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The Permanent Rank of a Matrix (Part Two)

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Abstract

Define the perrank of a matrix $M_{m\times n}$ to be the size of a largest square submatrix with nonzero permanent; if this is equal to m or n, then we say M has full perrank. We derive numerous results on the perrank of a matrix by methods of commutative algebra.

Keywords: permanent rank

Introduction

Coloring problems constitute a major area of research in discrete mathematics. The standard algebraic method, via the Combinatorial Nullstellensatz [1], gives rise to a matrix. If this matrix has full perrank, then a coloring exists. This paper is motivated by the Alon-Linial-Meshulam Conjecture A4.1, the Kahn-Tarsi Conjecture A2, and the Saxton-Yu Conjecture L7.1. All these conjectures establish sufficient conditions for a matrix to have full perrank.

We will demonstrate that perrank problems are closely connected to other areas of mathematics. For example, they are related to all three components of the Kähler package introduced in [2]: Lefschetz Theorem, Poincaré Duality, and Hodge Riemann Relation, as discussed in the first three sections of the paper. Interestingly, there is a single perrank result, Corollary R5, that relates to all three components of the Kähler package.

Section A presents an analogy between the perrank problem in positive characteristic and its counterpart in characteristic zero. Section D contains our main results in characteristic zero. Section C gives partial results on the Alon-Linial-Meshulam Conjecture in characteristic three.

In mathematics, when direct induction does not work, we often attempt to induct on a stronger or equivalent statement. By using matroid theory along with the minimum support function introduced in Section S, we derive an equivalent formulation of the Alon–Linial–Meshulam Conjecture that is suitable for induction: namely Conjecture A4.2. Professor Noga Alon notes that this is the correct approach. We were able to establish a weak version of Conjecture A4.1 in characteristic 3 using this approach in another paper: part three of the author's series *"The permanent rank of a matrix"*. It implies the Additive Basis Conjecture in the case Z_3, also gives an alternative proof of Jaeger's weak 3 Flow Conjecture (first proved by C. Thomassen in 2012). This paper is part two of the series, and part one was [7].

Numerous conjectures are discussed in this paper; those without names are made by the author with varying degrees of confidence; some of them are thought to be deep. For example, A7.1 is an innocent looking problem analogous to the matroid axiom, but very difficult.

During the study of the perrank problems, we derive several ancillary results that are not directly relevant to the main perrank problem, their proofs omitted to keep the paper from being too long. For example, R7.1 is a side result on eigenspace decomposition. We will post the proofs of these results at HAL, preprints.org or researchgate.net under the title "The permanent rank of a matrix (part 4)". Interested readers may search online, or contact the author via email.

Lefschetz Property

In sections L, D, S, we assume the ground field has characteristic zero.

For a matrix $M_{m\times n}$, define its *termrank* to be the maximum number of nonzero elements without any two in the same row or column, and define its *perrank* to be the order of the largest square submatrix with nonzero permanent. If its perrank or termrank is equal to $\min\{m,n\}$, then we say M has *full perrank* or *full termrank*.

Given a ground field, let A^n be the quotient of the polynomial ring in n variables $x_1, x_2, \cdots, x_n$ by the ideal generated by $x_1^2, x_2^2, \cdots, x_n^2$. The k-th degree component of the graded algebra A^n is denoted by A_k^n; we may omit n when there is no ambiguity.

For an ideal $J \subseteq A$, let $J_k = A_k \cap J$.

For $f \in A$, define $\mathrm{Ker}(f) = \{g : gf = 0\}$, $\mathrm{Im}(f) = \{fg : g \in A\}$.

It is easy to show that $\mathrm{Ker}(x-y) = \mathrm{Im}(x+y)$ for any pair of variables.

We introduce two operators. Let ∂_x and E_x be the quotient and remainder of formal division by x, and write $\partial_i := \partial_{x_i}$ for simplicity, similarly for E_i. For example, if $f = x_1x_2 + x_1x_3 + x_2x_3$, then $\partial_1 f = x_2 + x_3$ and $E_1 f = x_2x_3$. We use the letter E because E_i *eliminates* all terms containing x_i. The letter R is used for the remainder of another type of division in section C.

It is obvious that $E_i(fg) = (E_if)(E_ig)$, $\partial_i(fg) = (\partial_if)(E_ig) + (\partial_ig)(E_if)$, and $E_iE_j = E_jE_i$, $\partial_i\partial_j = \partial_j\partial_i$, $E_i\partial_j = \partial_jE_i$ (but $E_i\partial_i \neq E_i\partial_i$).

Define $E_J = \prod_{j\in J} E_j$, $\partial_J = \prod_{j\in J} \partial_j$, $X_J = \prod_{j\in J} x_j$.

If $\deg(f) = |J|$, then $\partial_J f$ is the coefficient of X_J in f.

For $u \in A_1$, define its *support* $\mathrm{supp}(u) = \{x : \partial_x u \neq 0\}$, $\mathrm{supp}(0) = \varnothing$; we say it has *full support* if $\partial_x u \neq 0$ for all x. An element of A_1 is also called a *linear form*.

Theorem L1 $\mathrm{Ker}_k(u^i) = 0$ for any linear form u with $|\mathrm{supp}(u)| \geq 2k + i$.

Proof. Suppose $f \in \mathrm{Ker}_k(u^i)$, fix any J with $|J| = k$, first assume $J = [k]$.

Take ∂_1 to $fu^i = 0$, we get $(\partial_1 f)E_1(u^i) + (i\partial_1 u)E_1(fu^{i-1}) = 0$; multiply by E_1u, we get $(\partial_1 f)(E_1u)^{i+1} = 0$; similarly $(\partial_{12}f)(E_{12}u)^{i+2} = 0, \cdots, (\partial_{[k]}f)(E_{[k]}u)^{i+k} = 0$.

$(E_Ju)^{i+k} \neq 0$ because $|\mathrm{supp}(E_Ju)| \geq i + k$, so $\partial_J f = 0$. Let J vary, $f = 0$. □

This theorem also follows from the Lefschetz property of algebra A, first proved by Stanley [3].

Label the rows and columns of $M_{m\times n}$ with variables $x_1, x_2, \cdots, x_m$ and $y_1, y_2, \cdots, y_n$ respectively, and view its rows and columns as linear forms in A^n and A^m. Then M has full perrank iff the product of its rows or columns is nonzero in A^n or A^m.

Corollary L2

Any matrix with at least $2k - 1$ nonzero entries in its k-th row has full perrank.

Exercise L3

If any one of the following holds, then $M = \begin{pmatrix} 1 & * & * \\ * & B_{n\times n} & C_{n\times n} \end{pmatrix}$ has full perrank.

(1) neither C nor B has zero element

(2) $\mathrm{per}(C) \neq 0 \neq \mathrm{per}(B)$

(3) $\mathrm{per}(C) \neq 0$, and B has no zero element

Corollary L4.1 $\mathrm{perrank}(M) \geq \mathrm{termrank}(M)/3$ for any matrix M.

Proof. Suppose $\mathrm{perrank}(M) = p$, $\mathrm{termrank}(M) = t$. Let $r_1, r_2, r_3 \cdots$ be the rows of M. Wlog assume $r_1r_2\cdots r_p \neq 0$, then $r_k \in \mathrm{Ker}(r_1r_2\cdots r_p)$ for all $k > p$. The rows r_k $(k > p)$ form a submatrix with termrank at least $t - p$, so a linear combination of them has support at least $t - p$. Theorem L1 gives $t - p \leq 2p$. □

The constant $1/3$ is a weak result, we believe $1/2$ is the truth as in the following conjecture made by Bryan Shader via private correspondence.

Conjecture L4.2 $\mathrm{perrank}(M) \geq \mathrm{termrank}(M)/2$ for any matrix M.

Next is a stronger version by the author.

Conjecture L5 $\mathrm{perrank}(M) \geq \mathrm{termrank}(M) - \mathrm{rank}(M)/2$ for any matrix M.

Conjecture L6 (Michael Tarsi)

For any square matrix M with full termrank, $(M\ \ M)$ has full perrank.

L6 and A2 appeared in the unpublished Master's thesis of Dan Hefetz (a student of Tarsi), in which he verified them respectively for 4×4 and 5×5 matrices.

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Main Conjecture L7.1 (author, 2001, unpublished)
For any matrices $M_{n\times n}$, $N_{n\times(n-1)}$ with full termrank, $(M\ \ N)$ has full perrank.

Conjecture L7.2 (David Saxton [4]) Suppose $u_i\ (1\le i\le h)$ are linear forms and $|\bigcup_{i\in S}\operatorname{supp}(u_i)|\ge 2|S|-1$ for any $S\subseteq[h]$, then $\prod_{i=1}^{h}u_i\ne 0$.

Conjecture L7.3 Suppose $u_i\ (1\le i\le h)$ are linear forms, $a_i>0$, and $|\bigcup_{i\in S}\operatorname{supp}(u_i)|\ge\sum_{i\in S}a_i+|S|-1$ for any $S\subseteq[h]$, then $\prod_{i=1}^{h}u_i^{a_i}\ne 0$.

The above three conjectures are actually equivalent in an algebraically closed field.

We now prove an easy case of L7.3. Four more special cases in appendix.

Observation If $|\{x : E_x f=0\}|>\deg(f)$, then $f=0$.

Case L8 $h=2$, write $u_1\ u_2\ a_1\ a_2$ as $u\ v\ i\ j$ for notation simplicity.

Proof. If $\operatorname{supp}(u)=\operatorname{supp}(v)$, for each variable x, take ∂_x to $u^iv^j=0$, we get $(i\partial_x u)E_x(u^{i-1}v^j)+(j\partial_x v)E_x(u^iv^{j-1})=0$. Multiply by E_xv, we get $E_x(u^{i-1}v^{j+1})=0$. By above observation, we have $u^{i-1}v^{j+1}=0$. Similarly $u^{i-2}v^{j+2}=0,\cdots$, eventually $v^{j+i}=0$, contradiction.

If $\operatorname{supp}(u)\ne\operatorname{supp}(v)$, wlog choose a variable x with $\partial_x u=0$, $\partial_x v\ne 0$. Take ∂_x to $u^iv^j=0$, we get $u^i(E_xv)^{j-1}=0$; contradiction if $j=1$, induction if $j>1$. □

Duality

Define operator D on A^n as a linear bijection by $D(X_J)=X_{[n]\setminus J}$ for all $J\subseteq[n]$. There is a canonical bilinear form on $A_k^n\times A_{n-k}^n$ by $\langle f,g\rangle := fg$.

Lemma D1 $\operatorname{Ker}_k(f)=\operatorname{Im}_{n-k}(f)^\perp$ for all $f\in A^n$.

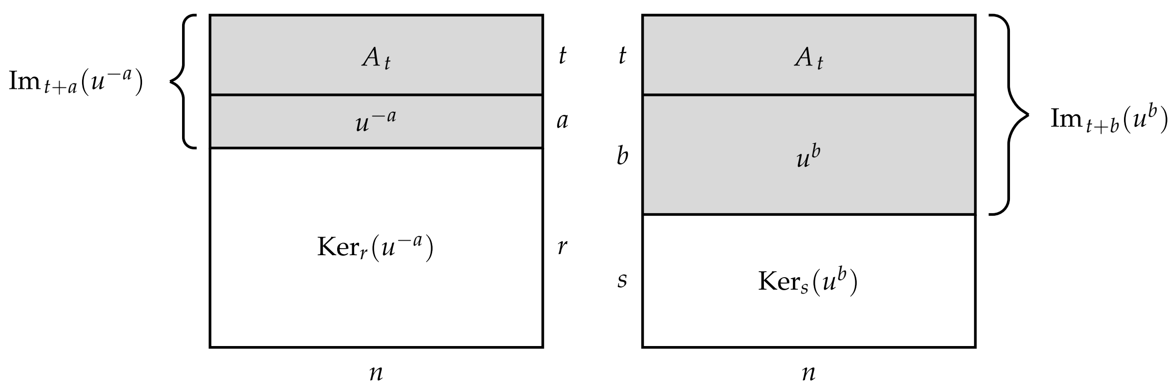
Proof. If $g\in\operatorname{Ker}_k(f)$, then $gh=0$ for any $h\in\operatorname{Im}_{n-k}(f)$, so $g\in\operatorname{Im}_{n-k}(f)^\perp$.

If $g\in A_k\backslash\operatorname{Ker}_k(f)$, then $gf\ne 0$; choose a term cX_J in gf, $c\ne 0$. We then have $gfD(X_J)=cX_{[n]}\ne 0$, so $g\notin\operatorname{Im}_{n-k}(f)^\perp$ since $fD(X_J)\in\operatorname{Im}_{n-k}(f)$. □

Define the *unit linear form* in A^n to be $e := x_1+x_2+\cdots+x_n$.
For a linear form $p=\sum p_ix_i$ in A^n with full support,
define its *inverse linear form* to be $p^{-1} := \sum p_i^{-1}x_i$ and let $p^{-k} := (p^{-1})^k$,
define *stretch operator* S_p on A^n as a ring automorphism by $S_p(x_i)=p_ix_i$.
We have $S_{p^{-1}}=(S_p)^{-1}$, and $D(S_pf)=(p_1p_2\cdots p_n)S_{p^{-1}}(Df)$ for all f.

Lemma D2 Suppose $u\in A^n$ is a linear form with full support, and parameters $a,b,r,s,t>0$ satisfy $n=r+s$, $r=b+t$, $s=a+t$, then we have bijections

$$\operatorname{Ker}_r(u^{-a})\overset{D}{\longleftrightarrow}\operatorname{Ker}_s(u^b)\quad\text{and}\quad\operatorname{Im}_{t+a}(u^{-a})\overset{D}{\longleftrightarrow}\operatorname{Im}_{t+b}(u^b)$$



Proof. Suppose $h\in\operatorname{Ker}_r(e^a)$, fix any $T\subseteq[n]$ with $|T|=t$. By the same method as in the proof of Theorem L1, we have $(\partial_T h)(E_Te)^{a+t}=0$, which means sum of all coefficients in $\partial_T h$ is zero, since $(r-t)+(a+t)=n-t$. Then

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$$\sum_{\substack{B\cap T=\varnothing\\|B|=b}} \partial_B(\partial_T h)=0 \longrightarrow \sum_{\substack{R\supseteq T\\|R|=r}} \partial_R h=0 \longrightarrow \sum_{\substack{S\cap T=\varnothing\\|S|=s}} \partial_S(Dh)=0=\sum_{\substack{S\cap T=\varnothing\\|S|=s}} \partial_S(E_T(Dh)),$$

i.e. $E_T((Dh)e^b)=0$. Let T vary, we have $(Dh)e^b=0$. Now suppose $f\in \mathrm{Ker}_r(u^{-a})$, $fu^{-a}=0\longrightarrow (S_u f)e^a=0 \longrightarrow D(S_u f)e^b=0\longrightarrow S_{u^{-1}}(Df)e^b=0\longrightarrow (Df)u^b=0$.

So D takes $\mathrm{Ker}_r(u^{-a})$ into $\mathrm{Ker}_s(u^b)$, done by symmetry and Lemma D1, because D preserves orthogonal complement. $\square$

We now present two applications of this lemma.

Theorem D3

$\mathrm{Ker}_k(u^i v^j)=0$ for any linear forms u,v in $A^{2k+i+j+1}$ with full support.

Proof. Induction on i. When $i=1$, suppose $f\in\mathrm{Ker}_k(uv^j)$, apply Theorem L1 twice, $fuv^j=0\longrightarrow fu=0\longrightarrow f=0$.

When $i>1$, apply D2 with $t=k+1$, $a=i-1$, $b=j$, $r=k+j+1$, $s=k+i$, we have $D(\mathrm{Ker}_{k+i}(v^j))=\mathrm{Ker}_{k+j+1}(v^{-(i-1)})$. Apply D2 with $t=k$, $a=j+1$, $b=i$, we have $D(\mathrm{Im}_{k+i}(u^i))=\mathrm{Im}_{k+j+1}(u^{-(j+1)})$.

Then $\mathrm{Ker}_k(u^iv^j)=0 \longleftrightarrow \mathrm{Im}_{k+i}(u^i)\cap\mathrm{Ker}_{k+i}(v^j)=0\longleftrightarrow$
$\mathrm{Im}_{k+j+1}(u^{-(j+1)})\cap\mathrm{Ker}_{k+j+1}(v^{-(i-1)})=0 \longleftrightarrow \mathrm{Ker}_k(v^{-(i-1)}\,u^{-(j+1)})=0.$ $\square$

Theorem D4

$\mathrm{Ker}_k(u^a v^b)=0$ for any linear forms u,v in A^{2k+a+b} with $\partial_i u/\partial_i v>0$ for all i.

Proof. First assume ground field is the real number field $\mathbb{R}$, and wma $v=e$.
Let $w=\sum\sqrt{\partial_i u}\,x_i$, then $S_w(w^a w^{-b})=u^a e^b$, need to show $\mathrm{Ker}_k(w^a w^{-b})=0$.
Suppose $f\in\mathrm{Ker}_k(w^aw^{-b})$, by above Lemma $D(fw^a)=gw^{-b}$ for some g,
then $fw^aD(fw^a)=fw^agw^{-b}=0\longrightarrow fw^a=0\longrightarrow f=0$.
When $\mathbb{R}$ is a subfield of the ground field, easily follows. $\square$

The following is our main result in characteristic zero.

Theorem D5 $\mathrm{Ker}_k(u^iv^j)=\mathrm{Ker}_k(u^i)+\mathrm{Ker}_k(v^j)$ for any linear forms u,v with $|\mathrm{supp}(u)|\geq i$, $|\mathrm{supp}(v)|\geq j$, and $|\mathrm{supp}(u)\cup\mathrm{supp}(v)|\geq 2k+i+j+1$.

Proof. Just need to show $\subseteq$ direction. Induction on k, $k=0$ follows from L8.
For fixed $k\geq 1$, induction on (i,j), the base case is when $i=0$ or $j=0$.
Suppose $f\in\mathrm{Ker}_k(u^iv^j)$. Let $S=\mathrm{supp}(u)\cup\mathrm{supp}(v)$, for any $x\notin S$, $fu^iv^j=0$ gives $(E_xf)u^iv^j=0=(\partial_xf)u^iv^j$, then $\partial_xf\in\mathrm{Ker}(u^i)+\mathrm{Ker}(v^j)$ by induction hypothesis. Since $f=(\partial_xf)x+E_xf$, it suffices to prove the statement for E_xf. So wma S is the entire universe of variables.

If both u and v has full support S, done by Theorem D3.
If neither u nor v has full support. Since $|\mathrm{supp}(u)\cup\mathrm{supp}(v)|\geq i+j+3$, either $|\mathrm{supp}(u)|\geq i+2$ or $|\mathrm{supp}(v)|\geq j+2$, wlog assume the latter, then pick $x\notin\mathrm{supp}(u)$.
If wlog, v has full support, u does not, then pick $x\notin\mathrm{supp}(u)$.
Write ∂_x, E_x as ∂, E for notation simplicity, wma $\partial v=1$.

take ∂ to $fu^iv^j=0$, we have $u^i(Ev)^{j-1}(jEf+Ev\partial f)=0$ (1)
By induction hypothesis of $(i,j-1,k)$,
we have $jEf+Ev\partial f\in\mathrm{Ker}(u^i)+\mathrm{Ker}(Ev^{j-1})\subseteq\mathrm{Ker}(u^i)+\mathrm{Ker}(v^j)$ (2)
here $\mathrm{Ker}(Ev^{j-1})\subseteq\mathrm{Ker}(v^j)$ because $v^j=(Ev+x)^j=Ev^{j-1}(Ev+jx)$.

Multiply (1) by Ev, we get $u^i(Ev)^{j+1}\partial f=0$. It is easy to verify u,Ev satisfy the support condition of induction hypothesis of $(i,j+1,k-1)$,
we have $\partial f=a+b$, where $a\in\mathrm{Ker}_{k-1}(u^i)$, $b\in\mathrm{Ker}_{k-1}(Ev^{j+1})$.
Then $b(Ev-jx)v^j=b(Ev-jx)Ev^{j-1}(Ev+jx)=bEv^{j+1}=0$
so $b(Ev-jx)\in\mathrm{Ker}(v^j)$, and also $a(Ev-jx)\in\mathrm{Ker}_k(u^i)$
so $\partial f(Ev-jx)\in\mathrm{Ker}(u^i)+\mathrm{Ker}(v^j)$ (3)
Finally (2)-(3) $f\in\mathrm{Ker}(u^i)+\mathrm{Ker}(v^j)$ $\square$

This theorem gives $h=2$ case of:

Conjecture D6.1 Suppose u_i $(1 \le i \le h)$ are linear forms, and $|\bigcup_{i \in S} \mathrm{supp}(u_i)| \ge 2k + 2|S| - 1$ for any $S \subseteq [h]$, then $\mathrm{Ker}_k(\prod_{i=1}^{h} u_i) = 0$.

Conjecture D6.2 Suppose u_i $(1 \le i \le h)$ are linear forms, $a_i > 0$, and $|\bigcup_{i \in S} \mathrm{supp}(u_i)| \ge 2k + \sum_{i \in S} a_i + |S| - 1$ for any $S \subseteq [h]$, then $\mathrm{Ker}_k(\prod_{i=1}^{h} u_i^{a_i}) = 0$.

Hodge Riemann Relation

In this section, we assume the ground field is $\mathbb{C}$, and $\mathbb{CP}^1$ is the complex projective line. For $f \in A$, define its *conjugate* $\bar{f}$ in the only plausible way.

Let e be the unit linear form, define $e_0 = 1$ and $e_i = e^i / i!$ when $i > 0$.

Lemma R1 $f = \sum_{i=0}^{\min\{m,n-m\}} (-1)^i D(f e_{n-m-i})\, e_{m-i}$ for all $f \in A^n_m$

Proof. Fix any $M \subseteq [n]$ with $|M| = m$. Fix i with $i \le m$ and $i \le n - m$.

$\partial_M(D(f e_{n-m-i}) e_{m-i}) = \sum_S \partial_S D(f e_{n-m-i}) = \sum_S \partial_{[n] \setminus S}(f e_{n-m-i}) = \sum_S \sum_T \partial_T f$

where S run over all i-set of M, and T run over all m-set of $[n] \setminus S$.

Now let i vary, coefficient of X_M in the RHS of this lemma is:

$\sum_S \sum_T (-1)^{|S|} \partial_T f = \sum_T \partial_T f \sum_S (-1)^{|S|}$ where $S \subseteq M$, $S \cap T = \varnothing$, $|T| = m$

For fixed $T \ne M$, $\sum_S (-1)^{|S|} = 0$. Take $T = M$, we get $\partial_M f$. □

Theorem R2 Suppose $n = 2k + h$, then the hermitian form $\langle f, g \rangle := (-1)^k f \bar{g} e^h$ defined on A^n_k is positive definite on $\mathrm{Ker}_k(e^{h+1})$.

This is known as the Hodge Riemann Relation.

Proof. By above Lemma, $f e^{h+1} = 0 \longrightarrow f = (-1)^k D(f e_h) \longrightarrow Df = (-1)^k f e_h$ then $\langle f, f \rangle = (-1)^k f e^h \bar{f} = h!\,(Df)\bar{f} \ge 0$, equality iff $f = 0$. □

Result R3.1 $\mathrm{Ker}_k(u^2 e) \cap \mathrm{Ker}_k(u e^2) = 0$ for any linear form u in A^{2k+3} with full support and $\partial_i u \in \mathbb{R}$ for all i.

Conjecture R3.2 $\mathrm{Ker}_k(u^2 e) \cap \mathrm{Ker}_k(u e^2) = 0$ for any linear form u in A^{2k+3} with $\partial_i u \ne \partial_j u$ for all $i \ne j$, where $k \ge 1$.

remark, S3 gives $\dim(\mathrm{Ker}_k(u^2 e) \cap \mathrm{Ker}_k(u e^2)) \le 1$.

Let $M = (\mathbb{CP}^1)^n$, it is a Kähler manifold whose cohomology ring is A^n with $A^n_k = H^{2k}(M, \mathbb{C}) = H^{k,k}(M)$. Let π_i be the projection of M to the i-th component, c_i be the Fubini-Study Kähler class of the i-th copy of $\mathbb{CP}^1$. We naturally identify $x_i \in A_1$ with the pullback $\pi_i^*(c_i)$. Then a linear form $u \in A_1$ is a Kähler class of M iff $\partial_i u > 0$ for all i.

Apply "Theorem A" in Dinh and Nguyen [5] to M with $p = q = k$, we get the following called the mixed Hodge Riemann Relation, which instantly gives D4.

Theorem R4 Suppose u_i $(0 \le i \le h)$ are Kähler classes in A^n and $n = 2k + h$, then the Hermitian form $\langle f, g \rangle := (-1)^k f \bar{g} \prod_{i=1}^{h} u_i$ defined on A^n_k is positive definite on $\mathrm{Ker}_k(\prod_{i=0}^{h} u_i)$. As a corollary, $\mathrm{Ker}_k(\prod_{i=1}^{h} u_i) = 0$.

Corollary R5 Suppose all elements of $P_{(2k+h) \times h}$ are positive real numbers, and $Q_{(2k+h) \times k}$ has full perrank, then $(P \;\; Q)$ has full perrank.

The case $h = 1$ also follows from Theorem L1, the case $h = 2$ also follows from Theorem D4.

Let u, v be linear forms in A^{2k+1} with full support, we work in A^{2k+2} with extra variable y. Point in $\mathbb{CP}^1$ is often written as (a, b), with $(a, b) = (ra, rb)$ for $r \ne 0$.

For $(a, b) \in \mathbb{CP}^1$, let $K(a, b) := E_y(\mathrm{Ker}_k((u + ay)(v + by)))$, well defined.

If $K(a, b) \ne 0$, then we say $K(a, b)$ is the eigenspace of eigenvalue (a, b).

Conjecture R6 For any linear forms $u, v \in A^{2k+1}$ with full support, there exist $(a, b) \in \mathbb{CP}^1$ such that $K(a, b) = 0$.

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Result R7.1 For any Kähler classes $u, v \in A^{2k+1}$, $\mathrm{Ker}_k(uv)$ is the direct sum of all eigenspaces and every eigenvalue (a,b) satisfies $b/a < 0$. Moreover:

Conjecture R7.2 If there is only one eigenvalue (a,b), then $bu + av = 0$.

Minimum Support

For subspaces $U \subseteq A_i$ and $V \subseteq A_j$, define UV to be the subspace of A_{i+j} spanned by $\{uv : u \in U,\ v \in V\}$. Define $\mathrm{Ker}(U) = \{f : fu = 0\ \ \forall u \in U\}$; $\mathrm{Im}(U)$ to be the ideal generated by U.

A subspace of A_1 is also called a *linear form space*. For a linear form space U, define its support $\mathrm{supp}(U) := \bigcup\limits_{u \in U} \mathrm{supp}(u)$; define its *minimum support function*

$\mathrm{ms}_i(U) := \min\{|\mathrm{supp}(V)| : V \text{ is a subspace of } U \text{ with } \dim(V) = i\}$

Theorem S1 If U is a linear form space with $\dim(U) = n$ and $\mathrm{ms}_1(U) \geq k$, then $\dim(U^k) = \binom{n+k-1}{k}$.

Proof. Obvious when $n = 1$ or $k = 1$, induction, now assume $n \geq 2$ and $k \geq 2$.

Choose a variable x and a linear basis $\{v_1 + x, v_2, v_3, \cdots, v_n\}$ of U with $\partial_x v_i = 0$ for all $1 \leq i \leq n$. Let $V_1 = \mathrm{span}(\{v_i : i \geq 1\})$ and $V_2 = \mathrm{span}(\{v_i : i \geq 2\})$. If $v_1 \in V_2$, then $x \in U$, contradict $\mathrm{ms}_1(U) \geq 2$. So $v_1 \notin V_2$, $\dim(V_1) = n$.

Let $d = (d_1, d_2, \cdots, d_n)$ and $D = \{d \in \mathbb{Z}^n : \sum\limits_{i=1}^{n} d_i = k,\ d_i \geq 0\}$, then $|D| = \binom{n+k-1}{k}$ and U^k is spanned by $\{(v_1 + x)^{d_1} \prod\limits_{i=2}^{n} v_i^{d_i} : d \in D\}$, need to show linear independence.

Suppose $\sum\limits_{d \in D} c_d\, (v_1 + x)^{d_1} \prod\limits_{i=2}^{n} v_i^{d_i} = 0 \qquad (*)$

Take ∂_x we get $\sum\limits_{d_1 \geq 1} (d_1 c_d)\, v_1^{d_1 - 1} \prod\limits_{i=2}^{n} v_i^{d_i} = 0$. Since $V_1 = E_x(U)$, $\mathrm{ms}_1(V_1) \geq k - 1$, by induction hypothesis of $(n, k-1)$, $\{v_1^{d_1 - 1} \prod\limits_{i=2}^{n} v_i^{d_i} : d \in D,\ d_1 \geq 1\}$ is a linear basis of V_1^{k-1}. So $c_d = 0$ for all d with $d_1 \geq 1$, substitute into $(*)$, we get $\sum\limits_{d_1 = 0} c_d \prod\limits_{i=2}^{n} v_i^{d_i} = 0$.

Since $V_2 \subseteq U$, $\mathrm{ms}_1(V_2) \geq k$, by induction hypothesis of $(n-1, k)$, $\{\prod\limits_{i=2}^{n} v_i^{d_i} : d \in D,\ d_1 = 0\}$ is a linear basis of V_2^k. So $c_d = 0$ for all d with $d_1 = 0$. $\square$

Result S2 Suppose U is a linear form space,

(1) if $\mathrm{ms}_1(U) = d + 1$, $0 \neq f \in U^d$, then $fx \neq 0$ for any variable x.

(2) if $\mathrm{ms}_1(U) = d + 2$, $0 \neq f \in U^d$, then $\mathrm{Ker}_1(f) = 0$.

Result S3 If U is a linear form space with $\dim(U) = 2$ and $\mathrm{ms}_1(U) = 2k$, then $\dim(\mathrm{Ker}_{k+1}(U)) \leq 1$.

The following are some miscellaneous conjectures.

Conjecture S4 $\mathrm{Ker}_n(f) = 0$ iff $\mathrm{Ker}_n(Df) = 0$ for all $f \in A_{2n}^{4n}$

Conjecture S5 For any linear forms u, v in $A^{2k+i+j+1}$ with full support, there exists a variable x such that $\mathrm{Ker}_k(E_x(u^i v^j)) = 0$.

It instantly implies D3, true when $i = j = k = 1$.

Conjecture S6 Suppose $g \in A_j$ is a product of j linear forms, $f \in A_k$ and u is a linear form. If $fg \neq 0 \neq gu^{2k+i+j}$, then $fgu^i \neq 0$.

In other words, $u^{2k+i+j} \neq 0$ implies $\mathrm{Ker}_k(u^i) = 0$ in Gorenstein algebra $A/\mathrm{Ker}(g)$.

It implies a special case of L7.3 and D6.2 in an algebraically closed field, when all linear forms have full support. The case $j = 1$ follows from D5. The condition that g is a product of linear forms is necessary, otherwise counter example exists with $i = 1$, $j = k = 3$. Follow up in A7.2.

Conjecture S7

Suppose $M_{n \times n}$ has full termrank, and the bottom square of $N_{n \times (n-1)}$ has nonzero permanent. Then there exists a permutation matrix P such that $(M\ \ N)P = (S\ \ T)$, where $\mathrm{per}(S_{n \times n}) \neq 0$ and the bottom square of $T_{n \times (n-1)}$ has full termrank.

It implies L7.1. Here the bottom square of an $n \times m$ tall matrix is the $m \times m$ submatrix consisting of its bottom m rows. Follow up in the next section.

Analogy

We make no assumption on field characteristic in this section.

The conjectural relation between perrank and termrank in characteristic zero is analogous to that between perrank and rank in positive characteristic.

The following two conjectures are analogous to L7.1 and L6 respectively.

Main Conjecture A1 (author, 2001)

For any matrices $M_{n\times n}$, $N_{n\times(n-1)}$ with full rank, $(M\ \ N)$ has full perrank.

Conjecture A2 For any nonsingular matrix M, $(M\ \ M)$ has full perrank.

This was made by Jeff Kahn [7] and Michael Tarsi (remark on L6), motivated by:

Conjecture A3 (Francois Jaeger [8]) For any nonsingular matrix M over $\mathbb{F}_5$, there exists a vector v over $\mathbb{F}_5$, such that neither v nor Mv has zero entries.

This Conjecture was generalized by Alon and Tarsi to $\mathbb{F}_p$ for any prime $p \geq 5$. Nagy and Pach [9] recently proved it for any prime $p > 80$.

It is easy to prove the analogue of L4.2, i.e. $\text{perrank}(M) \geq \text{rank}(M)/2$ for any matrix M, see [7]. Kisley and Shader [10] recently proved that there is essentially only one matrix to make equality hold.

It is also easy to prove the analogue of S7, but the analogue does not imply A1 although S7 implies L7.1.

Conjecture A4.1 (Alon, Linial and Meshulam [6]) Suppose $B_1, B_2, \cdots, B_q$ are nonsingular matrices over a field of characteristic $p \geq 3$, and $p \geq q \geq 2$, then

$M = \begin{pmatrix} B_1 & B_2 & \cdots & B_q \\ \vdots & \vdots & & \vdots \\ B_1 & B_2 & \cdots & B_q \end{pmatrix}$, where each row repeats $q-1$ times, has full perrank.

This was stated in [6] for $q = p$ only. When $q = p$, suppose $O_1, O_2, O_3, \cdots$ is a sequence of row operations that make B_1 into I. Now inside matrix M, perform O_1 on each $(B_1\ \ B_2\ \cdots\ B_p)$, then O_2 on each $(B_1\ \ B_2\ \cdots\ B_p)$, $\cdots$ until all B_1 become I. These operations change product of row linear forms of M by a factor $\det(B_1)^{p-1}$, so wma $B_1 = I$. Let $C = (B_2\ \cdots\ B_p)$, take product of row linear forms:

$$\prod_i (x_i + \sum_j c_{ij}y_j)^{p-1} = \prod_i ((p-1)x_i + \sum_j c_{ij}y_j) \prod_i (\sum_j c_{ij}y_j)^{p-2}$$

so $\begin{pmatrix} I & C \\ I & C \\ \vdots & \vdots \\ I & C \end{pmatrix}$ has full perrank, iff $\begin{pmatrix} (p-1)I & C \\ 0 & C \\ \vdots & \vdots \\ 0 & C \end{pmatrix}$ has, iff $\begin{pmatrix} C \\ \vdots \\ C \end{pmatrix}$ has.

So the case $q = p$ is equivalent to the case $q = p-1$. The same argument shows the case $q = t$ implies the case $q = t-1$, by taking $B_1 = I$.

The case $q = 2$ differs from our main conjecture A1 by one column.

Now we formulate an equivalent statement of this conjecture.

Conjecture A4.2 Suppose U is a linear form space over a field of characteristic $p \geq 3$, $\dim(U) = n$, and $\text{ms}_i(U) \geq pi$ for all $1 \leq i \leq n$, then $U^{(p-1)n} \neq 0$.

One direction, A4.2 implies the case $q = p$ of A4.1:

Let U be the linear form space spanned by the rows of $(B_1, B_2, \cdots, B_p)$, then U satisfies the condition of A4.2, and $U^{(p-1)n} \neq 0$ implies M has full perrank.

The other direction: Let $S = \text{supp}(U)$, choose a linear basis of U, we get a matrix $M_{n\times S}$. There is a matroid structure on S with rank function $r()$.

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For any $R \subseteq S$, let $r = r(R)$, which is the rank of the $n \times R$ submatrix. Perform row operations on M to make the bottom $n - r$ rows of the $n \times R$ submatrix zero. Let $V \subseteq U$ be the subspace spanned by the bottom $n - r$ rows of M. We have

$|S \backslash R| \geq |\text{supp}(V)| \geq p(n - r) = p(n - r(R))$. Then by Edmond's matroid base packing theorem, S contains p disjoint bases, and A4.1 gives A4.2. $\square$

Next is a slightly stronger version.

Conjecture A4.3 Suppose U is a linear form space over a field of characteristic $p \geq 3$, $\dim(U) = n$, and $\text{ms}_i(U) \geq pi - 1$ for all $1 \leq i \leq n$, then $U^{(p-1)n} \neq 0$.

This is analogous to L7.3, where $h = n$, $a_i = p - 1$, and the union of support in characteristic zero is analogous to the minimum support function in positive characteristic. We make this stronger version exactly because of the analogy.

Exercise A5

(1) $\dim(\text{Ker}_1(f)) \leq k$ for any $0 \neq f \in A_k$

(2) $\dim(\text{Ker}_k(U)) \leq 1$ for any linear form space U with dimension k

(3) what f, U makes equality hold ?

Corollary A6.1

$M_{n \times (2n-1)}$ has full perrank if every $n \times n$ submatrix of M is nonsingular.

Conjecture A6.2 We may replace $2n - 1$ above with $\lceil 3n/2 \rceil$.

When $n = 4$, verified by brute force if $M_{4 \times 6}$ contains at least one zero element.

If true, $\lceil 3n/2 \rceil$ seems sharp, easy example $M_{2k \times (3k-1)}$, but example $M_{(2k+1) \times (3k+1)}$ not yet found.

Let $\{v_i\}$ be a collection of linear forms (repetition allowed), define $V_I = \prod_{i \in I} v_i$ for index set I and $V_\varnothing = 1$. The following is analogous to matroid axiom.

Conjecture A7.1 If $|T| > 2|S|$ and $V_S \neq 0 \neq V_T$, then there exists $R \subseteq T \backslash S$ with $|R| = |T| - 2|S|$, such that $V_{S \cup R} \neq 0$.

seems extremely difficult, next is a slightly stronger version

Conjecture A7.2 Suppose $i, k \geq 1$, $|J| = j \geq 0$, $|H| = i + j + 2k$, $f \in A_k$ and $f V_J \neq 0 \neq V_J V_H$, then there exist $I \subseteq H$ with $|I| = i$, such that $f V_J V_I \neq 0$.

True when $i = 1$, $j = 0$ or 1, char $= 0$ or 3. When linear forms indexed by H are identical, it becomes S6.

We present some other ideas on A2.

Conjecture A8 Let $X_{n \times n} = (x_{ij})$ be a matrix of variables, then $\det(X)^n$ is in the ideal generated by permanents of all $n \times n$ submatrices of $(X \;\; X)$.

This was made by Bernd Sturmfels [11], and Richard Wilson in 1999 when the author was a postdoc at Caltech.

Let R and S be the polynomial ring in variables $x_1, x_2, \cdots, x_n$ and $y_1, y_2, \cdots, y_n$ respectively, J be the ideal generated by $x_1 x_2 \cdots x_n$ and I be the ideal generated by $y_1^3, y_2^3, \cdots, y_n^3$, then the following statements are equivalent:

(1) there exists a nonsingular $M_{n \times n}$, $(M \;\; M)$ does not have full perrank.

(2) there exists a nonsingular $M_{n \times n} = (a_{ij})$ such that $\prod_i (\sum_j a_{ij} y_j) \in I$

(3) there exists commutative diagram:

f is an automorphism

h is a homomorphism

q is the quotient map

$$\begin{tikzcd} R \arrow[r, "f"] \arrow[d, "q"] & S \arrow[d, "q"] \\ R/J \arrow[r, "h"] & S/I \end{tikzcd}$$

(4) there exists onto homomorphism $h : R/J \longrightarrow S/I$

(1) $\longleftrightarrow$ (2) easy. (2) $\longrightarrow$ (3) $\longrightarrow$ (4) trivial.

(4) $\longrightarrow$ (2) *Proof.* Let $\widetilde{x}_i = x_i + J$, $\widetilde{y}_i = y_i + I$. Let $Y = \{\widetilde{y}_1, \widetilde{y}_2, \cdots, \widetilde{y}_n\}$.

For each i, write $z_i = h(\widetilde{x}_i)$ as a polynomial of Y, let c_i and l_i be its constant and linear component, well defined. Let $Z = \{z_1, z_2, \cdots, z_n\}$, $L = \{l_1, l_2, \cdots, l_n\}$ and $M_{n \times n}$ be the coefficient matrix of L as linear combination of Y.

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Because S/I is graded, the linear component of any polynomial of Z, viewed as a polynomial in Y, must be a linear combination of L.

Since h is onto, every $\widetilde{y}_i$ is a polynomial of Z, it follows that M is nonsingular.

For each i, $l_i \neq 0$, the lowest degree component of z_i is either c_i or l_i. Because S/I is graded, we have $\prod_{i=1}^{n} \widetilde{x}_i = 0 \to \prod_{i=1}^{n} z_i = 0 \to \prod_{c_i \neq 0} c_i \prod_{c_i=0} l_i = 0 \to \prod_{i=1}^{n} l_i = 0$ $\square$

(2) can be stated as: orbit of $y_1y_2\cdots y_n$ intersects I under the action of general linear group. Does invariant theory help?

(3) A common technique to derive a contradiction from a commutative diagram is by applying a functor, such as homology. Does such a functor exist?

Characteristic Three

In this section, we assume the ground field has characteristic three.

Theorem C1

(1) $\mathrm{Ker}_k(u) = \mathrm{Im}_k(u^2)$ for any linear form u with $|\mathrm{supp}(u)| \geq 2k+1$.

(2) $\mathrm{Ker}_k(u^2) = \mathrm{Im}_k(u)$ for any linear form u with $|\mathrm{supp}(u)| \geq 2k+2$.

Proof. Since $u^3 = 0$, one direction is trivial. The other direction is by induction on k, easy to verify when $k = 1$.

Pick any $x \in \mathrm{supp}(u)$, write ∂_x, E_x as ∂, E for notation simplicity, wma $\partial u = 1$.

(1) Suppose $f \in \mathrm{Ker}_k(u)$, $fu = 0 \longrightarrow Ef + (\partial f)(Eu) = 0 \longrightarrow (\partial f)(Eu)^2 = 0$.

By induction hypothesis of (2), $\partial f = g(Eu)$ for some g, then

$f = Ef + (\partial f)x = (\partial f)(x - Eu) = -g(Eu)(Eu - x) = -g(Eu + x)^2 = -gu^2$

(2) Suppose $f \in \mathrm{Ker}_k(u^2)$, $fu^2 = 0 \longrightarrow (2Ef + (\partial f)Eu)Eu = 0$. By (1) we have $Ef - (\partial f)Eu = g(Eu)^2$ for some g, then

$f = Ef + (\partial f)x = (\partial f)(Eu + x) + g(Eu)^2 = (\partial f)u + gu(Eu - x) \in \mathrm{Im}(u)$. $\square$

Theorem C2 For any linear form u in A^n with full support,

(1) When $n = 2k+1$, $\mathrm{Ker}_i(u^2) = \mathrm{Im}_i(u)$ for all $i \neq k$

(2) When $n = 2k+1$, $\mathrm{Ker}_i(u) = \mathrm{Im}_i(u^2)$ for all $i \neq k+1$

(3) When $n = 2k+2$, $\mathrm{Ker}_i(u^2) = \mathrm{Im}_i(u)$ for all $i \neq k+1$

(4) When $n = 2k+2$, $\mathrm{Ker}_i(u) = \mathrm{Im}_i(u^2)$ for all $i \neq k+1$

If $i = k$ in (1) or $i = k+1$ in (2) (3) (4), then the corresponding $\mathrm{Ker}_i()$ strictly contains the corresponding $\mathrm{Im}_i()$, and their dimension differ by exactly one.

Proof. (1) If $i < k$, given by C1(2). If $i > k$, C1(1) gives $\mathrm{Ker}_{n-i}(u) = \mathrm{Im}_{n-i}(u^2)$, then $\mathrm{Ker}_i(u^2) = \mathrm{Im}_i(u)$ by Lemma D1. Same proof for (2) (3) (4).

For the second part of the theorem, induction on n, easy to verify when $n \leq 3$.

Pick any variable x, write ∂_x, E_x as ∂, E for notation simplicity, wma $\partial u = 1$.

Define $\theta : \mathrm{Ker}_{k+1}(u) \longrightarrow \mathrm{Ker}_k((Eu)^2) \cap A_k^{n-1}$ by $\theta(f) = \partial f$, it is one to one, and onto, $\theta^{-1}(g) = g(x - Eu)$, then easy to verify $\theta(\mathrm{Im}_{k+1}(u^2)) = \mathrm{Im}_k(Eu) \cap A_k^{n-1}$.

The $n = 2k+1$ case (2) follows from induction hypothesis $n = 2k$ case (3).

The $n = 2k+2$ case (4) follows from induction hypothesis $n = 2k+1$ case (1).

The $n = 2k+1$ case (1) and $n = 2k+2$ case (3) follow from Lemma D1. $\square$

It would be interesting to find a representative of the extra dimension in each case.

Conjecture C3 Suppose U is a linear form space with $\mathrm{ms}_i(U) \geq 3i - 1$ for all $1 \leq i \leq n$, where $n = \dim(U)$, then $U^{2n} \neq 0$.

It is easy to see that C3 is equivalent to A1 in characteristic three.

Let u be a linear form with $\partial_x u = c \neq 0$, write ∂_x, E_x as ∂, E for simplicity. For any f, introduce a new operation: divide f by u wrt variable x, as follows

$f = Ef + (\partial f)x = Ef + (\partial f)(u - Eu)c^{-1} = c^{-1}(\partial f)u + Ef - c^{-1}(\partial f)(Eu)$

and define $R_{(u,x)}f := Ef - c^{-1}(\partial f)(Eu)$ as the *remainder*.

It is obvious $\partial(Rf) = 0$ and $f - Rf \in \mathrm{Im}(u)$, this is what we need later.

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Conjecture C4 Suppose U is a linear form space with $\mathrm{ms}_i(U) \geq 3i-1+2k$ for all $1 \leq i \leq n$, where $n = \dim(U)$, then $\mathrm{Ker}_k(U^{2n}) = \mathrm{Im}_k(U)$.

We prove the case $n=2$ only. Attempts failed when $n=3$.

Proof. Observe $U^4 = \mathrm{span}(u^2v^2)$ for any linear basis $\{u,v\}$. Let $S = \mathrm{supp}(U)$, choose a basis $\{u,v\}$ and two variables x,y such that

Case (1) $\partial_x u = 1 = \partial_y v$ and $\partial_x v = 0 = \partial_y u$ if $\mathrm{ms}_1(U) = |S|-1$.

Case (2) $\partial_x u = 1 = \partial_y u$ and $\partial_x v = 0 = \partial_y v$ if $\mathrm{ms}_1(U) \leq |S|-2$.

When $k=0$, $u^2v^2 \neq 0$ because: Case (1) $\partial_x(u^2v^2) = 2(E_x u)v^2 \neq 0$ by C1(2); Case (2) $\partial_{xy}(u^2v^2) = 2v^2 \neq 0$. Induction on k, suppose $f \in \mathrm{Ker}_k(u^2v^2)$.

Case (1) Let $g = R_{(v,y)}(R_{(u,x)}f)$, then $gu^2v^2 = 0$, take ∂_{xy}, we get $gE_{xy}(uv) = 0$. Let $E := E_{xy}$. By C1(1), $g(Eu) = h(Ev)^2$ for some h, because $|\mathrm{supp}(Ev)| \geq 2k+3$. Multiply by $(Eu)^2$, we get $hE(u^2v^2) = 0$. $E(U)$ satisfies support condition of $k-1$, by induction hypothesis, we have $h = a(Eu) + b(Ev)$ for some a, b, then $g(Eu) = a(Eu)(Ev)^2$, that is, $g - a(Ev)^2 \in \mathrm{Ker}_k(Eu) = \mathrm{Im}_k((Eu)^2)$ by C1(1).

Finally $(Ev)^2 \in \mathrm{Im}(v)$, $(Eu)^2 \in \mathrm{Im}(u)$, $f - g \in \mathrm{Im}(U)$, done.

Case (2) Let $g = R_{(u,y)}f$, then $gu^2v^2 = 0$, take ∂_y, we get $g(E_y u)v^2 = 0$, take ∂_x, $(Eg + (\partial_x g)(Eu))v^2 = 0$, where $E := E_{xy}$. So $Eg + (\partial_x g)(Eu) \in \mathrm{Ker}_k(v^2) = \mathrm{Im}_k(v)$. Multiply by Eu, we get $(\partial_x g)E(u)^2v^2 = 0$. $E(U)$ satisfies support condition of $k-1$, by induction hypothesis, we have $\partial_x g = a(Eu) + bv$ for some a, b, then

$f = g = Eg + x(\partial_x g) = -(\partial_x g)(Eu) + x(\partial_x g) = -a(Eu)(Eu-x) = -a(Eu+x)^2$
$= -a(Eu+x-y)u = 0 \quad$ modulo $\mathrm{Im}(U)$. □

We say a linear form space U *covers* $(a_1, a_2, \cdots, a_k)$ if $\mathrm{ms}_i(U) \geq a_i \ \ \forall\ 1 \leq i \leq k$.

Theorem C5 If U is a linear form space covering $(2, 5+2k)$ with $\dim(U) = 2$, then $\mathrm{Ker}_k(U^4) = \sum \mathrm{Ker}_k(V^2)$, where V runs over one dimensional subspaces of U.

same proof as above, in the last step, $f = (\partial_x g)(y-x)$ modulo $\sum \mathrm{Ker}_k(V^2)$. □

We may generalize C5 to $\dim(U) = n$ to become a stronger conjecture than C4. Two similar cases, Case (2) easy to handle, Case (1) stuck at the inductive step.

Let U be a linear form space with $\dim(U) = n \geq k/2$, it is easy to see that $\dim(U^k) \leq |\{(d_1, d_2, \cdots, d_n) \in \mathbb{Z}^n : \sum_{i=1}^{n} d_i = k \text{ and } 0 \leq d_i \leq 2\}|$. We say U^k has *full dimension* if equality holds.

Result C6.1
U^k has full dimension for any linear form space U with $\mathrm{ms}_1(U) \geq k \leq 2\dim(U)$.

condition $\mathrm{ms}_1(U) \geq k$ seems too sufficient, so we make

Conjecture C6.2 U^k has full dimension for any linear form space U with
(1) U covers $(2, 5, 8, \cdots, 3h-1)$ when $k = 2h$, or
(2) U covers $(2, 5, 8, \cdots, 3h-1, 3h+1)$ when $k = 2h+1$

This is stronger than C3, true when $k \leq 4$, difficult when $k \geq 5$.

C1, C3, C4, C5, C6.1 are analogous to L1, L7.3, D6.2, D5, S1 respectively.

Appendix: Subcases of $h = 3$ Case of L7.3

Lemma L9.1 If $a^j b^{i-1} c^k = 0 = a^{j-1} b^i c^k$ and $|\mathrm{supp}(c)| \geq i + j + k$, for linear forms a, b, c, then $a^j b^i c^{k-1} = 0$.

Proof. For every $x \in \mathrm{supp}(c)$, take ∂_x to $a^j b^{i-1} c^k = 0$, we get
$(j\partial_x a)E_x(a^{j-1}b^{i-1}c^k) + (i-1)(\partial_x b)E_x(a^j b^{i-2} c^k) + (k\partial_x c)E_x(a^j b^{i-1} c^{k-1}) = 0$,
multiply by $E_x b$, we get $E_x(a^j b^i c^{k-1}) = 0$, so $a^j b^i c^{k-1} = 0$. □

Now assume $h = 3$, write $u_1\ u_2\ u_3\ a_1\ a_2\ a_3$ as $u\ v\ w\ i\ j\ k$ for notation simplicity.

Let $S = \mathrm{supp}(u)$, if $|S| \leq i+1$, then $|\mathrm{supp}(E_S v)| \geq j$, $|\mathrm{supp}(E_S w)| \geq k$ and $|\mathrm{supp}(E_S v) \cup \mathrm{supp}(E_S w)| \geq j+k+1$, Case L8 gives $E_S(v^j w^k) \neq 0$.

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If $|S| = i$, then u^i is a monomial, $u^i v^j w^k = u^i E_S(v^j w^k) \neq 0$. If $|S| = i+1$, pick any $x \in S$, then $E_x(u^i)$ is a monomial, $E_x(u^i v^j w^k) \neq 0$, so $u^i v^j w^k \neq 0$.

So wma $|\mathrm{supp}(u)| \geq i+2$ and $|\mathrm{supp}(v)| \geq j+2$ in L9.2 and L9.3 below.

Case L9.2 $k = 1$.

Proof. If there exist $x \in \mathrm{supp}(w)$ and $x \notin \mathrm{supp}(u) \cup \mathrm{supp}(v)$, then ∂_x reduces to Case L8, wma $\mathrm{supp}(w) \subseteq \mathrm{supp}(u) \cup \mathrm{supp}(v)$, then $|\mathrm{supp}(u) \cup \mathrm{supp}(v)| \geq i+j+3$.

$\mathrm{Ker}_1(u^i v^j) = \mathrm{Ker}_1(u^i) + \mathrm{Ker}_1(v^j) = 0$ by Theorem D5. □

Theorem D5 is not needed when $|\mathrm{supp}(w)| \geq i+j+2$.

If $|\mathrm{supp}(w)| \geq i+j+3$, for every $x \in \mathrm{supp}(w)$, take ∂_x to $u^i v^j w = 0$, we get
$(i\partial_x u)E_x(u^{i-1}v^j w) + (j\partial_x v)E_x(u^i v^{j-1} w) + (\partial_x w)E_x(u^i v^j) = 0$, multiply by $E_x(uv)$,
we get $E_x(u^{i+1}v^{j+1}) = 0$, so $u^{i+1}v^{j+1} = 0$, contradict Case L8.

If $|\mathrm{supp}(w)| = i+j+2$, pick a variable $x \notin \mathrm{supp}(w)$ and wlog assume $\partial_x v \neq 0$,
take ∂_x to $u^i v^j w = 0$, multiply by $E_x u$, we get $E_x(u^{i+1}v^{j-1}w) = 0$, along with $E_x(u^i v^j w) = 0$ gives $E_x(u^{i+1}v^j) = 0$ by Lemma L9.1, contradict Case L8.

Case L9.3 $k = 2$, $|\mathrm{supp}(w)| \geq i+j+3$, there exists a variable x such that $\partial_x w = 0$ and $\partial_x u \neq 0 \neq \partial_x v$.

Proof. If there exist $y \in \mathrm{supp}(w)$ and $y \notin \mathrm{supp}(u) \cup \mathrm{supp}(v)$, then ∂_y reduces to Case L9.2, wma $\mathrm{supp}(w) \subseteq \mathrm{supp}(u) \cup \mathrm{supp}(v)$, then $|\mathrm{supp}(u) \cup \mathrm{supp}(v)| \geq i+j+4$.

Take ∂_x to $u^i v^j w^2 = 0$, we get $E_x(u^{i+1}v^{j-1}w^2) = 0 = E_x(u^{i-1}v^{j+1}w^2)$.
$E_x(u^{i+1}v^{j-1}w^2) = 0$ and $E_x(u^i v^j w^2) = 0$ gives $E_x(u^{i+1}v^j w) = 0$,
$E_x(u^{i-1}v^{j+1}w^2) = 0$ and $E_x(u^i v^j w^2) = 0$ gives $E_x(u^i v^{j+1} w) = 0$,
$E_x(u^{i+1}v^j w) = 0$ and $E_x(u^i v^{j+1} w) = 0$ gives $E_x(u^{i+1}v^{j+1}) = 0$, contradict L8. □

Case L9.4 $i = j = 2$, $|\mathrm{supp}(w)| \leq k+4$.

Proof. For each variable x, wma $\{\partial_x u, \partial_x v, \partial_x w\}$ contains at most one zero entry, else if $\partial_x u$ or $\partial_x v$ is the only nonzero entry, then ∂_x reduces to Case L9.2; if $\partial_x w$ is the only nonzero entry, take ∂_x and induction on k.

There are at least $k+6$ variables, pick $x, y \notin \mathrm{supp}(w)$,
take ∂_x to $u^2 v^2 w^k = 0$, we get $E_x(u^3 v w^k) = 0 = E_x(u v^3 w^k)$
take ∂_y to them and to $u^2 v^2 w^k = 0$, wma $\partial_y u = -1$ and $\partial_y v = 1$
$U^3 W^k = 3U^2 V W^k = 3U V^2 W^k = V^3 W^k$, here $U = E_{xy} u$, V, W similarly,
take linear combinations, $(U - \lambda V)^3 W^k = 0$, $\lambda \in \{1, \theta, -\theta\}$, here $\theta = \sqrt{-1}$.

Let $S = \mathrm{supp}(U) \cup \mathrm{supp}(V)$, $T_\lambda = \{z \in S : \partial_z(U - \lambda V) = 0\}$. If $T_\lambda = \varnothing$ for some λ, done by L8 since $|\mathrm{supp}(U - \lambda V)| = |S| \geq k+4$. Assume T_λ is nonempty for all λ.

Let $T = \{z \in S : \partial_z W = 0\}$, wma $|\mathrm{supp}(w)| \geq k+2$, so $|T| \leq 2$. T can not intersect every T_λ since $T_1, T_\theta, T_{-\theta}$ pairwise disjoint, assume wlog $T \cap T_1 = \varnothing$, then $|\mathrm{supp}(U - V) \cup \mathrm{supp}(W)| = |S| \geq k+4$, by Case L8 we must have $|\mathrm{supp}(U - V)| \leq 2$, that is, $|S \backslash T_1| \leq 2$. It follows that $|T_\theta| = |T_{-\theta}| = 1$, and $S = T_1 \cup T_\theta \cup T_{-\theta}$. But if $T \cap T_{\pm\theta} = \varnothing$, then by the same argument $|T_1| = 1$, contradiction. So $T = T_\theta \cup T_{-\theta}$.

Finally easy to check this matrix has full perrank, need L8 twice. □
(illustration $k = 3$)

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & \theta & -\theta & -1 & * \\ 1 & 1 & 1 & 1 & 1 & \theta & -\theta & -1 & * \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ a & b & c & d & e & 0 & 0 & 0 & 0 \\ a & b & c & d & e & 0 & 0 & 0 & 0 \\ a & b & c & d & e & 0 & 0 & 0 & 0 \end{pmatrix}$$

Result Case L9.5 $\begin{pmatrix} \partial_x u & \partial_y u \\ \partial_x v & \partial_y v \end{pmatrix}$ is nonsingular for every pair of variables.

Conjecture L9.6
$u^2 v^2 w = 0 = u^2 v w^2$ does not hold for linear forms $u, v, w \in A^6$ with full support.

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