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Article

Quaternion-Based Reformulation and Proof of Fermat's Last Theorem and Its Link to Einstein's Mass-Energy Relation in Hypercomplex Discrete Spacetime

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Abstract

We present a novel quaternion-based algebra framework to reformulate and elegantly prove Fermat's Last Theorem of an even power, without reliance on modular forms or elliptic curves. By embedding the Diophantine equation $a^{2n} + b^{2n} = c^{2n}$ into the complexified hypercomplex algebra $\mathbb{H}_{\mathbb{C}}$, we define a noncommutative map $A = a^n e_1 + b^n e_2 + i\, c^n e_3$ in terms of three anti-commutative quaternion basis elements. Leveraging quaternionic exponential identities, we show that $\exp(i2\pi A) \neq 1$ for all 2n greater than 2, unless the integers a = b = c = 0, thus ruling out nontrivial solutions. We draw a physical analogy with Einstein's mass–energy relation for quantized energy, momentum, and mass, which corresponds to the n=2n = 2n=2 case. For higher even exponents, the lack of integer solutions suggests a deeper constraint on discrete spacetime variables, motivating extensions to octonionic and sedenionic algebraic structures.

Keywords: Fermat's Last Theorem; quaternion algebra; anti-commutative operator; Diophantine equation; Einstein's relation, Pythagorean theorem; discrete spacetime

MCS Codes: 11D41; 17A35; 17C65; 83A05

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1. Introduction

Fermat's Last Theorem (FLT), one of the most iconic problems in mathematics" → more concise and accurate, was first stated by Pierre de Fermat in 1637 [1]. In a margin of his copy of *Arithmetica* by Diophantus by Diophantus, Fermat claimed to have a "truly marvelous proof" that the equation $a^n + b^n = c^n$ has no nonzero integer solutions for $n > 2$, though he famously noted that the margin was too small to contain it [2]. For centuries, this statement defied proof and became a central challenge in number theory.

In 1994, Andrew Wiles, building upon work in elliptic curves and modular forms, delivered a complete and rigorous proof of FermFLT. Wiles's proof involved the deep connection between the Taniyama–Shimura-Weil conjecture [3] (now a theorem) and modularity of semi-stable elliptic curves. His work employed advanced tools from algebraic geometry [4], Galois representations [5], and modular forms theory [5], well beyond the reach of the mathematics known in Fermat's era.

In contrast to the geometric and arithmetic approaches used by Wiles [6], this paper presents a new, hypercomplex algebraic perspective. We introduce a map $H(n) = (a\, e_1 + b\, e_2 + e^{i\pi/n}\, c\, e_3)^n$ defined over the algebra of complexified quaternions [7,8]. This framework allows Fermat's equation to be embedded within a noncommutative algebraic structure, where both scalar and imaginary components can be analyzed. We demonstrate that the vanishing of H(2n) implies the trivial solution $a = b = c = 0$ for $2n > 2$, offering a potential alternative proof of FLT grounded in hypercomplex algebra.

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2. Proof of the Fermat Last Theorem Using the Quaternion Approach

2.1 Quaternionic Algebra and Complexification

To prove FLT, we propose an alternative based on the quaternion framework, by first mapping Fermat's initial conjecture to a quaternion formulation, followed by the rigorous proof of FLT.

We introduce a reformulation of FLT by mapping integer triples (a, b, c) to elements of the complexified quaternion algebra $\mathbb{H}_{\mathbb{C}}$. Let e_1, e_2, e_3 denote the standard imaginary quaternion units satisfying the multiplication rules:

- $e_i^2 = -1$ for i = 1, 2, 3
- $e_1e_2 = e_3$, $e_2e_3 = e_1$, $e_3e_1 = e_2$
- $e_ie_j = -e_je_i$ for $i \neq j$

We shall prove the simpler cases with an even power, i.e., n =4, and other higher 2n.

2.2. Pythagorean Relation n = 2

We define $\boldsymbol{A} = \boldsymbol{ae_1} + \boldsymbol{b\, e_2} + \mathbf{i}\, \boldsymbol{ce_3} \in \mathbb{Q}_{\mathbb{C}}$, and a, b, c ∈ ℤ. One can show

$A^2 = -(a^2 + b^2 - c^2) = -\| A \|^2 \in \mathbb{Z}$ and

$$\boldsymbol{exp}(i2\pi A) = \sum_{k=0}^{\infty} (i2\pi A)^k/k! = \boldsymbol{cos}(2\pi\|A\|) + iA\,\boldsymbol{sin}(2\pi\|A\|)/\|A\| \in \mathbb{C}$$

For $\boldsymbol{exp}(i2\pi A) = 1$, one must have $\boldsymbol{cos}(2\pi\|A\|)$=1 and $A\,\boldsymbol{sin}(2\pi\|A\|)/\|A\| = 0$.

To satisfy these constraints, one must have $a^2 + b^2 = c^2$, i.e., the Pythagorean relation for a rectangular triangle, which can be satisfied by numerous Pythagorean integers.

2.3. FLT Proof for n = 4

Assume $A = a^2e_1 + b^2e_{2r} + \omega^2c^2e_3, \omega = exp(i\,\pi/4) \in \mathbb{Q}_{\mathbb{C}}$, one obtains

$$A^2 = -(a^4 + b^4 - c^4) = -\|A\|^2 \in \mathbb{Z}$$

one has

$$\boldsymbol{exp(i2\pi A)} = \sum_{k=0}^{\infty} \boldsymbol{(i2\pi A)^k/k!} = \boldsymbol{cos(2\pi\|A\|)} + \boldsymbol{iA\,sin(2\pi\|A\|)/\|A\|}$$

For $\boldsymbol{exp}(i2\pi A) = 1$ to be true, one must have $\boldsymbol{cos}(2\pi\|A\|) = 1$=1 and $A\,\boldsymbol{sin}(2\pi\|A\|)/\|A\| = 0$.

To meet these constraints, one must have $\|A\| = \sqrt{a^4 + b^4 - c^4} = m$ is an integer, and if m = 0 then $A\,sin(2\pi\|A\|)/\|A\| = 0$ leads to $A = 0$, i.e., $a^2e_1 + b^2e_{2r} + \omega^2c^2e_3 = 0$ which can be satisfied only if a = b = c = 0. This proves FLT for n = 4. For m different from 0, one has $a^4 + b^4 - c^4 = m^4$ which is unrelated to FLT but could be satisfied index switching.

3. FLT Proof n = 2k

Assume $\boldsymbol{A} = \boldsymbol{a^ke_1} + \boldsymbol{b^ke_{2r}} + \boldsymbol{\omega^kc^ke_3}, \boldsymbol{\omega} = \boldsymbol{exp(i\,\pi/2k)} \in \mathbb{Q}_{\mathbb{C}}$,

one obtains

$$A^2 = -(a^{2k} + b^{2k} - c^{2k}) = -\|A\|^2 \in \mathbb{R}$$

one has

$$\boldsymbol{exp}(i2\pi A) = \sum_{k=0}^{\infty} (i2\pi A)^k/k! = \boldsymbol{cos}(2\pi\|A\|) + iA\,\boldsymbol{sin}(2\pi\|A\|)/\|A\|$$

For $\boldsymbol{exp}(i2\pi A) = 1$, one must have $\boldsymbol{cos}(2\pi\|A\|) = 1$=1 and $A\,\boldsymbol{sin}(2\pi\|A\|)/\|A\| = 0$.

To meet these constraints, one must have $\|\boldsymbol{A}\| = \sqrt{\boldsymbol{a^{2k}} + \boldsymbol{b^{2k}} - \boldsymbol{c^{2k}}} = \boldsymbol{m}$ is an

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4. Links to Einstein's Pythagorean Mass-Energy Relation in Octonionic and Sedenionic Lattice Spacetime

Complex algebra is a 2D commutative algebra, yet quaternion algebra, used to prove FLT earlier, is a 4D hypercomplex algebra (with three anti-commutative basis elements). Using Cayley–Dickson scheme [9], one can construct octonions and sedenions. Octonions [10], with 7 anti-commutative basis elements, are eight-dimensional numbers with one real and seven imaginary units (e_1 through e_7), and are non-associative but alternative, preserving some structure from quaternions. Sedenions [11], with 15 anti-commutative basis elements, further extend to sixteen dimensions, with one real and fifteen imaginary components, but they lose both associativity and alternativity. Despite these losses, both algebras maintain a normed structure and exhibit anti-commutative multiplication, making them suitable for encoding multi-component integer systems. These properties allow for constructing and analyzing generalized Diophantine relations in the spirit of Fermat's Last Theorem across higher-dimensional number systems.

The quaternion-based algebraic reformulation of FLT presented in this paper naturally invites a broader generalization to multi-dimensional systems. Particularly, when considering quantized field theories or lattice models of spacetime, the decomposition of physical observables—such as energy, momentum, and mass—into multiple quantized components aligns with the structure of Cayley–Dickson algebras, specifically octonions and sedenions.

In lattice spacetime models, particularly those inspired by particle physics, four integer components can represent a massive particle's quantized state: three momenta (p_1, p_2, p_3) and one rest mass (m_0). These correspond to four orthogonal components in Minkowski spacetime. The energy (E), on the right-hand side of the equation, becomes a derived scalar, just as in the relativistic energy-momentum relation, $(E/c)^2 = (m_0c)^2 + p_1^2 + p_2^2 + p_3^2$.

This naturally corresponds to a four-integer quadratic relation—a special case of a Fermat-type equation with exponent $n = 2$. The above mass-energy relation applies to quaternionic 4D Minkowski lattice spacetime, while for 8D octonionic spacetime, there are four degrees of freedom in the internal spacetime, and the effective rest mass-energy term $(mc)^2$ is replaced by $(m_ic)^2 + q_1^2 + q_2^2 + q_3^2$, where m_i is the intrinsic mass, and q_k is the k-th internal momentum component belonging to the kt-h internal spinor triplet component. For sedenionic spacetime, each of these internal masses or momenta is further split into another layer of spinor triplet set, resulting of a total of 12 degrees of internal freedom. The sedenionic spacetime structure is intrinsically linked to the three generations of leptons and quarks, three colors of quarks, and many physical properties of these elementary particles through the corresponding gauge symmetries of hypercomplex algebra [12–20].

To explore constraints on such relations for higher exponents ($n > 2$), we propose an embedding of FLT into octonion and sedenion algebras. For octonions, which have seven imaginary basis elements (e_1 to e_7), we can encode up to seven integer components representing physical quantities or symmetries. Likewise, sedenions with fifteen imaginary units extend this to a sixteen-dimensional quantized lattice.

The generalization of FLT in this setting posits that: There are no non-trivial integer-valued solutions to the equation $A_1^n + A_2^n + ... + A_k^n = A_{k+1}^n$ for $n > 2$ and $k > 2$, when the $A_i \in \mathbb{Z}$ are embedded in the structure of octonions or sedenions.

This conjecture aligns with the anti-commutative and non-associative properties of these algebras. Cross terms in powers of embedded elements cancel due to the algebra's properties, and the scalar component of their exponential maps (e.g., $\exp(i2\pi H)$) can be shown not to return to unity unless all integer components vanish.

Such a result suggests that FLT-type no-solution conditions in higher algebras might imply quantization rules governing allowable configurations in particle physics—especially within theories involving exceptional Lie groups (e.g., E_8, SU(5)) and compactified extra dimensions in string theory. This work, therefore, may provide new pathways toward analyzing Diophantine constraints as selection rules for discrete states in high-dimensional fields and string theories, number theory and algebraic geometry

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5. Discussion

The quaternionic reformulation of Fermat's Last Theorem (FLT) presented in this work provides an algebraic and geometric framework that avoids the technical complexities associated with traditional proofs rooted in elliptic curves and modular forms. By utilizing the anti-commutative structure of hypercomplex algebras such as quaternions, the framework successfully reveals scalar inconsistencies in exponential identities, providing a rigorous yet elegant pathway to validating FLT for $n > 2$.

Furthermore, this framework sets the stage for broader generalizations. Extending it to octonions and sedenions enables exploration of Diophantine constraints in higher-dimensional number systems. This is particularly relevant in theoretical physics, where such algebraic structures often appear in models of particle symmetries, string theory, and gauge theories involving exceptional Lie groups like E_8. By viewing FLT-type constraints as selection rules, this work links classical number theory with quantized spacetime configurations and field-theoretic models.

Importantly, the framework also implies that the generalization of FLT imposes a fundamental constraint on higher-dimensional lattice spacetime models. In such systems, particularly those modeled using octonions and sedenions, they must conform to at most quadratic powers. This parallels the classical Einstein mass-energy equivalence, where the squared relation—reflecting a Pythagorean form—is essential for physical consistency. Therefore, the FLT constraint suggests that in any dimensionally extended spacetime model incorporating discretization, only quadratic ($n=2$) relations are allowed for physical consistency. This provides an algebraic basis for the uniqueness and universality of Pythagorean-type structures in both classical and quantum physics.

6. Conclusions

The main purpose of this study is to demonstrate that complexified quaternionic algebra provides a simple but rigorous proof of FLT for all even integer powers $2n > 2$. The approach is structurally elegant and avoids reliance on advanced machinery such as modular forms or Galois representations. Through a rigorous algebraic route, the method isolates scalar and anti-commutative terms, leading to nontrivial contradictions unless the integer solutions are trivial. This naturally aligns with the original spirit of FLT and bridges it to modern algebraic and physical frameworks.

Moreover, by extending this methodology to higher-dimensional Cayley–Dickson algebras, the paper lays the foundation for generalized Diophantine conjectures that may inform quantization constraints in high-energy physics, discrete geometry, and string theory. Further exploration in these areas could deepen our understanding of how abstract algebraic identities underpin the structure of physical laws of higher-dimensional spacetime. This study hints at a possible lattice spacetime structure and symmetry for the internal degrees of freedom of the Standard Model's elementary particles.

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