Article

Not peer-reviewed version

Qualitative Properties of the Solutions to the Lane-Emden Equation in the Cylindrical Setup

Arsen Palestini * and Simone Recchi

Posted Date: 29 December 2023

doi: 10.20944/preprints202312.2238.v1

Keywords: Lane-Emden equation; polytrope; ODEs

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 29 December 2023 doi:10.20944/preprints202312.2238.v1

Article

Qualitative Properties of the Solutions to the Lane-Emden Equation in the Cylindrical Setup

Arsen Palestini [1,†] and Simone Recchi [2,†]

1 MEMOTEF, Sapienza University of Rome, Via del Castro Laurenziano 9, 00161 Rome, Italy; Arsen.Palestini@uniroma1.it
2 Urbangasse 6/3/41, 1170 Wien, Austria; recchisimo@outlook.it
* Correspondence: Arsen.Palestini@uniroma1.it
† These authors contributed equally to this work.

Abstract: Some qualitative results are provided on the cylindrical version of the Lane-Emden equation. Although the explicit form of the solution is not exactly known, we establish some properties on the critical points of the solution in some specific cases and we propose a technique to evaluate the distance between a pair of solutions in small intervals.

Keywords: Lane-Emden equation; polytrope; ODEs

MSC: 45J05; 44A45; 76E20; 85A99

1. Introduction

One of the most fascinating open problems in Applied Mathematics is the *Lane-Emden equation* together with its variation *Emden-Fowler equation*, initially proposed in 1870 by Jonathan Homer Lane, and subsequently extended by Robert Emden [1] in 1907, who aimed to model the dynamic behaviour of a nonrotating fluid subject to internal pressure and self-gravity. In order to briefly introduce the physical setting, the Lane-Emden equation originates from the combination of Poisson's equation and a generic polytropic state equation $P = K_N \rho^\gamma$, where P and ρ respectively are the pressure and the density of a fluid, K_N is a positive constant and $\gamma = 1 + \frac{1}{N}$ is the ratio of specific heats (see [2,3] for more details). After some simple manipulations, the Lane-Emden equation is derived having $\theta(x) = \left(\frac{\rho(x)}{\rho_c}\right)^{1/N}(x)$, where ρ_c is the density at the axis.

The Lane-Emden equation encountered a wide success especially in the 30s both in Physics, where Sir Ralph Howard Fowler [4,5] found and generalized further results and gave birth to Emden-Fowler equation, and in Astrophysics, where Chandrasekhar established the related spherical solutions in [6], whose first edition was published in 1939 and then was subsequently reprinted in 1967. Furthermore, Chandrasekhar and Fermi applied the Lane-Emden equation to isothermal filaments [7] some years after.

Successively, many contributions have been published on the equation, its several modified versions, and its applications. Christodoulou and Kazanas [8] derived exact asymmetric solutions of the Lane-Emden equation under rotation. A major result had already been provided by Jeremiah Paul Ostriker [9] in 1964: he was able to determine the solutions to the equation in closed form for cylindrical polytropes for parameter $N = 0$ (i.e. liquid cylinders), $N = 1$ and $N = \infty$ (i.e. cylinders with an isothermal perfect gas). In the astrophysical literature, a solution to the Lane-Emden equation is often called a *polytrope*. We are going to use such a denomination as well along the paper, keeping in mind that astrophysical models are out of our present scope. For those who are willing to develop an extensive knowledge of polytropes, the main textbook was published by Horedt [10] in 2004.

In recent years, the Lane-Emden equation has been widely studied in several versions, although it can be solved in closed form only in a few cases. An approach based on operational calculus, initially introduced by Adomian [11], has been outlined by Bengochea *et al.* in some recent works ([12–14]). In

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 29 December 2023 doi:10.20944/preprints202312.2238.v1

particular, in [12] a procedure is derived based on a linear operator acting on the set of all formal series which turns out to be helpful to solve several kinds of differential equations with variable coefficients, fractional differential equations and difference equations as well. Such approach is adopted in [13] to determine an algebraic solution to a specific version of the Lane-Emden equation. More recently, a numerical approximation algorithm has been proposed by [15].

Here is a basic summary of the present work:

- We reconstruct the extended derivations of the basic Lane-Emden equations in the basic scenarios;
- we outline the current state-of-the-art: explicit solutions, solution methods, cases in which the Lane-Emden equation is still unsolved;
- we identify a sequence of qualitative properties of the solutions in the cylindrical scenario. In particular, two distinct analyses are carried out depending on whether the critical exponent M is either odd or even;
- finally, we expose a relation which may be helpful to evaluate the distance between a pair of solutions in a small interval.

The remainder of this paper is as follows: in Section 2 we introduce the Lane-Emden differential equation together with some of its variations, whereas in Section 3 we present an overview of some known solutions, emphasizing the cases $N = 0$ and $N = 1$. Our main analytical results are collected in Section 4, where some qualitative properties are stated and demonstrated. Finally, our concluding remarks and some ideas on future developments can be read in Section 5.

2. The standard Lane-Emden equation

Firstly, we introduce the Lane-Emden equation in its well-known form:

$$\theta''(x) + k\frac{\theta'}{x} + \theta^N(x) = 0. \tag{1}$$

Based on the value of k, either we have the cylindrical setting if $k = 1$:

$$\theta''(x) + \frac{\theta'}{x} + \theta^N(x) = 0 \tag{2}$$

or we have the spherical setting if $k = 2$:

$$\theta''(x) + 2\frac{\theta'}{x} + \theta^N(x) = 0. \tag{3}$$

2.1. Initial conditions

The standard boundary conditions that form a Cauchy problem with a Lane-Emden equation are the specifications of the values of θ and θ' at 0, i.e. $\theta(0) = 1$ and $\theta'(0) = 0$. Namely, the value of θ at 0 is due to its construction, whereas the vanishing of its derivative at 0 indicates the absence of gravity in the cylinder's axis (see [9,16] for more explanation on the related physical motivations).

2.2. Modified versions of the Lane-Emden equation

Some Authors extend the form (1) to define other classes of Lane-Emden equations. For example, in [17] equation (1) is referred to as Lane-Emden equation of first kind (see [18]), whereas the Lane-Emden equation of the second kind has the following formulation[1]:

[1] In [18] a further version is mentioned, originating from a change of variable in (4), whose form is $\theta''(x) + k\frac{\theta'}{x} + e^{-\theta(x)} = 0$, and whose initial conditions are replaced with $\theta(0) = \theta'(0) = 0$.

$$\theta''(x) + k\frac{\theta'(x)}{x} + e^{\theta(x)} = 0, \tag{4}$$

which turns out to be the dynamics in isothermal cylinders (see [9] for the derivation of the hydrostatic problem).

We can establish a whole class of Lane-Emden problems employing the most general form as follows:

$$\theta''(x) + k\frac{\theta'(x)}{x} + f(\theta(x)) = 0, \tag{5}$$

It is also interesting to remark that (5) can be reformulated as an integro-differential equation. Multiplying the left hand side by x^k we have:

$$x^k\theta''(x) + kx^{k-1}\theta'(x) + x^k f(\theta(x)) = 0 \quad \Longleftrightarrow \quad \left(x^k\theta'(x)\right)' = -x^k f(\theta(x)),$$

which can be integrated on both sides, entailing:

$$\theta'(x) = -\frac{1}{x^k}\int_0^x t^k f(\theta(t))dt, \tag{6}$$

where the initial condition turns out to be

$$\lim_{x \longrightarrow 0} \frac{1}{x^k}\int_0^x t^k f(\theta(t))dt = 0,$$

which holds if and only if

$$\lim_{x \to 0} x f(\theta(x)) = 0,$$

by De L'Hospital's Theorem.

The form (6) is commonly used for numerical approximations of the solutions (see for example [17]). Perhaps the most relevant modification of the Lane-Emden equation is the Emden-Fowler equation (see Chandrasekhar [6] or Fowler's contributions [4,5]), i.e.

$$\frac{d}{dx}\left(x^\rho \frac{dy}{dx}\right) + x^\alpha y^\tau(x) = 0, \qquad x \geq 0, \tag{7}$$

where $\rho, \alpha \in \mathbb{R}$, $\tau \in \mathbb{R}_+$. Many papers contain a number of results for (7): a survey outlining results until 1975 is [19], whereas subsequent relevant papers are [20,21], and many others.

Such equation can be transformed into the modified form:

$$y''(x) - h(x)y^\tau(x) = 0, \qquad x \geq 0, \tag{8}$$

where $h(x)$ is a continuous and nonnegative function.

3. Exact solutions

In literature, the only known solutions to (2) in closed form are available for $N = 0$, $N = 1$ and $N = \infty$ (degenerate case). We are going to briefly outline the related polytropes and solution procedures for $N = 0$ and $N = 1$.

 doi:10.20944/preprints202312.2238.v1

3.1. Polytropes for N=0

The easiest case occurs[2] when $N = 0$ and we can trivially solve by separation of variables. Actually, in that case a generalization of (3) and (2) can be solved as well.

Proposition 1. *All generalized Lane-Emden equations of the following kind:*

$$\begin{cases} \frac{1}{x^k}\frac{d}{dx}\left(x^k\frac{d\theta}{dx}\right) + \theta^N(x) = 0 \\ \theta(0) = 1 \\ \theta'(0) = 0 \end{cases}, \tag{9}$$

can be solved for all $k \geq 0$ when $N = 0$ and the solution is the following family of parabolas:

$$\theta_k^*(x) = 1 - \frac{x^2}{2(k+1)}. \tag{10}$$

Proof. When $N = 0$, (9) amounts to:

$$\frac{1}{x^k}\frac{d}{dx}\left(x^k\frac{d\theta}{dx}\right) = -1 \iff \frac{d}{dx}\left(x^k\frac{d\theta}{dx}\right) = -x^k,$$

then after integrating both sides we have:

$$x^k\frac{d\theta}{dx} = -\frac{x^{k+1}}{k+1} + C_0 \iff \cdots \iff \theta(x) = -\frac{x^2}{2(k+1)} + \frac{C_0}{(-k+1)x^{k-1}} + C_1,$$

then applying the boundary conditions yields $C_0 = 0$ and $C_1 = 1$, leading to the following family of parabolas indexed by k: $\theta_k^*(x) = 1 - \frac{x^2}{2(k+1)}$. □

The respective polytropes for (2) and (3) are

$$\theta_1^*(x) = 1 - \frac{x^2}{4}, \qquad \theta_2^*(x) = 1 - \frac{x^2}{6}.$$

3.2. Polytropes for N=1

When $N = 1$, the polytrope of (3) is known as well. Expanding equation (3) yields:

$$\frac{1}{x^2}\left(2x\theta'(x) + x^2\theta''(x)\right) + \theta(x) = 0 \iff \theta''(x) + \frac{2}{x}\theta'(x) + \theta(x) = 0. \tag{11}$$

On the other hand, expanding the form (2) yields:

$$\frac{1}{x}\left(\theta'(x) + x\theta''(x)\right) + \theta(x) = 0 \iff \theta''(x) + \frac{1}{x}\theta'(x) + \theta(x) = 0. \tag{12}$$

In order to solve them, we assume a power series solution of the following kind (where $a_0 = 1$ because $\theta(0) = 1$):

$$\theta(x) = 1 + \sum_{j=1}^{\infty} a_j x^j. \tag{13}$$

---

[2] This case is far from reality in that $N = \frac{1}{\gamma - 1}$. Anyway we are going to outline the polytropes for completeness.

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 29 December 2023 doi:10.20944/preprints202312.2238.v1

Plugging (13) into (11) leads to:

$$\sum_{j=2}^{\infty}(j-1)ja_jx^{j-2} + 2\sum_{j=1}^{\infty} ja_jx^{j-2} + 1 + \sum_{j=1}^{\infty} a_jx^j = 0 \iff$$

$$\iff \frac{2a_1}{x} + 2a_2 + 4a_2 + 1 + \sum_{j=3}^{\infty}[((j-1)j + 2j)a_j + a_{j-2}]x^{j-2} = 0,$$

whose coefficients are supposed to verify:

$$a_1 = 0, \qquad a_2 = -\frac{1}{6}, \qquad a_j = -\frac{a_{j-2}}{j(j+1)},$$

hence the polytrope is

$$\theta^*(x) = 1 + \sum_{j=1}^{\infty} \frac{(-1)^j x^{2j}}{(2j+1)!} = \frac{\sin x}{x}. \tag{14}$$

An analogous procedure can be carried out to solve (12). Plugging (13) into (12) yields:

$$\sum_{j=2}^{\infty}(j-1)ja_jx^{j-2} + \sum_{j=1}^{\infty} ja_jx^{j-2} + 1 + \sum_{j=1}^{\infty} a_jx^j = 0 \iff$$

$$\iff \frac{a_1}{x} + 2a_2 + 2a_2 + 1 + \sum_{j=3}^{\infty}[((j-1)j + j)a_j + a_{j-2}]x^{j-2} = 0,$$

whose coefficients are:

$$a_1 = 0, \qquad a_2 = -\frac{1}{4}, \qquad a_j = -\frac{a_{j-2}}{j^2},$$

leading to the following polytrope:

$$\theta^*(x) = 1 + \sum_{j=1}^{\infty} \frac{(-1)^j x^{2j}}{((2j)!!)^2}. \tag{15}$$

The next Proposition intends to generalize the above findings as in Proposition 1.

Proposition 2. *All generalized Lane-Emden equations of the following kind:*

$$\begin{cases} \frac{1}{x^k}\frac{d}{dx}\left(x^k \frac{d\theta}{dx}\right) + \theta^N(x) = 0 \\ \theta(0) = 1 \\ \theta'(0) = 0 \end{cases},$$

can be solved for all $k \geq 0$ when $N = 1$ and the solution is the following family of power series:

$$\theta_k^*(x) = 1 + \sum_{j=1}^{\infty} \frac{(-1)^j x^{2j}}{(2j)!!(2j-1+k)!!}. \tag{16}$$

Proof. Expanding the equation leads to

$$\theta''(x) + \frac{k}{x}\theta'(x) + \theta(x) = 0. \tag{17}$$

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 29 December 2023 doi:10.20944/preprints202312.2238.v1

Employing the above method, we obtain:

$$\frac{ka_1}{x} + 2a_2 + 2ka_2 + 1 + \sum_{j=3}^{\infty}[((j-1)j + kj)a_j + a_{j-2}]x^{j-2} = 0,$$

leading to the following coefficients:

$$a_1 = 0, \qquad a_2 = -\frac{1}{2+2k}, \qquad a_j = -\frac{a_{j-2}}{j(j+k-1)},$$

consequently the polytrope is (16). □

4. Analytical properties in the cylindrical scenario

In this Section, we are going to establish some qualitative properties of the solutions to (2) after setting $k = 1$. We begin from some elementary analytical result, then we proceed to provide some insights on the graph of the involved functions. From now on, we will indicate with $\theta^*_M(x)$ the solution to the cylindrical Lane-Emden equation for $N = M$.

Proposition 3. *For all $M \geq 0$, we have that $(\theta^*_M)''(0) = -\frac{1}{2}$.*

Proof. It is elementary to collect terms in (2) as follows:

$$(\theta^*_M)''(x) + \frac{(\theta^*_M)'(x)}{x} + (\theta^*_M)^M(x) = 0 \qquad \Longleftrightarrow \qquad \frac{((\theta^*_M)'(x)x)'}{x} = -(\theta^*_M)^M(x).$$

Now call $F_M(x) = (\theta^*_M)'(x)x$, whose derivatives respectively are

$$F'_M(x) = (\theta^*_M)''(x)x + (\theta^*_M)'(x), \qquad F''_M(x) = (\theta^*_M)'''(x)x + 2(\theta^*_M)''(x).$$

Since $F'_M(0) = (\theta^*_M)'(0) = 0$ and by the initial condition $\theta^*_M(0) = 1$, we can deduce that

$$\lim_{x \longrightarrow 0} \frac{F'_M(x)}{x} = -1$$

but the above limit is equal to $F''_M(0)$ by De L'Hospital's Theorem, hence

$$F''_M(0) = 2(\theta^*_M)''(0) = -1,$$

which implies $(\theta^*_M)''(0) = -\frac{1}{2}$. □

It is simple to check that the same procedure illustrated in Proposition 3 can be extended to calculate the higher order derivatives of the solution at zero. Although we are not going to develop this argument, the implementation of this method might provide an approximation series of the solution in a neighbourhood of the origin.

As an illustrative example of the method, we can check the value of the third derivative at 0. Since $F''(x) = x(\theta^*_M)'''(x) + 2(\theta^*_M)''(x)$, differentiating the right-hand side as well yields:

$$x(\theta^*_M)'''(x) + 2(\theta^*_M)''(x) = -(\theta^*_M)^M(x) - Mx(\theta^*_M)^{M-1}(x)(\theta^*_M)'(x) \qquad \Longrightarrow$$

$$\Longrightarrow \qquad (\theta^*_M)'''(x) = -\frac{2(\theta^*_M)''(x) + (\theta^*_M)^M(x)}{x} - M(\theta^*_M)^{M-1}(x)(\theta^*_M)'(x).$$

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 29 December 2023 doi:10.20944/preprints202312.2238.v1

Subsequently, evaluating both sides at 0 entails:

$$(\theta_M^*)'''(0) = \lim_{x\longrightarrow 0}\left(-\frac{2(\theta_M^*)''(x) + (\theta_M^*)^M(x)}{x} - M(\theta_M^*)^{M-1}(x)(\theta_M^*)'(x)\right),$$

then, by De L'Hospital's Theorem, we obtain

$$(\theta_M^*)'''(0) = -2(\theta_M^*)'''(0) \qquad \Longrightarrow \qquad (\theta_M^*)'''(0) = 0.$$

Furthermore, Proposition 3 establishes that for all M, $\theta_M^*(x)$ is concave in a neighbourhood of 0. As a matter of fact, he solution that we explicitly know for $M = 0$ is a parabola having a decreasing and concave behaviour for $x > 0$. As is well-known, if the function admits no inflection points for $x > 0$, this is a sufficient condition to guarantee the existence of a zero x_M^*. When $M = 0$, $x_1^* = 2$.

The following results intend to establish some further qualitative properties of $\theta_M^*(x)$ which are verified for all $M \geq 1$.

Proposition 4. *If $\theta_M^*(x)$ admits at least a positive zero for $M \geq 1$, and x_M is the smallest zero of $\theta_M^*(x)$, then one of the following conditions holds:*

1. $\theta_M^*(x_M) = (\theta_M^*)'(x_M) = (\theta_M^*)''(x_M) = \cdots = (\theta_M^*)^{(k)}(x_M) = 0$ *for all* $k \in \mathbb{Z}_+$;
2. *the function* $\theta_M^*(x)$ *admits at least one inflection point* $F(x_F, y_F)$ *such that* $0 < x_F < x_M$.

Proof. If we call $F_M(x) = (\theta_M^*)'(x)x$, it is easy to note that $F_M'(x_M) = 0$ by construction. Since $F_M'(x_M) = (\theta_M^*)''(x_M)x_M + (\theta_M^*)'(x_M) = 0$, two cases may occur. Either both first and second derivative vanish at x_M, and consequently all the derivatives of any order vanish at x_M, or $x_M = -\frac{(\theta_M^*)'(x_M)}{(\theta_M^*)''(x_M)}\theta_M^*(x_M)$, which can only hold if the second order derivative changed its sign in the interval $(0, x_M)$, meaning that the graph has an inflection point at $x_F < x_M$. □

Now we are going to provide some further insights on the behavior of the solution by separating the two circumstances where M is odd and M is even, because some relevant differences occur. The role of possible inflection points, zeros and stationary points will be analyzed in detail.

4.1. Qualitative behavior if M is odd

The presence of a stationary point, i.e. either a maximum or a minimum point, when M is odd, is an interesting issue. If we suppose that $\theta_M^*(x)$ admits one stationary point x^* such that $(\theta_M^*)'(x^*) = 0$, in the main equation we would have

$$(\theta_M^*)''(x^*) = -(\theta_M^*)^M(x^*).$$

If $\theta_M^*(x^*) > 0$, this point can only be a local maximum, by the negativity of the second order derivative. Vice versa, if $\theta_M^*(x^*) < 0$, it is a local minimum, and clearly $x_F < x_M < x^*$.

The above considerations establish that if M is odd, $(\theta_M^*)(x)$ can only have a maximum point having both positive coordinates. On the other hand, any local minimum has a negative image, hence there are always at least one inflection point and a zero between each maximum and minimum point.

4.2. Qualitative behavior if M is even

If M is even, i.e a positive integer greater or equal to 2, the results are slightly different with respect to the previous case. Suppose that $x^* > 0$ is the first stationary point for $\theta_M^*(x)$. If $\theta_M^*(x^*) > 0$, the negativity of the second order derivative implies that such a point is a local maximum, but this holds true even if $\theta_M^*(x^*) < 0$. Namely, a stationary point can only be a maximum point, consequently there can only be one maximum point, after which the solution decreases asymptotically.

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 29 December 2023 doi:10.20944/preprints202312.2238.v1

No oscillating behaviour is feasible in this case, as occurs in the easiest case $M = 0$, where the polytrope is monotonically decreasing. There may be some changes in the convexity/concavity form of the graph, but the beahaviour is unambiguously decreasing.

4.3. Evaluation of the difference between two solutions

Call θ_M^*, θ_P^* the solutions for any $M, P \in \mathbb{Z}_+$, where $M \neq P$. We posit $k = 1$, i.e. we are in the cylindrical setup. By (1), we have that:

$$\begin{cases} (\theta_M^*)''(x) + \dfrac{(\theta_M^*)'(x)}{x} + (\theta_M^*)^M(x) = 0 \\ \\ (\theta_P^*)''(x) + \dfrac{(\theta_P^*)'(x)}{x} + (\theta_P^*)^P(x) = 0 \end{cases} \iff \begin{cases} \dfrac{((\theta_M^*)'(x)x)'}{x} = -(\theta_M^*)^M(x) \\ \\ \dfrac{((\theta_P^*)'(x)x)'}{x} = -(\theta_P^*)^P(x) \end{cases}.$$

Call now $F_M(x) = (\theta_M^*)'(x)x$ and $F_P(x) = (\theta_P^*)'(x)x$, so that we obtain the following dynamic system:

$$\begin{cases} F_M'(x) = -x(\theta_M^*)^M(x) \\ \\ F_P'(x) = -x(\theta_P^*)^P(x) \end{cases},$$

endowed with the initial conditions: $F_M(0) = 0$, $F_P(0) = 0$. Subtracting the left hand sides, we have:

$$(F_M(x) - F_P(x))' = x\left[(\theta_P^*)^P(x) - (\theta_M^*)^M(x)\right],$$

from which we have, after integrating both sides:

$$F_M(x) - F_P(x) = \int_0^x t\left[(\theta_P^*)^P(t) - (\theta_M^*)^M(t)\right] dt,$$

i.e.

$$(\theta_M^*(x) - \theta_P^*(x))' = \frac{\int_0^x t\left[(\theta_P^*)^P(t) - (\theta_M^*)^M(t)\right] dt}{x},$$

then, integrating both sides again:

$$\theta_M^*(x) - \theta_P^*(x) = \int_0^x \left[\frac{\int_0^t s\left[(\theta_P^*)^P(s) - (\theta_M^*)^M(s)\right] ds}{t}\right] dt. \tag{18}$$

In (18), the difference between solutions is on the left-hand side, whereas the difference between their powers is in the double integral in the right-hand side. This relation can be employed to identify an approximation method for the polytropes, with the help of the above considerations on the qualitative behavior of the solutions.

5. Concluding remarks and discussion

We identified some properties of the solutions of the Lane-Emden in the cylindrical framework, especially taking into account the critical points and their possible positions. The determination of solutions in closed form remains a difficult issue, anyway some insights are provided which may lead to a more precise approximation. A future development of the present work may concern the realization and the computational optimization of a suitable algorithm to constructively approximate the real solution.

Author Contributions: Conceptualization, A.P. and S.R.; methodology, A.P. and S.R.; software, A.P. and S.R.; validation, A.P. and S.R.; formal analysis, A.P. and S.R.; investigation, A.P. and S.R.; resources, A.P. and S.R.; data curation, A.P. and S.R.; writing—original draft preparation, A.P. and S.R.; writing—review and editing, A.P.

and S.R.; visualization, A.P. and S.R.; supervision, A.P. and S.R.; project administration, A.P. and S.R.; funding acquisition, A.P. and S.R. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Emden, R. *Gaskugeln*, BG Teubner, 1907.
2. Recchi, S., A. Hacar, Palestini, A. Nonisothermal filaments in equilibrium. *Astron. and Astroph.* **2013**, *558*, 27.
3. Recchi, S., Hacar, A., Palestini, A. On the equilibrium of rotating filaments. *Month. Not. of the Royal Astron. Soc.* **2014**, *444(2)*, 1775–1782.
4. Fowler, R.H. The form near infinity of real continuous solutions of a certain differential equation of second order. *Quart. J. of Math.* **1914**, *45*, 289–350.
5. Fowler, R.H. Further studies of Emden's and similar differential equations. *Quart. J. of Math.* **1931**, *1*, 259–288.
6. Chandrasekhar, S. *An Introduction to the Study of Stellar Structure*, Chicago: University of Chicago Press, 1939; pp. 84-103.
7. Chandrasekhar, S., Fermi, E. Problems of Gravitational Stability in the Presence of a Magnetic Field. *Astroph. J.* **1953**, *118*, 115–141.
8. Christodoulou, D. M., Kazanas, D. Exact Axisymmetric Solutions of the 2-D Lane-Emden Equations with Rotation. *J. Mod. Phys.* **2016**, *7*, 2177–2187.
9. Ostriker, J. The equilibrium of polytropic and isothermal cylinders. *The Astroph. J.* **1964**, *140*, 1056.
10. Horedt, G.P. *Polytropes: applications in astrophysics and related fields* **2004**,*306*, Springer Science & Business Media.
11. Adomian, G. A new approach to nonlinear partial differential equations. *J. of Math. Anal. and Appl.* **1984**, *102*, 420–434.
12. Bengochea, G., Verde-Star, L. Linear algebraic foundations of the operational calculi. *Adv. in Appl. Math.* **2011**, *47(2)*, 330–351.
13. Bengochea, G. Algebraic approach to the Lane-Emden equation. *Appl. Math. and Comp.* **2014**, *232*, 424—430.
14. Bengochea, G., Verde-Star, L. An operational approach to the Emden-Fowler equation. *Math. Meth. in the Appl. Sci.* **2015**, *38(18)* 4630-–4637.
15. Arqub, O. A., Osman, M. S., Abdel-Aty, A. H., Mohamed, A. B. A., Momani, S. A numerical algorithm for the solutions of ABC singular Lane–Emden type models arising in astrophysics using reproducing kernel discretization method. *Mathematics* **2020**, *8(6)*, 923.
16. Iacono, R., De Felice, M. Constructing analytic approximate solutions to the Lane-Emden equations. *Phys. Lett. A* **2015**, *379*, 1802–1807.
17. Sahu, P. K., Ray, S. S. Numerical solutions for Volterra integro-differential forms of Lane-Emden equations of first and second kind using Legendre multi-wavelets. *El. J. of Diff. Eq.* **2015**, *28*, 1–11.
18. Wazwaz, A. M., Rach, R., Duan, J. S. Adomian decomposition method for solving the Volterra integral form of the Lane–Emden equations with initial values and boundary conditions. *Appl. Math. and Comp.* **2013**, *219(10)*, 5004–5019.
19. Wong, J. S. On the generalized Emden–Fowler equation. *Siam Rev.* **1975**, *17(2)*, 339–360.
20. Habets, P., Zanolin, F. Upper and lower solutions for a generalized Emden-Fowler equation. *J. Math. Anal. Appl.* **1994**, *181(3)*, 684–700.
21. Zhang, Y. Positive solutions of singular sublinear Emden-Fowler boundary value problems. *J. of Math. Anal. and Appl.* **1994**, *185*, 215–222.
22. Lane, H.J. On the Theoretical Temperature of the Sun, under the Hypothesis of a gaseous Mass maintaining its Volume by its internal Heat, and depending on the laws of gases as known to terrestrial Experiment. *Amer. J. of Science* **1870)**, *148*, 57–74.