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Article

Fundamental Dual-Phase Theory of Identical Particles: The Existence Conditions for Non-Abelian Anyons in 2D Topology (Series II)

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Abstract

This work uncovers an essential dual-phase structure of identical particles, revealing a fundamental limitation in existing quantum statistical theories. For a considerable period, it was commonly assumed that an interchange of two particles would be accompanied by a fixed phase factor. Here, the author discovers that the exchange process inherently involves two distinct phases: a relative phase α and an exchange phase β , rigorously related through $\alpha = \beta^{-1} = e^{\pm i\theta}$. Specifically, $\alpha = \beta = 1$ characterizes bosons, while $\alpha = \beta = -1$ describes fermions. This finding necessitates a radical change in handling identical particles. Within this new framework, clockwise exchange yields $\alpha = e^{i\theta}$ and $\beta = e^{-i\theta}$, while counter-clockwise exchange gives $\alpha = e^{-i\theta}$ and $\beta = e^{i\theta}$. Crucially, both scenarios, despite their topological differences, are physically equivalent, governed by the universal constraint $\alpha\beta = 1$ for all identical particles. This framework not only fundamentally changes the conceptual basis of quantum statistics but also provides critical theoretical constraints on the existence conditions of non-Abelian anyons in two-dimensional topological systems.

Keywords: identical particles; quantum statistics; exchange phase; non-abelian anyons

1. Introduction

This is the second installment of the series on the theory of identical particles. In part I [1], we introduced an operator formalism to address the symmetry challenges [2–4] inherent to identical particles, rigorously demonstrating the physical equivalence between the operator formalism [1] and the wave function formalism. In particular, the operator formalism offers a more efficient alternative to traditional wave function techniques. For instance, while the wave function formalism [4–7] requires constructing explicit symmetric or antisymmetric wave functions, the operator formalism solves this challenge directly through the application of an operator. In the present work, our focus shifts to the phase dynamics of identical particles, with the goal of addressing quantum phase-related issues in many-body systems and statistical mechanics [8–12]. Building on the operator formalism established in Series I, this work extends the existing theory to the foundational phase dynamics of identical particles, ultimately providing a more rigorous theoretical foundation for quantum many-body systems.

The statistical behavior of identical particles is essentially characterized by the phase factor $e^{i\theta}$ acquired by the wave function upon particle exchange. In conventional three-dimensional systems, this phase is restricted to discrete values: $\theta = 0$ for bosons and $\theta = \pi$ for fermions, giving rise to the well-established Bose-Einstein and Fermi-Dirac statistics. A profound generalization emerges in two-dimensional systems [13–17], where the phase θ can assume any continuous value between 0 and π , defining the novel particles known as anyons [18,19]. Unlike their bosonic and fermionic counterparts, the exchange statistics of anyons are not described by a unique phase. Crucially, the acquired phase depends on both the direction of the exchange rotation and the number of times one particle circulates around another. This path dependence implies that, under parity or time-reversal transformations [20–22], a system of anyons with statistical phase $e^{i\theta}$ is transformed into a system

with phase $e^{-i\theta}$, a distinct physical state except for special cases of bosons and fermions. This Abelian framework naturally extends to an even more exotic possibility: non-Abelian braiding statistics [23–28]. Here, the act of exchanging particles is represented not by commutative phase factors, but by non-commutative matrices that operate in the subspace of degenerate ground states [11,29–31]. The statistical properties of these exotic quasiparticles are profoundly intertwined with topology. In a system of non-Abelian anyons, each braiding operation is interpreted as a topological transformation, where different sequences of exchanges correspond to distinct paths in the configuration space. Owing to this global topological nature, the resulting transformations are remarkably robust against local perturbations. This inherent topological stability forms the cornerstone of their proposed application in fault-tolerant topological quantum computation [32–36], where quantum information is encoded in the non-local state space of anyons and processed through their braiding operations. Despite considerable theoretical and experimental advances in the pursuit of non-Abelian anyons, conclusive verification of non-Abelian anyons remains an outstanding challenge. This enduring elusiveness is exemplified by the ongoing debate over key experimental signatures, as evidenced by studies of intricate interferometric patterns in fractional quantum Hall systems [37–42] and zero-bias conductance peaks in topological superconductors [43–48]. These results remain subject to alternative interpretations and have not yet been universally accepted as definitive proof [49–53].

This persistent ambiguity stems from a fundamental shortcoming in the prevailing theoretical framework for lower-dimensional systems: its reliance on a single path-dependent exchange phase [54–56]. This conventional approach is insufficient to fully capture the physical intricacies of particle exchange, as it inherently conflates topological distinction with direct observability. To address this inadequacy, the theory has to invoke symmetry-breaking effects, such as those imposed by external magnetic fields or geometric constraints, which may not genuinely reflect the intrinsic statistics of identical particles. At the core of this ambiguity lies a critical conceptual conflation in quantum theory regarding the phases associated with particle exchange. We often implicitly assume, or explicitly conflate, that the relative phase α and the exchange phase β represent the same physical quantity. To better elucidate the distinction between these two phases, we can adopt the phases from those of bosonic and fermionic systems. The quantum state for such particles is

$$\begin{aligned}\psi(r_1, r_2) &= C[\psi_1(r_1)\psi_2(r_2) \pm \psi_1(r_2)\psi_2(r_1)] \\ &= C[\psi_1(r_1)\psi_2(r_2) + \alpha\psi_1(r_2)\psi_2(r_1)]\end{aligned}\quad (1)$$

where the phase factor $\alpha = \pm 1$ is introduced as a relative phase. Applying the exchange operator $\hat{P}$ to this state yields

$$\begin{aligned}\hat{P}\psi(r_1, r_2) &= C[\psi_1(r_2)\psi_2(r_1) \pm \psi_1(r_1)\psi_2(r_2)] \\ &= \pm C[\psi_1(r_1)\psi_2(r_2) \pm \psi_1(r_2)\psi_2(r_1)] \\ &= \beta\psi(r_1, r_2)\end{aligned}\quad (2)$$

where $\beta = \pm 1$ is the eigenvalue of $\hat{P}$, defined as the exchange phase. For bosons and fermions, the mathematics straightforwardly shows that $\alpha = \beta = \pm 1$. This has led to the widespread practice of treating these two phases as synonymous, a convention deeply entrenched in foundational texts [7,57–60]. However, this mathematical equivalence obscures a deeper physical question: are α and β fundamentally the same, or do they originate from distinct aspects of quantum behavior? This conceptual conflation directly leads to several unresolved questions. What is the precise relationship between the relative phase, the exchange phase, and the statistical phase? Do they represent the same physical concept, or do they capture different layers of quantum phase? And ultimately, which of these phases governs the emergent behavior of many-body systems?

Yet, despite the significant progress made in existing theoretical frameworks, a clear and conclusive resolution to these questions remains conspicuously absent within the framework of modern quantum theory. While traditional frameworks, such as Bose–Einstein and Fermi–Dirac statistics, em-

ploy probabilistic arguments to successfully predict thermodynamic behaviour [61–63], they provide no microscopic insight into the phase dynamics of individual particles during exchange. Conversely, the topological approach, which focuses on path topology in parameter space [64–66], captures global geometric properties but fails to explicitly resolve the relationships among different phase types or their interplay in many-body contexts. The persistence of these unresolved questions sustains a fundamental ambiguity that ultimately hinders a complete and unified understanding of quantum statistics. It is therefore a primary objective of this work to address these fundamental questions, as their resolution is indispensable for deepening our understanding of the conceptual foundations of quantum statistics. We now rigorously examine the physical interrelations among these phases to resolve this foundational discrepancy.

2. Rigorous Derivation of the Relationship Between α and β

In classical mechanics, particles remain distinguishable despite identical physical properties, as their distinct trajectories allow for unambiguous identification. The scenario in quantum mechanics is fundamentally different, owing to the fact that quantum particles are described by wave functions. The permutation of any two indistinguishable particles is considered physically equivalent [58,67]. Consider a state of two identical particles, denoted as $|r_1, r_2\rangle$. Exchanging their coordinates yields an exchange state $|r_2, r_1\rangle$, which differs from the original state by at most a phase factor α [7,68–70]. This relationship is expressed as

$$|r_1, r_2\rangle = \alpha |r_2, r_1\rangle. \quad (3)$$

The relative phase α must satisfy $|\alpha| = 1$ and can be expressed as $\alpha = e^{\pm i\theta}$, which implies that

$$\alpha\alpha^{-1} = 1, \quad \alpha^{-1} = \alpha^*. \quad (4)$$

Multiplying both sides of Eq. (3) by α^{-1} gives:

$$\alpha^{-1}|r_1, r_2\rangle = |r_2, r_1\rangle. \quad (5)$$

We define an exchange operator $\hat{R}$ by its action on the basis states such that:

$$\hat{R}|r_1, r_2\rangle = |r_2, r_1\rangle. \quad (6)$$

The operator $\hat{R}$ here is fundamentally equivalent to the conventional exchange operator $\hat{P}$, as both describe the exchange of particle positions. Notably, $\hat{P}$ is associated with the permutation group where two consecutive exchanges always restore the initial state [6,13], whereas $\hat{R}$ is relevant to both two- and three-dimensional systems. To avoid ambiguity, it is convenient to use the operator $\hat{R}$ in two-dimensional systems, where an exchange can be implemented by clockwise or counterclockwise rotations of the particles around each other, corresponding to distinct topological operations in the braid group [71]. Here, two consecutive exchanges do not necessarily restore the initial state, reflecting the non-trivial topology of particle trajectories [8,9,13]. Now, substituting Eq. (6) into Eq. (5) yields:

$$\alpha^{-1}|r_1, r_2\rangle = \hat{R}|r_1, r_2\rangle = |r_2, r_1\rangle. \quad (7)$$

It shows that α^{-1} is the eigenvalue of $\hat{R}$, which implies that α in Eq. (3) is not the eigenvalue of the exchange operator $\hat{R}$ as traditionally believed. Suppose that the operator $\hat{R}$ acting on the wave function results in a phase factor β [6,7,58]:

$$\hat{R}|r_1, r_2\rangle = \beta|r_1, r_2\rangle \quad (8)$$

where β is the exchange phase, the eigenvalue of $\hat{R}$. Comparing Eq. (7) and Eq. (8), we obtain:

$$\alpha^{-1}|r_1, r_2\rangle = \hat{R}|r_1, r_2\rangle = \beta|r_1, r_2\rangle. \quad (9)$$

This leads to the fundamental relation between the relative phase and the exchange phase:

$$\alpha^{-1} = \beta = e^{\mp i\theta} \quad (10)$$

which implies that the relative phase α and the exchange phase β are distinct yet related phase factors. It is noteworthy that the definitions $\hat{R}|r_1, r_2\rangle = \beta|r_1, r_2\rangle$ can be derived from $|r_1, r_2\rangle = \alpha|r_2, r_1\rangle$ due to their deep physical interconnection. Specifically, the equation $\hat{R}|r_1, r_2\rangle = \beta|r_1, r_2\rangle$ follows directly from Eq. (7), which itself originates from $|r_1, r_2\rangle = \alpha|r_2, r_1\rangle$. One may find that we can get it if we just take $\alpha^{-1} = \beta$ into Eq. (7). The relationship derived above started from the definition of the relative phase. This fundamental inverse relationship can be equivalently derived by starting from the exchange phase β , offering an alternative perspective that further reveals their intrinsic connection. Beginning with the eigenvalue definition of the exchange operator from Eqs. (6) and (8), we can have:

$$\hat{R}|r_1, r_2\rangle = |r_2, r_1\rangle = \beta|r_1, r_2\rangle. \quad (11)$$

Since β is a phase factor ($|\beta| = 1$), it follows that $\beta\beta^{-1} = 1$. Multiplying both sides of Eq. (11) by β^{-1} yields:

$$\beta^{-1}\hat{R}|r_1, r_2\rangle = \beta^{-1}|r_2, r_1\rangle = |r_1, r_2\rangle. \quad (12)$$

A direct comparison with Eq. (3), immediately reveals that β^{-1} and α represent the same phase factors. From Eqs. (3) and (12), we can get

$$\beta^{-1}|r_2, r_1\rangle = |r_1, r_2\rangle = \alpha|r_2, r_1\rangle. \quad (13)$$

Therefore, we again arrive at the fundamental reciprocal relationship:

$$\beta^{-1} = \alpha = e^{\pm i\theta}. \quad (14)$$

We get the same result as Eq. (10), providing clear confirmation of the inverse relationship between α and β . This relationship highlights that particle exchange involves two interdependent phases. Note that the definition $|r_1, r_2\rangle = \alpha|r_2, r_1\rangle$ can also be obtained from $\hat{R}|r_1, r_2\rangle = \beta|r_1, r_2\rangle$. It is clearly to see that $|r_1, r_2\rangle = \alpha|r_2, r_1\rangle$ emerges naturally from Eq. (12), as $\beta^{-1}|r_2, r_1\rangle = |r_1, r_2\rangle$ is identical to $|r_1, r_2\rangle = \alpha|r_2, r_1\rangle$ if we take $\beta^{-1} = \alpha$ in Eq. (12). Both derivations converge to the same fundamental relationship: $\alpha = \beta^{-1} = e^{\pm i\theta}$, where θ is the statistical parameter defining the particle types. Furthermore, this inverse relationship can be expressed in an operator form. Substituting $\alpha = \beta^{-1}$ back into Eq. (12) gives:

$$\beta^{-1}\hat{R}|r_1, r_2\rangle = \alpha\hat{R}|r_1, r_2\rangle = |r_1, r_2\rangle. \quad (15)$$

Since the interchange of two identical particles involves two distinct phases, the relative phase α and the exchange phase β depend on the direction in which one particle winds around the other. It suggests that the fundamental entity governing interchange is not the exchange operator $\hat{R}$ alone, but a combination of the exchange operation and its associated phase. Therefore, we introduce a phase operator $\hat{\Lambda}$, derived from Eq. (15), to characterize the phase evolution of identical particles, defined as:

$$\hat{\Lambda} = \alpha\hat{R}. \quad (16)$$

This serves as a key conservation formula for identical particles, encapsulating the full phase dynamics during particle interchange. Compared with the exchange operator $\hat{R}$, $\hat{\Lambda}$ more explicitly captures the essence of particle interchange. In Series I, we demonstrated that $\hat{\Lambda}$ determines the symmetry properties of bosons and fermions. By definition, $\hat{\Lambda}$ is a function of the relative phase α and the exchange operator $\hat{R}$, integrating the variations of both α and β during particle exchange. Applying Eq. (15) again, we have:

$$\hat{\Lambda}|r_1, r_2\rangle = |r_1, r_2\rangle. \quad (17)$$

The phase operator $\hat{\Lambda}$ has a universal eigenvalue of $\lambda = 1$ for all identical particles, a key property that directly implies its time conservation. This conservation is path-independent (Figure 1), in contrast to the individual phases α and β , which may vary with the exchange path. Given that $\hat{P}\hat{H}(r_1, r_2) = \hat{H}(r_2, r_1)$ and $\hat{R}\hat{H}(r_1, r_2) = \hat{H}(r_2, r_1)$, the Hamiltonian $\hat{H}$ remains unchanged under an arbitrary permutation. Thus, $\hat{R}$ commutes with $\hat{H}$, leading to:

$$[\hat{H}, \hat{R}] = 0. \quad (18)$$

Since α is a constant phase factor for a given type of particle, the phase operator $\hat{\Lambda}$ also commutes with $\hat{H}$:

$$[\hat{H}, \hat{\Lambda}] = [\hat{H}, \alpha\hat{R}] = 0. \quad (19)$$

This commutation relation confirms that $\hat{\Lambda}$, $\hat{R}$, and $\hat{H}$ share a common set of eigenfunctions. The existence of the conserved phase operator $\hat{\Lambda}$ with eigenvalue 1 is a profound result, generalizing the theory of quantum statistics and providing a unified foundation for understanding bosons, fermions, and anyons within a single framework.

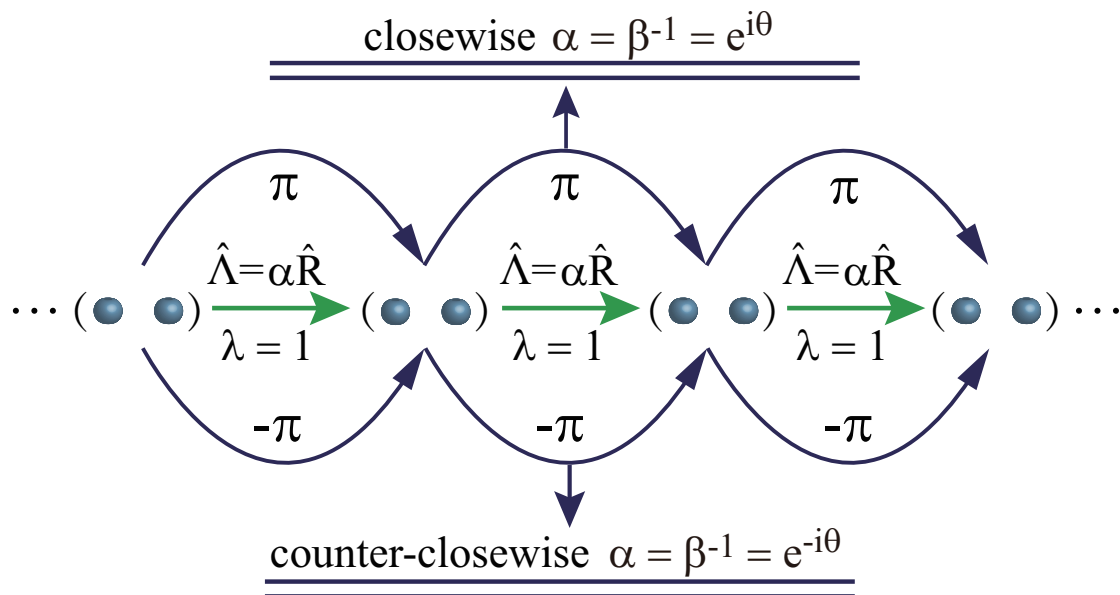


Figure 1. Schematic of the interchange of two identical particles. Although different exchange paths may lead to distinct values of the relative phase α and the exchange phase β , the conserved quantity $\hat{\Lambda}$, which captures the fundamental and universal nature of identical-particle exchange, remains strictly path-independent.

3. Dynamical Dual-Phase Generation from Topology and Time Evolution

The dual-phase structure finds its origin in the interplay between the topological geometry of the configuration space and the unitary process of time evolution. In quantum mechanics, the Feynman path integral formulation assigns a phase factor to each path connecting initial and final points in spacetime [72]. Paths within the same homotopy class, those that can be continuously deformed into one another, must yield the same phase factor to maintain consistency of the quantum amplitude [66,73,74]. For identical particles, the configuration described by (r_1, r_2) is physically indistinguishable from that described by (r_2, r_1) , differing only by particle labeling. This indistinguishability implies that the configuration space of n particles in d dimensions is not the Cartesian product $\mathbb{R}^{dn}$, but rather the quotient space $C_n = (\mathbb{R}^{dn} - \Delta) / S_n$, where Δ denotes the set of points where two or more particles coincide, and S_n is the symmetric group acting by permuting particle labels. This identification results in a multiply-connected space with non-contractible loops. Such non-contractible loops are

precisely the paths that cannot be shrunk to a point without crossing Δ , and they play a critical role in distinguishing particle types [13,75]. In two dimensions, paths can wind around specific singular locations with a well-defined sense of winding. For example, counterclockwise loops around the origin carry a distinct winding sense compared to clockwise ones. Consequently, the acquired phase may depend on both the winding number and the orientation of the path. In particular, the operation of exchanging two particles, which corresponds to a half-circle rotation (by an angle π) about one another [75,76], induces a statistical phase factor $e^{\pm i\theta}$ in the wave function [21,75]. The effect can be described algebraically using an operator $\hat{R}$. Its action on a two-particle state yields $\hat{R}|\mathbf{r}_1, \mathbf{r}_2\rangle = \beta|\mathbf{r}_1, \mathbf{r}_2\rangle$, where β represents the fundamental exchange phase. Repeated application of $\hat{R}$ then reveals the cumulative phase effect:

$$\hat{R}^n|\mathbf{r}_1, \mathbf{r}_2\rangle = \beta(n)|\mathbf{r}_1, \mathbf{r}_2\rangle \quad (20)$$

where the phase factor $\beta(n)$ is determined by the topological properties of the particle exchange path, characterized by its winding direction and the integer n , which counts the number of exchanges. It is expressed as:

$$\beta(n) = e^{\mp in\theta} \quad (21)$$

where the $\mp$ sign corresponds to clockwise or counterclockwise winding. In the configuration space, paths between initial and final positions fall into infinitely many homotopy classes. The integer n (where $n = 0$ corresponds to no particle exchange) can be indexed by $n = 1, 2, 3, \dots$. For bosonic systems, $\beta(n) = 1$ for all n , whereas for fermionic systems, $\beta(n) = (-1)^n$ with even and odd values of n corresponding to phase factors of $+1$ and -1 respectively. Now, consider the relationship between successive exchange operations. Note that the state $\hat{R}^n|\mathbf{r}_1, \mathbf{r}_2\rangle$ and the subsequent exchange state $\hat{R}^{n+1}|\mathbf{r}_1, \mathbf{r}_2\rangle$ are physically equivalent, they can differ only by a phase factor α :

$$\hat{R}^n|\mathbf{r}_1, \mathbf{r}_2\rangle = \alpha\hat{R}^{n+1}|\mathbf{r}_1, \mathbf{r}_2\rangle. \quad (22)$$

Given that $\hat{R}^{n+1}|\mathbf{r}_1, \mathbf{r}_2\rangle = \beta(n)\hat{R}|\mathbf{r}_1, \mathbf{r}_2\rangle$, Eq. (22) can be rewritten as:

$$\hat{R}^n|\mathbf{r}_1, \mathbf{r}_2\rangle = \alpha\hat{R}^{n+1}|\mathbf{r}_1, \mathbf{r}_2\rangle = \alpha\beta(n)\hat{R}|\mathbf{r}_1, \mathbf{r}_2\rangle. \quad (23)$$

Applying Eq. (20) to both sides of Eq. (23) further gives

$$\beta(n)|\mathbf{r}_1, \mathbf{r}_2\rangle = \alpha\beta(n+1)|\mathbf{r}_1, \mathbf{r}_2\rangle = \beta(n)\alpha\beta|\mathbf{r}_1, \mathbf{r}_2\rangle. \quad (24)$$

This leads to the fundamental phase constraint:

$$\alpha\beta = 1 \quad (25)$$

which is identical to Eq. (14). This solution directly yields the dual-phase structure: $\alpha = \beta^{-1} = e^{\mp i\theta}$. Additionally, from Eq. (24), the relative phase α can be alternatively defined as

$$\alpha = \frac{\beta(n)}{\beta(n+1)} = e^{\pm i\theta}. \quad (26)$$

This formulation reveals the physical equivalence between the relations $\beta(n+1)|\mathbf{r}_1, \mathbf{r}_2\rangle = \beta(n)|\mathbf{r}_2, \mathbf{r}_1\rangle$ and $|\mathbf{r}_1, \mathbf{r}_2\rangle = \alpha|\mathbf{r}_2, \mathbf{r}_1\rangle$. Furthermore, applying Eq. (24) enables the derivation of the fundamental exchange phase β :

$$\beta = \frac{\beta(n+1)}{\beta(n)} = e^{\mp i\theta}. \quad (27)$$

The relative phase α and the exchange phase β are intrinsically tied to the number of exchange trajectories. As indicated by Eqs. (26) and (27), these definitions establish a unified framework for different particle statistics. For bosonic systems, we obtain $\alpha = \beta = 1$ since $\beta(n) = \beta(n+1) = 1$,

corresponding to symmetric wave functions. For fermionic systems, the antisymmetry requirement leads to $\alpha = \beta = -1$ with $\beta(n) = -\beta(n+1)$. Anyonic statistics emerge as the continuous interpolation between these two limits, where clockwise particle exchange yields $\alpha = \beta^{-1} = e^{i\theta}$ with $\beta(n)/\beta(n+1) = e^{i\theta}$, while counter-clockwise exchange produces $\alpha = \beta^{-1} = e^{-i\theta}$ with $\beta(n)/\beta(n+1) = e^{-i\theta}$. This result is consistent with the previous analysis, essentially because $\beta(n)$ acts as a global phase. Using the relation $\hat{R}^n |\mathbf{r}_1, \mathbf{r}_2\rangle = \beta(n) |\mathbf{r}_1, \mathbf{r}_2\rangle$, Eq. (23) can be rewritten as:

$$\beta(n) |\mathbf{r}_1, \mathbf{r}_2\rangle = \alpha \beta(n) \hat{R} |\mathbf{r}_1, \mathbf{r}_2\rangle = \alpha \beta(n) |\mathbf{r}_2, \mathbf{r}_1\rangle. \quad (28)$$

The phase factor $\beta(n)$ can be eliminated from both sides of this equation, which naturally leads to the fundamental relations Eq. (3): $|\mathbf{r}_1, \mathbf{r}_2\rangle = \alpha |\mathbf{r}_2, \mathbf{r}_1\rangle$. Furthermore, Eq. (28) also implies that

$$|\mathbf{r}_1, \mathbf{r}_2\rangle = \alpha \hat{R} |\mathbf{r}_1, \mathbf{r}_2\rangle = \Lambda |\mathbf{r}_1, \mathbf{r}_2\rangle, \quad (29)$$

which is consistent with Eqs. (16) and (17) from the earlier discussion. This shows that $n+1$ applications of the exchange operator effectively reduce to a single exchange operation once the accumulated phase $\beta(n)$ is eliminated. It appears that the system's time evolution corresponds to the continuous deformation of trajectories in this space, without altering the fundamental nature of the physics. To further elucidate the phase evolution under continuous exchanges in a specific direction, we examine the application of the operator $\hat{\Lambda}$ that encapsulates the exchange process. From Eq. (29), this operator satisfies the relation:

$$\Lambda^n |\mathbf{r}_1, \mathbf{r}_2\rangle = \Lambda^{n-1} |\mathbf{r}_1, \mathbf{r}_2\rangle = \cdots = |\mathbf{r}_1, \mathbf{r}_2\rangle. \quad (30)$$

This recursive relation demonstrates the fact that the state is invariant under the combined operation of phase application and exchange. Furthermore, starting from the relation $\alpha \hat{R} = \Lambda$, we can derive the effect of n such operations:

$$\alpha(n) \hat{R}^n |\mathbf{r}_1, \mathbf{r}_2\rangle = \alpha(n-1) \hat{R}^{n-1} |\mathbf{r}_1, \mathbf{r}_2\rangle = \cdots = |\mathbf{r}_1, \mathbf{r}_2\rangle, \quad (31)$$

where $\alpha(n)$ denotes the accumulated relative phase. This result indicates that the product $\alpha(n) \hat{R}^n$ acts as the identity operator on the two-particle state, independent of the number of exchanges n . To further clarify the relationship between phase accumulation and exchange history, we multiply both sides of Eq. (20) by $\beta^{-1}(n)$, obtaining:

$$\beta^{-1}(n) \hat{R}^n |\mathbf{r}_1, \mathbf{r}_2\rangle = |\mathbf{r}_1, \mathbf{r}_2\rangle. \quad (32)$$

A direct comparison between Eq. (31) and Eq. (32) reveals a basic correspondence:

$$\alpha(n) = \beta^{-1}(n) = e^{\pm i n \theta}. \quad (33)$$

This relation shows that for a continuous sequence of n exchanges in a specific direction, the accumulated exchange phase $\beta(n)$ and the accumulated relative phase $\alpha(n)$ are generated simultaneously throughout the exchange process of identical particles. This generalizes the fundamental dual-phase relation to an arbitrary number of exchanges as shown in Figure 2, demonstrating that the mutual inversion between the accumulated relative phase and the accumulated exchange phase is dynamically preserved throughout the entire braiding history. Crucially, as the exchange count n varies smoothly with time, both phases co-evolve continuously with n . This smooth evolution ensures that the fundamental relation $\alpha(n)\beta(n) = 1$ is maintained at all times. Thus, the continuous accumulation of $\beta(n)$ not only encodes the complete winding history of the particles but also dictates, through this mutual constraint, the simultaneous evolution of $\alpha(n)$. This intertwined dynamics reflects the underlying topological structure of the braiding operations.

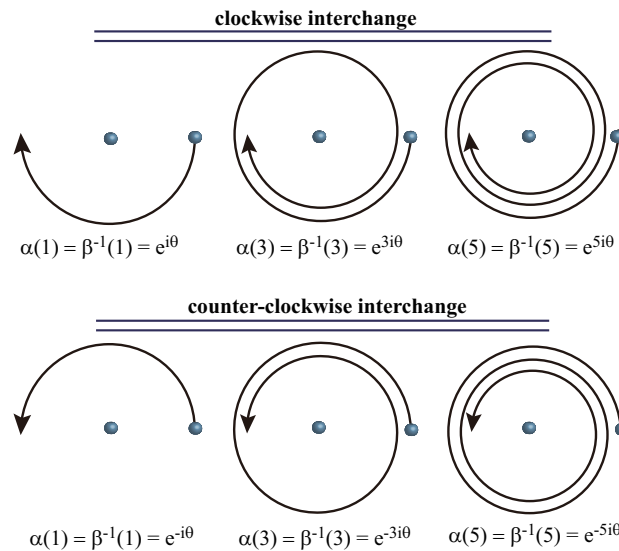


Figure 2. Schematic illustration of the clockwise and counterclockwise particle exchange rotations, demonstrating the dynamic accumulation of both the exchange phase $\beta(n)$ and the relative phase $\alpha(n)$.

4. Phase Dynamics in Bosonic and Fermionic Systems

Bosons and fermions represent two fundamental aspects of identical particles in quantum mechanics. The wave function of such systems inherently depends on both spatial ($\mathbf{r}$) and spin (σ) degrees of freedom [6,7]. As established in our previous work, a dual-phase structure naturally arises in systems of identical particles, encompassing relative phases (α , α_L , α_S), exchange phases (β , β_L , β_S), and the corresponding phase operators ($\hat{\Lambda}$, $\hat{\Lambda}_L$, $\hat{\Lambda}_S$) that interconnect them. In Series I, we demonstrated that the operators $\hat{\Lambda}_L$ and $\hat{\Lambda}_S$ emerge directly from the wave functions of bosons and fermions [1]. Specifically, the spatial part of the wave function adheres to the form:

$$\begin{aligned}
 \psi(\mathbf{r}_1, \mathbf{r}_2) &= C_L [\psi_1(\mathbf{r}_1)\psi_2(\mathbf{r}_2) + \alpha_L \psi_1(\mathbf{r}_2)\psi_2(\mathbf{r}_1)] \\
 &= C_L [\psi_1(\mathbf{r}_1)\psi_2(\mathbf{r}_2) + \alpha_L \hat{P}_L \psi_1(\mathbf{r}_1)\psi_2(\mathbf{r}_2)] \\
 &= C_L (1 + \alpha_L \hat{P}_L) \psi_1(\mathbf{r}_1)\psi_2(\mathbf{r}_2) \\
 &= C_L (1 + \hat{\Lambda}_L) \psi_1(\mathbf{r}_1)\psi_2(\mathbf{r}_2),
 \end{aligned} \tag{34}$$

where C_L is a normalization coefficient and the spatial relative phase $\alpha_L = \pm 1$. The state is anti-symmetric if $\alpha_L = -1$ and symmetric if $\alpha_L = +1$. Similarly, the spin wave function is expressed as:

$$\begin{aligned}
 \chi(\sigma_1, \sigma_2) &= C_S [\chi_1(\sigma_1)\chi_2(\sigma_2) + \alpha_S \chi_1(\sigma_2)\chi_2(\sigma_1)] \\
 &= C_S [\chi_1(\sigma_1)\chi_2(\sigma_2) + \alpha_S \hat{P}_S \chi_1(\sigma_1)\chi_2(\sigma_2)] \\
 &= C_S [\chi_1(\sigma_1)\chi_2(\sigma_2) + \hat{\Lambda}_S \chi_1(\sigma_1)\chi_2(\sigma_2)] \\
 &= C_S (1 + \hat{\Lambda}_S) \chi_1(\sigma_1)\chi_2(\sigma_2),
 \end{aligned} \tag{35}$$

which accounts for the four possible spin configurations [1]. Here $\alpha_S = \pm 1$ and C_S is the normalization constant, and $\chi_1(\sigma_1)$, $\chi_2(\sigma_2)$ represent spin-up or spin-down states. From Eqs. (34) and (35), we obtain:

$$\hat{\Lambda}_L = \alpha_L \hat{P}_L, \tag{36}$$

$$\hat{\Lambda}_S = \alpha_S \hat{P}_S. \tag{37}$$

These represent special cases of Eq. (16) which serves as a universal formula for all identical particles. The total permutation operator $\hat{P}$ satisfies $\hat{P} = \hat{P}_L \hat{P}_S$ [1]. The complete phase operator is given by:

$$\hat{\Lambda} = \hat{\Lambda}_L \hat{\Lambda}_S = \alpha_L \hat{P}_L \alpha_S \hat{P}_S = \alpha \hat{P} \quad (38)$$

which corresponds to Eq. (16) for $\theta = 0$ and $\theta = \pi$. Here, the total relative phase $\alpha = \alpha_L \alpha_S = \pm 1$, with $\alpha = +1$ for bosons and $\alpha = -1$ for fermions. The operator $\hat{\Lambda}$ enables a unified description of symmetric and antisymmetric basis states, significantly simplifying many-body treatments [2,4,77]. Finally, the total wave function is given by:

$$\begin{aligned} \Phi &= \psi(\mathbf{r}_1, \mathbf{r}_2) \chi(\sigma_1, \sigma_2) \\ &= \hat{A} \psi_1(\mathbf{r}_1) \psi_2(\mathbf{r}_2) \chi_1(\sigma_1) \chi_2(\sigma_2) \end{aligned} \quad (39)$$

where the symmetry component $\hat{A} = C(1 + \hat{\Lambda}_L)(1 + \hat{\Lambda}_S)$ has an eigenvalue $4C$ ($C = C_L C_S$) and represents a conserved quantity [1]. Crucially, the symmetry properties of the wave function are entirely governed by the operator $\hat{\Lambda}$, which unifies both the relative phases α and the permutation operator $\hat{P}$. The space permutation operator $\hat{P}_L$ and the spin permutation operator $\hat{P}_S$ respectively satisfy:

$$\hat{P}_S |\sigma_1, \sigma_2\rangle = |\sigma_2, \sigma_1\rangle = \beta_S |\sigma_1, \sigma_2\rangle, \quad (40)$$

$$\hat{P}_L |\mathbf{r}_1, \mathbf{r}_2\rangle = |\mathbf{r}_2, \mathbf{r}_1\rangle = \beta_L |\mathbf{r}_1, \mathbf{r}_2\rangle. \quad (41)$$

Here, both β_S and β_L can only take values $+1$ and -1 . This can be seen by examining the squared permutation operator: applying $\hat{P}_S^2$ to the spin state $|\sigma_1, \sigma_2\rangle$ yields $\hat{P}_S^2 |\sigma_1, \sigma_2\rangle = \hat{P}_S |\sigma_2, \sigma_1\rangle = |\sigma_1, \sigma_2\rangle$ through double exchange, while simultaneously $\hat{P}_S^2 |\sigma_1, \sigma_2\rangle = \beta_S \hat{P}_S |\sigma_1, \sigma_2\rangle = \beta_S^2 |\sigma_1, \sigma_2\rangle$. These two results together imply $\beta_S^2 = 1$ and consequently $\beta_S = \pm 1$. The same argument applies to the spatial permutation, giving $\beta_L = \pm 1$. The total permutation operator $\hat{P} = \hat{P}_L \hat{P}_S$ therefore satisfies:

$$\begin{aligned} \hat{P} |\mathbf{r}_1, \mathbf{r}_2\rangle |\sigma_1, \sigma_2\rangle &= \hat{P}_L |\mathbf{r}_1, \mathbf{r}_2\rangle \hat{P}_S |\sigma_1, \sigma_2\rangle \\ &= \beta_L |\mathbf{r}_1, \mathbf{r}_2\rangle \beta_S |\sigma_1, \sigma_2\rangle, \end{aligned} \quad (42)$$

$$\begin{aligned} \hat{P} |\mathbf{r}_1, \mathbf{r}_2\rangle |\sigma_1, \sigma_2\rangle &= |\mathbf{r}_2, \mathbf{r}_1\rangle |\sigma_2, \sigma_1\rangle \\ &= \beta |\mathbf{r}_1, \mathbf{r}_2\rangle |\sigma_1, \sigma_2\rangle, \end{aligned} \quad (43)$$

from which it follows that $\beta = \beta_L \beta_S = \pm 1$. For bosons, the spin and spatial components must share the same symmetry, implying $\alpha = \alpha_L \alpha_S = +1$ and $\beta = \beta_L \beta_S = +1$. For fermions, the spin and spatial components must have opposite symmetry, giving $\alpha = \alpha_L \alpha_S = -1$ and $\beta = \beta_L \beta_S = -1$. Note that the relative phases satisfy $|\sigma_1, \sigma_2\rangle = \alpha_S |\sigma_2, \sigma_1\rangle$ and $|\mathbf{r}_1, \mathbf{r}_2\rangle = \alpha_L |\mathbf{r}_2, \mathbf{r}_1\rangle$. In fact, the spin relations $|\sigma_1, \sigma_2\rangle = \alpha_S |\sigma_2, \sigma_1\rangle$ and $|\sigma_2, \sigma_1\rangle = \beta_S |\sigma_1, \sigma_2\rangle$, along with the spatial relations $|\mathbf{r}_1, \mathbf{r}_2\rangle = \alpha_L |\mathbf{r}_2, \mathbf{r}_1\rangle$ and $|\mathbf{r}_2, \mathbf{r}_1\rangle = \beta_L |\mathbf{r}_1, \mathbf{r}_2\rangle$, are mutually derivable, as discussed earlier. Using the identities $\alpha_S \alpha_S^{-1} = 1$ and $\alpha_L \alpha_L^{-1} = 1$, exchange operations lead to the fundamental relations:

$$\alpha_S^{-1} |\sigma_1, \sigma_2\rangle = |\sigma_2, \sigma_1\rangle, \quad (44)$$

$$\alpha_L^{-1} |\mathbf{r}_1, \mathbf{r}_2\rangle = |\mathbf{r}_2, \mathbf{r}_1\rangle, \quad (45)$$

By comparing them with Eqs. (40) and (41), we can obtain:

$$\alpha_S^{-1} |\sigma_1, \sigma_2\rangle = |\sigma_2, \sigma_1\rangle = \beta_S |\sigma_1, \sigma_2\rangle, \quad (46)$$

$$\alpha_L^{-1} |\mathbf{r}_1, \mathbf{r}_2\rangle = |\mathbf{r}_2, \mathbf{r}_1\rangle = \beta_L |\mathbf{r}_1, \mathbf{r}_2\rangle, \quad (47)$$

which then yields $\alpha_S^{-1} = \beta_S$ and $\alpha_L^{-1} = \beta_L$. Evidently, these are special cases of Eq. (10) or Eq. (14). Noting that $\alpha_S^{-1} = \alpha_S$ and $\alpha_L^{-1} = \alpha_L$, we obtain

$$\alpha_S = \beta_S = \pm 1, \quad (48)$$

$$\alpha_L = \beta_L = \pm 1, \quad (49)$$

$$\alpha = \beta = \pm 1. \quad (50)$$

The eigenvalues of $\hat{\Lambda}$, $\hat{\Lambda}_L$, and $\hat{\Lambda}_S$ can be obtained by using these formulas. Operating on $|\mathbf{r}_1, \mathbf{r}_2\rangle|\sigma_1, \sigma_2\rangle$ with these operators we see that

$$\begin{aligned} \hat{\Lambda}_S |\mathbf{r}_1, \mathbf{r}_2\rangle|\sigma_1, \sigma_2\rangle &= |\mathbf{r}_1, \mathbf{r}_2\rangle \alpha_S \hat{P}_S |\sigma_1, \sigma_2\rangle \\ &= \alpha_S \beta_S |\mathbf{r}_1, \mathbf{r}_2\rangle|\sigma_1, \sigma_2\rangle \\ &= |\mathbf{r}_1, \mathbf{r}_2\rangle|\sigma_1, \sigma_2\rangle, \end{aligned} \quad (51)$$

$$\begin{aligned} \hat{\Lambda}_L |\mathbf{r}_1, \mathbf{r}_2\rangle|\sigma_1, \sigma_2\rangle &= \alpha_L \hat{P}_L |\mathbf{r}_1, \mathbf{r}_2\rangle|\sigma_1, \sigma_2\rangle \\ &= \alpha_L \beta_L |\mathbf{r}_1, \mathbf{r}_2\rangle|\sigma_1, \sigma_2\rangle \\ &= |\mathbf{r}_1, \mathbf{r}_2\rangle|\sigma_1, \sigma_2\rangle, \end{aligned} \quad (52)$$

$$\begin{aligned} \hat{\Lambda} |\mathbf{r}_1, \mathbf{r}_2\rangle|\sigma_1, \sigma_2\rangle &= \hat{\Lambda}_L \hat{\Lambda}_S |\mathbf{r}_1, \mathbf{r}_2\rangle|\sigma_1, \sigma_2\rangle \\ &= \hat{\Lambda}_L |\mathbf{r}_1, \mathbf{r}_2\rangle \hat{\Lambda}_S |\sigma_1, \sigma_2\rangle \\ &= |\mathbf{r}_1, \mathbf{r}_2\rangle|\sigma_1, \sigma_2\rangle. \end{aligned} \quad (53)$$

Their eigenvalues are all 1, confirming the relations $\alpha_S \beta_S = 1$, $\alpha_L \beta_L = 1$, and $\alpha \beta = 1$. The conserved eigenvalue $\lambda = 1$ for $\hat{\Lambda}$ reflects the fundamental invariance of particle exchange phases under time evolution. The complete set of phase parameters and symmetry properties for bosonic and fermionic systems is systematically summarized in Table 1, providing a comprehensive framework for understanding quantum statistical behavior.

Table 1. Dual-Phase Structure and Symmetry Properties of Bosons and Fermions

Property	Fermions	Bosons
Relative phase α	$\alpha = \alpha_L \alpha_S = -1$	$\alpha = \alpha_L \alpha_S = 1$
Exchange phase β	$\beta = \beta_L \beta_S = -1$	$\beta = \beta_L \beta_S = 1$
Total phase operator $\hat{\Lambda}$	$\hat{\Lambda} = \alpha \hat{P} = \hat{\Lambda}_L \hat{\Lambda}_S$	$\hat{\Lambda} = \alpha \hat{P} = \hat{\Lambda}_L \hat{\Lambda}_S$
Spatial phase operator $\hat{\Lambda}_L$	$\hat{\Lambda}_L = \alpha_L \hat{P}_L$	$\hat{\Lambda}_L = \alpha_L \hat{P}_L$
Spin phase operator $\hat{\Lambda}_S$	$\hat{\Lambda}_S = \alpha_S \hat{P}_S$	$\hat{\Lambda}_S = \alpha_S \hat{P}_S$
Spatial wave function $\psi(\mathbf{r}_1, \mathbf{r}_2)$	$C_L(1 + \hat{\Lambda}_L)\psi_1(\mathbf{r}_1)\psi_2(\mathbf{r}_2)$	$C_L(1 + \hat{\Lambda}_L)\psi_1(\mathbf{r}_1)\psi_2(\mathbf{r}_2)$
Spin wave function $\chi(\sigma_1, \sigma_2)$	$C_S(1 + \hat{\Lambda}_S)\chi_1(\sigma_1)\chi_2(\sigma_2)$	$C_S(1 + \hat{\Lambda}_S)\chi_1(\sigma_1)\chi_2(\sigma_2)$
Relationship between α_L and β_L	$\alpha_L = \beta_L = \pm 1$	$\alpha_L = \beta_L = \pm 1$
Relationship between α_S and β_S	$\alpha_S = \beta_S = \pm 1$	$\alpha_S = \beta_S = \pm 1$
Relationship between α and β	$\alpha = \beta = -1$	$\alpha = \beta = 1$

5. Discussion

5.1. Dual-Phase Structure of Identical Particle

It demonstrates that the exchange dynamics of identical particles is governed by a dual-phase structure, comprising the relative phase α and the exchange phase β . This dual-phase structure offers a basis for understanding the exchange symmetry of identical particles and reveals a limitation of conventional quantum statistical theories, which traditionally assume that particle exchange can be fully described by a single phase. The relative phase α represents the phase difference between the two equivalent states of identical particles, such that $|\mathbf{r}_1, \mathbf{r}_2\rangle = \alpha |\mathbf{r}_2, \mathbf{r}_1\rangle$, while the exchange phase β describes the action of the exchange operator $\hat{R}$ on the quantum state: $\hat{R}|\mathbf{r}_1, \mathbf{r}_2\rangle = \beta |\mathbf{r}_1, \mathbf{r}_2\rangle$. These two phases are intrinsically linked through the relation $\alpha = \beta^{-1} = e^{\pm i\theta}$ where θ is the statistical angle,

reflecting the continuous change of symmetry. This relation underpins a continuous spectrum of quantum statistics, where bosons ($\alpha = \beta = 1$) and fermions ($\alpha = \beta = -1$) emerge as special discrete cases.

This dual-phase structure emerges naturally from the topology of the configuration space, revealing it to be an intrinsic geometric property of identical particles. We establish a profound connection between the topological structure of the configuration space and the dynamical generation of dual phases that characterize particle statistics. Under time evolution, the phases accumulate continuously, with the number of exchanges n encoded in the accumulated phase $\beta(n) = e^{\mp i n \theta}$. From this dynamical perspective, the two phases can be equivalently defined as $\alpha = \beta(n) / \beta(n + 1)$ and $\beta = \beta(n + 1) / \beta(n)$. Under this new definition, exchanging the two particles yields consistent results with the earlier formulation: a clockwise exchange gives $\alpha = \beta^{-1} = e^{i \theta}$, while a counter-clockwise exchange gives $\alpha = \beta = e^{-i \theta}$. The phase operator $\hat{\Lambda} = \alpha \hat{R}$ unifies both phases and their operations within a single framework. Despite this directional difference, both processes are physically equivalent for all identical particles.

5.2. Challenges to the Conventional Existence of Non-Abelian Anyons

The conventional understanding and experimental pursuit of non-Abelian anyons are fundamentally based on the assumption that breaking parity (P) or time-reversal (T) symmetry is necessary. This requirement arises because, in established topological theories, clockwise and counter-clockwise exchanges are treated as topologically distinct operations, each expected to produce physically distinguishable phase factors [11,20,76]. This symmetry-breaking condition has become a cornerstone in the search for non-Abelian anyons, directly informing theoretical model construction and guiding experimental efforts in systems such as fractional quantum Hall states [56,78–80] and topological superconductors [81–85].

The inherent dual-phase structure of identical particles, however, fundamentally challenges this paradigm. We have rigorously demonstrated that particle exchange is intrinsically governed by two phases, the relative phase α and the exchange phase β , linked by an inverse relationship such that $\alpha \beta = 1$. This universal constraint implies that the physical outcome of an exchange is independent of its direction. As illustrated in Figure 3, the limitation of conventional theory stems directly from its reliance on a single-phase description, which artificially necessitates the violation of discrete symmetries. In contrast, the dual-phase structure harmonizes topological path dependence through a conserved quantity $\hat{\Lambda}$ which remains invariant under a change of exchange direction. Thus, despite topological differences between clockwise and counter-clockwise braids, they produce identical physical effects at the observable level.

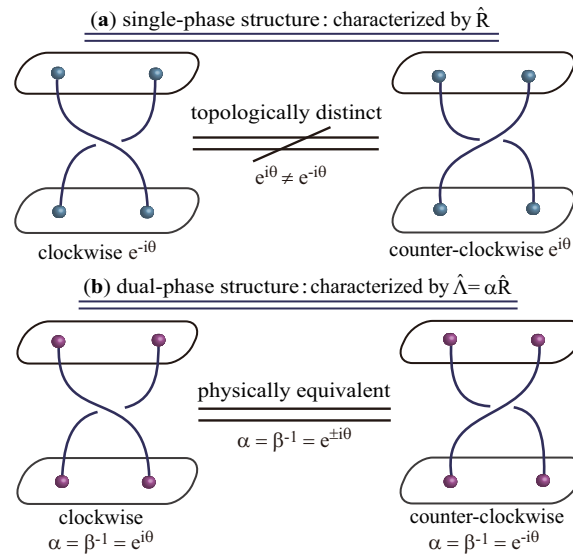


Figure 3. Comparison between conventional anyon statistics and the dual-phase framework. **(a)** Single-phase structure: Under P or T transformations, the exchange phase transforms as $e^{\pm i\theta} \rightarrow e^{\mp i\theta}$, making clockwise and counter-clockwise exchanges physically distinct. This implies that anyonic statistics fundamentally violate parity and time reversal symmetries. **(b)** Dual-phase structure: Clockwise rotation results in $\alpha = e^{i\theta}$ and $\beta = e^{-i\theta}$, while counter-clockwise rotation gives $\alpha = e^{-i\theta}$ and $\beta = e^{i\theta}$. Despite their topological differences, the two exchange directions are physically equivalent, a property captured by a path-independent conserved quantity $\hat{\Lambda}$.

This demonstration that topological distinctions do not necessarily translate into observable physical differences, offers a convincing explanation for the long-standing difficulty in experimentally identifying non-Abelian anyons. Within this new framework, the symmetry-breaking signatures previously deemed essential for their detection may fundamentally lack observable effects. The dual-phase property of identical particles does not merely tweak the existing picture of non-Abelian anyons, but poses a profound challenge to their conventional existence theorem [11,23,28]. It demands a fundamental transformation in both theoretical and experimental approaches, underscoring that topological distinctions in exchange paths need not give rise to measurable physical differences. This insight could help resolve why non-Abelian anyons have remained beyond experimental reach.

5.3. The $\hat{\Lambda}$ -Operator Representation for Quantum Many-Body Systems

In contrast to traditional theoretical approaches, the treatment of quantum many-body problems becomes greatly streamlined within the dual-phase framework. This simplification arises from the fact that the dual-phase structure, an inherent property of all identical particles, is intrinsically linked to the operator formalism developed in Series I. Guided by the basic characteristics of the operator $\hat{\Lambda}$, one can obtain two fully equivalent descriptions, namely $E = \langle \Phi | \hat{H} | \Phi \rangle = \langle \psi | \hat{O} | \psi \rangle$, for representing systems of identical particles: the wave function formalism $E = \langle \Phi | \hat{H} | \Phi \rangle$, where the symmetry of identical particles is embedded in the wave function Φ ; and the operator formalism $E = \langle \psi | \hat{O} | \psi \rangle$, where the symmetry of identical particles is entirely encoded within the transformed Hamiltonian operator $\hat{O} = \hat{\Lambda} \hat{H} \hat{\Lambda}$. Here, $\hat{\Lambda} = C(1 + \hat{\Lambda}_L)(1 + \hat{\Lambda}_S)$ as given in Eq. (39) is a Hermitian operator that comprehensively encapsulates all symmetric information for any pair of bosons or fermions. In this $\hat{\Lambda}$ -operator representation, the wave function and operator formalisms are fully equivalent and mutually derivable, thereby offering significant flexibility in technical applications. For a system of n particles, the total energy can be expressed exactly as $E = \sum_{i < j} \langle \psi_{ij} | \hat{O}_{ij} | \psi_{ij} \rangle = \sum_{i < j} \langle \Phi_{ij} | \hat{H}_{ij} | \Phi_{ij} \rangle$ [1]. Since each $\hat{O}$ contains the complete symmetric information for the particle pair (i, j) , the collective set $\{\hat{O}_{ij}\}$ provides a complete description of the symmetry properties of a system of n bosons or n fermions. For instance, in an n -electron system, this approach allows the direct acquisition of the common eigenfunctions corresponding to the full set of operators $\hat{H}$, $\hat{L}^2$, $\hat{S}^2$, $\hat{L}_z$, and $\hat{S}_z$. This is primarily because the symmetry of an n -electron system can be fully and accurately described by the operators

$\hat{O}$. In contrast, traditional wave function methods, with their inherent limitations [2–5], are incapable of achieving such a comprehensive and symmetric representation. The $\hat{A}$ -operator representation forms the foundation for the operator method developed in Series I, and thus offers a unified and powerful framework for many-body systems, possessing significant advantages over conventional techniques such as Slater determinants [4,86] or second quantization [7,59]. It eliminates the need for explicit construction of symmetric wave functions, embeds all symmetry requirements exactly within the operators $\hat{O}$, and allows the n-particle wave function to be expressed in a simple product form of elementary wave functions, rather than a complex, explicitly symmetrized entangled state. This formalism substantially improves computational efficiency and energy accuracy, and delivers complete and exact symmetric information of the system.

The dual-phase structure, an inherent property of all identical particles, has been shown to greatly simplify the theoretical description of bosonic and fermionic systems. When extending this framework to anyon systems, however, one must address additional challenges—particularly those related to path-dependent properties [21,75]. These aspects introduce deeper layers of topological and statistical complexity [87–91]. Nonetheless, the dual-phase construction offers a promising foundation for tackling these intriguing problems, providing a unified point of departure for future studies on fractional statistics. It is important to note that the fundamental questions regarding the relationship between the relative phase, the exchange phase, and the statistical phase, and their ultimate role in governing many-body behavior [92–99], extend beyond the scope of the present work. The exploration of the statistical phase and its profound connection to anyonic physics, building upon the established dual-phase foundation, constitutes a major focus of our ongoing research and will be addressed systematically in subsequent series of this study.

Data Availability Statement: The data that support the findings of this study are available within the article.

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