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Biased Stochastic Process of Randomly Moving Particles with Constant Average Velocities

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Abstract

In a randomly moving particle swarm with fixed kinetic energy, the particle speeds follow the Maxwell distribution. In a certain period, the moving directions of particles in a sub-particle swarm may aggregate. Thus, the movements of the particles have the characteristics of biased stochastic movement. Regarding the biased particle swarm formed by a series of randomly moving particles (with a uniform average velocity c) with a greater probability of moving in a certain direction and the same probability of moving in other directions, there is a certain group velocity u in this direction, while the diffusion rate in other directions is slower than that of unbiased moving particles with the same average speed c. Moreover, the degree of slowing follows the Lorentz-like factor $\frac{\sqrt{c^2-u^2}}{c}$. In this article, the characteristics of this kind of biased random process are deduced starting from a biased random walk by using probability theory, and the expression of the Ito equation is provided. This article is expected to provide a reference to understand the nature of the special relativity effect.

Keywords: Biased Stochastic Process; Randomly Moving Particles; Special Relativity Effect; Lorentz-like factor

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Introduction

Most random phenomena in nature are biased,[1] and so is random motion. Regarding a random moving particle swarm, the motion characteristics of its sub-particle swarm are usually biased. Among these biased random motions, several special motion processes deserve our special attention. These special motions include particles that have a greater probability of moving in a certain direction and the same probability of moving in other directions;[2] particles that have greater probabilities of moving in the directions toward a point and the same probability of moving in other directions;[3] and biased random processes of particles that can produce rotation.[4] The study of this kind of phenomenon generally starts with the simplest biased random walk.

Biased-randomized processes are currently popular research.[5] Different biased characteristics will produce different forms of random processes. For example:

$$\mathrm{d}x(t) = vS_t\mathrm{d}t + \sigma S_t\mathrm{d}w(t), \tag{1}$$

where $w(t)$ is a Brownian motion process. In Eq. 1, when the drift (v) and diffusion (σ) terms grow synchronously according to S_t, geometric Brownian motion will be produced. There will be fractional Brownian motion when particles follow the motion law in Eq. 2.

$$\mathrm{d}x(t) = v\mathrm{d}t + \sigma\sqrt{\frac{s^{2h} + t^{2h} - |t-s|^{2h}}{2}}\mathrm{d}w(t), \tag{2}$$

where h is the Hurst index; and s and t denote the different times. In addition, there are another simpler cases in which particles with the same jumping speed have a greater probability of moving in a certain direction and the same probability of moving in other directions. This is a common phenomenon in physics. In a system with fixed kinetic energy, we believe that the distribution of the particle speed is not easy to change, and the directions of motions can easily change. In the system in which the motion law of particles follows Eq.

3:

$$\mathrm{d}x(t) = v\mathrm{d}t + \sigma\mathrm{d}w(t), \tag{3}$$

the kinetic energy of the system will increase with an increasing drift term σ. It is obvious that such a system does not reflect real scenarios. In contrast, when the drift term v increases, it may be a better solution to reduce the diffusion term σ accordingly. However, such concerns of this situation are often ignored. There are some quantitative relationships when these concerns are denoted with biased stochastic processes: the diffusion rate of biased stochastic processes of particles with determined jump speed is slower than that of unbiased random walk with the same jump speed. The degree of slowing down is determined by the Lorentz-like factor (it is a function of the particle swarm velocity in this direction, and the group velocity is a function of the biased random probability of particles in this direction). This kind of problem is similar to the problem that the particle speed or time slows down in a moving frame in special relativity. Although these problems have been ignored for a long time, this is a very important issue, that can provide an understanding of the essence behind the special relativity effect. Some researchers[6–9] demonstrated the diffusion process (random walk) with a special relativistic effect. However, it is the diffusion or random walk of particles when their speed is affected by the special relativity effect. The case involved here is that the particle velocity is determined, and the particle velocity or diffusion in the biased random particle swarm formed by them is restricted by the Lorentz factor. They are not the same kind of problem.

Our previous work[1] proved the special-relativity-like effect of this kind of biased random motion based on the uniform speed (the speed of each particle is c) of randomly moving particles. Starting from a random walk, this article proved that this kind of biased random motion still has a special-relativity-like effect based on particles whose speeds follow a Maxwell distribution and whose average speeds are equal. Furthermore, the corresponding

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Ito process equation is also given, and the motion process of related examples is simulated. This article will provide clues for further understanding the relationship between stochastic processes and the special relativity effect.

Results and Discussion

Biased Random Walk

There are different situations for biased random walk. We will only discuss that the moving probability of particles in one of the 6 directions in 3-dimensional space is p and in the other 5 directions is $\frac{1-p}{5}$ (where $\frac{1}{6} < p < 1$. Another interesting situation occurs when $0 < p < \frac{1}{6}$; however, this will not be studied here). If the jumping speed of particles is c, then the group velocity of all particles is

$$u = \frac{6p-1}{5}c. \tag{4}$$

This is equivalent to the form $\{X(t), Y(t), Z(t), t \in T\}$, where $X(t), Y(t)$ and $Z(t)$ are 1-dimensional random walks with a jump speed of $\frac{c}{\sqrt{3}}$. We established a 3-dimensional rectangular coordinate system and set its (1, 1, 1) direction parallel to the direction of group velocity u. Thus, the 1-dimensional random walk model on the x-axis of the three equivalent coordinate axes is

$$P_{x,\mathrm{bia}} = \begin{cases} \frac{1+u/c}{2}, & n > 0, \\ \frac{1-u/c}{2}, & n < 0, \\ 0, & \text{otherwise}, \end{cases} \tag{5}$$

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where $n \in \mathbb{N}$. The corresponding unbiased 1-dimensional random walk model on the x-axis of three equivalent coordinate axes is

$$P_{x,\text{unbia}} = \begin{cases} \dfrac{1}{2}, & n > 0, \\ \dfrac{1}{2}, & n < 0, \\ 0, & \text{otherwise}, \end{cases} \tag{6}$$

where $n \in \mathbb{N}$. The relative standard deviation of each axis is

$$\frac{\sigma_{\text{bia}}}{\sigma_{\text{unbia}}} = -\frac{12}{25}(p-1)(3p+2). \tag{7}$$

The norm of the 3-dimensional random walk vector formed by $X(t), Y(t)$ and $Z(t)$ follows the discrete Maxwell distribution. The average speed of this 3-dimensional vector is proportional to the standard deviation of its component vectors on three coordinate axes.[2] Therefore, the moving speed of the biased 3-dimensional random walk is $\dfrac{\sqrt{c^2-u^2}}{c}$ of that in the unbiased case. The biased case is obviously a process of slowing down. However, regarding the continuous time stochastic process, it is not easy to prove it in this way. We can transpose to the following strategy.

Motion Law of the Stochastic Process of Randomly Moving Particles Following the Maxwell Distribution

When the speeds of particles in particle swarm $\mathcal{A}$ follow Maxwell distribution with an average value of c, the expression of the scale parameter of this Maxwell distribution is $\dfrac{1}{2}\sqrt{\dfrac{\pi}{2}}c$. We can divide the particles in $\mathcal{A}$ into sub-particle groups $\mathcal{A}_i$ with velocity vectors terminating on a series of spheres with radius r_i $(i = 1, 2, 3, \cdots)$ according to the speeds, and the velocity vector terminals of particles in $\mathcal{A}_i$ are uniformly distributed on the sphere with radius r_i. The numbers of particles in particle swarm $\mathcal{A}$ on these spheres follow a Maxwell distribution according to r (when the scale parameter is $\frac{1}{2}\sqrt{\frac{\pi}{2}}\ c$, $\overline{r} = c$). Therefore, the case[1] in which the particle speeds are strictly c can be regarded as a special case when $r_i = c$. The motion characteristics of a series of particles with velocity vector terminals on the sphere with radius r_i $(i = 1, 2, 3, \cdots)$ are gathered together, which is the motion characteristics of particles whose speeds follow a Maxwell distribution. The numbers of particles in $\mathcal{A}$ and $\mathcal{A}_i$ are large and uniformly distributed. When a subgroup $\mathcal{A}$ of the parent particle swarm following a Maxwell distribution moving along the z-axis (the reference frame is based on the parent particle swarm) with an average speed of u, it is equivalent to each sub-particle group $\mathcal{A}_i$ $(i = 1, 2, 3, \cdots)$ moves along the z-axis with the speed $\frac{r_i\, u}{c}$.

If r follows Maxwell distribution with scale parameter λ, ar follows Maxwell distribution with scale parameter $a\lambda$, and the mean value of the Maxwell distribution is linear relationship with the scale parameter. Then, the speed of all particles in particle swarm $\mathcal{A}$ follow a Maxwell distribution with the mean value u on the z-axis, and the standard deviations σ_i $(i = 1, 2, 3, \cdots)$ of the mixed distributions of all sphere layers also follow a Maxwell distribution. For the case on the x- or y-axis, we can regard the particles in $\mathcal{A}_i$ as a particle swarm possessing a mixed distribution with weight $w = \frac{c+u}{2c}$. According to the method similar to my previous work,[1] it can be proved that the ratio of the standard deviation of the mixed distribution on the x- or y-axis to that of the unbiased moving particle on that axis satisfies the relationship $\frac{\sqrt{c^2 - u^2}}{c}$, and the ratio of the standard deviation on the z-axis also satisfies this relationship (see Part 2 of the Supplementary Information for the detailed Mathematica code).

Ito Equation of Biased Stochastic Processes

The above results are written in the form of the Ito equation

$$\mathrm{d}x(t) = v\mathrm{d}t + \sigma\sqrt{1 - v^2}\, \mathrm{d}w(t), \tag{8}$$

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where v is the drift speed; $w(t)$ is a Brownian motion process; and σ is its standard deviation. Then, the slice distribution of Eq. 8 is

$$\frac{\mathrm{e}^{\frac{(x-tv)^2}{2\sigma^2 t(v^2-1)}}}{\sqrt{2\pi}\sigma\sqrt{t(1-v^2)}}. \tag{9}$$

Moreover, the forward equation of Eq. 8 is

$$\frac{\partial p(x,t)}{\partial t} = \frac{\sigma^2(1-v^2)}{2}\frac{\partial^2 p(x,t)}{\partial x^2} - v\frac{\partial p(x,t)}{\partial x}, \tag{10}$$

where $p(x,t)$ is the probability of a particle at position x and time t. This is also a drift-diffusion (or advection-diffusion) equation. Only the diffusion coefficient of this equation is limited by the advection term.

This result (3-dimensional biased Brownian motion) can be simulated by 3 1-dimensional Brownian motions $\{X(t), Y(t), Z(t), t \in T\}$. Here, the trajectories of 500 particles randomly moving for 50 s with a time step of 0.1 s are presented in Fig. 1. There is no significant difference from the situation that is unbiased and not affected by the special-relativity-like effect.

Conclusions

Regarding a particle swarm formed by a series of randomly moving particles (with uniform average speed c) with a greater probability of moving in a certain direction and the same probability of moving in other directions, there is a certain group velocity u in this direction, while the diffusion rate in other directions is slower than that of unbiased moving particles with the same average velocity c. Moreover, the degree of slowing follows the Lorentz-like factor. In this article, the relationship between the special-relativity-like effect and the biased stochastic process of randomly moving particles is explained in detail, and the Ito equation

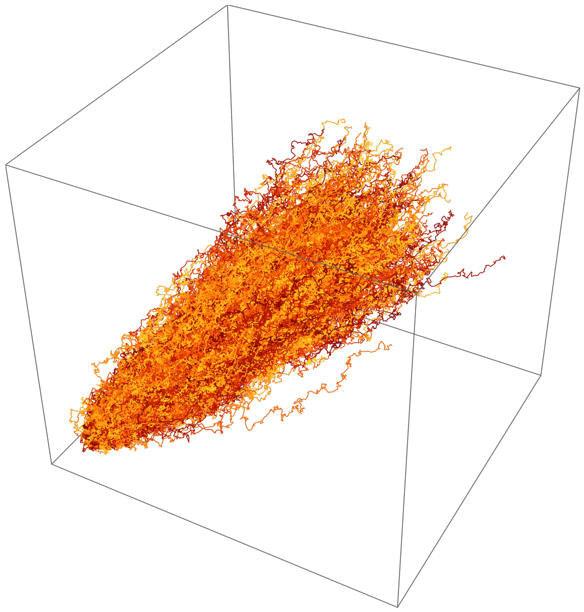

Figure 1: Simulation results of the trajectories of 500 particles randomly moving for 50 s ($v = 0.6$ and $\sigma = 1$).

of such a process is given.

Acknowledgement

The authors are thankful for Wolfram Inc..

Supporting Information Available

Mathematica code for necessary calculation process and graphics. The following file is available free of charge.

- Supporting Information

References

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Appendix:

In[ ]:=

Supplementary Information
(Mathematica v13.1.0 code of TraditionalForm)

In[ ]:=

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NOTE:
1. The "Euclid Math One" regular and bold fonts are needed to display the contents correctly in this Notebook.
2. If there is no special case, the Mathematica code starts with gray "*In*[∘]:=" and is bold by default according to Mathematica's rules.

Part 1. Necessary Calculation Processes

In[∘]:= $\mathbf{Simplify}\left[\dfrac{\mathbf{Variance}\left[\mathbf{RandomWalkProcess}\left[\frac{1+\frac{6p-1}{5}}{2}, \frac{1-\frac{6p-1}{5}}{2}\right][t]\right]}{\mathbf{Variance}\left[\mathbf{RandomWalkProcess}\left[\frac{1}{2}\right][t]\right]}\right]$

Out[∘]= $-\frac{12}{25}\left(3p^2-p-2\right)$

In[∘]:= $\mathbf{Simplify}\left[\left(\dfrac{\sqrt{c^2-\left(\frac{6p-1}{5}c\right)^2}}{c}\right)^2, \mathbf{Assumptions} -> c>0\right]$

Out[∘]= $-\frac{12}{25}\left(3p^2-p-2\right)$

Part 2. Process of Obtaining the Lorentz Factor for Randomly Moving Particles with Speed Following Maxwell Distribution

In[∘]:= **PDF[TransformedDistribution[*a r*, {*r* ≈ MaxwellDistribution[*λ*]}], *x*]**

Out[∘]= $\begin{cases} \dfrac{\sqrt{\frac{2}{\pi}}\, x^2\, e^{-\frac{x^2}{2a^2\lambda^2}}}{a^3\lambda^3} & x>0 \\ 0 & \text{True} \end{cases}$

In[∘]:= **PDF[MaxwellDistribution[*a λ*], *x*]**

Out[∘]= $\begin{cases} \dfrac{\sqrt{\frac{2}{\pi}}\, x^2\, e^{-\frac{x^2}{2a^2\lambda^2}}}{a^3\lambda^3} & x>0 \\ 0 & \text{True} \end{cases}$

In[∘]:= **Mean[MaxwellDistribution[*λ*]]**

Out[∘]= $2\sqrt{\frac{2}{\pi}}\,\lambda$

For the case on the x- or y-axis, we can regard the particles in $\mathcal{A}_i$ as a particle swarm possessing a mixed distribution with weight $w=\dfrac{c+u}{2c}$ (this code takes approximately 36 seconds).

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In[∘]:= $\mathcal{D}$ = TransformedDistribution[r Cos[θ] Sin[ArcCos[η]], {$\theta \approx$ UniformDistribution[{−π, π}],
$\eta \approx$ UniformDistribution[{−1, 1}], $r \approx$ MaxwellDistribution[$\frac{1}{2}\sqrt{\frac{\pi}{2}}\, c$]}];
$\mathcal{D}_1$ = TransformedDistribution[r Cos[θ] Sin[ArcCos[η]], {$\theta \approx$ UniformDistribution[{−π, π}],
$\eta \approx$ UniformDistribution[{$\frac{u}{c}$, 1}], $r \approx$ MaxwellDistribution[$\frac{1}{2}\sqrt{\frac{\pi}{2}}\, c$]}];
$\mathcal{D}_2$ = TransformedDistribution[r Cos[θ] Sin[ArcCos[η]], {$\theta \approx$ UniformDistribution[{−π, π}],
$\eta \approx$ UniformDistribution[{−1, $\frac{u}{c}$}], $r \approx$ MaxwellDistribution[$\frac{1}{2}\sqrt{\frac{\pi}{2}}\, c$]}];
$w = \frac{c+u}{2\,c}$;
$\mathcal{D}$12 = MixtureDistribution[{w, 1 − w}, {$\mathcal{D}_1$, $\mathcal{D}_2$}];
σ_u = Simplify[StandardDeviation[$\mathcal{D}$12], Assumptions → 0 < u < c]

Out[∘]= $$\frac{1}{2}\sqrt{\frac{\pi}{2}}\sqrt{c^2-u^2}$$

In[∘]:= Simplify[σ_u / StandardDeviation[$\mathcal{D}$], Assumptions → 0 < u < c]

Out[∘]= $$\frac{\sqrt{c^2-u^2}}{c}$$

When c = 10 and u = 6, the distribution of $\mathcal{D}$12 on the x- or y-axes is like this (this code takes approximately 18 seconds):

```
In[∘]:= c = 10;
u = 6;
data = RandomVariate[𝒟12, 30 000 000];
𝒟0 = SmoothKernelDistribution[data, {"Adaptive", Automatic, Automatic}];
s2 = Plot[PDF[𝒟0, x], {x, −20, 20}, PlotRange → {{−21, 21}, {0, 0.091}},
    PlotStyle → {Blue, Thickness → 0.004}, AxesLabel → {HoldForm[Speed], HoldForm[Probability Density]},
    AxesStyle → Directive[Black, Thickness → 0.0018], TicksStyle → Directive[Black, Thickness → 0.0014],
    LabelStyle → Directive[Black, FontFamily → "Arial", FontSize → 15]]
```

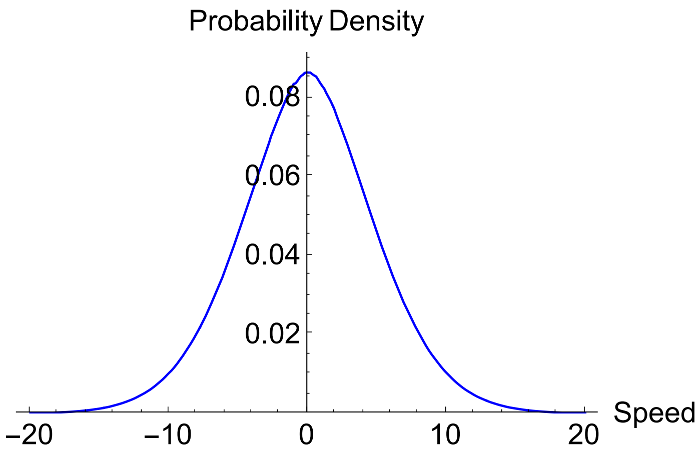
Out[∘]=


Figure S1 | Simulated probability density of the mixed distribution $\mathcal{D}$12 when $\bar{r} = c$ = 10 and u = 6.

For the case on the z-axis:

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In[∘]:= $w = \frac{c+u}{2c}$;

$\mathcal{D}_3$ = TruncatedDistribution$\left[\left\{r\,\frac{u}{c}, r\right\}\right.$, UniformDistribution[{−r, r}]$\left.\right]$;

$\mathcal{D}_4$ = TruncatedDistribution$\left[\left\{-r, r\,\frac{u}{c}\right\}\right.$, UniformDistribution[{−r, r}]$\left.\right]$;

$\mathcal{D}34$ = MixtureDistribution[{w, 1 − w}, {$\mathcal{D}_3$, $\mathcal{D}_4$}];

Simplify[StandardDeviation[$\mathcal{D}34$], Assumptions → $r > r\,\frac{u}{c} > 0$]

Out[∘]= $\frac{r\sqrt{1-\frac{u^2}{c^2}}}{\sqrt{3}}$

In[∘]:= StandardDeviation[UniformDistribution[{−r, r}]]

Out[∘]= $\frac{r}{\sqrt{3}}$

In[∘]:= $\mathcal{D}_{mz}$ = TransformedDistribution$\left[\frac{r\sqrt{1-\frac{u^2}{c^2}}}{\sqrt{3}}, r \approx \text{MaxwellDistribution}\left[\frac{1}{2}\sqrt{\frac{\pi}{2}}\,c\right]\right]$;

$\mathcal{D}_z$ = TransformedDistribution$\left[\frac{r}{\sqrt{3}}, r \approx \text{MaxwellDistribution}\left[\frac{1}{2}\sqrt{\frac{\pi}{2}}\,c\right]\right]$;

Simplify[Mean[$\mathcal{D}_{mz}$] / Mean[$\mathcal{D}_z$], Assumptions → c > 0]

Out[∘]= $\sqrt{1-\frac{u^2}{c^2}}$

Part 3. Calculation Process about Ito Equation

In[∘]:= Clear[v]

proc = ItoProcess$\left[d\,x[t] == v\,d\,t + \sqrt{1-v^2}\;\sigma\,d\,w[t], x[t], \{x, 0\}, t, w \approx \text{WienerProcess}[]\right]$;

PDF[proc[t], x] // Simplify

PDF$\left[\text{WienerProcess}\left[v, \sqrt{1-v^2}\;\sigma\right][t], x\right]$ // Simplify

SliceDistribution$\left[\text{WienerProcess}\left[v, \sqrt{1-v^2}\;\sigma\right], t\right]$

Out[∘]= $\frac{e^{\frac{(x-t\,v)^2}{2\,\sigma^2\,t\,(v^2-1)}}}{\sqrt{2\,\pi}\;\sqrt{\sigma^2\,(-t)\,(v^2-1)}}$

Out[∘]= $\frac{e^{\frac{(x-t\,v)^2}{2\,\sigma^2\,t\,(v^2-1)}}}{\sqrt{2\,\pi}\;\sigma\;\sqrt{t}\;\sqrt{1-v^2}}$

Out[∘]= NormalDistribution$\left[t\,v, \sigma\,\sqrt{t}\;\sqrt{1-v^2}\right]$

```
In[∘]:= proc = ItoProcess[dx[t] == v dt + Sqrt[1 - v^2] σ dw[t], x[t], {x, 0}, t, w ≈ WienerProcess[]];
        proc["KolmogorovForwardEquation"] // TraditionalForm
```

Out[∘]//TraditionalForm=

$$p^{(0,1)}(x, t) = \frac{1}{2}\,\frac{\partial^2\left(\left(1 - v^2\right)\sigma^2\, p(x, t)\right)}{\partial x\,\partial x} - \frac{\partial\left(v\, p(x, t)\right)}{\partial x}$$

Part 4. Figures Used in the Main Text

NOTE: To run these codes correctly, the contents in "**MyDirection = ****" in the next cell should be modified. It is similar to **MyDirection** = "**/Users/yourdirection/**". Then, run it (Shift+Enter) beforehand.

```
In[∘]:= MyDirection = "/Users/gotall/Library/Mobile
            Documents/com~apple~CloudDocs/SPaper/Normal Paper/conti/zhengshi/LaTeX/";
        Protect[MyDirection];
        Off[General::wrsym];
```

```
## ## ## ## ## ## ## ## ## ## ## Figure1 ## ## ## ## ## ## ## ## ## ## ## ## ## ## ##
```

```
In[∘]:= v = 0.6;
        σ = 1;
        proc = WienerProcess[v, Sqrt[1 - v^2] σ];
        SeedRandom[123];
        sample = Table[RandomFunction[proc, {0, 50, 0.1}, 3]["ValueList"]ᵀ, {500}];
        figure1 = Graphics3D@Table[{ColorData["SolarColors"][RandomReal[]], Line@sample[[i]]}, {i, 500}];
        Export[MyDirection <> "figure1.png", figure1, Background → None, ImageResolution → 700];
```

```
## ## ## ## ## ## ## ## ## ## ## Figure1 ## ## ## ## ## ## ## ## ## ## ## ## ## ## ##
```