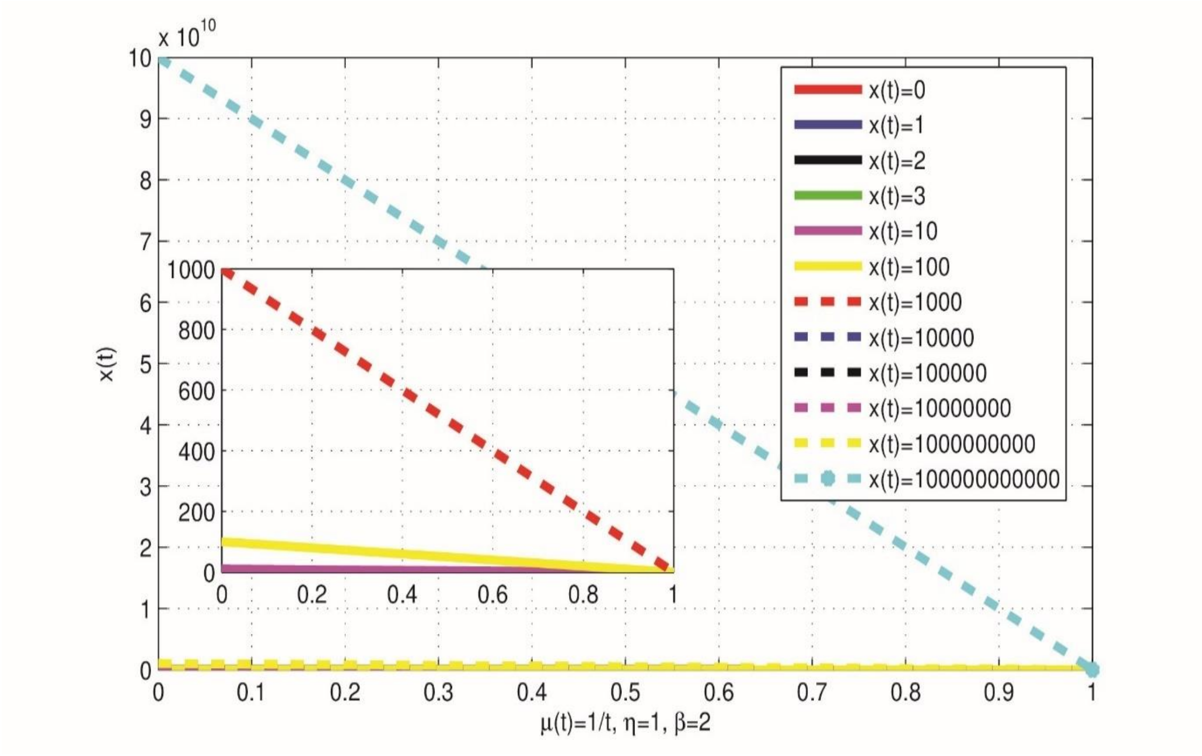

Figure 3.

4. Closing Remarks with Next Phase of Research

In this work, a challenging topic in queueing theory is examined; more precisely, the underlying queue's state variable is determined. The article offers a solution to this issue by utilizing a pointwise stationary fluid flow approximation (PSFFA) technique to formulate the non-stationary $M/D/1$ queueing system. Future work will concentrate on resolving open research issues and investigating applications of non-stationary queues in other scientific areas. The study also examines the effects of time, combined with and queueing parameters on the underlying queue's stability dynamics.

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