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Multi-Objective Optimization by Means of Pairs of Functions

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Abstract: We introduce and discuss a generalization of the classical multi-objective optimization to pairs of functions. This procedure is referred to as bi-multi-objective optimization. A justification of this general optimization procedure is presented, related both to multi-objective optimization under ambiguity concerning individual preferences and to Pareto optimality for a family of preferences with nontransitive indifference. Incidentally, the binary relation naturally associated to a bi-multi-objective optimization problem is represented by a finite bi-multi-utility, which generalizes to the nontransitive case the classical finite multi-utility representation. An important application is presented to Markowitz portfolio selection under ambiguity concerning both the vector of returns and the covariance matrix.

Keywords: Pareto optimal elemen; Bi-multi-objective optimization; interval order; ambiguity; Markowitz portfolio selection

MSC: 90C29 (Primary); 91B10 (Secondary)

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1. Introduction

It is very well known that *multi-objective optimization* (see e.g. Miettinen 1999 and Ehrgott 2005) is the most popular tool which allows to choose among different options in the presence of many *agents* (or *criteria*) represented by finitely many real-valued functions u_i ($i \in \{1, ..., m\}$), which have to be maximized all at the same time by following the approach introduced by Pareto 1896, and therefore called *Pareto optimality*. Such notion appears in Debreu 1954 and Mas-Colell et al. 1995 in the framework of general equilibrium. Multi-objective optimization appears in different applications of mathematics, such as insurance theory (see e.g. Asimit et al. 2017), design engineering (see e.g. Das 1999), economics and risk-sharing (see e.g. Chateauneuf et al. 2015, Chanas and Kuchta 1996 and Barrieu and Scandolo 2008), and portfolio selection (see e.g. Fliege and Werner 2014 and Xidonas et al. 2017).

Recently, Bevilacqua et al. 2018 approached the classical multi-objective optimization problem by referring to the naturally associated *preorders*, in such a way that, since the Pareto optimal elements are precisely the maximal elements of such preorders, the classical results concerning for example the existence of maximal elements for not necessarily total preorders on compact topological spaces can be used (see e.g. the famous results of Bergstrom 1975, Ward 1954 and Rodríguez-Palmero and García-Lapresta 2002, and the recent results in Bosi and Zuanon 2017).

In this paper we introduce a generalization of multi-objective optimization, called *bi-multi-objective optimization*. Such a problem has the form

$$\max_{x \in X}[(u_1(x), v_1(x)), ..., (u_m(x), v_m(x))] = \max_{x \in X}(\mathbf{u}(x), \mathbf{v}(x)), \quad m \geq 2,$$

where the pair (u_i, v_i) of real-valued functions on the set X associated with the i-th individual satisfies, for every $i \in \{1, ..., m\}$, the condition $u_i \leq v_i$, and a point $x_0 \in X$ is said to be *Pareto optimal* if for no $x \in X$ it occurs that $u_i(x_0) \leq v_i(x)$ for all $i \in \{1, ..., m\}$ and $v_{\bar{i}}(x_0) < u_{\bar{i}}(x)$ for at least one index $\bar{i}$.

Needless to say, bi-multi-objective optimization coincides with the classical multi-objective optimization when $u_i = v_i$ for all $i \in \{1, ..., m\}$.

We present an interpretation of bi-multi-objective optimization based on *decision theory*, since to every pair (u_i, v_i) we can associate the *interval order* $\precsim_i$ on X defined, for all $x \in X$, by

$$x \precsim_i y \Leftrightarrow u_i(x) \leq v_i(y).$$

Hence, the appearance and use of bi-multi-objective optimization can be related to the *intransitivity of the indifference* of the individual interval orders (see e.g. Bosi and Zuanon 2014ab).

Incidentally, we observe that we can associate to every bi-multi-objective optimization problem $\max_{x \in X}(\mathbf{u}(x), \mathbf{v}(x))$ the reflexive binary relation $\precsim$ on X defined by

$$x \precsim y \Leftrightarrow [u_i(x) \leq v_i(y) \quad \forall i \in \{1, ..., m\}].$$

This is a finite *bi-multi-utility representation* of a not necessarily transitive binary relation $\precsim$ on a set X, which is performed by means of a finite family $\{(u_i, v_i)\}_{i=1,...m}$ of pairs (u_i, v_i) of real valued functions on X such that $u_i \leq v_i$ for all $i \in \{1, ..., m\}$. Needless to say, this bi-multi-utility representation generalizes to the nontransitive case the classical finite *multi-utility representation* $(u_i\}_{i=1,...m}$ of a preorder $\precsim$ on X (see e.g. Bevilacqua et al. 2018 and Kaminski 2007), according to which

$$x \precsim y \Leftrightarrow [u_i(x) \leq u_i(y) \quad \forall i \in \{1, ..., m\}].$$

Another interpretation of the bi-multi-utility optimization consists of *ambiguity* concerning the individual preferences, in the spirit of Haskell et al. 2016. Indeed, $(\mathbf{u}, \mathbf{v})$ can be interpreted as a range of *utility functions* $\mathbf{w}$ such that $\mathbf{u} \leq \mathbf{w} \leq \mathbf{v}$. The set of all the weak Pareto optimal solutions to the bi-multi-objective optimization problem corresponding to the pair $(\mathbf{u}, \mathbf{v})$ include all the weak Pareto optimal solutions to the multi-objective optimization problems corresponding to all the functions $\mathbf{w}$ between $\mathbf{u}$ and $\mathbf{v}$ (see Theorem 1 below).

Compared to other interval optimization methods proposed in the literature (see e.g. Ishibuchi and Tanaka 1990, Chateauneuf et al. 2015 and Wu 2009), our approach appears simpler, since it does not require any particular choice among the possible orderings of intervals. On the other hand, bi-multi-objective optimization, which can be also applied to portfolio choice, is close to multi-objective optimization with interval-valued objective functions (see e.g. Wu et al. 2013).
The paper is structured as follows. Section 2 contains the notation, the basic definitions and the preliminary results. Section 3 presents the characterization of the solutions to the bi-multi-objective optimization problem in terms of the individual interval orders and of the maximal elements of the naturally associated reflexive binary relation. Section 4 contains an application to Markowitz portfolio selection under ambiguity. The section of the conclusions finishes the paper.

2. Notation and Preliminaries

The classical definitions relative to multi-objective optimization that we are going to present are the same as those found in Miettinen 1999 and Ehrgott 2005.

Definition 1. *The* multi-objective optimization problem *(MOP) is formulated by means of the standard notation*[1]

$$\max_{x \in X} [u_1(x), ..., u_m(x)] = \max_{x \in X} \mathbf{u}(x), \quad m \geq 2, \tag{1}$$

where X is the choice set *(or the* decision space*), u_i is the* decision function *(in this case a* utility function*) associated with the i-th individual (or criterion), and $\mathbf{u} : X \mapsto \mathbb{R}^m$ is the vector-valued function defined by $\mathbf{u}(x) = (u_1(x), ..., u_m(x))$ for all $x \in X$.*

Definition 2. *Consider the multi-objective optimization problem (1). Then a point $x_0 \in X$ is said to be*

1. Pareto optimal *with respect to the function $\mathbf{u} = (u_1, ..., u_m) : X \mapsto \mathbb{R}^m$ if for no $x \in X$ it occurs that $u_i(x_0) \leq u_i(x)$ for all $i \in \{1, ..., m\}$ and at the same time $u_{\bar{i}}(x_0) < u_{\bar{i}}(x)$ for at least one index $\bar{i}$;*
2. weakly Pareto optimal *with respect to the function $\mathbf{u} = (u_1, ..., u_m) : X \mapsto \mathbb{R}^m$ if for no $x \in X$ it occurs that $u_i(x_0) < u_i(x)$ for all $i \in \{1, ..., m\}$.*

Following Bevilacqua et al. 2018, Definition 2.3, the set of all (weakly) Pareto optimal elements with respect to the function $\mathbf{u} = (u_1, ..., u_m) : X \mapsto \mathbb{R}^m$ will be denoted by $X^{Par}_{\mathbf{u}}$ ($X^{wPar}_{\mathbf{u}}$, respectively). It is clear that $X^{Par}_{\mathbf{u}} \subset X^{wPar}_{\mathbf{u}}$ for every positive integer m, every nonempty set X and every function $\mathbf{u} = (u_1, ..., u_m) : X \mapsto \mathbb{R}^m$.

We now introduce the *bi-multi-objective optimization problem*, and then the associated concepts of *Pareto optimal* and *weakly Pareto optimal point*.

Definition 3. *The* bi-multi-objective optimization problem *(BIMOP) is formulated as follows:*

$$\max_{x \in X} [(u_1(x), v_1(x)), ..., (u_m(x), v_m(x))] = \max_{x \in X} (\mathbf{u}(x), \mathbf{v}(x)), \quad m \geq 2, \tag{2}$$

where the pair (u_i, v_i) of decision functions associated with the i-th individual satisfies, for every $i \in \{1, ..., m\}$, the condition $u_i \leq v_i$.

Definition 4. *Consider the bi-multi-objective optimization problem (2). Then a point $x_0 \in X$ is said to be*

1. Pareto optimal *with respect to the function $(\mathbf{u}, \mathbf{v}) : X \mapsto \mathbb{R}^m \times \mathbb{R}^m$ if for no $x \in X$ it occurs that $u_i(x_0) \leq v_i(x)$ for all $i \in \{1, ..., m\}$ and at the same time $v_{\bar{i}}(x_0) < u_{\bar{i}}(x)$ for at least one index $\bar{i}$;*
2. weakly Pareto optimal *with respect to the function $(\mathbf{u}, \mathbf{v}) : X \mapsto \mathbb{R}^m \times \mathbb{R}^m$ if for no $x \in X$ it occurs that $v_i(x_0) < u_i(x)$ for all $i \in \{1, ..., m\}$.*

Definition 5. The set of all (weakly) Pareto optimal elements with respect to the function $(\mathbf{u}, \mathbf{v}) : X \mapsto \mathbb{R}^m \times \mathbb{R}^m$ will be denoted by $X^{Par}_{(\mathbf{u},\mathbf{v})}$ ($X^{wPar}_{(\mathbf{u},\mathbf{v})}$, respectively).

Remark 1. *It is clear that $X^{wPar}_{(\mathbf{u},\mathbf{v})} \supset X^{Par}_{(\mathbf{u},\mathbf{v})}$. Indeed, consider any element $x_0 \notin X^{wPar}_{(\mathbf{u},\mathbf{v})}$. Then it happens that, for some element $x \in X$, $u_i(x_0) \leq v_i(x_0) < u_i(x) \leq v_i(x)$ for all $i \in \{1, ..., m\}$, which clearly implies that $x_0 \notin X^{Par}_{(\mathbf{u},\mathbf{v})}$, since $u_i(x_0) < v_i(x)$ for all $i \in \{1, ..., m\}$ and at the same time $v_{\bar{i}}(x_0) < u_{\bar{i}}(x)$ for some $\bar{i}$.*

Remark 2. *Everyone immediately checks that*

$$\max_{x \in X} \mathbf{u}(x) = \max_{x \in X} (\mathbf{u}(x), \mathbf{u}(x)),$$

i.e., problem (2) coincides with problem (1) in case that $u_i = v_i$ for every $i \in \{1, ..., m\}$.

[1] Needless to say, this formulation of the multi-objective optimization problem is equivalent, “mutatis mutandis", to $\min_{x \in X} [f_1(x), ..., f_m(x)] = \min_{x \in X} \mathbf{f}(x),\ m \geq 2$.

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Example 1. *Consider the case when $X = [0,2]$ (i.e., X is the closed real interval with extremes 0 and 2), $m = 2$, $u_1(x) = x$, $v_1(x) = x + 1$, $u_2(x) = x^2 - 2$, $v_2(x) = x$. Observe that $u_i \leq v_i$ for $i = 1,2$, so that we can consider the bi-multi-objective optimization problem*

$$\max_{x\in[0,2]} [(u_1(x), v_1(x)), (u_2(x), v_2(x))] = \max_{x\in[0,2]} [(x, x+1), (x^2 - 2, x)].$$

We have that 2 is the unique Pareto optimal point, since for every $x_0 \in [0,2[$ it happens that $x_0 \leq 2 + 1 = 3$, and on the other hand $x_0 < 2^2 - 2 = 2$. The set of all weakly Pareto optimal points is $[1,2]$, since $x_0 + 1 < 2$ if and only if $x_0 < 1$, while $x_0 < 2^2 - 2$ for every $x_0 < 2$.

3. Bi-Multi-Objective Optimization

An important result can be established, which relates the weakly Pareto optimal solutions to the bi-multi-objective optimization problem with respect to the function $(\mathbf{u}(x), \mathbf{v}(x))$ and the weakly Pareto optimal solutions to the multi-objective optimization problem with respect to any function $\mathbf{w} = (w_1, ..., w_m)$ such that $u_i \leq w_i \leq v_i$ for every $i \in \{1, ..., m\}$.

Theorem 1. *Consider the bi-multi-objective optimization problem (2) with respect to the function $(\mathbf{u}(x), \mathbf{v}(x))$ and let $\mathbf{w} : X \mapsto \mathbb{R}^m$ be any function such that $u_i \leq w_i \leq v_i$ for every $i \in \{1, ..., m\}$. Then $X_{\mathbf{w}}^{wPar} \subset X_{(\mathbf{u},\mathbf{v})}^{wPar}$.*

Proof. Consider the function $(\mathbf{u}(x), \mathbf{v}(x)) : X \mapsto \mathbb{R}^m \times \mathbb{R}^m$ with $u_i \leq v_i$ for every $i \in \{1, ..., m\}$, and let $\mathbf{w} : X \mapsto \mathbb{R}^m$ be any function such that $u_i \leq w_i \leq v_i$ for every $i \in \{1, ..., m\}$. By contraposition, if $x_0 \in X$ is such that $x_0 \notin X_{(\mathbf{u},\mathbf{v})}^{wPar}$, then, according to Definition 4, it happens that there exists $x \in X$ such that, for every $i \in \{1, ..., m\}$, $w_i(x_0) \leq v_i(x_0) < u_i(x) \leq w_i(x)$, implying that $w_i(x_0) < w_i(x)$ for every $i \in \{1, ..., m\}$. This means that $x_0 \notin X_{\mathbf{w}}^{wPar}$ (see Definition 2). This consideration completes the proof. □

Remark 3. *The above Theorem 1 can be interpreted in terms of* ambiguity *concerning the criteria to be adopted in a classical multi-objective optimization problem. Indeed, the weakly Pareto optimal solutions to a bi-multi-objective optimization problem (2) include all the weakly Pareto optimal solutions to every multi-objective optimization problem (1) corresponding to a set of utilities $\mathbf{w}$ in the assigned* range $(\mathbf{u}, \mathbf{v})$.

Remark 4. *Since in problem (2) we require that $u_i \leq v_i$ for every $i \in \{1, ..., m\}$, we have that m (possibly degenerate) closed real intervals $[u_i(x), v_i(x)]$ are naturally associated to every $x \in X$.*

In order to present a characterization and interpretation of the bi-multi-objective optimization problem (2) based on decision theory, let us introduce some classical definitions relative to binary relations and in particular to *interval orders*. While these definitions are classical (see e.g. Fishburn 1985), the reader can refer for example to Bosi and Zuanon 2014ab for a deeper discussion concerning the existence of representations by means of pairs of *upper semicontinuous* real valued functions.

In the sequel, the symbol $\precsim$ will stand for a *reflexive* binary relation on a set X (i.e., $x \precsim x$ for every $x \in X$). For all $x, y \in X$, $x \precsim y$ has to be read as "the alternative y is at least as preferable as the alternative x". The *strict part* (or *asymmetric part*) of a binary relation $\precsim$ will be denoted by $\prec$ (i.e., for all $x, y \in X$, $x \prec y$ if and only if $(x \precsim y)$ and not$(y \precsim x)$). The *indifference relation* $\sim$ associated to $\precsim$ is defined, for all $x, y \in X$, as $x \sim y$ if and only if $(x \precsim y)$ *and* $(y \precsim x)$). Notice that $\sim$ is an *equivalence* on X when $\precsim$ is *transitive* (i.e., for all $x, y, z \in X$, $(x \precsim y)$ *and* $(y \precsim z) \Rightarrow x \precsim z$).

Definition 6. *A* preorder $\precsim$ *on an arbitrary nonempty set X is a binary relation on X which is reflexive and transitive.*

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Definition 7. *An* interval order $\precsim$ *on an arbitrary nonempty set* X *is a binary relation on* X *which is reflexive and in addition verifies the following condition for all* $x, y, z, w \in X$*:*

$$(x \precsim z) \text{ and } (y \precsim w) \Rightarrow (x \precsim w) \text{ or } (y \precsim z).$$

This latter property is often referred to as the *Ferrers property* (see e.g. Bosi and Zuanon 2014a).

Since it is easily seen that an interval order is *total* (i.e., for all $x, y \in X$ either $x \precsim y$ or $y \precsim x$), we have that actually $x \prec y$ if and only if $not(y \precsim x)$ $(x, y \in X)$ when $\precsim$ is an interval order. It is well known that an interval order $\precsim$ is not transitive in general, while its strict part $\prec$ is always transitive.

We recall that a total preorder $\precsim$ on a set X is *represented* by a real-valued function u on X if, for all $x \in X$,

$$x \precsim y \Leftrightarrow u(x) \leq u(y).$$

In this case u is said to be a *utility function* for $\precsim$.

Definition 8. *A pair* (u, v) *of real-valued functions on* X *is said to* represent *an interval order* $\precsim$ *on* X *if, for all* $x, y \in X$*,*

$$x \precsim y \Leftrightarrow u(x) \leq v(y).$$

It is clear that, if (u, v) is a representation of an interval order $\precsim$ on X, then $u(x) \leq v(x)$ for every $x \in X$ due to the fact that $\precsim$ is reflexive. Hence, we have that $u \leq v$.

The classical definition of a *maximal element* is needed.

Definition 9. *Let* $\precsim$ *be a reflexive binary relation on a set* X*. A point* $x_0 \in X$ *is said to be a* maximal element of $(X, \precsim)$ *if for no* $x \in X$ *it happens that* $x_0 \prec x$*.*

It should be noted that actually a point $x_0 \in X$ is a maximal element for an interval order $\precsim$ on a set X if and only if $x \precsim x_0$ for every $x \in X$.

Let us introduce the concept of Pareto optimality with respect to a (finite) family of preferences (see e.g. d'Aspremont and Gevers 2002).

Definition 10. *A point* $x_0 \in X$ *is said to be* Pareto optimal with respect to the family $\{\precsim_i\}_{i \in \{1,...,m\}}$ of interval orders *if for no point* $x \in X$ *it occurs that* $x_0 \precsim_i x$ *for all* $i \in \{1, ..., m\}$*, with at least one index* $\bar{i}$ *such that* $x_0 \prec_{\bar{i}} x$*.*

We are now ready to present a characterization of the solutions to the bi-multi-objective optimization problem.

Theorem 2. *Consider the bi-multi-objective optimization problem (2) with respect to the function* $(\mathbf{u}, \mathbf{v}) : X \mapsto \mathbb{R}^m \times \mathbb{R}^m$ *with* $u_i \leq v_i$ *for every* $i \in \{1, ..., m\}$*. Then the following conditions are equivalent on a point* $x_0 \in X$*:*

1. $x_0 \in X^{Par}_{(\mathbf{u},\mathbf{v})}$*;*
2. x_0 *is Pareto optimal with respect to the family* $\{\precsim_i\}_{i \in \{1,...,m\}}$ *of interval orders on* X *such that, for every* $i \in \{1, ..., m\}$*,* $\precsim_i$ *is represented by the pair* (u_i, v_i)*;*
3. x_0 *is a maximal element of the binary relation* $\precsim = \bigcap_{i=1}^m \precsim_i$ *where, for every* $i \in \{1, ..., m\}$*, the interval order* $\precsim_i$ *is represented by the pair* (u_i, v_i)*;*
4. x_0 *is a maximal element for the reflexive binary relation* $\precsim$ *on* X *defined as follows for all* $x, y \in X$*:*

$$x \precsim y \Leftrightarrow [u_i(x) \leq v_i(y) \quad \forall i \in \{1, ..., m\}]. \tag{3}$$

Proof. In order to show that $1 \Rightarrow 2$, just consider that if x_0 is not Pareto optimal with respect to the given family $\{\precsim_i\}_{i \in \{1,...,m\}}$ of interval orders on X, then there exists $x \in X$ such that $u_i(x_0) \leq v_i(x)$ for all $i \in \{1, ..., m\}$, with at least one index $\bar{i}$ such that $v_{\bar{i}}(x_0) < u_{\bar{i}}(x)$, which precisely means that

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$x_0 \notin X^{Par}_{(\mathbf{u},\mathbf{v})}$.
The proofs that $2 \Rightarrow 3$ and $3 \Rightarrow 4$ are rather simple and therefore they are left to the reader.
Finally, to show that $4 \Rightarrow 1$, consider that if $x_0 \notin X^{Par}_{(\mathbf{u},\mathbf{v})}$, then the existence of $x \in X$ such that $u_i(x_0) \leq v_i(x) \Leftrightarrow x_0 \precsim_i x$ for all $i \in \{1, ..., m\}$ and at the same time $v_{\bar{i}}(x_0) < u_{\bar{i}}(x) \Leftrightarrow x_0 \prec_{\bar{i}} x$ for at least one index $\bar{i}$ precisely means that x_0 is not a maximal element of the binary relation $\precsim$ defined in (3). This consideration completes the proof. □

Remark 5. *It is clear that the finite representation (3) of a (reflexive) binary relation $\precsim$ generalizes the classical finite* multi-utility representation of a preorder *(see e.g. Kaminski 2007 and Bevilacqua et al. 2018) according to which, for a given function $\mathbf{u} : X \mapsto \mathbb{R}^m$ and for all $x, y \in X$,*

$$x \precsim y \Leftrightarrow [u_i(x) \leq u_i(y) \quad \forall i \in \{1, ..., m\}]. \tag{4}$$

Compared to the this latter kind of representation, which is possible only when $\precsim$ is a preorder, the representation (3) has the advantage that it is compatible with intransitive indifference, *i.e. situations when there exist $x, y, z \in X$ such that $x \sim y$ and $y \sim z$, but not$(x \sim z)$. Needless to say, representation (3) coincides with the representation of an interval order by means of two real-valued functions in the case when $m = 1$ (see Definition 8).*

As an immediate corollary of Theorem 2 (see Remark 2), we get a characterization of the solutions to the multi-objective optimization problem.

Corollary 1. *Consider the multi-objective optimization problem (1) with respect to the function $\mathbf{u} : X \mapsto \mathbb{R}^m$. Then the following conditions are equivalent on a point $x_0 \in X$:*

1. *$x_0 \in X^{Par}_{\mathbf{u}}$;*
2. *x_0 is Pareto optimal with respect to the family $\{\precsim_i\}_{i \in \{1,...,m\}}$ of total preorders on X such that, for every $i \in \{1, ..., m\}$, $\precsim_i$ is represented by the function u_i;*
3. *x_0 is a maximal element of the binary relation $\precsim = \bigcap_{i=1}^{m} \precsim_i$ where, for every $i \in \{1, ..., m\}$, the total preorder $\precsim_i$ is represented by function u_i;*
4. *x_0 is a maximal element for the preorder $\precsim$ on X defined as follows for all $x, y \in X$:*

$$x \precsim y \Leftrightarrow [u_i(x) \leq u_i(y) \quad \forall i \in \{1, ..., m\}]. \tag{5}$$

We finish this section by presenting a topological condition guaranteeing the existence of solutions to the bi-multi-objective optimization problem.

We first recall that a real-valued function u on a topological space (X, τ) is said to be *upper semicontinuous* if $u^{-1}(]-\infty, \alpha[) = \{x \in X : u(x) < \alpha\}$ is an open set for all $\alpha \in \mathbb{R}$. The well known *Weierstrass extreme value theorem* guarantees that an upper semicontinuous real-valued function attains its maximum on a *compact* topological space.

We now present a sufficient condition for the existence of a Pareto optimal solution for the bi-multi-objective optimization problem (2). As a corollary, we obtain a well known result concerning the existence of a Pareto optimal solution to the multi-objective optimization problem.

Theorem 3. *$X^{Par}_{(\mathbf{u},\mathbf{v})} \neq \emptyset$ provided that X is endowed with a compact topology τ and one of the following conditions is verified:*

1. *the functions u_i ($i \in \{1, ..., m\}$) are upper semicontinuous;*
2. *the functions v_i ($i \in \{1, ..., m\}$) are upper semicontinuous.*

Proof. 1. For every $i \in \{1, ..., m\}$, consider a point $x_i \in \arg\max\, u_i$. These points exist since τ is a compact topology on X, and all the functions u_i are upper semicontinuous on the topological space

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(X, τ). Further, let the index $i^* \in \{1, ..., m\}$ be such that $u_{i^*}(x_{i^*}) = \max\{u_1(x_1), u_2(x_2), ..., u_m(x_m)\}$. We have that $x_{i^*} \in X^{Par}_{(\mathbf{u},\mathbf{v})}$, since for no $x \in X$ and $i \in \{1, ..., m\}$ it may happen that $u_{i^*}(x_{i^*}) \leq v_{i^*}(x_{i^*}) < u_i(x)$.

2. We proceed in a perfectly analogous way, by defining the index $i^* \in \{1, ..., m\}$ in such a way that $v_{i^*}(x_{i^*}) = \max\{v_1(x_1), v_2(x_2), ..., v_m(x_m)\}$, with $x_i \in \arg\max\, v_i$ for every $i \in \{1, ..., m\}$. We have that $x_{i^*} \in X^{Par}_{(\mathbf{u},\mathbf{v})}$. □

Corollary 2. *(Ehrgott 2005, Theorem 2.19) $X^{Par}_{\mathbf{u}} \neq \emptyset$ provided that X is endowed with a compact topology τ and the real-valued functions u_i ($i \in \{1, ..., m\}$) are all upper semicontinuous.*

Proof. This is a particular application of the above Theorem 3, when $\mathbf{u} = \mathbf{v}$. □

4. An Application to Markowitz Portfolio Selection

The Markowitz 1952 portfolio selection problem in the absence of short sales appears in the following form when expressed in terms of a multi-objective optimization problem:

$$\max_{x \in X}[\mu^T x, -x^T \Sigma x], \tag{6}$$

where $m = 2$ is the number of criteria, $X = \{x \in \mathbb{R}^n_+ : \sum_{i=1}^n x_i = 1\}$ is the set of all portfolios, $n > 0$ is the number of risky assets considered, $\mu \in \mathbb{R}^n$ is a vector of expected returns, and Σ is a covariance matrix of the returns (see e.g. Fliege and Werner 2014). It can be therefore generalized by the following bi-multi-objective optimization problem:

$$\max_{x \in X}[(\mu_1^T x, \mu_2^T x), (-x^T \Sigma_2 x, -x^T \Sigma_1 x)], \tag{7}$$

where $\mu_1^T \leq \mu_2^T$ are two vectors of returns, and $\Sigma_1 \leq \Sigma_2$ are two covariance matrices (see also Wu et al. 2013).

From Theorem 1, we can now state the following proposition.

Proposition 1. *Given two vectors $\mu_1^T \leq \mu_2^T$ of expected returns, and given two covariance matrices $\Sigma_1 \leq \Sigma_2$, consider the Markowitz bi-multi-objective optimization problem*

$$\max_{x \in X}[(\mu_1^T x, \mu_2^T x), (-x^T \Sigma_2 x, -x^T \Sigma_1 x)], \quad X = \{x \in \mathbb{R}^n_+ : \sum_{i=1}^n x_i = 1\} \tag{8}$$

We have that, for every vector of expected returns $\mu_1^T \leq \mu^T \leq \mu_2^T$ and for every covariance matrix $\Sigma_1 \leq \Sigma \leq \Sigma_2$, the weakly Pareto optimal solutions to the corresponding multi-objective optimization problem

$$\max_{x \in X}[\mu^T x, -x^T \Sigma x], \quad X = \{x \in \mathbb{R}^n_+ : \sum_{i=1}^n x_i = 1\}, \tag{9}$$

are also weakly Pareto optimal for the above bi-multi-objective problem.

Therefore, it is easy to incorporate ambiguity concerning both the vector of returns and the covariance matrix in the classical Markowitz portfolio selection problem, since our bi-multi-objective optimization model takes into account a range of expected returns and covariance matrices.

5. Conclusions

The bi-multi-objective optimization problem is an extension of the classical multi-objective optimization problem, which on one hand is simple to implement, and on the other hand can be

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considered interesting since it can be thought of as a useful tool for incorporating ambiguity in the individual preferences.

In this paper, we have presented the basic theory concerning this optimization problem, whose associated Pareto optimal elements can be interpreted as the maximal elements of the intersection of the interval orders representing the individual preferences.

Hopefully, we shall develop the full potential of our proposal in a future paper.

Conflicts of Interest: The authors guarantee that they have no conflict of interest.

References

A.V. Asimit, V. Bignozzi, K. C. Cheung, J. Hu and E.-S. Kim, Robust and Pareto optimality of insurance contracts, *European J. Oper. Res.*,**262** (2017), 720-732.

P. Barrieu and G. Scandolo, General Pareto optimal allocations and applications to multi-period risks, *Astin Bull.* **38** (2008), 105-136.

T.C. Bergstrom, Maximal elements of acyclic relations on compact sets, *J. Econom. Theory* **10** (1975), 403-404.

P. Bevilacqua, G. Bosi, M. Zuanon, Multiobjective optimization, scalarization and maximal elements of preorders, *Abstr. Appl. Anal.*, Volume 2018 (2018), Article ID 3804742, 6 pages.

G. Bosi, M. Zuanon, Upper semicontinuous representations of interval orders, *Math. Social Sci.* **60** (2014), 60-63.

G. Bosi, M. Zuanon, Conditions for the Upper Semicontinuous Representability of Preferences with Nontransitive Indifference, *Theoretical Economics Letters* **4** (2014), 371-377.

G. Bosi and M. Zuanon, Maximal elements of quasi upper semicontinuous preorders on compact spaces, *Econ. Theory Bull.* **5** (2017), 109-117.

A. Chateauneuf, M. Mostoufi and D. Vyncke, Multivariate risk sharing and the derivation of individually rational Pareto optima, *Math. Social Sci.* **74** (2015), 73-78.

S. Chanas and D. Kuchta, Multiobjective programming in optimization of interval objective functions - A generalized approach, *European J. Oper. Res.* **94** (1996), 594-598.

I. Das, A preference ordering among various Pareto optimal alternatives, *Structural Optimization* **18** (1999), 30-35.

C. d'Aspremont and L. Gevers, Social welfare functionals and interpersonal comparability, in *Handbook of Social Choice and Welfare*, Volume I, Chapter 10, (2002), 459-541.

G. Debreu, Valuation Equilibrium and Pareto Optimum, *Proc. Natl. Acad. Sci. USA* **40** (1954) 588-592.

M. Ehrgott, Multicriteria Optimization, *Lectures notes in economics and mathematical systems*. Berlin, Heidelberg, Germany: Springer, 2005.

P.C. Fishburn (1985). *Interval Orders and Interval Graphs*. Wiley, New York.

J. Fliege and R. Werner, Robust multiobjective optimization & applications in portfolio optimization, *European J. Oper. Res.* **234** (2014), 422-433.

W. B. Haskell, L. Fu and M. Dessouky, Ambiguity in risk preferences in robust stochastic optimization, *European J. Oper. Res.* **254** (2016), 214-225.

H. Ishibuchi and H. Tanaka, Multiobjective programming in optimization of the interval objective function, *European J. Oper. Res.* **48** (1990), 219-225.

B. Kaminski, On quasi-orderings and multi-objective functions, *European J. Oper. Res.* **177** (2007), 1591-1598.

H. Markowitz,, Portfolio selection, *Journal of Finance* **7** (1952), 77-91.

A. Mas-Colell, M.D. Whinston, and J.R. Green (1995). *Microeconomic Theory*, Oxford University Press.

K. Miettinen, Nonlinear Multiobjective Optimization, Kluwer Academic, Boston, Massachusetts, 1999.

V. Pareto, *Manual d'économie politique*, F. Rouge, Lausanne, 1896.

C. Rodríguez-Palmero and J.-L. García-Lapresta, Maximal elements for irreflexive binary relations on compact sets, *Math. Social Sci.* **43** (2002), 45-60.

L.E. Ward, Jr., Partially ordered topologicl spaces, *Proc. Amer. Math. Soc.* **5** (1954), 144-161.

H.-C. Wu, The Karush–Kuhn–Tucker optimality conditions in multiobjective programming problems with interval-valued objective functions, *European J. Oper. Res.* **196** (2009), 49-60.

M. Wu, D.-W. Kong, J.-P. Xu and N.-J. Huang, On interval portfolio selection problem, *Fuzzy Optim. Decis. Mak.* **12** (2013), 289-304.

 doi:10.20944/preprints202312.0596.v1

P. Xidonas, G, Mavrotas, C. Hassapis and C. Zopounidis, Robust multiobjective portfolio optimization: A minimax regret approach, *European J. Oper. Res.* **262** (2017) 299-305.