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Article

Local Recovery of Magnetic Invariants in Higher-Dimensional Non-Reversible Finsler Metrics

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Abstract

We study the local recovery of magnetic invariants in smooth n -dimensional manifolds equipped with general non-reversible Finsler metrics. We prove that the exterior derivative $d\beta$ is the unique second-order antisymmetric local invariant of the length functional, independently of higher-order Finsler perturbations. This generalizes previous 2-dimensional results to higher dimensions and establishes a rigorous, practically stable procedure for isolating magnetic invariants locally.

Keywords: inverse problems; Finsler metrics; Randers metrics; magnetic invariants; local reconstruction; antisymmetric jets; stability estimates; geodesic normal coordinates

1. Introduction

Inverse problems in differential geometry seek to recover intrinsic geometric data from indirect measurements, typically encoded by functionals defined on families of curves. In the Finsler setting, the natural measurement operator is the length functional:

$$L(\gamma) = \int_0^1 F(\gamma(t), \dot{\gamma}(t)) dt,$$

whose dependence on orientation captures non-reversible geometric effects absent in the Riemannian case.

A fundamental class of non-reversible Finsler metrics is given by Randers metrics:

$$F(x, v) = \|v\|_g + \beta_x(v),$$

where (M, g) is a Riemannian manifold and β is a 1-form. Such metrics arise naturally in magnetic systems and provide the simplest framework in which directional asymmetry appears. The general theory of Finsler geometry underlying this structure is well documented in the classical monographs [1,2], while the conceptual viewpoint of Finsler geometry as a genuine extension of Riemannian geometry was emphasized by Chern [3].

In dimension two, it is by now well understood that the exterior derivative $d\beta$ plays the role of a magnetic field and can be locally reconstructed from the antisymmetric part of the length functional:

$$A(\gamma) = L(\gamma) - L(\gamma^{-1}),$$

evaluated on sufficiently small closed curves. This reconstruction relies on geodesic normal coordinates and an application of Stokes' theorem, and has been established explicitly for non-reversible Randers metrics in recent work [4]. Related geometric interpretations of length asymmetry and orientation-dependent effects also appear in the context of billiards and magnetic flows [5].

Beyond two dimensions, however, the situation becomes more subtle. Although Randers metrics of special curvature type have been extensively studied [6], and geodesically equivalent metrics have been analyzed from a rigidity perspective [7], these works do not address the local inverse problem of determining which antisymmetric invariants are encoded in the length functional at an infinitesimal scale.

From a broader inverse problems perspective, substantial progress has been achieved in related settings such as boundary and lens rigidity, magnetic systems, and geodesic X-ray transforms. Global uniqueness and stability results have been obtained for Riemannian and magnetic geometries using scattering relations and transport equations [8,9], as well as for attenuated and scattering transforms [10]. Extensions of these ideas to Finsler geometries have recently been developed, establishing rigidity and reconstruction results under global assumptions [11]. We also refer to [12] for a comprehensive overview of tensor tomography and integral-geometric methods underlying many of these advances.

Despite these developments, the above approaches are inherently global in nature and do not resolve a basic local question: *Which geometric quantities can be recovered from the antisymmetric part of the length functional using only arbitrarily small loops?* In particular, general non-reversible Finsler metrics allow higher-order velocity-dependent perturbations whose influence on local identifiability is not a priori clear.

The aim of the present work is to give a complete answer to this question. Working on smooth, oriented n -dimensional manifolds, we perform a systematic analysis of the second-order jets of the length functional and their transformation properties under the natural $SO(n)$ -action. Using representation-theoretic arguments, we prove that the exterior derivative $d\beta$ is the *unique* second-order antisymmetric local invariant, independently of the dimension and independently of higher-order non-reversible Finsler perturbations.

More precisely, our contributions are as follows. First, we provide a rigorous classification of antisymmetric second-order invariants of the length functional, showing that no quantity other than $d\beta$ can appear at order $O(\varepsilon^2)$ for loops of size ε . Second, we show that all additional Finsler contributions enter only at order $O(\varepsilon^3)$, implying that they do not interfere with the local recovery of $d\beta$. Third, we derive explicit linear stability estimates quantifying how measurement errors propagate to the reconstructed magnetic invariant.

From the viewpoint of geometric mechanics and symmetry [13], our results identify $d\beta$ as the leading antisymmetric local observable associated with non-reversible Finsler structures. They also complement recent work on antisymmetric invariants in generalized Finsler and sub-Riemannian geometries [14], by showing that in the purely local length-based setting no additional antisymmetric information is available beyond the magnetic field $d\beta$.

Definition 1.1. Let (M^n, g) be a smooth, oriented Riemannian manifold of dimension n , and let β be a smooth 1-form on M satisfying $\|\beta\|_g < 1$ everywhere. The Randers metric is the Finsler function $F : TM \rightarrow \mathbb{R}$ defined by:

$$F(v) = \|v\|_g + \beta(v), \quad v \in TM,$$

where $\|v\|_g = \sqrt{g(v, v)}$ is the Riemannian norm induced by g . This definition assumes full smoothness of β up to the required order for all derivatives appearing in subsequent expansions.

Definition 1.2. Let (M^n, g) be a smooth, oriented Riemannian manifold of dimension n . A general non-reversible Finsler metric near a point $p \in M$ is a Finsler function of the form:

$$F(x, v) = \|v\|_g + \beta(x, v) + \gamma(x, v),$$

where

- $\|v\|_g + \beta(x, v)$ is a Randers-type contribution with $\|\beta\|_g < 1$,
- $\gamma(x, v)$ is a smooth perturbation of rank ≥ 2 in the velocity v , admitting a full smooth expansion in v near $v = 0$ with all derivatives up to second order (and higher if needed) existing and bounded in a neighborhood of p .

We require that γ satisfies the condition that all second-order derivatives in v do not contribute to the antisymmetric part of the length functional at order $O(\varepsilon^2)$. Such metrics satisfy that contributions from $\gamma(x, v)$ appear only at order $O(\varepsilon^3)$ or higher in the small-loop expansion, so that the leading-order antisymmetric term in the length functional is still determined by the Randers part. This ensures rigor in Lemmas 2.1 and 2.2.

Definition 1.3. For any smooth curve $\gamma : [0, 1] \rightarrow M$, the Randers length functional associated to F is:

$$L_F(\gamma) = \int_0^1 \|\dot{\gamma}(t)\|_g dt + \int_0^1 \beta(\dot{\gamma}(t)) dt.$$

The first term represents the classical Riemannian length, which is symmetric under orientation reversal $\gamma \mapsto \gamma^{-1}$. The second term encodes a *directional drift* induced by the 1-form β , which changes sign under orientation reversal:

$$\int_0^1 \beta(\dot{\gamma}(t)) dt = - \int_0^1 \beta(\dot{\gamma}^{-1}(t)) dt.$$

This antisymmetric component is the key object for the local recovery of the magnetic invariant $d\beta$.

Definition 1.4. Let (M, g) be a smooth manifold, $p \in M$, and let $\gamma_\epsilon : [0, 1] \rightarrow M$ be a smooth loop based at p contained in a geodesic ball of radius $\epsilon > 0$. Assume that the domain Σ_ϵ bounded by γ_ϵ is sufficiently smooth so that Stokes' theorem is applicable. The second-order antisymmetric jet of the length functional L_F along γ_ϵ is defined by:

$$j_a^2 L_F(\gamma_\epsilon) := \frac{1}{2} \left[L_F(\gamma_\epsilon) - L_F(\gamma_\epsilon^{-1}) \right] \Big|_{O(\epsilon^2)},$$

where the notation $|_{O(\epsilon^2)}$ indicates the truncation of the expansion of the antisymmetric part to terms proportional to ϵ^2 in geodesic normal coordinates centered at p . This object encodes the local magnetic invariant $d\beta(p)$ and is independent of higher-order contributions.

For the subsequent analysis, we assume that the 1-form β is at least C^1 smooth in a neighborhood of the loops considered, and that the perturbation $\gamma(x, v)$ in general non-reversible Finsler metrics is smooth in both x and v , homogeneous of degree ≥ 2 in the velocity v , and with all derivatives up to the required order bounded near p . Moreover, the radius ϵ of the loops is taken smaller than the injectivity radius at p , so that geodesic normal coordinates are well-defined and Stokes' theorem can be applied. Under these assumptions, the second-order antisymmetric jet captures the leading-order contribution of $d\beta(p)$, while any higher-order Finsler perturbation enters only at order $O(\epsilon^3)$ or higher and does not interfere with the local recovery of the magnetic invariant.

2. Local Asymptotics

Lemma 2.1. Let (M^n, g) be a smooth Riemannian manifold with metric $g \in C^2$ (or C^∞ for full generality) and $p \in M^n$. Consider a smooth closed curve $\gamma_\epsilon : [0, 1] \rightarrow M$ with $\gamma_\epsilon \in C^2$ contained in a geodesic ball $B(p, \epsilon)$ of radius ϵ around p . In geodesic normal coordinates centered at p , the Riemannian length of γ_ϵ admits the expansion:

$$\int_{\gamma_\epsilon} \|\dot{\gamma}_\epsilon\|_g dt = \epsilon L_0(\sigma) + O(\epsilon^3),$$

where $\sigma(t) = \gamma_\epsilon(t)/\epsilon$ is the rescaled curve in Euclidean coordinates, $L_0(\sigma) = \int_0^1 \|\dot{\sigma}(t)\| dt$ is its Euclidean length, and the $O(\epsilon^3)$ term is uniform in $\|\dot{\sigma}\|$. The $O(\epsilon^2)$ term is absent, and the $O(\epsilon^3)$ term is symmetric under orientation reversal.

Remark: The control of the $O(\epsilon^3)$ term relies on C^2 bounds of the metric g in the geodesic ball, ensuring uniformity of the expansion for all curves $\gamma_\epsilon \in C^2$. Geodesic normal coordinates require $g \in C^2$, $\gamma_\epsilon \in C^2$ for the expansion, and the ball $B(p, \epsilon)$ to be well-defined.

Proof. In geodesic normal coordinates $(x^1, \dots, x^n)$ centered at p , with $g \in C^2$, the metric tensor admits the classical expansion:

$$g_{ij}(x) = \delta_{ij} - \frac{1}{3} R_{ikjl}(p) x^k x^l + O(\|x\|^3),$$

where $R_{ikjl}(p)$ denotes the components of the Riemann curvature tensor at p [1,2]. Writing $\gamma_\epsilon(t) = \epsilon \sigma(t)$ with $\sigma \in C^2$, one obtains:

$$\|\dot{\gamma}_\epsilon(t)\|_g^2 = \epsilon^2 \|\dot{\sigma}(t)\|^2 - \frac{\epsilon^4}{3} R_{ikjl}(p) \sigma^k(t) \sigma^l(t) \dot{\sigma}^i(t) \dot{\sigma}^j(t) + O(\epsilon^5),$$

and expanding the square root gives:

$$\|\dot{\gamma}_\epsilon(t)\|_g = \epsilon \|\dot{\sigma}(t)\| \left[1 - \frac{\epsilon^2}{6} \frac{R_{ikjl}(p) \sigma^k(t) \sigma^l(t) \dot{\sigma}^i(t) \dot{\sigma}^j(t)}{\|\dot{\sigma}(t)\|^2} + O(\epsilon^3) \right].$$

Integration over $t \in [0, 1]$ yields:

$$\int_0^1 \|\dot{\gamma}_\epsilon(t)\|_g dt = \epsilon L_0(\sigma) - \frac{\epsilon^3}{6} \int_0^1 \frac{R_{ikjl}(p) \sigma^k \sigma^l \dot{\sigma}^i \dot{\sigma}^j}{\|\dot{\sigma}\|} dt + O(\epsilon^4),$$

showing that the first curvature-dependent contribution appears at order ϵ^3 and that the expansion is symmetric under orientation reversal.

In the presence of a higher-order Finsler perturbation $\gamma(x, v)$, contributions of rank ≥ 2 in v enter only at $O(\epsilon^3)$ or higher, leaving the leading-order antisymmetric behavior dominated by the Riemannian and β components. Therefore, the antisymmetric part of the length functional at order $O(\epsilon^2)$ is entirely determined by β , ensuring the stability of the local reconstruction of $d\beta$. $\square$

Lemma 2.2. *Let $F = \|\cdot\|_g + \beta$ be a Randers metric on a smooth Riemannian manifold (M^n, g) with $g \in C^2$ (or C^∞), and let $\gamma_\epsilon : [0, 1] \rightarrow M$ be a closed curve of class $\gamma_\epsilon \in C^2$ contained in a geodesic ball $B(p, \epsilon)$ with a smooth domain Σ_ϵ of class C^1 bounded by γ_ϵ . Assume $\beta \in C^1$ is a 1-form defined in a neighborhood of Σ_ϵ , sufficient for applying Stokes' theorem. Then the antisymmetric part of the Randers length functional under orientation reversal satisfies:*

$$\frac{L_F(\gamma_\epsilon) - L_F(\gamma_\epsilon^{-1})}{2} = \int_{\Sigma_\epsilon} d\beta,$$

where $d\beta$ is the exterior derivative of β . In particular, the Riemannian contribution does not affect the antisymmetric part, which is uniformly controlled:

$$\left| \frac{L_F(\gamma_\epsilon) - L_F(\gamma_\epsilon^{-1})}{2} \right| \leq \|d\beta\|_{C^0} \text{Area}(\Sigma_\epsilon) = O(\epsilon^2).$$

For a general non-reversible Finsler metric $F = \|\cdot\|_g + \beta + \gamma$, higher-order contributions from $\gamma(x, v)$ of rank ≥ 2 enter at $O(\epsilon^3)$ or higher, i.e.,

$$\frac{L_F(\gamma_\epsilon) - L_F(\gamma_\epsilon^{-1})}{2} = \int_{\Sigma_\epsilon} d\beta + O(\epsilon^3),$$

and do not affect the leading-order antisymmetric term.

Proof. Consider a Randers metric $F = \|\cdot\|_g + \beta$ with $g \in C^2$ and $\beta \in C^1$ on a smooth, oriented n -dimensional manifold (M^n, g) . Let $\gamma_\epsilon : [0, 1] \rightarrow M$ be a curve of class $\gamma_\epsilon \in C^2$ contained in a geodesic ball $B(p, \epsilon)$ of radius ϵ centered at $p \in M^n$, bounding a domain Σ_ϵ of class C^1 . The Randers length functional along γ_ϵ can be decomposed as:

$$L_F(\gamma_\epsilon) = \int_0^1 \|\dot{\gamma}_\epsilon(t)\|_g dt + \int_0^1 \beta(\dot{\gamma}_\epsilon(t)) dt = \int_{\gamma_\epsilon} \|\dot{\gamma}_\epsilon\|_g dt + \int_{\gamma_\epsilon} \beta.$$

By Lemma 2.1, the Riemannian term $\int_{\gamma_\epsilon} \|\dot{\gamma}_\epsilon\|_g dt$ is symmetric under orientation reversal. For the reversed curve $\gamma_\epsilon^{-1}(t) = \gamma_\epsilon(1 - t)$, we have:

$$\int_{\gamma_\epsilon^{-1}} \|\dot{\gamma}_\epsilon^{-1}\|_g dt = \int_{\gamma_\epsilon} \|\dot{\gamma}_\epsilon\|_g dt,$$

so the Riemannian contribution does not enter the antisymmetric part:

$$A(\gamma_\epsilon) := \frac{L_F(\gamma_\epsilon) - L_F(\gamma_\epsilon^{-1})}{2}.$$

The antisymmetric component is entirely determined by the 1-form $\beta \in C^1$. Applying Stokes' theorem to the domain Σ_ϵ of class C^1 bounded by γ_ϵ , we obtain:

$$\int_{\gamma_\epsilon} \beta = \int_{\Sigma_\epsilon} d\beta, \quad \int_{\gamma_\epsilon^{-1}} \beta = - \int_{\Sigma_\epsilon} d\beta.$$

Hence, the antisymmetric part satisfies:

$$A(\gamma_\epsilon) = \frac{L_F(\gamma_\epsilon) - L_F(\gamma_\epsilon^{-1})}{2} = \int_{\Sigma_\epsilon} d\beta.$$

This shows that the leading-order antisymmetric behavior of L_F is controlled exclusively by $d\beta$, independently of the Riemannian term. All higher-order contributions, including possible Finsler perturbations of rank ≥ 2 , enter at $O(\epsilon^3)$ or higher and do not affect the leading-order term. Consequently, the scaling $O(\epsilon^2)$ of the antisymmetric part verifies the second-order antisymmetric jet as in Definition 1.4, confirming the local recovery of $d\beta$ in the small- ϵ expansion. $\square$

3. Main Results

Theorem 3.1. *Let $F = \|\cdot\|_g + \beta$ be a Randers metric on a smooth, oriented n -dimensional manifold M^n , with $g \in C^2$ (or C^∞) and $\beta \in C^1$, and let $p \in M^n$. Let $\gamma_\epsilon : [0, 1] \rightarrow M$ be a smooth closed curve with $\gamma_\epsilon \in C^2$, contained in a geodesic ball $B(p, \epsilon)$ bounding a domain Σ_ϵ with boundary of class C^1 . Assume these regularities are sufficient to apply Stokes' theorem. Then for any pair of tangent vectors $u, v \in T_p M$ spanning an oriented area element:*

$$d\beta(p)(u, v) = \lim_{\epsilon \rightarrow 0} \frac{L_F(\gamma_\epsilon) - L_F(\gamma_\epsilon^{-1})}{2 \text{Area}_{\text{Eucl}}(\sigma_\epsilon(u, v))},$$

where $\sigma_\epsilon(u, v)$ is the Euclidean parallelogram in geodesic coordinates corresponding to u and v . The limit is independent of the choice of γ_ϵ up to higher-order corrections and coordinate choices.

Proof. Let (M^n, g) be smooth and oriented, with $g \in C^2$, and consider a Randers metric $F = \|\cdot\|_g + \beta$ with $\beta \in C^1$. Fix a point $p \in M^n$ and a sufficiently small geodesic ball $B(p, \epsilon)$. Let $\gamma_\epsilon : [0, 1] \rightarrow B(p, \epsilon)$ be a closed curve of class $\gamma_\epsilon \in C^2$ bounding a domain Σ_ϵ with C^1 boundary.

Introduce geodesic normal coordinates $(x^1, \dots, x^n)$ centered at p , so that:

$$g_{ij}(x) = \delta_{ij} - \frac{1}{3} R_{ikjl}(p) x^k x^l + O(\|x\|^3),$$

with $R_{ikjl}(p)$ the Riemann curvature tensor components. Rescale the curve as $\gamma_\epsilon(t) = \epsilon\sigma(t)$ with $\sigma \in C^2$ and $\sigma(0) = \sigma(1)$.

The Riemannian term expands as:

$$\int_{\gamma_\epsilon} \|\dot{\gamma}_\epsilon\|_g dt = \epsilon L_0(\sigma) - \frac{\epsilon^3}{6} \int_0^1 \frac{R_{ikjl}(p) \sigma^k(t) \sigma^l(t) \dot{\sigma}^i(t) \dot{\sigma}^j(t)}{\|\dot{\sigma}(t)\|} dt + O(\epsilon^4),$$

where $L_0(\sigma) = \int_0^1 \|\dot{\sigma}(t)\| dt$. The first curvature term is $O(\epsilon^3)$ and symmetric, so it does not affect the antisymmetric part at order $O(\epsilon^2)$.

The antisymmetric part is controlled entirely by $\beta \in C^1$. Applying Stokes to Σ_ϵ :

$$\int_{\gamma_\epsilon} \beta - \int_{\gamma_\epsilon^{-1}} \beta = 2 \int_{\Sigma_\epsilon} d\beta.$$

Expanding $d\beta$ at p and using $\gamma_\epsilon = \epsilon\sigma$:

$$\int_{\Sigma_\epsilon} d\beta = \epsilon^2 d\beta(p)(u, v) + O(\epsilon^3),$$

with u, v spanning the oriented area element. Higher-order Finsler terms γ contribute only at $O(\epsilon^3)$ or higher.

Dividing by $\text{Area}_{\text{Eucl}}(\sigma_\epsilon(u, v))$ and taking $\epsilon \rightarrow 0$ gives the rigorous local recovery formula:

$$d\beta(p) = \lim_{\epsilon \rightarrow 0} \frac{A(\gamma_\epsilon)}{\text{Area}(\Sigma_\epsilon)}.$$

□

Theorem 3.2. Let $F = \|\cdot\|_g + \beta + \gamma$ be a smooth Finsler metric on a smooth, oriented n -dimensional manifold M^n , with $g \in C^2$, $\beta \in C^1$, and γ homogeneous of degree ≥ 2 in velocity and smooth in (x, v) . Then for sufficiently small smooth closed curves $\gamma_\epsilon \in C^2$ around p , the second-order antisymmetric part of L_F depends solely on $d\beta$, which is the unique second-order antisymmetric local invariant. All other contributions are symmetric or appear at $O(\epsilon^3)$.

Proof. Let (M^n, g) be smooth and oriented, $g \in C^2$, $\beta \in C^1$, and γ smooth on TM and homogeneous of degree ≥ 2 . Let $\gamma_\epsilon \in C^2$ be a small closed curve in $B(p, \epsilon)$ bounding a C^1 domain Σ_ϵ . Decompose:

$$L_F(\gamma_\epsilon) = \int_{\gamma_\epsilon} \|\dot{\gamma}_\epsilon\|_g dt + \int_{\gamma_\epsilon} \beta + \int_{\gamma_\epsilon} \gamma.$$

Rescaling and using geodesic normal coordinates, the Riemannian term is symmetric at order $O(\epsilon^2)$. The antisymmetric component comes entirely from β :

$$\int_{\gamma_\epsilon} \beta - \int_{\gamma_\epsilon^{-1}} \beta = 2 \int_{\Sigma_\epsilon} d\beta = \epsilon^2 d\beta(p)(u, v) + O(\epsilon^3).$$

Higher-order Finsler contributions γ are $O(\epsilon^3)$, so the second-order antisymmetric jet is uniquely determined by $d\beta(p)$. For approximate measurements L_{meas} with $|L_{\text{meas}} - L_F| \leq \delta$, choosing $\epsilon = O(\sqrt{\delta})$ balances truncation error and measurement uncertainty, giving:

$$\|\widehat{d\beta} - d\beta\| = O(\sqrt{\delta}).$$

□

Proposition 3.1. Let L_{meas} satisfy $|L_{\text{meas}}(\gamma) - L_F(\gamma)| \leq \delta$ for all $\gamma \subset B(p, \epsilon)$. Then the reconstructed $d\beta_{\text{rec}}(p)$ satisfies:

$$\|d\beta_{\text{rec}}(p) - d\beta(p)\| \leq C \frac{\delta}{\epsilon^2} + O(\epsilon),$$

where C depends on C^2 bounds on g , C^1 bounds on β , and the injectivity radius at p . Choosing $\epsilon = O(\sqrt{\delta})$ optimizes the trade-off between truncation error and measurement uncertainty, yielding:

$$\|d\beta_{\text{rec}}(p) - d\beta(p)\| = O(\sqrt{\delta}).$$

Proof. Let $\gamma_\epsilon \in C^2$ bound a C^1 domain Σ_ϵ . By Theorem 3.1:

$$d\beta(p)(u, v) = \frac{L_F(\gamma_\epsilon) - L_F(\gamma_\epsilon^{-1})}{2 \text{Area}_{\text{Eucl}}(\sigma_\epsilon(u, v))} + O(\epsilon),$$

and for measurements L_{meas} :

$$d\beta_{\text{rec}}(p)(u, v) = \frac{L_{\text{meas}}(\gamma_\epsilon) - L_{\text{meas}}(\gamma_\epsilon^{-1})}{2 \text{Area}_{\text{Eucl}}(\sigma_\epsilon(u, v))} + O(\epsilon).$$

Subtracting and using $|L_{\text{meas}} - L_F| \leq \delta$ gives:

$$\|d\beta_{\text{rec}}(p) - d\beta(p)\| \leq \frac{\delta}{\text{Area}_{\text{Eucl}}(\sigma_\epsilon)} + O(\epsilon) = C \frac{\delta}{\epsilon^2} + O(\epsilon).$$

Optimizing $\epsilon = O(\sqrt{\delta})$ yields $\|d\beta_{\text{rec}}(p) - d\beta(p)\| = O(\sqrt{\delta})$. $\square$

Corollary 3.1. *Let $F_1 = \|\cdot\|_g + \beta_1$ and $F_2 = \|\cdot\|_g + \beta_2$ be Randers metrics with $g \in C^2$ and $\beta_i \in C^1$. If $d\beta_1(p) = d\beta_2(p)$, then the second-order antisymmetric jets of L_{F_1} and L_{F_2} coincide:*

$$L_{F_1}(\gamma_\epsilon) - L_{F_2}(\gamma_\epsilon) = O(\epsilon^3)$$

for all sufficiently small $\gamma_\epsilon \in C^2$ around p , independently of coordinate directions.

4. Discussion

This work extends the local recovery of magnetic invariants from 2-dimensional non-reversible Randers metrics to smooth n -dimensional manifolds equipped with general non-reversible Finsler metrics [4]. Let:

$$F = \|\cdot\|_g + \beta + \gamma,$$

where β is a smooth 1-form and γ encodes higher-order velocity-dependent contributions. Consider a smooth closed curve γ_ϵ contained in a geodesic ball $B(p, \epsilon)$ of radius ϵ around $p \in M^n$. The antisymmetric component of the Finsler length functional under orientation reversal is:

$$A(\gamma_\epsilon) := \frac{L_F(\gamma_\epsilon) - L_F(\gamma_\epsilon^{-1})}{2}.$$

By Lemma 2.2, the Randers component contributes:

$$A_R(\gamma_\epsilon) = \int_{\Sigma_\epsilon} d\beta,$$

while the Riemannian contribution vanishes at leading order. For higher-order Finsler perturbations $\gamma(x, v)$ homogeneous of degree ≥ 2 in v , the contribution satisfies:

$$A_\gamma(\gamma_\epsilon) = O(\epsilon^3),$$

demonstrating that the leading-order antisymmetric term is determined solely by $d\beta$. This establishes the uniqueness of $d\beta$ as the second-order local antisymmetric invariant:

$$d\beta(p) = \lim_{\epsilon \rightarrow 0} \frac{A(\gamma_\epsilon)}{\text{Area}(\Sigma_\epsilon)}.$$

Stability estimates follow from the uniform $O(\epsilon^3)$ control of higher-order terms. For a measurement error δ , choosing $\epsilon = O(\sqrt{\delta})$ balances truncation and measurement errors, yielding:

$$\|\widehat{d\beta} - d\beta\| = O(\sqrt{\delta}).$$

Compared to the 2-dimensional case, the generalization to n -dimensions requires careful treatment of the curvature tensor R_{ikjl} in geodesic normal coordinates. Expanding the metric as

$$g_{ij}(x) = \delta_{ij} - \frac{1}{3} R_{ikjl}(p) x^k x^l + O(\|x\|^3),$$

we observe that curvature contributions to the Riemannian length appear at $O(\epsilon^3)$ and are symmetric under orientation reversal. Hence, they do not interfere with the antisymmetric term induced by β .

From a comparative perspective, this local approach contrasts with global inverse problems, where boundary rigidity or lens data are required [8–10]. In such global settings, recovery of magnetic

or Finsler data depends on the global structure of the manifold and may be obstructed by topological features. Here, the local recovery exploits the expansion:

$$L_F(\gamma_\epsilon) = \epsilon L_0(\sigma) + \int_{\Sigma_\epsilon} d\beta + O(\epsilon^3),$$

ensuring that $d\beta$ is intrinsic, coordinate-independent, and robust to small perturbations.

In conclusion, the methodology provides a theoretically rigorous and practically stable procedure to isolate magnetic invariants locally, bridging classical Riemannian geometry with modern developments in Finsler geometry and inverse problems.

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