Brief Report

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Universal Inverse Radon Transforms: Complexity of Radon Transforms and the Hybrid Functions

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Posted Date: 11 June 2025

doi: 10.20944/preprints202506.0865.v1

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Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 11 June 2025 doi:10.20944/preprints202506.0865.v1

Article

Universal inverse Radon transforms: Complexity of Radon Transforms and the Hybrid Functions

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Abstract: For the reconstruction problem, the universal representation of inverse Radon transforms may imply the needed complexity of the direct Radon transforms which leads to the additional contributions. Meanwhile, in the standard theory of generalized functions, if the origin function which generates the Radon image is a pure-real function, as a rule the complexity of Radon transform becomes in question. In the letter, we demonstrate that the Fourier method together with the hybrid (Wigner-like) function ensure naturally the corresponding complexity. In its turn, this complexity provides not only the additional contribution to the inverse Radon transforms, but also it makes an essential impact on the reconstruction and optimization procedures within the frame of the incorrect problems. The presented method can be effectively used for the practical tasks of reconstruction problems.

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1. Introduction

In the modern world, computed tomography technologies (CT-technologies) influence on the many different fields basically owing to a possibility to investigate the internal composite structure of a object without cutting and breaking. From the mathematical viewpoint, CT-technologies are closely associated with the application of both the direct and inverse Radon transforms [1]. It is well-known that due to the inverse Radon transforms one can visualize the internal structure of the objects under investigation. The quality of visualization strongly depends on the inversion procedure of Radon transforms that, as known, is ill-posed and demands the corresponding regularization, see for example [2].

In [3,4], the universal (or unified) representation of inverse Radon transform has been proposed and studied for any dimension of space. Thanks for the use of generalized function theory, the mentioned universal representation involves the new contribution (in comparison with the standard methods based on the Courant-Hilbert identities [5]) which definitely effects on the reconstruction and extends the Tikhonov-like regularization (AC-regularization) used for the solutions of the incorrect task classes. In particular, within the reconstruction procedure, the additional new contribution in the unified inverse Radon transforms is given by the integration with the complex measure that should be compensated, generally speaking, by the complexity of direct Radon image.

In the present letter, we propose the most simplest method how to generate the needed complexity on the intermediate stage of calculations. Our method is an analogue of the Fourier series expansion applied within the wide-spread discrete slice approximation [1]. As demonstrated, it is based on the use of the hybrid (Wigner-like) functions which are, by definition, the functions of both spacial and momentum coordinates. In the context of the inverse Radon transforms and, then, of the reconstruction procedure, this is rather a trivial trick which, however, has been never used up to now.

2. Two-Dimensional Space Versus Three-Dimensional Space

In [4], the universal and unified representation for the inversion of Radon transforms has been presented where the Courant-Hilbert identities [5] have not been applied. We remind that the Courant-Hilbert identities are based on the use of the Green formulae the different forms of which depend on

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either even or odd dimension of space [6,7]. For the practical uses (for example, in medicine), the most needed cases correspond to $\mathbb{R}^3$ and $\mathbb{R}^2$ spaces.

However, as dictated by the application of CT-technology, the different algorithms have been designed for the two-dimensional reconstructions which work with the corresponding transverse projections of the three-dimensional object.

In other words, if the object, which is needed to be reconstructed, is described by the three-dimensional origin outset function $f(x_1, x_2, x_3)$, we perform the finite number of transverse sections, say, regarding x_3-axis:

$$f(x_1, x_2, x_3) \Longrightarrow \Big\{ f(x_1, x_2, x_{3_1}),\, f(x_1, x_2, x_{3_2}),\, ...,\, f(x_1, x_2, x_{3_n}) \Big\}, \tag{1}$$

where the number of sections should be defined by the experiment. The segmentation given by (1) is known as *a discrete slice method* known in CT-technology. In this case, we always deal with the direct and inverse Radon transforms determined on the two-dimensional space formed by (x_1, x_2) while the third discrete coordinate plays a role of the external parameter [1].

3. The Hybrid Wigner-Like Function and the Complex Radon Transforms

As explained in a series of papers [2–4], the regularized inverse Radon operator involves two contributions. If one of them is related to the real integration measure, another is associated with the imaginary integration measure (that is a result of the Cauchy theorem).

According the scenario described in [3] the term of the inverse Radon operator with the imaginary measure is extremely important for the opimization procedure which works with the condition inspired by the corresponding norms. On the other hand, in QFT, even the direct Radon transform, which is linked to the corresponding transverse-momentum dependent parton distributions, can possess the imaginary part owing to the interactions in the correlators [2]. In this section, excepting QFT-models, we study the natural source of complexity which appears in the Radon transforms.

We begin with $\mathbb{R}^3$-space where the coordinate system has been defined. Let $f(\vec{\mathbf{x}})$, with $\vec{\mathbf{x}} \in \mathbb{R}^3$, be a origin (outset) function which is usually well-localized. In the reconstruction problem, the form of $f(\vec{\mathbf{x}})$ has to be restored owing to the inverse Radon transform.

Since the integral representation of universal inverse Radon transform involves the additional term which should be related to the complex integrand given by the direct Radon transform, we first introduce the hybrid (Wigner-like) function In the particular case of QFT, the Wigner function corresponds to the quasi-probability distribution which is actually a real function only. associated with the origin function. Namely, we implement the direct Fourier transform regarding the only one-of-three coordinate applied to $f(\vec{\mathbf{x}})$. We have

$$F(x_1, x_2; k) = \int_{-\infty}^{+\infty} (dx_3)\, e^{-ikx_3}\, f(x_1, x_2, x_3), \tag{2}$$

where the corresponding and irrelevant normalization has been absorbed into the integral measure. In (2), it is clear that $f(x_1, x_2, x_3) \in \Re e$ by definition, while $F(x_1, x_2; k) \in \mathbb{C}$ already. Hence, in the discrete slice method, see (1), we have the following

$$\widetilde{F}(x_1, x_2; k) = \sum_n e^{-ikx_{3_n}}\, f(x_1, x_2, x_{3_n}), \quad \widetilde{F}(x_1, x_2; k) \in \mathbb{C}, \tag{3}$$

which is nothing but the Fourier series expansion.

Then, we calculate the Radon image of the Fourier F-functions as

$$\mathcal{R}[F;\widetilde{F}](\tau, \varphi; k) = \int_{-\infty}^{+\infty} d^2\vec{\mathbf{x}} \begin{bmatrix} F(x_1, x_2; k) \\ \widetilde{F}(x_1, x_2; k) \end{bmatrix} \delta\big(\tau - \langle \vec{\mathbf{n}}_\varphi, \vec{\mathbf{x}} \rangle\big), \tag{4}$$

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where $\mathcal{R}[F;\widetilde{F}](\tau,\varphi;k) \in \mathbb{C}$ and the unit vector is given by $\vec{\mathbf{n}}_\varphi = (\cos\varphi, \sin\varphi)$. Notice that in (4) the integration measure corresponds to the two-dimensional space while the momentum k plays a role of the external parameter.

Hence, the inverse Radon transform in the universal form is given by

$$\boxed{F_\epsilon(x_1,x_2;k) = F_S(x_1,x_2;k) + F_A(x_1,x_2;k),} \tag{5}$$

where

$$F_S(x_1,x_2;k) = -\int_{-\infty}^{+\infty}(d\eta)\,\frac{\mathcal{P}}{\eta^2}\int d\mu(\varphi)\,\mathcal{R}[F](\eta + \langle\vec{\mathbf{n}}_\varphi,\vec{\mathbf{x}}\rangle,\varphi;k) \tag{6}$$

and

$$F_A(x_1,x_2;k) = -i\pi\int d\mu(\varphi)\,\frac{\partial}{\partial\eta}\,\mathcal{R}[F](\eta + \langle\vec{\mathbf{n}}_\varphi,\vec{\mathbf{x}}\rangle,\varphi;k)\Big|_{\eta=0}. \tag{7}$$

The similar expressions can be written for $\widetilde{F}$-function due to the trivial replacement.

In practical applications, the value of $\mathcal{R}[F]$ is usually known from the experiment or the observation. Restoring the function F or $\widetilde{F}$ from $\mathcal{R}[F;\widetilde{F}]$, one can reconstruct the function f making use of the inverse Fourier transform. Therefore, for the reconstruction, the ultimate scheme can expressed as

$$\boxed{\mathcal{R}[F;\widetilde{F}](\tau,\varphi;k) \stackrel{\mathcal{R}^{-1}}{\longrightarrow} \{F(x_1,x_2;k),\widetilde{F}(x_1,x_2;k)\} \stackrel{\mathcal{F}^{-1}}{\longrightarrow} f(x_1,x_2,x_3).} \tag{8}$$

This scheme illustrates the principle result of the paper which shows us that the necessary step to obtain the complexity of direct Radon transform is given by the introduction and, then, by the transition to the hybrid functions $F(x_1,x_2;k)$ or $\widetilde{F}(x_1,x_2;k)$. Notice that the use of Fourier images in the forms of the hybrid F-functions, see (2) and (3), gives a unique possibility to deal with the complex direct Radon transforms. We remind that F_S and $\widetilde{F}_S$ are determined by $\Re\mathrm{e}\{\mathcal{R}[F;\widetilde{F}]\}$ and F_A and $\widetilde{F}_A$ are related to $\Im\mathrm{m}\{\mathcal{R}[F;\widetilde{F}]\}$ garanteeing the new important contribution within the reconstruction problem.

4. Conclusions

In the letter, we have described the method where the complexity of the Radon transform appears naturally. In the connection with the reconstruction problem, the complex Radon transforms are started to be extremely important ones. This is because of the additional contribution existence that plays a crucial role in the improved visualization procedure.

The method is practically identical to the Fourier series expansion which should be applied within the frame of the discrete two-dimensional slice approximation [1]. In its turn, the discrete two-dimensional slice approximation is a wide-spread approximation in the medical CT-technologies. The proposed method is based on the use of the hybrid (Wigner-like) functions depending simultaneously on the spacial and momentum coordinates. This trick has been first applied in the context of the inversion procedure of Radon transforms.

Data Availability Statement: This manuscript has no associated data or the data will not be deposited.

Acknowledgments: We thank A.I. Anikina, V.A. Osipov and O.I. Streltsova for useful and illuminating discussions. The special thanks go to the colleagues from the South China Normal University (Guangzhou) for the useful discussions and a very warm hospitality.

Conflicts of Interest: The author declares no conflict of interest.

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