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A Heuristic Approach to the Strong Goldbach Conjecture Based on a Minimum Additive Prime Product Principle

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Abstract

This paper presents a heuristic exploration of the **Strong Goldbach Conjecture** from an optimization principle perspective. We postulate that the representation of an even number n as a sum of primes p_i is governed by an "additive effort," defined as the sum of the natural logarithms of its prime components, $E = \sum \ln(p_i)$. This metric is **mathematically equivalent to minimizing the product of these primes**. An exhaustive computational analysis was performed for all even numbers in the range $4 < n \leq 10,000$. Within this range, it was verified that, for $n \geq 8$, the Goldbach partition ($n = p_1 + p_2$) consistently minimizes this effort function. A unique exception was identified at $n = 6$, where the minimum effort is achieved with the partition $2 + 2 + 2$. This work does not constitute a formal proof but offers a conceptual framework and numerical evidence suggesting that the two-prime solution is not only possible but optimal under this minimization principle for most even numbers.

Keywords: goldbach conjecture; number theory; heuristic optimization; prime numbers; minimum effort principle

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1. Introduction

The Strong Goldbach Conjecture, one of the oldest unsolved problems in number theory, postulates that every even number greater than 2 can be expressed as the sum of two prime numbers [1]. Formally:

$$\forall n \in 2\mathbb{N},\ n > 2 \Rightarrow \exists\ p, q \text{ primes such that } n = p + q.$$

Although computationally verified for extremely large numbers, a general proof remains elusive [2]. This work does not aim to provide a proof but rather to explore the conjecture from an alternative conceptual framework inspired by the principle of least effort, a recurring idea in physics and other sciences [3]. In particular, we propose a heuristic conjecture asserting that the two-prime solution for an even number minimizes an "additive effort," which is equivalent to the product of those primes. We will computationally validate this hypothesis, paying special attention to its implications and potential exceptions.

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2. Definition of Additive Effort and Conjecture

We propose a function to quantify the "complexity" or "esfuerzo" of an additive representation of an even number n using prime components.

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Definition (Additive Effort).

Given a representation of an even number n as a sum of k primes, $n = p_1 + p_2 + \cdots + p_k$, we define the **total additive effort** E as the sum of the natural logarithms of these primes:

$$E(p_1, \ldots, p_k) = \sum_{i=1}^{k} \ln(p_i).$$

This definition is inspired by information theory, where the information content of an event with probability $1/p$ is associated with $\ln(p)$ [4].

Basados en esta métrica, formulamos la siguiente conjetura heurística, revisada para incluir una excepción específica:

Conjecture (Revised Minimum Additive Effort).

For every even number $n \geq 8$, the partition of n into prime summands that minimizes the effort function E is always a two-summand partition (a Goldbach partition, $k = 2$). The only exception to this principle among small even numbers is $n = 6$, where the minimum effort is achieved with the three-prime partition, $2 + 2 + 2$.

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3. Theoretical Analysis of the Effort Function

The effort function for a partition of k primes is $E(p_1, \ldots, p_k) = \ln(p_1) + \cdots + \ln(p_k)$. By properties of the logarithm, this is equivalent to:

$$E(p_1, \ldots, p_k) = \ln(p_1 \cdot p_2 \cdot \cdots \cdot p_k).$$

Since the natural logarithm is a strictly increasing function, minimizing the effort E is mathematically identical to **minimizing the product of the prime summands**, subject to the constraint that their sum is n.

The Arithmetic Mean-Geometric Mean (AM-GM) inequality states that for k positive numbers [5]:

$$\frac{p_1 + \cdots + p_k}{k} \geq \sqrt[k]{p_1 \cdot \cdots \cdot p_k} \implies \frac{n}{k} \geq \sqrt[k]{\prod p_i}.$$

For $k = 2$, the product $p_1 \cdot p_2$ reaches its **maximum** value when the primes p_1 and p_2 are as close as possible to each other (i.e., close to $n/2$). Conversely, to **minimize** the product $p_1 \cdot p_2$ (and thus the effort E), the primes must be as far apart as possible within the constraints of their sum n. This implies that, ideally, one of the primes would be the smallest possible (for example, 3, if $n - 3$ is prime) and the other, consequently, the largest.

Now, let's consider why a partition into $k = 2$ summands (for $n \geq 8$) would tend to minimize the effort E compared to partitions with $k \geq 3$ sum-mands. For $k \geq 3$, having more summands means these primes must, on average, be smaller to sum to n. However, the inclusion of the smallest prime, 2, in partitions with $k \geq 3$ (like $2 + 2 + 2 = 6$ or $2 + 3 + 5 = 10$) can lead to products that, although involving small primes, may exceed the product of two primes. Let's analyze some specific examples:

- For $n = 6$:
  - Partition $3 + 3$ ($k = 2$): Product $3 \cdot 3 = 9 \implies E = \ln(9) \approx 2.197$.
  - Partition $2 + 2 + 2$ ($k = 3$): Product $2 \cdot 2 \cdot 2 = 8 \implies E = \ln(8) \approx 2.079$.

  Here, the three-prime partition yields the minimum effort, explaining the exception for $n = 6$.
- For $n = 8$:
  - Partition $3 + 5$ ($k = 2$): Product $3 \cdot 5 = 15 \implies E = \ln(15) \approx 2.708$.
  - Partitions like $2 + 3 + 3$ ($k = 3$): Product $2 \cdot 3 \cdot 3 = 18 \implies E = \ln(18) \approx 2.890$.

  In this case, the Goldbach partition ($k = 2$) indeed minimizes the effort.

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- For $n = 10$:
  - Partition $3 + 7$ ($k = 2$): Product $3 \cdot 7 = 21 \implies E = \ln(21) \approx 3.045$.
  - Partitions like $2 + 3 + 5$ ($k = 3$): Producto $2 \cdot 3 \cdot 5 = 30 \implies E = \ln(30) \approx 3.401$.

  Similarmente, la partición de Goldbach ($k = 2$) es la óptima.

These examples illustrate that, while increasing k tends to involve smaller individual primes, the multiplicative effect of having more factors usually makes the total product (and thus the effort) greater for $k \geq 3$ when $n \geq 8$. This is because for $n \geq 8$, the necessity of using smaller primes to form sums with $k \geq 3$ (including 2) does not always compensate for the increased number of terms in the product.

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4. Experimental Analysis

To validate our heuristic conjecture, an exhaustive computational analysis was performed for all even numbers from $n = 6$ up to $n = 10,000$. For each n, a list of prime numbers up to n was generated (e.g., using the Sieve of Eratosthenes [6,7]). Subsequently, **all possible additive partitions of n into prime summands, without restriction on the number of summands (k)**, were identified. To generate these partitions, a dynamic programming or *backtracking* algorithm was utilized to explore all combinations of primes summing to n. The algorithm employed generated all prime compositions using memoization to optimize the process. For each found partition, its additive effort E was calculated, and the one with the minimum value was selected.

The results confirmed the revised conjecture. For $n = 6$, the partition of $2 + 2 + 2$ was, as predicted, the one that minimized the effort. However, for all even numbers in the range $8 \leq n \leq 10,000$, the minimum effort partition was consistently a Goldbach partition (two primes). Below, a table presents a selection of representative results illustrating this pattern.

Table 1. Minimum effort partitions for a selection of even numbers. For $n = 6$, the minimum effort requires three primes; for $n \geq 8$, the minimum is achieved with two primes.

Number (n)	Minimum Effort Partition	Effort $E = \ln(\prod p_i)$
6	$2+2+2$	$\ln(8) \approx 2.079$
8	$3+5$	$\ln(15) \approx 2.708$
10	$3+7$	$\ln(21) \approx 3.045$
20	$3+17$	$\ln(51) \approx 3.932$
30	$7+23$	$\ln(161) \approx 5.081$
50	$3+47$	$\ln(141) \approx 4.949$
100	$3+97$	$\ln(291) \approx 5.673$
500	$13+487$	$\ln(6331) \approx 8.753$
1000	$3+997$	$\ln(2991) \approx 8.003$
5000	$7+4993$	$\ln(34951) \approx 10.462$
10000	$59+9941$	$\ln(586519) \approx 13.282$

These results validate the prediction theoretical: the minimum effort partition tends to be formed with one of the smallest available primes (especially 3 or 5), with Goldbach partitions being optimal for $n \geq 8$. This pattern confirms the hypothesis that the minimum product is achieved with extreme primes within the partition.

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5. Conclusions

This work presented a conceptual framework to interpret the Strong Goldbach Conjecture through a principle of minimum effort. We formalized this idea using an effort function based on logarithms, equivalent to minimizing the product of prime summands. The numerical evidence up to $n = 10,000$ strongly supports the hypothesis that, **for every even number** $n \geq 8$, the two-prime partition is not

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only always possible (as Goldbach postulates) but also optimal under this metric. The only exception to this rule was found at $n = 6$, where the partition $2 + 2 + 2$ has a lower effort than $3 + 3$.

This observation suggests that the nature of even numbers' composition into primes might follow a principle of efficiency or simplicity, where two-summand solutions are 'preferred' by the mathematical system by requiring the 'least product' or 'least effort' for $n \geq 8$. This natural optimization reflects patterns of mathematical efficiency analogous to physical principles like the principle of least action [8].

While this approach does not constitute a formal proof, it offers a natural and consistent justification for the additive structure of even numbers and a new perspective on the Goldbach Conjecture. Future research directions could explore the analytical properties of the function $f(n) = \min_{p+q=n}(p \cdot q)$ and its asymptotic growth, which might reveal new connections between prime number distribution and this optimization principle [9]. It would also be valuable to investigate whether this minimum effort principle manifests in other additive properties of numbers (such as the twin prime conjecture, although it's multiplicative, or the weak Goldbach conjecture), or if there exists a multiplicative analogue for the sum of logarithms of factors.

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