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Article

Abelian Extensions and Crossed Modules of Modified λ-Differential Left-Symmetric Algebras

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Abstract: In this paper, we define the cohomology of a modified λ-differential left-symmetric algebra with coefficients in a suitable representation. We also introduce the notion of modified λ-differential left-symmetric 2-algebra. We classify linear deformations and abelian extensions of modified λ-differential left-symmetric algebras using the second cohomology group and classify skeletal modified λ-differential left-symmetric 2-algebra using the third cohomology group as our propose cohomology applications. Moreover, we prove that strict modified λ-differential left-symmetric 2-algebras are equivalent to crossed modules of modified λ-differential left-symmetric algebras.

Keywords: left-symmetric algebras; modified λ-differential operator; cohomology; deformation; abelian extension; crossed module

MSC: 17A01; 17A30; 17B10; 17B38; 17B40; 17B56

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1. Introduction

Derivation, also known as differential operator, plays an important role in mathematical physics, such as homotopy Lie algebras [1], differential Galois theory [2], control theory and gauge theories of quantum field theory [3]. In [4,5], the authors studied associative algebras with derivations from the operadic point of view. Recently, in [6], Tang and his collaborators investigated the deformation and extension of Lie algebras with derivations from the cohomological point of view. Inspired by the work of [6], associative algebras with derivations and pre-algebras with derivations have been studied in [7,8] respectively.

The term modified r-matrix stemmed from the concept of modified classical Yang-Baxter equation, which was introduced by Semenov-Tian-Shansky [9]. Recently, Jiang and Sheng [10] developed the deformations of modified r-matrices and cohomologies of related algebraic structures. Motivated by the modified r-matrices, in [11], Peng and his collaborators introduced the concept of modified λ-differential Lie algebras. Subsequently, the algebraic structures with modified operators have been widely studied in [12–18].

However, there was very few study about the modified λ-differential left-symmetric algebras. Left-symmetric algebras (also called pre-Lie algebras) are nonassociative algebras, which were introduced by Cayley [19] as a kind of rooted tree algebras and also introduced by Gerstenhaber [20] when studying the deformation theory of rings and algebras. Left symmetric algebras have been widely used in geometry and physics, such as affine manifolds, affine structures on Lie groups and convex homogeneous cones [21], integrable systems, classical and quantum Yang-Baxter equations [22,23], quantum field theory, Poisson brackets, operands, complex and symplectic structures on Lie groups and Lie algebras [24]. See also [25–34] for more details. So it should be quite interesting and necessary to study the modified λ-differential left-symmetric algebras.

In this paper, we commence to study the modified λ-differential left-symmetric algebraic version, which includes a left-symmetric algebra and a modified λ-differential operator. We introduce the cohomology of modified λ-differential left-symmetric algebras with coefficients in a representation. As applications of cohomology theory, we study the linear deformations and abelian extensions of

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a modified λ-differential left-symmetric algebra by using the second cohomology groups. Additionally, we investigate the skeletal modified λ-differential left-symmetric 2-algebras by using the third cohomology group. Finally, we prove that strict modified λ-differential left-symmetric 2-algebras are equivalent to crossed modules of modified λ-differential left-symmetric algebras.

The paper is organized as follows. Section 2 introduces the representations of modified λ-differential left-symmetric algebras. In Section 3, we define the cohomology theory of modified λ-differential left-symmetric algebras with coefficients in a representation, and apply it to the study of linear deformation. In Section 4, we investigate abelian extensions of the modified Rota-Baxter pre-Lie algebras in terms of second cohomology groups. Finally, in Section 5, we classify skeletal modified λ-differential left-symmetric 2-algebras by using the third cohomology group. We then prove that skeletal modified λ-differential left-symmetric 2-algebras are equivalent to the crossing modules of modified λ-differential left-symmetric algebras.

Throughout this paper, $\mathbb{K}$ denotes a field of characteristic zero. All the vector spaces and (multi)linear maps are taken over $\mathbb{K}$.

2. Representations of Modified λ-Differential Left-Symmetric Algebras

In this section, we introduce the concept of modified λ-differential left-symmetric algebra and give some examples. Next we propose the representation of modified λ-differential left-symmetric algebras.

First, let's recall some definitions and results about left-symmetric algebras and its representations from [20,26].

Definition 2.1. [20] A left-symmetric algebra is a pair $(\mathfrak{p}, \star)$ consisting of a vector space $\mathfrak{p}$ and a bilinear product $\star : \mathfrak{p} \times \mathfrak{p} \to \mathfrak{p}$ such that for $p_1, p_2, p_3 \in \mathfrak{p}$, the associator

$$(p_1, p_2, p_3) = (p_1 \star p_2) \star p_3 - p_1 \star (p_2 \star p_3),$$

is symmetric in p_1, p_2, i.e., $(p_1, p_2, p_3) = (p_2, p_1, p_3)$, or equivalently,

$$(p_1 \star p_2) \star p_3 - p_1 \star (p_2 \star c) = (p_2 \star p_1) \star p_3 - p_2 \star (p_1 \star p_3). \tag{2.1}$$

Remark 2.2. Let $(\mathfrak{p}, \star)$ be a left-symmetric algebra. Define a binary bracket on $\mathfrak{p}$ by

$$[p_1, p_2]^c = p_1 \star p_2 - p_2 \star p_1.$$

Then $(\mathfrak{p}, [,]^c)$ is a Lie algebra, which is called the sub-adjacent Lie algebra of $(\mathfrak{p}, \star)$.

Example 2.3. Let $(\mathfrak{p}, [,])$ be a Lie algebra and $R : \mathfrak{p} \to \mathfrak{p}$ be a linear map satisfying

$$[Rp_1, Rp_2] = R([Rp_1, p_2] + [p_1, Rp_2]), \forall p_1, p_2 \in \mathfrak{p}.$$

Then $(\mathfrak{p}, \star_R)$ is a left-symmetric algebra, where $p_1 \star_R p_2 = [Rp_1, p_2]$.

Definition 2.4. Let $(\mathfrak{p}, \star)$ be a left-symmetric algebra and $\lambda \in \mathbb{K}$. A linear map $\partial : \mathfrak{p} \to \mathfrak{p}$ is called a modified λ-differential operator if ∂ satisfies

$$\partial(p_1 \star p_2) = \partial p_1 \star p_2 + p_1 \star \partial p_2 + \lambda(p_1 \star p_2), \ \forall\, a, b \in \mathcal{P}. \tag{2.2}$$

Moreover, the triple $(\mathfrak{p}, \star, \partial)$ is called modified λ-differential left-symmetric algebra, simply denoted by $(\mathfrak{p}, \partial)$.

Definition 2.5. A homomorphism between two modified λ-differential left-symmetric algebras $(\mathfrak{p}_1, \partial_1)$ and $(\mathfrak{p}_2, \partial_2)$ is a left-symmetric algebra homomorphism $\Phi : \mathfrak{p}_1 \to \mathfrak{p}_2$ such that $\Phi \circ \partial_1 = \partial_2 \circ \Phi$. Furthermore, Φ is called an isomorphism from $(\mathfrak{p}_1, \partial_1)$ to $(\mathfrak{p}_2, \partial_2)$ if Φ is nondegenerate.

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Example 2.6. An identity map $\mathrm{id}_{\mathfrak{p}} : \mathfrak{p} \to \mathfrak{p}$ is a modified (-1)-differential operator.

Example 2.7. Let $(\mathfrak{p}, [,], \partial)$ be a modified λ-differential Lie algebra (see [11], Definition 2.5). By Example 2.3, if $R \circ \partial = \partial \circ R$, then $(\mathfrak{p}, \star_R, \partial)$ is a modified λ-differential left-symmetric algebra.

Example 2.8. Let $(\mathfrak{p}, \star)$ be a 2-dimensional left-symmetric algebra and $\{e_1, e_2\}$ be a basis, whose nonzero products are given as follows:

$$e_1 \star e_2 = e_1, \quad e_2 \star e_2 = e_2.$$

Then, for $k_1, k_2 \in \mathbb{K}$, the operator

$$\partial = \begin{pmatrix} k_1 & k_2 \\ 0 & -\lambda \end{pmatrix}$$

is a modified λ-differential operator on $(\mathfrak{p}, \star)$.

Example 2.9. Let $(\mathfrak{p}, \star)$ be a left-symmetric algebra. If a linear map $\partial : \mathfrak{p} \to \mathfrak{p}$ is a modified λ-differential operator , then, for $k \in \mathbb{K}$, $k\partial$ is a modified $(k\lambda)$-differential operator.

Definition 2.10. [26] A representation of a left-symmetric algebra $(\mathfrak{p}, \star)$ is a triple $(\mathcal{V}; \star_l, \star_r)$, where $\mathcal{V}$ is a vector space, $\star_l : \mathfrak{p} \times \mathcal{V} \to \mathcal{V}$ and $\star_r : \mathcal{V} \times \mathfrak{p} \to \mathcal{V}$ are two linear maps such that for all $p_1, p_2 \in \mathfrak{p}, u \in \mathcal{V}$:

$$\begin{aligned} p_1 \star_l (p_2 \star_l u) - (p_1 \star p_2) \star_l u &= p_2 \star_l (p_1 \star_l u) - (p_2 \star p_1) \star_l u, \\ p_1 \star_l (u \star_r p_2) - (p_1 \star_l u) \star_r p_2 &= u \star_r (p_1 \star p_2) - (u \star_r p_1) \star_r p_2. \end{aligned}$$

Definition 2.11. A representation of a modified λ-differential left-symmetric algebra $(\mathfrak{p}, \star, \partial)$ is a quadruple $(\mathcal{V}; \star_l, \star_r, \partial_{\mathcal{V}})$, where $(\mathcal{V}; \star_l, \star_r)$ is a representation of the left-symmetric algebra $(\mathfrak{p}, \star, \partial)$ and $\partial_{\mathcal{V}} : \mathcal{V} \to \mathcal{V}$ is a linear map such that for all $p_1 \in \mathfrak{p}, u \in \mathcal{V}$:

$$\partial_{\mathcal{V}}(p_1 \star_l u) = \partial p_1 \star_l u + p_1 \star_l \partial_{\mathcal{V}} u + \lambda(p_1 \star_l u), \tag{2.3}$$

$$\partial_{\mathcal{V}}(u \star_r p_1) = \partial_{\mathcal{V}} u \star_r p_1 + u \star_r \partial p_1 + \lambda(u \star_r p_1), \tag{2.4}$$

For example, given a modified λ-differential left-symmetric algebra $(\mathfrak{p}, \star, \partial)$, there is a natural adjoint representation on itself. The corresponding representation maps $\star_l, \star_r$ and $\partial_{\mathcal{V}}$ are given by $\star_l = \star_r = \star$ and $\partial_{\mathcal{V}} = \partial$.

Proposition 2.12. *Let $(\mathfrak{p}, \star, \partial)$ be a modified λ-differential left-symmetric algebra and $(\mathcal{V}; \star_l, \star_r, \partial_{\mathcal{V}})$ be a representation of it. Then $(\mathcal{V}; \star_l, \star_r, \partial_{\mathcal{V}})$ is a representation of $(\mathfrak{p}, \star, \partial)$ if and only if $\mathfrak{p} \oplus \mathcal{V}$ is a modified λ-differential left-symmetric algebra with the following maps:*

$$\begin{aligned} &(p_1 + u_1) \star_{\ltimes} (p_2 + u_2) := p_1 \star p_2 + p_1 \star_l u_2 + u_1 \star_r p_2, \\ &\partial \oplus \partial_{\mathcal{V}}(p_1 + u_1) = \partial p_1 + \partial_{\mathcal{V}} u_1, \end{aligned}$$

for $p_1, p_2 \in \mathfrak{p}$ and $u_1, u_2 \in \mathcal{V}$. In the case, the modified λ-differential left-symmetric algebra $\mathfrak{p} \oplus \mathcal{V}$ is called a semidirect product of $\mathfrak{p}$ and $\mathcal{V}$, denoted by $\mathfrak{p} \ltimes \mathcal{V} = (\mathfrak{p} \oplus \mathcal{V}, \star_{\ltimes}, \partial \oplus \partial_{\mathcal{V}})$.

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Proof. Firstly, it is easy to verify that $(\mathfrak{p}\oplus\mathcal{V},\star_{\ltimes})$ is a left-symmetric algebra. Furthermore, for any $p_1,p_2\in\mathfrak{p}$ and $u_1,u_2\in\mathcal{V}$, by Equations (2.2)- (2.4) we have

$$\begin{aligned}
&\partial\oplus\partial_{\mathcal{V}}((p_1+u_1)\star_{\ltimes}(p_2+u_2))\\
&=\partial\oplus\partial_{\mathcal{V}}(p_1\star p_2+p_1\star_l u_2+u_1\star_r p_2)\\
&=\partial(p_1\star p_2)+\partial_{\mathcal{V}}(p_1\star_l u_2+u_1\star_r p_2)\\
&=\partial p_1\star p_2+p_1\star\partial p_2+\lambda(p_1\star p_2)+\partial p_1\star_l u_2+p_1\star_l\partial_{\mathcal{V}}u_2+\lambda(p_1\star_l u_2)\\
&\quad+\partial_{\mathcal{V}}u_1\star_r p_2+u_1\star_r\partial p_2+\lambda(u_1\star_r p_2)\\
&=\partial\oplus\partial_{\mathcal{V}}(p_1+u_1)\star_{\ltimes}(p_2+u_2)+(p_1+u_1)\star_{\ltimes}\partial\oplus\partial_{\mathcal{V}}(p_2+u_2)+\lambda((p_1+u_1)\star_{\ltimes}(p_2+u_2)).
\end{aligned}$$

Hence, $(\mathfrak{p}\oplus\mathcal{V},\star_{\ltimes},\partial\oplus\partial_{\mathcal{V}})$ is a modified λ-differential left-symmetric algebra.

Conversely, suppose $(\mathfrak{p}\oplus\mathcal{V},\star_{\ltimes},\partial\oplus\partial_{\mathcal{V}})$ is a modified λ-differential left-symmetric algebra, then for any $p\in\mathfrak{p}$ and $u\in\mathcal{V}$, we have

$$\begin{aligned}
&\partial\oplus\partial_{\mathcal{V}}((p+0)\star_{\ltimes}(0+u))\\
&\quad=\partial\oplus\partial_{\mathcal{V}}(p+0)\star_{\ltimes}(0+u)+(p+0)\star_{\ltimes}\partial\oplus\partial_{\mathcal{V}}(0+u)+\lambda((p+0)\star_{\ltimes}(0+u)),\\
&\partial\oplus\partial_{\mathcal{V}}((0+u)\star_{\ltimes}(p+0))\\
&\quad=\partial\oplus\partial_{\mathcal{V}}(0+u)\star_{\ltimes}(p+0)+(0+u)\star_{\ltimes}\partial\oplus\partial_{\mathcal{V}}(p+0)+\lambda((0+u)\star_{\ltimes}(p+0)),
\end{aligned}$$

which implies that $\partial_{\mathcal{V}}(p\star_l u)=\partial p\star_l u+p\star_l\partial_{\mathcal{V}}u+\lambda(p\star_l u)$ and $\partial_{\mathcal{V}}(u\star_r p)=\partial_{\mathcal{V}}u\star_r p+u\star_r\partial p+\lambda(u\star_r p)$. Therefore, $(\mathcal{V};\star_l,\star_r,\partial_{\mathcal{V}})$ is a representation of $(\mathfrak{p},\star,\partial)$. □

3. Cohomology of Modified λ-Differential Left-Symmetric Algebras

In this section, we define the cohomology of a modified λ-differential left-symmetric algebra with coefficients in its representation.

Let us recall the cohomology theory of left-symmetric algebras in [32]. Let $(\mathfrak{p},\star)$ be a left-symmetric algebra and $(\mathcal{V};\star_l,\star_r)$ be a representation of it. Denote the $n-$cochains of $\mathfrak{p}$ with coefficients in representation $\mathcal{V}$ by $\mathfrak{C}^n_{\mathrm{LSA}}(\mathfrak{p},\mathcal{V}):=\mathrm{Hom}(\mathfrak{p}^{\otimes n},\mathcal{V})$.

The coboundary map $\delta:\mathfrak{C}^n_{\mathrm{LSA}}(\mathfrak{p},\mathcal{V})\to\mathfrak{C}^{n+1}_{\mathrm{LSA}}(\mathfrak{p},\mathcal{V})$, for $p_1,\cdots,p_{n+1}\in\mathfrak{p}$ and $\theta\in\mathfrak{C}^n_{\mathrm{LSA}}(\mathfrak{p},\mathcal{V})$, as

$$\begin{aligned}
&\delta\theta(p_1,\dots,p_{n+1})\\
&=\sum_{i=1}^{n}(-1)^{i+1}p_i\star_l\theta(p_1,\dots,\widehat{p}_i,\dots,p_{n+1})+\sum_{i=1}^{n}(-1)^{i+1}\theta(p_1,\dots,\widehat{p}_i,\dots,p_n,p_i)\star_r p_{n+1}\\
&\quad-\sum_{i=1}^{n}(-1)^{i+1}\theta(p_1,\dots,\widehat{p}_i,\dots,p_n,p_i\star p_{n+1})\\
&\quad+\sum_{1\le i<j\le n}(-1)^{i+j}\theta(p_i\star p_j-p_j\star p_i,p_1,\dots,\widehat{p}_i,\dots,\widehat{p}_j,\dots,p_{n+1}).
\end{aligned}\tag{3.1}$$

Then, it was proved that $\delta\circ\delta=0$. Let us denote by $\mathcal{H}^{\bullet}_{\mathrm{LSA}}(\mathfrak{p},\mathcal{V})$, the cohomology group associated to the cochain complex $(\mathfrak{C}^{\bullet}_{\mathrm{LSA}}(\mathfrak{p},\mathcal{V}),\delta)$.

For any $n\ge1$, we define a linear map $\Gamma:\mathfrak{C}^n_{\mathrm{LSA}}(\mathfrak{p},\mathcal{V})\to\mathfrak{C}^n_{\mathrm{LSA}}(\mathfrak{p},\mathcal{V})$ by

$$(\Gamma\theta)(p_1,\dots,p_n)=\sum_{i=1}^{n}\theta(p_1,\dots,\partial p_i,\dots,p_n)+(n-1)\lambda\theta(p_1,\dots,p_n)-\partial_{\mathcal{V}}\theta(p_1,\dots,p_n).\tag{3.2}$$

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Lemma 3.1. *The map* Γ *is a cochain map, i.e.,* $\Gamma \circ \delta = \delta \circ \Gamma$*. In other words, the following diagram is commutative:*

$$\begin{array}{ccc} \mathfrak{C}^{n}_{\mathrm{LSA}}(\mathfrak{p}, \mathcal{V}) & \xrightarrow{\delta} & \mathfrak{C}^{n+1}_{\mathrm{LSA}}(\mathfrak{p}, \mathcal{V}) \\ \downarrow{\scriptstyle \Gamma} & & \downarrow{\scriptstyle \Gamma} \\ \mathfrak{C}^{n}_{\mathrm{LSA}}(\mathfrak{p}, \mathcal{V}) & \xrightarrow{\delta} & \mathfrak{C}^{n+1}_{\mathrm{LSA}}(\mathfrak{p}, \mathcal{V}). \end{array}$$

Proof. For any $\theta \in \mathfrak{C}^{n}_{\mathrm{LSA}}(\mathfrak{p}, \mathcal{V})$ and $p_1, \dots, p_{n+1} \in \mathfrak{p}$, we have

$$\begin{aligned} &\Gamma(\delta\theta)(p_1, \dots, p_{n+1}) \\ =& \sum_{i=1}^{n+1} (\delta\theta)(p_1, \dots, \partial p_i, \dots, p_{n+1}) + n\lambda(\delta\theta)(p_1, \dots, p_{n+1}) - \partial_{\mathcal{V}}(\delta\theta)(p_1, \dots, p_{n+1}) \end{aligned} \tag{3.3}$$

and

$$\begin{aligned} &\delta(\Gamma\theta)(p_1, \dots, p_{n+1}) \\ =& \sum_{i=1}^{n} (-1)^{i+1} p_i \star_l (\Gamma\theta)(p_1, \dots, \widehat{p_i}, \dots, p_{n+1}) + \sum_{i=1}^{n} (-1)^{i+1} (\Gamma\theta)(p_1, \dots, \widehat{p_i}, \dots, p_n, p_i) \star_r p_{n+1} \\ &- \sum_{i=1}^{n} (-1)^{i+1} (\Gamma\theta)(p_1, \dots, \widehat{p_i}, \dots, p_n, p_i \star p_{n+1}) \\ &+ \sum_{1 \le i < j \le n} (-1)^{i+j} (\Gamma\theta)(p_i \star p_j - p_j \star p_i, p_1, \dots, \widehat{p_i}, \dots, \widehat{p_j}, \dots, p_{n+1}). \end{aligned} \tag{3.4}$$

By Equations (2.1)-(2.4) and further expanding Equations (3.3) and (3.4), we have (3.3)=(3.4). Therefore, $\Gamma \circ \delta = \delta \circ \Gamma$. $\square$

Definition 3.2. Let $(\mathfrak{p}, \star, \partial)$ be a modified λ-differential left-symmetric algebra and $(\mathcal{V}; \star_l, \star_r, \partial_{\mathcal{V}})$ be a representation of it. We define the cochain complex $(\mathcal{C}^{\bullet}_{\mathrm{MDLSA}}(\mathfrak{p}, \mathcal{V}), \mathfrak{D})$ of $(\mathfrak{p}, \star, \partial)$ with coefficients in $(\mathcal{V}; \star_l, \star_r, \partial_{\mathcal{V}})$ to the negative shift of the mapping cone of Γ, that is, let

$$\mathcal{C}^{1}_{\mathrm{MDLSA}}(\mathfrak{p}, \mathcal{V}) = \mathfrak{C}^{1}_{\mathrm{LSA}}(\mathfrak{p}, \mathcal{V}) \text{ and } \mathcal{C}^{n}_{\mathrm{MDLSA}}(\mathfrak{p}, \mathcal{V}) := \mathfrak{C}^{n}_{\mathrm{LSA}}(\mathfrak{p}, \mathcal{V}) \oplus \mathfrak{C}^{n-1}_{\mathrm{LSA}}(\mathfrak{p}, \mathcal{V}), \forall n \geq 2,$$

and the coboundary map $\mathfrak{D} : \mathcal{C}^{1}_{\mathrm{MDLSA}}(\mathfrak{p}, \mathcal{V}) \to \mathcal{C}^{2}_{\mathrm{MDLSA}}(\mathfrak{p}, \mathcal{V})$ is given by

$$\mathfrak{D}(\theta) = (\delta\theta, -\Gamma\theta), \forall \theta \in \mathcal{C}^{1}_{\mathrm{MDLSA}}(\mathfrak{p}, \mathcal{V});$$

for $n \geq 2$, the coboundary map $\mathfrak{D} : \mathcal{C}^{n}_{\mathrm{MDLSA}}(\mathfrak{p}, \mathcal{V}) \to \mathcal{C}^{n+1}_{\mathrm{MDLSA}}(\mathfrak{p}, \mathcal{V})$ is given by

$$\mathfrak{D}(\theta_1, \theta_2) = (\delta\theta_1, \delta\theta_2 + (-1)^n \Gamma\theta_1), \forall (\theta_1, \theta_2) \in \mathcal{C}^{n}_{\mathrm{MDLSA}}(\mathfrak{p}, \mathcal{V}).$$

The cohomology of $(\mathcal{C}^{\bullet}_{\mathrm{MDLSA}}(\mathfrak{p}, \mathcal{V}), \mathfrak{D})$, denoted by $\mathcal{H}^{\bullet}_{\mathrm{MDLSA}}(\mathfrak{p}, \mathcal{V})$, is called the cohomology of the modified λ-differential left-symmetric algebra $(\mathfrak{p}, \star, \partial)$ with coefficients in $(\mathcal{V}; \star_l, \star_r, \partial_{\mathcal{V}})$. In particular, when $(\mathcal{V}; \star_l, \star_r, \partial_{\mathcal{V}}) = (\mathfrak{p}; \star_l = \star_r = \star, \partial)$, we just denote $(\mathcal{C}^{\bullet}_{\mathrm{MDLSA}}(\mathfrak{p}, \mathfrak{p}), \mathfrak{D})$, $\mathcal{H}^{\bullet}_{\mathrm{MDLSA}}(\mathfrak{p}, \mathfrak{p})$ by $(\mathcal{C}^{\bullet}_{\mathrm{MDLSA}}(\mathfrak{p}), \mathfrak{D})$, $\mathcal{H}^{*}_{\mathrm{MDLSA}}(\mathfrak{p})$ respectively, and call them the cochain complex, the cohomology of modified λ-differential left-symmetric algebra $(\mathfrak{p}, \star, \partial)$ respectively.

It is obvious that there is a short exact sequence of cochain complexes:

$$0 \to \mathfrak{C}^{n-1}_{\mathrm{LSA}}(\mathfrak{p}, \mathcal{V}) \longrightarrow \mathcal{C}^{n}_{\mathrm{MDLSA}}(\mathfrak{p}, \mathcal{V}) \longrightarrow \mathfrak{C}^{n}_{\mathrm{LSA}}(\mathfrak{p}, \mathcal{V}) \to 0.$$

It induces a long exact sequence of cohomology groups:

$$\cdots \to \mathcal{H}^{n}_{\mathrm{MDLSA}}(\mathfrak{p}, \mathcal{V}) \to \mathcal{H}^{n}_{\mathrm{LSA}}(\mathfrak{p}, \mathcal{V}) \to \mathcal{H}^{n+1}_{\mathrm{MDLSA}}(\mathfrak{p}, \mathcal{V}) \to \mathcal{H}^{n+1}_{\mathrm{LSA}}(\mathfrak{p}, \mathcal{V}) \to \cdots.$$

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At the end of this section, we use the established cohomology theory to characterize linear deformations of modified λ-differential left-symmetric algebras.

Definition 3.3. Let $(\mathfrak{p}, \star, \partial)$ be a modified λ-differential left-symmetric algebra. If for all $t \in \mathbb{K}$, $(\mathfrak{p}[[t]]/(t^2), \star_t = \star + t\star_1, \partial_t = \partial + t\partial_1)$ is still a modified λ-differential left-symmetric algebra over $\mathbb{K}[[t]]/(t^2)$, where $(\star_1, \partial_1) \in \mathcal{C}^2_{\mathrm{MDLSA}}(\mathfrak{p})$. We say that $(\star_1, \partial_1)$ generates a linear deformation of a modified λ-differential left-symmetric algebra $(\mathfrak{p}, \star, \partial)$.

Proposition 3.4. *If $(\star_1, \partial_1)$ generates a linear deformation of a modified λ-differential left-symmetric algebra $(\mathfrak{p}, \star, \partial)$, then $(\star_1, \partial_1)$ is a 2-cocycle of the modified λ-differential left-symmetric algebra $(\mathfrak{p}, \star, \partial)$.*

Proof. If $(\star_1, \partial_1)$ generates a linear deformation of a modified λ-differential left-symmetric algebra $(\mathfrak{p}, \star, \partial)$, then for any $p_1, p_2, p_3 \in \mathfrak{p}$, we have

$$\begin{aligned}(p_1 \star_t p_2) \star_t p_3 - p_1 \star_t (p_2 \star_t c) =& (p_2 \star_t p_1) \star_t p_3 - p_2 \star_t (p_1 \star_t p_3),\\ \partial_t(p_1 \star_t p_2) =& \partial_t p_1 \star_t p_2 + p_1 \star_t \partial_t p_2 + \lambda(p_1 \star_t p_2).\end{aligned}$$

Comparing coefficients of t^1 on both sides of the above equations, we have

$$\begin{aligned}&(p_1 \star_1 p_2) \star p_3 + (p_1 \star p_2) \star_1 p_3 - p_1 \star (p_2 \star_1 p_3) - p_1 \star_1 (p_2 \star p_3)\\ &= (p_2 \star_1 p_1) \star p_3 + (p_2 \star p_1) \star_1 p_3 - p_2 \star_1 (p_1 \star p_3) - p_2 \star (p_1 \star_1 p_3)\end{aligned}$$

and

$$\partial_1(p_1 \star p_2) + \partial(p_1 \star_1 p_2) = \partial p_1 \star_1 p_2 + \partial_1 p_1 \star p_2 + p_1 \star \partial_1 p_2 + p_1 \star_1 \partial p_2 + \lambda p_1 \star_1 p_2.$$

Note that the first equation is exactly $\delta\star_1 = 0$ and that second equation is exactly to $\delta\partial_1 + \Gamma\star_1 = 0$. Therefore, $\mathfrak{D}(\star_1, \partial_1) = (\delta\star_1, \delta\partial_1 + \Gamma\star_1) = 0$, that is, $(\star_1, \partial_1)$ is a 2-cocycle. $\square$

Definition 3.5. Let $(\mathfrak{p}[[t]]/(t^2), \star_t = \star + t\star_1, \partial_t = \partial + t\partial_1)$ and $(\mathfrak{p}[[t]]/(t^2), \star'_t = \star + t\star'_1, \partial'_t = \partial + t\partial'_1)$ be two linear deformations of modified λ-differential left-symmetric algebra $(\mathfrak{p}, \star, \partial)$. We call them equivalent if there exists $\Phi_1 : \mathfrak{p} \to \mathfrak{p}$ such that $\Phi_t = \mathrm{id}_{\mathfrak{p}} + t\Phi_1$ is a homomorphism from $(\mathfrak{p}[[t]]/(t^2), \star'_t, \partial'_t)$ to $(\mathfrak{p}[[t]]/(t^2), \star_t, \partial_t)$, i.e., for all $p_1, p_1 \in \mathfrak{p}$, the following equations hold:

$$\Phi_t(p_1 \star'_t p_2) = \Phi_t(p_1) \star_t \Phi_t(p_2), \tag{3.5}$$
$$\Phi_t(\partial'_t p_1) = \partial_t \Phi_t(p_1). \tag{3.6}$$

Proposition 3.6. *If two linear deformations $(\mathfrak{p}[[t]]/(t^2), \star_t = \star + t\star_1, \partial_t = \partial + t\partial_1)$ and $(\mathfrak{p}[[t]]/(t^2), \star'_t = \star + t\star'_1, \partial'_t = \partial + t\partial'_1)$ are equivalent, then $(\star_1, \partial_1)$ and $(\star'_1, \partial'_1)$ are in the same cohomology class of $\mathcal{H}^2_{\mathrm{MDLSA}}(\mathfrak{p})$.*

Proof. Let $\Phi_t : (\mathfrak{p}[[t]]/(t^2), \star'_t, \partial'_t) \to (\mathfrak{p}[[t]]/(t^2), \star_t, \partial_t)$ be an isomorphism. Expanding the equations and collecting coefficients of t, we get from Equations (3.5) and (3.6):

$$\begin{aligned}p_1 \star'_1 p_2 - p_1 \star_1 p_2 = \Phi_1(p_1) \star p_2 + p_1 \star \Phi_1(p_2) - \Phi_1(p_1 \star p_2) =& \delta\Phi_1(p_1, p_2),\\ \partial'_1 p_1 - \partial_1 p_1 = \partial\Phi_1(p_1) - \Phi_1(\partial p_1) =& -\Gamma\Phi_1(p_1),\end{aligned}$$

that is, $(\star'_1, \partial'_1) - (\star_1, \partial_1) = (\delta\Phi_1, -\Gamma\Phi_1) = \mathfrak{D}(\Phi_1) \in \mathcal{B}^2_{\mathrm{MDLSA}}(\mathfrak{p})$. So, $(\star_1, \partial_1)$ and $(\star'_1, \partial'_1)$ are in the same cohomology class of $\mathcal{H}^2_{\mathrm{MDLSA}}(\mathfrak{p})$. $\square$

Remark 3.7. If $(\mathfrak{p}[[t]]/(t^2), \star_t, \partial_t)$ is further equivalent to the undeformed deformation $(\mathfrak{p}[[t]]/(t^2), \star, \partial)$, we call the linear deformation $(\mathfrak{p}[[t]]/(t^2), \star_t = \star + t\star_1, \partial_t = \partial + t\partial_1)$ of a modified λ-differential left-symmetric algebra $(\mathfrak{p}, \star, \partial)$ is trivial.

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4. Abelian Extensions of Modified λ-Differential Left-Symmetric Algebra

In this section, we study abelian extensions of a modified λ-differential left-symmetric algebra. It is proved that they are classified by the second cohomology group.

Definition 4.1. Let $(\mathfrak{p},\star,\partial)$ be a modified λ-differential left-symmetric algebra and $(\mathcal{V},\star_{\mathcal{V}},\partial_{\mathcal{V}})$ an abelian modified λ-differential left-symmetric algebra with the trivial product $\star_{\mathcal{V}}$. An abelian extension $(\hat{\mathfrak{p}},\hat{\star},\hat{\partial})$ of $(\mathfrak{p},\star,\partial)$ by $(\mathcal{V},\star_{\mathcal{V}},\partial_{\mathcal{V}})$ is a short exact sequence of morphisms of modified Rota-Baxter pre-Lie algebras

$$0 \longrightarrow (\mathcal{V},\star_{\mathcal{V}},\partial_{\mathcal{V}}) \stackrel{\mathrm{i}}{\longrightarrow} (\hat{\mathfrak{p}},\hat{\star},\hat{\partial}) \stackrel{\mathrm{p}}{\longrightarrow} (\mathfrak{p},\star,\partial) \longrightarrow 0, \tag{4.1}$$

that is, there exists a commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{V} & \stackrel{\mathrm{i}}{\longrightarrow} & \hat{\mathfrak{p}} & \stackrel{\mathrm{p}}{\longrightarrow} & \mathfrak{p} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \partial_{\mathcal{V}}} & & \downarrow{\scriptstyle \hat{\partial}} & & \downarrow{\scriptstyle \partial} & & \\ 0 & \longrightarrow & \mathcal{V} & \stackrel{\mathrm{i}}{\longrightarrow} & \hat{\mathfrak{p}} & \stackrel{\mathrm{p}}{\longrightarrow} & \mathfrak{p} & \longrightarrow & 0, \end{array}$$

such that $\hat{\partial}u = \partial_{\mathcal{V}}u$ and $u\hat{\star}v = 0$, for $u,v \in \mathcal{V}$, i.e., $\mathcal{V}$ is an abelian ideal of $\hat{\mathfrak{p}}$.

A section of an abelian extension $(\hat{\mathfrak{p}},\hat{\star},\hat{\partial})$ of $(\mathfrak{p},\star,\partial)$ by $(\mathcal{V},\star_{\mathcal{V}},\partial_{\mathcal{V}})$ is a linear map $\mathrm{s}:\mathfrak{p}\to\hat{\mathfrak{p}}$ such that $\mathrm{p}\circ\mathrm{s} = \mathrm{id}_{\mathfrak{p}}$.

Definition 4.2. Let $(\hat{\mathfrak{p}}_1,\hat{\star}_1,\hat{\partial}_1)$ and $(\hat{\mathfrak{p}}_2,\hat{\star}_2,\hat{\partial}_2)$ be two abelian extensions of $(\mathfrak{p},\star,\partial)$ by $(\mathcal{V},\star_{\mathcal{V}},\partial_{\mathcal{V}})$. They are said to be isomorphic if there exists a modified λ-differential left-symmetric algebra isomorphism $\Phi : (\hat{\mathfrak{p}}_1,\hat{\star}_1,\hat{\partial}_1) \to (\hat{\mathfrak{p}}_2,\hat{\star}_2,\hat{\partial}_2)$, such that the following diagram is commutative:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & (\mathcal{V},\star_{\mathcal{V}},\partial_{\mathcal{V}}) & \stackrel{\mathrm{i}_1}{\longrightarrow} & (\hat{\mathfrak{p}}_1,\hat{\star}_1,\hat{\partial}_1) & \stackrel{\mathrm{p}_1}{\longrightarrow} & (\mathfrak{p},\star,\partial) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \mathrm{id}_{\mathcal{V}}} & & \downarrow{\scriptstyle \Phi} & & \downarrow{\scriptstyle \mathrm{id}_{\mathfrak{p}}} & & \\ 0 & \longrightarrow & (\mathcal{V},\star_{\mathcal{V}},\partial_{\mathcal{V}}) & \stackrel{\mathrm{i}_2}{\longrightarrow} & (\hat{\mathfrak{p}}_2,\hat{\star}_2,\hat{\partial}_2) & \stackrel{\mathrm{p}_2}{\longrightarrow} & (\mathfrak{p},\star,\partial) & \longrightarrow & 0. \end{array} \tag{4.2}$$

We will show that isomorphism classes of abelian extensions of $(\mathfrak{p},\star,\partial)$ by $(\mathcal{V},\star_{\mathcal{V}},\partial_{\mathcal{V}})$ are in bijection with the second cohomology group $\mathcal{H}^2_{\mathrm{MDLSA}}(\mathfrak{p},\mathcal{V})$

Let $(\hat{\mathfrak{p}},\hat{\star},\hat{\partial})$ be an abelian extension of a modified λ-differential left-symmetric algebra $(\mathfrak{p},\star,\partial)$ by $(\mathcal{V},\star_{\mathcal{V}},\partial_{\mathcal{V}})$ and $\mathrm{s}:\mathfrak{p}\to\hat{\mathfrak{p}}$ be a section of it. For any $p\in\mathfrak{p}, u\in\mathcal{V}$, define $\star_l : \mathfrak{p}\times\mathcal{V}\to\mathcal{V}$ and $\star_r : \mathcal{V}\times\mathfrak{p}\to\mathcal{V}$ respectively by

$$p \star_l u = \mathrm{s}(p)\hat{\star}u, \quad u \star_r p = u\hat{\star}\mathrm{s}(p).$$

We further define linear maps $\omega : \mathfrak{p}\times\mathfrak{p}\to\mathcal{V}$ and $\chi : \mathfrak{p}\to\mathcal{V}$ respectively by

$$\begin{aligned} \omega(p_1,p_2) &= \mathrm{s}(p_1)\hat{\star}\mathrm{s}(p_2) - \mathrm{s}(p_1\star p_2), \\ \chi(p_1) &= \hat{\partial}\mathrm{s}(p_1) - \mathrm{s}(\partial p_1), \quad \forall p_1,p_2\in\mathfrak{p}. \end{aligned}$$

Obviously, $\hat{\mathfrak{p}}$ is isomorphic to $\mathfrak{p}\oplus\mathcal{V}$ as vector spaces. Transfer the modified λ-differential left-symmetric algebra structure on $\hat{\mathfrak{p}}$ to that on $\mathfrak{p}\oplus\mathcal{V}$, we obtain a modified λ-differential left-symmetric algebra $(\mathfrak{p}\oplus\mathcal{V},\star_\omega,\partial_\chi)$, where $\star_\omega$ and ∂_χ are given by

$$\begin{aligned} (p_1+u_1)\star_\omega(p_2+u_2) &= p_1\star p_2 + p_1\star_l u_2 + u_1\star_r p_2 + \omega(p_1,p_2), \\ \partial_\chi(p_1+u_1) &= \partial p_1 + \chi(p_1) + \partial_{\mathcal{V}}u_1, \ \forall p_1,p_2\in\mathfrak{p},\ u_1,u_2\in\mathcal{V}. \end{aligned}$$

Moreover, we get an abelian extension

$$0 \longrightarrow (\mathcal{V}, \star_{\mathcal{V}}, \partial_{\mathcal{V}}) \xrightarrow{\mathrm{i}} (\mathfrak{p} \oplus \mathcal{V}, \star_{\omega}, \partial_{\chi}) \xrightarrow{\mathrm{p}} (\mathfrak{p}, \star, \partial) \longrightarrow 0 \tag{4.3}$$

which is easily seen to be isomorphic to the original one (4.1).

Proposition 4.3. *With the above notations, $(\mathcal{V}; \star_l, \star_r, \partial_{\mathcal{V}})$ is a representation of the modified λ-differential left-symmetric algebra $(\mathfrak{p}, \star, \partial)$.*

Proof. For any $p_1, p_2 \in \mathfrak{p}$ and $u \in \mathcal{V}$, by $\mathcal{V}$ is an abelian ideal of $\hat{\mathfrak{p}}$ and $\mathrm{s}(p_1 \star p_2) - \mathrm{s}(p_1)\hat{\star}\mathrm{s}(p_2) \in \mathcal{V}$, we have

$$\begin{aligned} p_1 \star_l (p_2 \star_l u) - (p_1 \star p_2) \star_l u &= \mathrm{s}(p_1)\hat{\star}(\mathrm{s}(p_2)\hat{\star}u) - \mathrm{s}(p_1 \star p_2)\hat{\star}u \\ &= \mathrm{s}(p_1)\hat{\star}(\mathrm{s}(p_2)\hat{\star}u) - (\mathrm{s}(p_1)\hat{\star}\mathrm{s}(p_2))\hat{\star}u \\ &= \mathrm{s}(p_2)\hat{\star}(\mathrm{s}(p_1)\hat{\star}u) - (\mathrm{s}(p_2)\hat{\star}\mathrm{s}(p_1))\hat{\star}u \\ &= p_2 \star_l (p_1 \star_l u) - (p_2 \star p_1) \star_l u. \end{aligned}$$

It is similar to see $p_1 \star_l (u \star_r p_2) - (p_1 \star_l u) \star_r p_2 = u \star_r (p_1 \star p_2) - (u \star_r p_1) \star_r p_2$. Hence, this shows that $(\mathcal{V}; \star_l, \star_r)$ is a representation of the left-symmetric algebra $(\mathfrak{p}, \star)$.

Moreover, by $\hat{\partial}\mathrm{s}(p_1) - \mathrm{s}(\partial p_1) \in \mathcal{V}$, we have

$$\begin{aligned} \partial_{\mathcal{V}}(p_1 \star_l u) &= \partial_{\mathcal{V}}(\mathrm{s}(p_1)\hat{\star}u) = \hat{\partial}(\mathrm{s}(p_1)\hat{\star}u) \\ &= \hat{\partial}\mathrm{s}(p_1)\hat{\star}u + \mathrm{s}(p_1)\hat{\star}\hat{\partial}u + \lambda(\mathrm{s}(p_1)\hat{\star}u) \\ &= \mathrm{s}(\partial p_1)\hat{\star}u + \mathrm{s}(p_1)\hat{\star}\partial_{\mathcal{V}}u + \lambda(\mathrm{s}(p_1)\hat{\star}u) \\ &= \partial p_1 \star_l u + p_1 \star_l \partial_{\mathcal{V}}u + \lambda(p_1 \star_l u). \end{aligned}$$

By the same token, $\partial_{\mathcal{V}}(u \star_r p_1) = \partial_{\mathcal{V}}u \star_r p_1 + u \star_r \partial p_1 + \lambda(u \star_r p_1)$. Hence, $(\mathcal{V}; \star_l, \star_r, \partial_{\mathcal{V}})$ is a representation of $(\mathfrak{p}, \star, \partial)$. □

Proposition 4.4. *With the above notation, the pair (ω, χ) is a 2-cocycle of the modified λ-differential left-symmetric algebra $(\mathfrak{p}, \star, \partial)$ with coefficients in the representation $(\mathcal{V}; \star_l, \star_r, \partial_{\mathcal{V}})$.*

Proof. By $(\mathfrak{p} \oplus \mathcal{V}, \star_\omega, \partial_\chi)$ is a modified λ-differential left-symmetric algebra, for any $p_1, p_2, p_3 \in \mathfrak{p}$ and $u_1, u_2, u_3 \in \mathcal{V}$, we have

$$\begin{aligned} &((p_1 + u_1) \star_\omega (p_2 + u_2)) \star_\omega (p_3 + u_3) - (p_1 + u_1) \star_\omega ((p_2 + u_2) \star_\omega (p_3 + u_3)) \\ &= ((p_2 + u_2) \star_\omega (p_1 + u_1)) \star_\omega (p_3 + u_3) - (p_2 + u_2) \star_\omega ((p_1 + u_1) \star_\omega (p_3 + u_3)), \\ &\partial_\chi((p_1 + u_1) \star_\omega (p_2 + u_2)) \\ &= \partial_\chi(p_1 + u_1) \star_\omega (p_2 + u_2) + (p_1 + u_1) \star_\omega \partial_\chi(p_2 + u_2) + \lambda(p_1 + u_1) \star_\omega (p_2 + u_2). \end{aligned}$$

Furthermore, the above two equations are equivalent to the following equations:

$$\begin{aligned} &\omega(p_1, p_2) \star_r p_3 + \omega(p_1 \star p_2, p_3) - p_1 \star_l \omega(p_2, p_3) - \omega(p_1, p_2 \star p_3) \\ &= \omega(p_2, p_1) \star_r p_3 + \omega(p_2 \star p_1, p_3) - p_2 \star_l \omega(p_1, p_3) - \omega(p_2, p_1 \star p_3), \end{aligned} \tag{4.4}$$

$$\begin{aligned} &\chi(p_1 \star p_2) + \partial_{\mathcal{V}}\omega(p_1, p_2) \\ &= \chi(p_1) \star_r p_2 + \omega(\partial p_1, p_2) + p_1 \star_l \chi(p_2) + \omega(p_1, \partial p_2) + \lambda\omega(p_1, p_2). \end{aligned} \tag{4.5}$$

Using Equations (4.4) and (4.5), we have $\delta\omega = 0$ and $\delta_M\chi + \Gamma\omega = 0$ respectively. Therefore, $\mathfrak{D}(\omega, \chi) = (\delta\omega, \delta\chi + \Gamma\omega) = 0$, that is, (ω, χ) is a 2-cocycle. □

Let's now study the influence of different choices of sections.

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Proposition 4.5. *Let $(\hat{\mathfrak{p}}, \hat{\star}, \hat{\partial})$ be an abelian extension of a modified λ-differential left-symmetric algebra $(\mathfrak{p}, \star, \partial)$ by $(\mathcal{V}, \star_{\mathcal{V}}, \partial_{\mathcal{V}})$ and $\mathrm{s} : \mathfrak{p} \to \hat{\mathfrak{p}}$ be a section of it.*

(i) Different choices of the section s *give the same representation on $(\mathcal{V}, \partial_{\mathcal{V}})$. Moreover, isomorphic abelian extensions give rise to the same representation of $(\mathfrak{p}, \star, \partial)$.*

(ii) The cohomology class (ω, χ) of does not depend on the choice of s.

Proof. (i) Let $\mathrm{s}' : \mathfrak{p} \to \hat{\mathfrak{p}}$ be another section of $(\hat{\mathfrak{p}}, \hat{\star}, \hat{\partial})$ and $(\mathcal{V}; \star_l', \star_r', \partial_{\mathcal{V}})$ be another representation of $(\mathfrak{p}, \star, \partial)$ constructed using the section s'. By $\mathrm{s}(p_1) - \mathrm{s}'(p_1) \in \mathcal{V}$ for $p_1 \in \mathfrak{p}$, then we have

$$p_1 \star_l u - p_1 \star_l' u = \mathrm{s}(p_1)\hat{\star}u - \mathrm{s}'(p_1)\hat{\star}u = (\mathrm{s}(p_1) - \mathrm{s}'(p_1))\hat{\star}u = 0,$$

which implies that $\star_l = \star_l'$. Similarly, there is also $\star_r = \star_r'$. Thus, different choices of the section s give the same representation on $(\mathcal{V}, \partial_{\mathcal{V}})$.

Moreover, let $(\hat{\mathfrak{p}}_1, \hat{\star}_1, \hat{\partial}_1)$ and $(\hat{\mathfrak{p}}_2, \hat{\star}_2, \hat{\partial}_2)$ be two isomorphic abelian extensions of $(\mathfrak{p}, \star, \partial)$ by $(\mathcal{V}, \star_{\mathcal{V}}, \partial_{\mathcal{V}})$ with the associated isomorphism $\Phi : (\hat{\mathfrak{p}}_1, \hat{\star}_1, \hat{\partial}_1) \to (\hat{\mathfrak{p}}_2, \hat{\star}_2, \hat{\partial}_2)$ such that the diagram in (4.2) is commutative. Let $\mathrm{s}_1 : \mathfrak{p} \to \hat{\mathfrak{p}}_1$ and $\mathrm{s}_2 : \mathfrak{p} \to \hat{\mathfrak{p}}_2$ be two sections of $(\hat{\mathfrak{p}}_1, \hat{\star}_1, \hat{\partial}_1)$ and $(\hat{\mathfrak{p}}_2, \hat{\star}_2, \hat{\partial}_2)$ respectively. By Proposition 4.3, we have $(\mathcal{V}; \star_l^1, \star_r^1, \partial_{\mathcal{V}})$ and $(\mathcal{V}; \star_l^2, \star_r^2, \partial_{\mathcal{V}})$ are their representations respectively. Define $\mathrm{s}_1' : \mathfrak{p} \to \hat{\mathfrak{p}}_1$ by $\mathrm{s}_1' = \Phi^{-1} \circ \mathrm{s}_2$. As $\mathrm{p}_2 \circ \Phi = \mathrm{p}_1$, we have

$$\mathrm{p}_1 \circ \mathrm{s}_1' = (\mathrm{p}_2 \circ \Phi) \circ (\Phi^{-1} \circ \mathrm{s}_2) = \mathrm{id}_{\mathfrak{p}}.$$

So, we obtain that s_1' is a section of $(\hat{\mathfrak{p}}_1, \hat{\star}_1, \hat{\partial}_1)$. By Φ is an isomorphism of modified λ-differential left-symmetric algebras such that $\Phi|_{\mathcal{V}} = \mathrm{id}_{\mathcal{V}}$, for any $p \in \mathfrak{p}$ and $u \in \mathcal{V}$, we have

$$p \star_l^1 u = \mathrm{s}_1'(p)\hat{\star}_1 u = \Phi^{-1} \circ \mathrm{s}_2(p)\hat{\star}_1 u = \Phi^{-1}(\mathrm{s}_2(p)\hat{\star}_2 u) = p \star_l^2 u,$$

which implies that $\star_l^1 = \star_l^2$. Similarly, there is also $\star_r^1 = \star_r^2$. Thus, isomorphic abelian extensions give rise to the same representation of $(\mathfrak{p}, \star, \partial)$.

(ii) Let $\mathrm{s}' : \mathfrak{p} \to \hat{\mathfrak{p}}$ be another section of $(\hat{\mathfrak{p}}, \hat{\star}, \hat{\partial})$, by Proposition 4.4, we get another corresponding 2-cocycle (ω', χ'). Define $\tau : \mathfrak{p} \to \mathcal{V}$ by $\tau(p_1) = \mathrm{s}(p_1) - \mathrm{s}'(p_1)$, for any $p_1, p_2 \in \mathfrak{p}$, we have

$$\begin{aligned}
\omega(p_1, p_2) &= \mathrm{s}(p_1)\hat{\star}\mathrm{s}(p_2) - \mathrm{s}(p_1 \star p_2) \\
&= (\mathrm{s}'(p_1) + \tau(p_1))\hat{\star}(\mathrm{s}'(p_2) + \tau(p_2)) - (\mathrm{s}'(p_1 \star p_2) + \tau(p_1 \star p_2)) \\
&= \mathrm{s}'(p_1)\hat{\star}\mathrm{s}'(p_2) + \mathrm{s}'(p_1)\hat{\star}\tau(p_2) + \tau(p_1)\hat{\star}\mathrm{s}'(p_2) + \tau(p_1)\hat{\star}\tau(p_2) - \mathrm{s}'(p_1 \star p_2) - \tau(p_1 \star p_2) \\
&= \mathrm{s}'(p_1)\hat{\star}\mathrm{s}'(p_2) - \mathrm{s}'(p_1 \star p_2) + p_1 \star_l \tau(p_2) + \tau(p_1) \star_r p_2 - \tau(p_1 \star p_2) \\
&= \omega'(p_1, p_2) + \delta\tau(p_1, p_2), \\
\chi(p_1) &= \hat{\partial}\mathrm{s}(p_1) - \mathrm{s}(\partial p_1) \\
&= \hat{\partial}(\mathrm{s}'(p_1) + \tau(p_1)) - (\mathrm{s}'(\partial p_1) + \tau(\partial p_1)) \\
&= \hat{\partial}\mathrm{s}'(p_1) + \hat{\partial}\tau(p_1) - \mathrm{s}'(\partial p_1) - \tau(\partial p_1) \\
&= \hat{\partial}\mathrm{s}'(p_1) - \mathrm{s}'(\partial p_1) + \partial_{\mathcal{V}}\tau(p_1) - \tau(\partial p_1) \\
&= \chi'(p_1) - \Gamma\tau(p_1).
\end{aligned}$$

Hence, $(\omega, \chi) - (\omega', \chi') = (\delta\tau, -\Gamma\tau) = \mathfrak{D}(\tau) \in \mathcal{B}^2_{\mathrm{MDLSA}}(\mathfrak{p}, \mathcal{V})$, that is (ω, χ) and (ω', χ') are in the same cohomological class in $\mathcal{H}^2_{\mathrm{MDLSA}}(\mathfrak{p}, \mathcal{V})$. $\square$

Next we are ready to classify abelian extensions of a modified λ-differential left-symmetric algebra.

Theorem 4.6. *Abelian extensions of a modified λ-differential left-symmetric algebra $(\mathfrak{p}, \star, \partial)$ by $(\mathcal{V}, \star_{\mathcal{V}}, \partial_{\mathcal{V}})$ are classified by the second cohomology group $\mathcal{H}^2_{\mathrm{MDLSA}}(\mathfrak{p}, \mathcal{V})$.*

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Proof. Assume that $(\hat{\mathfrak{p}}_1, \hat{\star}_1, \hat{\partial}_1)$ and $(\hat{\mathfrak{p}}_2, \hat{\star}_2, \hat{\partial}_2)$ be two isomorphic abelian extensions of $(\mathfrak{p}, \star, \partial)$ by $(\mathcal{V}, \star_{\mathcal{V}}, \partial_{\mathcal{V}})$ with the associated isomorphism $\Phi : (\hat{\mathfrak{p}}_1, \hat{\star}_1, \hat{\partial}_1) \to (\hat{\mathfrak{p}}_2, \hat{\star}_2, \hat{\partial}_2)$ such that the diagram in (4.2) is commutative. Let s_1 be a section of $(\hat{\mathfrak{p}}_1, \hat{\star}_1, \hat{\partial}_1)$. As $\mathrm{p}_2 \circ \Phi = \mathrm{p}_1$, we have

$$\mathrm{p}_2 \circ (\Phi \circ \mathrm{s}_1) = \mathrm{p}_1 \circ \mathrm{s}_1 = \mathrm{id}_{\mathfrak{p}}.$$

So we obtain that $\Phi \circ \mathrm{s}_1$ is a section of $(\hat{\mathfrak{p}}_2, \hat{\star}_2, \hat{\partial}_2)$. Denote $\mathrm{s}_2 := \Phi \circ \mathrm{s}_1$. Since Φ is an isomorphism of modified λ-differential left-symmetric algebras such that $\Phi|_{\mathcal{V}} = \mathrm{id}_{\mathcal{V}}$, we have

$$\begin{aligned}
\omega_2(p_1, p_2) &= \mathrm{s}_2(p_1)\hat{\star}_2\mathrm{s}_2(p_2) - \mathrm{s}_2(p_1 \star p_2) \\
&= \Phi \circ \mathrm{s}_1(p_1)\hat{\star}_2\Phi \circ \mathrm{s}_1(p_2) - \Phi \circ \mathrm{s}_1(p_1 \star p_2) \\
&= \Phi(\mathrm{s}_1(p_1)\hat{\star}_1\mathrm{s}_1(p_2) - \mathrm{s}_1(p_1 \star p_2)) \\
&= \Phi(\omega_1(p_1, p_2)) \\
&= \omega_1(p_1, p_2)
\end{aligned}$$

and

$$\begin{aligned}
\chi_2(p_1) &= \hat{\partial}\mathrm{s}_2(p_1) - \mathrm{s}_2(\partial p_1) \\
&= \hat{\partial}(\Phi \circ \mathrm{s}_1(p_1)) - \Phi \circ \mathrm{s}_1(\partial p_1) \\
&= \hat{\partial}(\mathrm{s}_1(p_1)) - \mathrm{s}_1(\partial p_1) \\
&= \chi_1(p_1).
\end{aligned}$$

Thus, isomorphic abelian extensions gives rise to the same element in $\mathcal{H}^2_{\mathrm{MDLSA}}(\mathfrak{p}, \mathcal{V})$.

Conversely, given two 2-cocycles (ω_1, χ_1) and (ω_2, χ_2) , we can construct two abelian extensions $(\mathfrak{p} \oplus \mathcal{V}, \star_{\omega_1}, \partial_{\chi_1})$ and $(\mathfrak{p} \oplus \mathcal{V}, \star_{\omega_2}, \partial_{\chi_2})$ via (4.3). If they represent the same cohomology class in $\mathcal{H}^2_{\mathrm{MDLSA}}(\mathfrak{p}, \mathcal{V})$, then there exists $\tau : \mathfrak{p} \to \mathcal{V}$ such that

$$(\omega_1, \chi_1) - (\omega_2, \chi_2) = \mathfrak{D}(\tau).$$

We define $\Phi_\tau : \mathfrak{p} \oplus \mathcal{V} \to \mathfrak{p} \oplus \mathcal{V}$ by $\Phi_\tau(p_1 + u) := p_1 + \tau(p_1) + u,\ p_1 \in \mathfrak{p}, u \in \mathcal{V}$. Then it is easy to verify that Φ_τ is an isomorphism of these two abelian extensions $(\mathfrak{p} \oplus \mathcal{V}, \star_{\omega_1}, \partial_{\chi_1})$ and $(\mathfrak{p} \oplus \mathcal{V}, \star_{\omega_2}, \partial_{\chi_2})$ such that the diagram in (4.2) is commutative. □

5. Skeletal Modified λ-Differential Left-Symmetric Algebras and Crossed Modules

In this section, we introduce the notion of modified λ-differential left-symmetric 2-algebras and show that skeletal modified λ-differential left-symmetric 2-algebras are classified by 3-cocycles of modified λ-differential left-symmetric algebras.

We first recall the definition of left-symmetric 2-algebras from [33], which is a categorization of a left-symmetric algebra.

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A left-symmetric 2-algebra is a quintuple $(\mathfrak{p}_0, \mathfrak{p}_1, d, l_2, l_3)$, where $d : \mathfrak{p}_1 \to \mathfrak{p}_0$ is a linear map, $l_2 : \mathfrak{p}_i \times \mathfrak{p}_j \to \mathfrak{p}_{i+j}$ are bilinear maps and $l_3 : \mathfrak{p}_0 \times \mathfrak{p}_0 \times \mathfrak{p}_0 \to \mathfrak{p}_1$ is a trilinear map, such that for any $p_1, p_2, p_3, p_4 \in \mathfrak{p}_0$ and $u, v \in \mathfrak{p}_1$, the following equations are satisfied:

$$dl_2(p_1, u) = l_2(p_1, d(u)), \tag{5.1}$$
$$dl_2(u, p_1) = l_2(d(u), p_1), \tag{5.2}$$
$$l_2(d(u), v) = l_2(u, d(v)), \tag{5.3}$$
$$dl_3(p_1, p_2, p_3) = l_2(p_1, l_2(p_2, p_3)) - l_2(l_2(p_1, p_2), p_3) - l_2(p_2, l_2(p_1, p_3)) + l_2(l_2(p_2, p_1), p_3), \tag{5.4}$$
$$l_3(p_1, p_2, d(u)) = l_2(p_1, l_2(p_2, u)) - l_2(l_2(p_1, p_2), u) - l_2(p_2, l_2(p_1, u)) + l_2(l_2(p_2, p_1), u), \tag{5.5}$$
$$l_3(d(u), p_2, p_3) = l_2(u, l_2(p_2, p_3)) - l_2(l_2(u, p_2), p_3) - l_2(p_2, l_2(u, p_3)) + l_2(l_2(p_2, u), p_3), \tag{5.6}$$
$$\begin{aligned} &l_2(p_1, l_3(p_2, p_3, p_4)) - l_2(p_2, l_3(p_1, p_3, p_4)) + l_2(p_3, l_3(p_1, p_2, p_4)) + l_2(l_3(p_2, p_3, p_1), p_4) \\ &- l_2(l_3(p_1, p_3, p_2), p_4) + l_2(l_3(p_1, p_2, p_3), p_4) - l_3(p_2, p_3, l_2(p_1, p_4)) + l_3(p_1, p_3, l_2(p_2, p_4)) \\ &- l_3(p_1, p_2, l_2(p_3, p_4)) - l_3(l_2(p_1, p_2) - l_2(p_2, p_1), p_3, p_4) + l_3(l_2(p_1, p_3) - l_2(p_3, p_1), p_2, p_4) \\ &- l_3(l_2(p_2, p_3) - l_2(p_3, p_2), p_1, p_4) = 0. \end{aligned} \tag{5.7}$$

Motivated by [33] and [34], we propose the concept of a modified λ-differential left-symmetric 2-algebra.

Definition 5.1. A modified λ-differential left-symmetric 2-algebra consists of a left-symmetric 2-algebra $\mathcal{P} = (\mathfrak{p}_0, \mathfrak{p}_1, d, l_2, l_3)$ and a modified λ-differential 2-operator $\widetilde{\partial} = (\partial_0, \partial_1, \partial_2)$ on $\mathcal{P}$, where $\partial_0 : \mathfrak{p}_0 \to \mathfrak{p}_0$, $\partial_1 : \mathfrak{p}_1 \to \mathfrak{p}_1$ and $\partial_2 : \mathfrak{p}_0 \times \mathfrak{p}_0 \to \mathfrak{p}_1$, for any $p_1, p_2, p_3 \in \mathfrak{p}_0, u \in \mathfrak{p}_1$, satisfying the following equations:

$$\partial_0 \circ d = d \circ \partial_1, \tag{5.8}$$
$$d\partial_2(p_1, p_2) + \partial_0 l_2(p_1, p_2) = l_2(\partial_0 p_1, p_2) + l_2(p_1, \partial_0 p_2) + \lambda l_2(p_1, p_2), \tag{5.9}$$
$$\partial_2(p_1, d(u)) + \partial_1 l_2(p_1, u) = l_2(\partial_0 p_1, u) + l_2(p_1, \partial_1 u) + \lambda l_2(p_1, u), \tag{5.10}$$
$$\partial_2(d(u), p_2) + \partial_1 l_2(u, p_2) = l_2(\partial_1 u, p_2) + l_2(u, \partial_0 p_2) + \lambda l_2(u, p_2), \tag{5.11}$$
$$\begin{aligned} &l_2(p_1, \partial_2(p_2, p_3)) - l_2(p_2, \partial_2(p_1, p_3)) + l_2(\partial_2(p_2, p_1), p_3) - l_2(\partial_2(p_1, p_2), p_3) \\ &- \partial_2(p_2, l_2(p_1, p_3)) + \partial_2(p_1, l_2(p_2, p_3)) - \partial_2(l_2(p_1, p_2) - l_2(p_2, p_1), p_3) + l_3(\partial_0 p_1, p_2, p_3) \\ &+ l_3(p_1, \partial_0 p_2, p_3) + l_3(p_1, p_2, \partial_0 p_3) + 2\lambda l_3(p_1, p_2, p_3) - \partial_1 l_3(p_1, p_2, p_3) = 0. \end{aligned} \tag{5.12}$$

We denote a modified λ-differential left-symmetric 2-algebra by $(\mathcal{P}, \widetilde{\partial})$.

A modified λ-differential left-symmetric 2-algebra is said to be skeletal (resp. strict) if $d = 0$ (resp. $l_3 = 0, \partial_2 = 0$).

First we have the following trivial example of strict modified λ-differential left-symmetric 2-algebra.

Example 5.2. For any modified λ-differential left-symmetric algebra $(\mathfrak{p}, \star, \partial)$, $(\mathfrak{p}_0 = \mathfrak{p}_1 = \mathfrak{p}, d = 0, l_2 = \star, \partial_0 = \partial_1 = \partial)$ is a strict modified λ-differential left-symmetric 2-algebra.

Proposition 5.3. *Let $(\mathcal{P}, \widetilde{\partial})$ be a modified λ-differential left-symmetric 2-algebra.*

(i) If $(\mathcal{P}, \widetilde{\partial})$ is skeletal or strict, then $(\mathfrak{p}_0, \star_0, \partial_0)$ is a modified λ-differential left-symmetric algebra, where $p_1 \star_0 p_2 = l_2(p_1, p_2)$ for $p_1, p_2 \in \mathfrak{p}_0$.

(ii) If $(\mathcal{P}, \widetilde{\partial})$ is strict, then $(\mathfrak{p}_1, \star_1, \partial_1)$ is a modified λ-differential left-symmetric algebra, where $u \star_1 v = l_2(d(u), v) = l_2(u, d(v))$ for $u, v \in \mathfrak{p}_1$.

(iii) If $(\mathcal{P}, \widetilde{\partial})$ is skeletal or strict, then $(\mathfrak{p}_1; \star_l, \star_r, \partial_1)$ is a representation of $(\mathfrak{p}_0, \star_0, \partial_0)$, where $p_1 \star_l u = l_2(p_1, u)$ and $u \star_r p_1 = l_2(u, p_1)$ for $p_1 \in \mathfrak{p}_0, u \in \mathfrak{p}_1$.

Proof. From Equations (5.1)-(5.6) and (5.8)-(5.11), (i),(ii) and (iii) can be obtained by direct verification. □

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Theorem 5.4. *There is a one-to-one correspondence between skeletal modified λ-differential left-symmetric 2-algebras and 3-cocycles of modified λ-differential left-symmetric algebras.*

Proof. Let $(\mathcal{P}, \widetilde{\partial})$ be a modified λ-differential left-symmetric 2-algebra. By Proposition 5.3, we can consider the cohomology of modified λ-differential left-symmetric algebra $(\mathfrak{p}_0, \star_0, \partial_0)$ with coefficients in the representation $(\mathfrak{p}_1; \star_l, \star_r, \partial_1)$. For any $p_1, p_2, p_3, p_4 \in \mathfrak{p}_0$, by Equation (5.7), we have

$$\begin{aligned}
&\delta l_3(p_1, p_2, p_3, p_4)\\
=&l_2(p_1, l_3(p_2, p_3, p_4)) - l_2(p_2, l_3(p_1, p_3, p_4)) + l_2(p_3, l_3(p_1, p_2, p_4)) + l_2(l_3(p_2, p_3, p_1), p_4)\\
&- l_2(l_3(p_1, p_3, p_2), p_4) + l_2(l_3(p_1, p_2, p_3), p_4) - l_3(p_2, p_3, l_2(p_1, p_4)) + l_3(p_1, p_3, l_2(p_2, p_4))\\
&- l_3(p_1, p_2, l_2(p_3, p_4)) - l_3(l_2(p_1, p_2) - l_2(p_2, p_1), p_3, p_4) + l_3(l_2(p_1, p_3) - l_2(p_3, p_1), p_2, p_4)\\
&- l_3(l_2(p_2, p_3) - l_2(p_3, p_2), p_1, p_4)\\
=&0.
\end{aligned}$$

By Equations (3.2) and (5.12), there holds that

$$\begin{aligned}
&(\delta\partial_2 + \Gamma l_3)(p_1, p_2, p_3) = \delta\partial_2(p_1, p_2, p_3) + \Gamma l_3(p_1, p_2, p_3)\\
=&p_1 \star_l \partial_2(p_2, p_3) - p_2 \star_l \partial_2(p_1, p_3) + \partial_2(p_2, p_1) \star_r p_3 - \partial_2(p_1, p_2) \star_r p_3\\
&- \partial_2(p_2, p_1 \star p_3) + \partial_2(p_1, p_2 \star p_3) - \partial_2(p_1 \star p_2 - p_2 \star p_1, p_3) + l_3(\partial_0 p_1, p_2, p_3)\\
&+ l_3(p_1, \partial_0 p_2, p_3) + l_3(p_1, p_2, \partial_0 p_3) + 2\lambda l_3(p_1, p_2, p_3) - \partial_1 l_3(p_1, p_2, p_3)\\
=&l_2(p_1, \partial_2(p_2, p_3)) - l_2(p_2, \partial_2(p_1, p_3)) + l_2(\partial_2(p_2, p_1), p_3) - l_2(\partial_2(p_1, p_2), p_3)\\
&- \partial_2(p_2, l_2(p_1, p_3)) + \partial_2(p_1, l_2(p_2, p_3)) - \partial_2(l_2(p_1, p_2) - l_2(p_2, p_1), p_3) + l_3(\partial_0 p_1, p_2, p_3)\\
&+ l_3(p_1, \partial_0 p_2, p_3) + l_3(p_1, p_2, \partial_0 p_3) + 2\lambda l_3(p_1, p_2, p_3) - \partial_1 l_3(p_1, p_2, p_3)\\
=&0.
\end{aligned}$$

Thus, $\mathfrak{D}(l_3, \partial_2) = (\delta l_3, \delta\partial_2 + \Gamma l_3) = 0$, which implies that $(l_3, \partial_2) \in \mathcal{C}^3_{\mathrm{MDLSA}}(\mathfrak{p}_0, \mathfrak{p}_1)$ is a 3-cocycle of modified λ-differential left-symmetric algebra $(\mathfrak{p}_0, \star_0, \partial_0)$ with coefficients in the representation $(\mathfrak{p}_1; \star_l, \star_r, \partial_1)$.

Conversely, assume that $(l_3, \partial_2) \in \mathcal{C}^3_{\mathrm{MDLSA}}(\mathfrak{p}, \mathcal{V})$ is a 3-cocycle of modified λ-differential left-symmetric algebra $(\mathfrak{p}, \star, \partial)$ with coefficients in the representation $(\mathcal{V}; \star_l, \star_r, \partial_{\mathcal{V}})$. Then $(\mathcal{P}, \widetilde{\partial})$ is a skeletal modified λ-differential left-symmetric 2-algebra, where $\mathcal{P} = (\mathfrak{p}_0 = \mathfrak{p}, \mathfrak{p}_1 = \mathcal{V}, d = 0, l_2, l_3)$ and $\widetilde{\partial} = (\partial_0 = \partial, \partial_1 = \partial_{\mathcal{V}}, \partial_2)$ with $l_2(p_1, p_2) = p_1 \star p_2, l_2(p_1, u) = p_1 \star_l u, l_2(u, p_1) = u \star_r p_1$ for any $p_1, p_2 \in \mathfrak{p}_0, u \in \mathfrak{p}_1$. □

Next we introduce the concept of crossed modules of modified λ-differential left-symmetric algebras, which are equivalent to skeletal modified λ-differential left-symmetric 2-algebras.

Definition 5.5. A crossed module of modified λ-differential left-symmetric algebras is a quadruple $((\mathfrak{p}_0, \star_0, \partial_0), (\mathfrak{p}_1, \star_1, \partial_1), d, (\star_l, \star_r))$, where $(\mathfrak{p}_0, \star_0, \partial_0)$ and $(\mathfrak{p}_1, \star_1, \partial_1)$ are modified λ-differential left-symmetric algebras, $d : \mathfrak{p}_1 \to \mathfrak{p}_0$ is a homomorphism of modified λ-differential left-symmetric algebras and $(\mathfrak{p}_1, \star_l, \star_r, \partial_1)$ is a representation of $(\mathfrak{p}_0, \star_0, \partial_0)$, for any $p \in \mathfrak{p}_0, u, v \in \mathfrak{p}_1$, satisfying the following equations:

$$d(p \star_l u) = p \star_0 d(u), d(u \star_r p) = d(u) \star_0 p, \tag{5.13}$$
$$d(u) \star_l v = u \star_r d(v) = u \star_1 v. \tag{5.14}$$

Theorem 5.6. *There is a one-to-one correspondence between skeletal modified λ-differential left-symmetric 2-algebras and crossed modules of modified λ-differential left-symmetric algebras.*

Proof. Let $(\mathcal{P}, \widetilde{\partial}) = ((\mathfrak{p}_0, \mathfrak{p}_1, d, l_2, l_3 = 0), (\partial_0, \partial_1, \partial_2 = 0))$ be a skeletal modified λ-differential left-symmetric 2-algebra. By Proposition 5.3, we construct a crossed module of modified λ-differential

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left-symmetric algebra $((\mathfrak{p}_0,\star_0,\partial_0),(\mathfrak{p}_1,\star_1,\partial_1),d,(\star_l,\star_r))$, where, $p_1\star_0 p_2 = l_2(p_1,p_2), u_1\star_1 u_2 = l_2(d(u_1),u_2) = l_2(u_1,d(u_2)), p_1\star_l u_1 = l_2(p_1,u_1)$ and $u_1\star_r p_1 = l_2(u_1,p_1)$, for $p_1,p_2\in\mathfrak{p}_0, u_1,u_2\in\mathfrak{p}_1$.

Conversely, a crossed module of modified λ-differential left-symmetric algebra $((\mathfrak{p}_0,\star_0,\partial_0),(\mathfrak{p}_1,\star_1,\partial_1),d,(\star_l,\star_r))$ gives rise to a strict modified λ-differential left-symmetric 2-algebra $(\mathcal{P},\widetilde{\partial}) = ((\mathfrak{p}_0,\mathfrak{p}_1,d,l_2,l_3=0),(\partial_0,\partial_1,\partial_2=0))$, where $l_2:\mathfrak{p}_i\times\mathfrak{p}_j\to\mathfrak{p}_{i+j}$ are given by

$$l_2(p_1,p_2)=p_1\star_0 p_2,\ \ l_2(u_1,u_2)=p_1\star_1 p_2,\ \ l_2(p_1,u_1)=p_1\star_l u_1,\ \ l_2(u_1,p_1)=u_1\star_r p_1,$$

for all $p_1,p_2\in\mathfrak{p}_0, u_1,u_2\in\mathfrak{p}_1$. Direct verification shows that $((\mathfrak{p}_0,\mathfrak{p}_1,d,l_2,l_3=0),(\partial_0,\partial_1,\partial_2=0))$ is a strict modified λ-differential left-symmetric 2-algebra. □

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