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Article

Hadamard's Variational Formula for Simple Eigenvalues

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Abstract: We study Hadamard's variational formula for simple eigenvalues under dynamical and conformal deformations. Particularly, harmonic convexity of the first eigenvalue of the Laplacian under the mixed boundary condition is established for two-dimensional domain, which implies several new inequalities.

Keywords: eigenvalue problem; perturbation theory of linear operators; domain deformation; Hadamard's variational formula; Garabadian-Schiffer's formula

MSC: 35J25; 35R35

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1. Introduction

The purpose of the present paper is to derive Hadamard's variational formulae for simple eigenvalue of the Laplacian and is to derive several new inequalities concerning the first eigenvalue concerning the mixed boundary condition.

In the previous work [9] we studied Hadamard's variational formula for general domain deformation and extended the results [3] on two or three-space dimensions under normal perturbations of the domain. There, we develop an abstract theory of perturbation of self-adjoint operators, refining the argument in [6].

To be precise, let $(X, |\ |)$ and $(V, \|\ \|)$ be a pair of Hilbert spaces over $\mathbb{R}$ with compact embedding $V \hookrightarrow X$. Henceforth, $C > 0$ denotes a generic constant. Let $A_t : V \times V \to \mathbb{R}$ and $B_t : X \times X \to \mathbb{R}$ for $t \in I = (-\varepsilon_0, \varepsilon_0)$, $\varepsilon_0 > 0$, be symmetric bilinear forms satisfying

$$|A_t(u,v)| \leq C\|u\|\|v\|,\ A_t(u,u) \geq \delta\|u\|^2, \quad u,v \in V \tag{1}$$

and

$$|B_t(u,v)| \leq C|u||v|,\ B_t(u,u) \geq \delta|u|^2, \quad u,v \in X \tag{2}$$

for some $\delta > 0$. We take the abstract eigenvalue problem

$$u \in V,\ B_t(u,u) = 1, \quad A_t(u,v) = \lambda B_t(u,v),\ \forall v \in V, \tag{3}$$

which ensures a sequence of eigenvalues denoted by

$$0 < \lambda_1(t) \leq \lambda_2(t) \leq \cdots \to +\infty.$$

The associated normalized eigenfunctions,

$$u_1(t), u_2(t), \cdots,$$

furthermore, form a complete ortho-normal system in X, provided with the inner product induced by $B_t = B_t(\cdot,\cdot)$. Hence it holds that

$$B_t(u_i(t), u_j(t)) = \delta_{ij},\ A_t(u_j(t), v) = \lambda_j B_t(u_j(t), v)$$

for any $v \in V$ and $i, j = 1, 2, \cdots$.

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The abstract theory developed in [9] is stated as follows. First, if

$$\lim_{h\to 0}\sup_{\|u\|,\|v\|\le 1}|A_{t+h}(u,v)-A_t(u,v)|=0$$
$$\lim_{h\to 0}\sup_{|u|,|v|\le 1}|B_{t+h}(u,v)-B_t(u,v)|=0$$

holds for fixed $t\in I$, we obtain

$$\lim_{h\to 0}\lambda_j(t+h)=\lambda_j(t)$$

at this t, where $j=1,2,\cdots$ are arbitrary (Theorem 8 of [9]). Second, if there exist bilinear forms $\dot{A}_t : V\times V\to\mathbb{R}$ and $\dot{B}_t : X\times X\to\mathbb{R}$ for any $t\in I$ such that

$$|\dot{A}_t(u,v)|\le C\|u\|\|v\|, \quad u,v\in V$$
$$|\dot{B}_t(u,v)|\le C|u||v|, \quad u,v\in X, \tag{4}$$

and if it holds that

$$\lim_{h\to 0}\frac{1}{h}\sup_{\|u\|,\|v\|\le 1}\left|\left(A_{t+h}-A_t-h\dot{A}_t\right)(u,v)\right|=0$$
$$\lim_{h\to 0}\frac{1}{h}\sup_{|u|,|v|\le 1}\left|\left(B_{t+h}-B_t-h\dot{B}_t\right)(u,v)\right|=0, \tag{5}$$

for fixed $t\in I$, we obtain the existence of the unilateral derivatives

$$\dot{\lambda}_j^{\pm}(t)=\lim_{h\to\pm 0}\frac{1}{h}\{\lambda_j(t+h)-\lambda_j(t)\} \tag{6}$$

at this t, where $j=1,2,\cdots$ are arbitrary (Theorem 12 of [9]). If inequalities (5) are valid to any $t\in I$ and it also holds that

$$\lim_{h\to 0}\sup_{\|u\|_V,\|v\|_V\le 1}\left|\dot{A}_{t+h}(u,v)-\dot{A}_t(u,v)\right|=0$$
$$\lim_{h\to 0}\sup_{|u|_X,|v|_X\le 1}\left|\dot{B}_{t+h}(u,v)-\dot{B}_t(u,v)\right|=0 \tag{7}$$

for the specified t, furthermore, the above unilateral derivatives satisfy

$$\lim_{h\to\pm 0}\dot{\lambda}_j^{\pm}(t+h)=\dot{\lambda}_j^{\pm}(t)$$

(Theorem 13 of [9]).

We assume, furthermore,

$$\lambda_{k-1}(t)<\lambda_k(t)\le\cdots\le\lambda_{k+m-1}(t)<\lambda_{k+m}(t), \quad \forall t\in I \tag{8}$$

for some $k,m=1,2,\cdots$, under the agreement of $\lambda_0(t)=-\infty$. Assume, also that (5) and (7) hold for any $t\in I$. Then, there exists a family of C^1 curves denoted by $\tilde{C}_i$, $k\le i\le k+m-1$, made by a rearrangement of

$$\{C_j \mid k\le j\le k+m-1\}$$

at most countably many times in I, where

$$C_j=\{\lambda_j(t)\mid t\in I\}, \quad k\le j\le k+m-1$$

(Theorem 3, Theorem 14 of [9]).

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Third, if we have the other bilinear forms $\ddot{A}_t : V \times V \to \mathbb{R}$ and $\ddot{B}_t : X \times X \to \mathbb{R}$ satisfying

$$\begin{aligned} |\ddot{A}_t(u,v)| &\le C\|u\|\|v\|, \quad u,v \in V \\ |\ddot{B}_t(u,v)| &\le C|u||v|, \quad u,v \in X \end{aligned} \tag{9}$$

for any $t \in I$, and

$$\begin{aligned} \lim_{h\to 0} \frac{1}{h^2} \sup_{\|u\|,\|v\|\le 1} \left| \left(A_{t+h} - A_t - h\dot{A}_t - \frac{h^2}{2}\ddot{A}_t \right)(u,v) \right| &= 0 \\ \lim_{h\to 0} \frac{1}{h^2} \sup_{|u|,|v|\le 1} \left| \left(B_{t+h} - B_t - h\dot{B}_t - \frac{h^2}{2}\ddot{B}_t \right)(u,v) \right| &= 0 \end{aligned} \tag{10}$$

for the fixed t, then there are

$$\ddot{\lambda}_j^{\pm}(t) = \lim_{h\to 0} \frac{2}{h^2}(\lambda_j(t+h) - \lambda_j(t) - h\dot{\lambda}_j^{\pm}(t))$$

for this t, where $j = 1, 2, \cdots$ are arbitrary (Remark 12 of [9]). These $\ddot{\lambda}_j^{\pm}(t)$, furthermore, satisfy

$$\lim_{h\to\pm 0} \ddot{\lambda}_j^{\pm}(t+h) = \ddot{\lambda}_j^{\pm}(t),$$

if inequalities (10) are valid to any $t \in I$ and it holds that

$$\begin{aligned} \lim_{h\to 0} \sup_{\|u\|,\|v\|\le 1} \left| \ddot{A}_{t+h}(u,v) - \ddot{A}_t(u,v) \right| &= 0 \\ \lim_{h\to 0} \sup_{|u|,|v|\le 1} \left| \ddot{B}_{t+h}(u,v) - \ddot{B}_t(u,v) \right| &= 0 \end{aligned} \tag{11}$$

for the above specified t (Theorem 23 of [9]). Furthermore, if (10) and (11) hold for any $t \in I$ and if it holds that (8), then the above described family of curves $\tilde{C}_i$, $k \le i \le k+m-1$, are C^2 (Theorem 24 of [9]).

Finally, these derivatives $\dot{\lambda}_j^{\pm}(t)$ and $\ddot{\lambda}_j^{\pm}(t)$ for $k \le j \le k+m-1$ are characterized as the eigenvalues of associated eigenvalue problems on the m-dimensional space, $\langle u_j(t) \mid k \le j \le k+m-1 \rangle$ (Theorem 12 and Theorem 15 of [9]). Henceforth, we assume (1), (2), (4), (5), (7), (9), (10), and (11) for any $t \in I$.

If $m = 1$ holds in (8), for example, we obtain

$$\dot{\lambda} \equiv \dot{\lambda}_k^{\pm}(t) = (\dot{A}_t - \lambda\dot{B}_t)(\phi_t, \phi_t) \tag{12}$$

and

$$\ddot{\lambda} \equiv \ddot{\lambda}_k^{\pm}(t) = (\ddot{A}_t - \lambda\ddot{B}_t - 2\dot{\lambda}B_t)(\phi_t, \phi_t) - 2C_t^{\lambda}(\gamma(\phi_t), \gamma(\phi_t)),$$

where $\lambda = \lambda_k(t)$, $\phi_t = u_k(t)$, and $C_t^{\lambda} = A_t - \lambda B_t$. Here, $w = \gamma(u) \in V_1$ is defined for $u \in V$ by

$$C_t^{\lambda}(w,v) = -\dot{C}_t^{\lambda,\dot{\lambda}}(u,v), \quad \forall v \in V_1,$$

where $V_1 = PV$, $P = I - R$, and $R : X \to V_0 = \langle \phi_t \rangle$ is the orthogonal projection, and

$$\dot{C}_t^{\lambda,\dot{\lambda}} = \dot{A}_t - \lambda\dot{B}_t - \dot{\lambda}B_t.$$

Thus we obtain the following theorem because the bilinear form C_t^{λ} is non-negative definite on $V \times V$ if $\lambda = \lambda_1$.

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Theorem 1. *If $k = m = 1$ holds in (8), there arises that*

$$\dot{\lambda} = \dot{A} - \lambda\dot{B}, \quad \ddot{\lambda} \leq \ddot{A} - \lambda\ddot{B} - 2\dot{\lambda}\dot{B}, \tag{13}$$

where

$$\dot{A} = \dot{A}_t(\phi_t, \phi_t), \ \ddot{A} = \ddot{A}_t(\phi_t, \phi_t), \ \dot{B} = \dot{B}_t(\phi_t, \phi_t), \ \ddot{B} = \ddot{B}_t(\phi_t, \phi_t) \tag{14}$$

for $\phi_t = u_k(t)$.

Since

$$\frac{d^2}{dt^2}\frac{1}{\lambda} = -\frac{\ddot{\lambda}\lambda^2 - 2\lambda\dot{\lambda}^2}{\lambda^2}$$

Theorem 1 implies the following result.

Theorem 2. *If $k = m = 1$ and*

$$2\dot{A}^2 + \lambda^2\ddot{B} \geq \lambda(\ddot{A} + 2\dot{A}\dot{B}), \tag{15}$$

it holds that

$$\frac{d^2}{dt^2}\frac{1}{\lambda} \geq 0. \tag{16}$$

Harmonic convexity of the first eigenvalue, inequality (16), was noticed by [3] for the Dirichlet problem of Laplacian under the conformal deformations of the domain in two space dimension. Here we calculate the values $\dot{A}$, $\ddot{A}$, $\dot{B}$, $\ddot{B}$ and the validity of (15) under general setting of the deformations of the domain, and then turn to the dynamical and the conformal deformations. Taking preliminaries in §2, thus, we show the results on dynamical and conformal deformations in §3 (Theorem 5 and Theorem 6) and §4 (Theorem 8 and Theorem 9), respectively. As applications, we show several new inequalities on the first eigenvalue of the two-dimensional problem.

2. General Deformations around $t = 0$

Let Ω be a bounded Lipschitz domain in n-dimensional Euclidean space $\mathbb{R}^n$ for $n \geq 2$. Suppose that its boundary $\partial\Omega$ is divided into two relatively open disconnected sets γ_0 and γ_1, satisfying

$$\overline{\gamma_0} \cup \overline{\gamma_1} = \partial\Omega, \quad \overline{\gamma_0} \cap \overline{\gamma_1} = \emptyset. \tag{17}$$

We study the eigenvalue problem of the Laplacian with mixed boundary condition,

$$-\Delta u = \lambda u \text{ in } \Omega, \quad u = 0 \text{ on } \gamma_0, \quad \frac{\partial u}{\partial \nu} = 0 \text{ on } \gamma_1, \tag{18}$$

where

$$\Delta = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}$$

and ν denotes the outer unit normal vector on $\partial\Omega$. This problem takes the weak form, finding u satisfying

$$u \in V, \ B(u,u) = 1, \quad A(u,v) = \lambda B(u,v), \ \forall v \in V \tag{19}$$

defined for

$$A(u,v) = \int_\Omega \nabla u \cdot \nabla v \, dx, \quad u,v \in V \tag{20}$$

and

$$B(u,v) = \int_\Omega uv \, dx, \quad u,v \in X, \tag{21}$$

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where
$$X = L^2(\Omega), \quad V = \{v \in H^1(\Omega) \mid v|_{\gamma_0} = 0\}. \tag{22}$$
This V is a closed subspace of $H^1(\Omega)$ under the norm
$$\|v\| = \left(\int_\Omega |\nabla v|^2 + v^2\, dx\right)^{1/2}.$$
The above reduction of (18) to (19)-(22) is justified via the trace operator to the boundary since Ω is a bounded Lipschitz domain (Theorem 2 of [8]).

To confirm the well-posedness of (3), we note, first, that if $\gamma_0 \neq \emptyset$, there is a coercivity of $A : V \times V \to \mathbb{R}$, which means the existence of $\delta > 0$ such that
$$A(v,v) \geq \delta\|v\|^2, \quad \forall v \in V. \tag{23}$$
If $\gamma_0 = \emptyset$ we replace A by $A + B$, denoted by $\tilde{A}$. Then this $\tilde{A} : V \times V \to \mathbb{R}$ is coercive, and the eigenvalue problem
$$u \in V,\ B(u,u) = 1, \quad \tilde{A}(u,v) = \tilde{\lambda}B(u,v),\ \forall v \in V,$$
is equivalent to (3) by $\tilde{\lambda} = \lambda + 1$. Hence we can assume (1), using this reduction if it is necessary.

If the bounded domain $\Omega \subset \mathbf{R}^n$ is provided with the cone property, the inclusion $H^1(\Omega) \hookrightarrow L^2(\Omega)$ is compact by Rellich-Kondrachov's theorem [1]. Thus there is a sequence of eigenvalues to (3), denoted by
$$0 < \lambda_1 \leq \lambda_2 \leq \cdots \to +\infty.$$
The associated eigenfunctions, $u_1, u_2, \cdots$, furthermore, form a complete ortho-normal system in X, provided with the inner product induced by $B = B(\cdot,\cdot)$:
$$B(u_i,u_j) = \delta_{ij}, \quad A(u_j,v) = \lambda_j B(u_j,v),\ \forall v \in V, \quad i,j = 1,2,\cdots.$$

The j-th eigenvalue of (3) is given by the mini-max principle
$$\lambda_j = \min_{L_j} \max_{v \in L_j \setminus \{0\}} R[v] = \max_{W_j} \min_{v \in W_j \setminus \{0\}} R[v], \tag{24}$$
where
$$R[v] = \frac{A(v,v)}{B(v,v)}$$
is the Rayleigh quotient, and $\{L_j\}$ and $\{W_j\}$ denote the families of all subspaces of V with dimension and codimension j and $j-1$, respectively.

The following well-known fact is valid without the smoothness of $\partial\Omega$. The proof given in Appendix A for completeness.

Theorem 3. *If $\Omega \subset \mathbf{R}^n$ is a bounded Lipschitz domain, the first eigenvalue λ_1 to (3) formulated to $A(u,v)$ and $B(u,v)$ defined by (20)-(22) is simple.*

Coming back to the Lipschitz bounded domain Ω, we introduce its deformation as follows. Let
$$T_t : \Omega \to \Omega_t = T_t(\Omega), \quad t \in (-\varepsilon_0, \varepsilon_0) \tag{25}$$
be a family of bi-Lipschitz homeomorphisms. We assume that $T_t x$ is continuous in t uniformly in $x \in \Omega$, and continue to use the following definition as in [9].

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Definition 1. *The family $\{T_t\}$ of bi-Lipschitz homeomorphisms is said to be p-differentiable in t for $p \geq 1$, if $T_t x$ is p-times differentiable in t for any $x \in \Omega$ and the mappings*

$$\frac{\partial^\ell}{\partial t^\ell} DT_t,\ \frac{\partial^\ell}{\partial t^\ell}(DT_t)^{-1} : \Omega \to M_n(\mathbb{R}), \quad 0 \leq \ell \leq p$$

are uniformly bounded in $(x,t) \in \Omega \times (-\varepsilon_0, \varepsilon_0)$, where DT_t denotes the Jacobi matrix of $T_t : \Omega \to \Omega_t$ and $M_n(\mathbb{R})$ stands for the set of real $n \times n$ matrices. This $\{T_t\}$ is furthermore said to be continuously p-differentiable in t if it is p-differentiable and the mappings

$$t \in (-\varepsilon_0, \varepsilon_0) \mapsto \frac{\partial^\ell}{\partial t^\ell} DT_t,\ \frac{\partial^\ell}{\partial t^\ell}(DT_t)^{-1} \in L^\infty(\Omega \to M_n(\mathbb{R})), \quad 0 \leq \ell \leq p$$

are continuous.

Putting

$$T_t(\gamma_i) = \gamma_{it}, \quad i = 0, 1, \tag{26}$$

in (18), we introduce the other eigenvalue problem

$$-\Delta u = \lambda u \text{ in } \Omega_t, \quad u = 0 \text{ on } \gamma_{0t}, \quad \frac{\partial u}{\partial \nu} = 0 \text{ on } \gamma_{1t}, \tag{27}$$

which is reduced to finding

$$u \in V_t,\ \int_{\Omega_t} u^2\, dx = 1, \quad \int_{\Omega_t} \nabla u \cdot \nabla v\, dx = \lambda \int_{\Omega_t} uv\, dx,\ \forall v \in V_t \tag{28}$$

for

$$V_t = \{v \in H^1(\Omega_t) \mid v|_{\gamma_{0t}} = 0\}. \tag{29}$$

Let $\lambda_j(t)$ be the j-th eigenvalue of the eigenvalue problem (27). Then Lemma 7 of [9] ensures that the eigenvalue problem (28)-(29) is reduced to

$$u \in V,\ B_t(u,u) = 1, \quad A_t(u,v) = \lambda B_t(u,v),\ \forall v \in V \tag{30}$$

by the transformation of variables $y = T_t x$, where

$$\begin{aligned} B_t(u,v) &= \int_\Omega uva_t\, dx, \quad u,v \in X \\ A_t(u,v) &= \int_\Omega Q_t[\nabla u, \nabla v] a_t\, dx, \quad u,v \in V \end{aligned} \tag{31}$$

for V and X defined by (22), and

$$a_t = \det DT_t, \quad Q_t = (DT_t)^{-1}(DT_t)^{-1T}. \tag{32}$$

Recall that $\Omega \subset \mathbb{R}^n$ is a bounded Lipshitz domain, and let $T_t : \Omega \to \Omega_t = T_t\Omega$, $t \in I = (-\varepsilon_0, \varepsilon_0)$ be a family of twice continuously differentiable bi-Lispchitz transformations. Then the abstract theory described in §1 is applicable with

$$\dot{A}_t(u,v) = \int_\Omega \frac{\partial}{\partial t}(Q_t[\nabla u, \nabla v] a_t) dx, \quad \dot{B}_t(u,v) = \int_\Omega uv\, \frac{\partial a_t}{\partial t}\, dx$$

and

$$\ddot{A}_t(u,v) = \int_\Omega \frac{\partial^2}{\partial t^2}(Q_t[\nabla u, \nabla v] a_t) dx, \quad \ddot{B}_t(u,v) = \int_\Omega uv\, \frac{\partial^2 a_t}{\partial t^2}\, dx$$

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Henceforth, we write

$$G[\xi,\eta] = \xi^T G\eta, \quad \xi,\eta \in \mathbb{R}^n$$

for the symmetric matrix $G \in M_n(\mathbb{R})$, where $\cdot^T$ denotes the transpose of vectors or matrices. The unit matrix is denoted by $E \in M_n(\mathbb{R})$, and

$$G : F = \sum_{i,j=1}^{n} g_{ij} f_{ij}$$

for $G = (g_{ij})$ and $F = (f_{ij}) \in M_n(\mathbf{R})$. Let, furthermore, $I : \Omega \to \Omega$ be the identity mapping.

Theorem 4. *Assume*

$$T_t = I + tS + \frac{t^2}{2}R + o(t^2), \quad t \to 0 \tag{33}$$

uniformly on Ω, where $S, R : \Omega \to \mathbb{R}^n$ are Lipschitz continuous vector fields. Then, if $m = 1$ in (8) for $t = 0$, it holds that

$$\begin{aligned}
\dot{A} &= \int_\Omega (-(DS^T + DS) + (\nabla\cdot S)E)[\nabla\varphi, \nabla\varphi] \\
\ddot{A} &= \int_\Omega ((2(DS)^2 - DR)^T + (2(DS)^2 - DR) \\
&\quad +2(DS)(DS)^T - (\nabla\cdot S)(DS^T + DS) \\
&\quad +(\nabla\cdot R + (\nabla\cdot S)^2 - DS^T : DS)E)[\nabla\phi, \nabla\phi]\, dx \\
\dot{B} &= \int_\Omega (\nabla\cdot S)\phi^2\, dx \\
\ddot{B} &= \int_\Omega (\nabla\cdot R + (\nabla\cdot S)^2 - DS^T : DS)\phi^2\, dx,
\end{aligned}$$

where $\dot{A}$, $\ddot{A}$, $\dot{B}$, and $\ddot{B}$ are defined by (14) with $t = 0$, and $\phi = \phi_0$.

Proof: First, since

$$DT_t = E + tDS + \frac{t^2}{2}DR + o(t^2)$$

uniformly on Ω, it holds that

$$(DT_t)^{-1} = E - tDS^T + \frac{t^2}{2}(2(DS)^2 - DR) + o(t^2)$$

uniformly on Ω, which implies

$$a_t = \det DT_t = 1 + t\nabla\cdot S + \frac{t^2}{2}(\nabla\cdot R + (\nabla\cdot S)^2 - DS^T : DS) + o(t^2)$$

uniformly on Ω (Lemma 5 and Lemma 6 of [8]).

Second, we have

$$Q_t[\nabla u, \nabla v] = (DT_t^{-1}\nabla u, DT_t^{-1T}\nabla v), \quad u, v \in V,$$

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using the standard L^2 inner product $(\ ,\ )$, and therefore, it follows that

$$
\begin{aligned}
&\frac{\partial Q_t}{\partial t}[\nabla u, \nabla v]\Big|_{t=0} \\
&\quad = \Big(\frac{\partial}{\partial t} DT_t^{-1T}\nabla u, DT_t^{-1T}\nabla v\Big) + \Big(DT_t^{-1T}\nabla u, \frac{\partial}{\partial t} DT_t^{-1T} Dv\Big)\Big|_{t=0} \\
&\quad = (-DS^T\nabla u, \nabla v) + (\nabla u, -DS^T\nabla v) = -(DS^T + DS)[\nabla u, \nabla v]
\end{aligned}
$$

and

$$
\begin{aligned}
\frac{\partial^2 Q_t}{\partial t^2}[\nabla u, \nabla v]\Big|_{t=0} &= \Big(\frac{\partial^2}{\partial t^2} DT_t^{-1T}\nabla u, DT_t^{-1T}\nabla v\Big) \\
&\quad + 2\Big(\frac{\partial}{\partial t} DT_t^{-1T}\nabla u, \frac{\partial}{\partial t} DT_t^{-1T}\nabla v\Big) + \Big(DT_t^{-1T}\nabla u, \frac{\partial^2}{\partial t^2} DT_t^{-1T} Dv\Big)\Big|_{t=0} \\
&= (2(DS)^2 - DR)^T\nabla u, \nabla v) + 2(DS^T\nabla u, DS^T\nabla v) \\
&\quad + (\nabla u, (2(DS)^2 - DR)^T\nabla v) \\
&= ((2(DS)^2 - DR)^T + (2(DS)^2 - DR) + 2DSDS^T)[\nabla u, \nabla v].
\end{aligned}
$$

By these equalities, finally, we obtain

$$
\begin{aligned}
\frac{\partial}{\partial t}(Q_t[\nabla u, \nabla v] a_t)\Big|_{t=0} &= \frac{\partial Q_t}{\partial t}[\nabla u, \nabla v] a_t + Q_t[\nabla u, \nabla v]\frac{\partial a_t}{\partial t}\Big|_{t=0} \\
&= -(DS^T + DS)[\nabla u, \nabla v] + (\nabla u, \nabla v)\nabla \cdot S
\end{aligned}
$$

and

$$
\begin{aligned}
\frac{\partial^2}{\partial t^2}(Q_t[\nabla u, \nabla v] a_t)\Big|_{t=0} &= \frac{\partial^2 Q_t}{\partial t^2}[\nabla u, \nabla v] a_t \\
&\quad + 2\frac{\partial Q_t}{\partial t}[\nabla u, \nabla v]\frac{\partial a_t}{\partial t} + Q_t[\nabla u, \nabla v]\frac{\partial^2 a_t}{\partial t^2}\Big|_{t=0} \\
&= -((2(DS)^2 - DR) + (2(DS)^2 - DR) + 2DSDS^T)[\nabla u, \nabla v] \\
&\quad - 2(DS^T + DS)[\nabla u, \nabla v](\nabla \cdot S) \\
&\quad + (\nabla u, \nabla v)(\nabla \cdot R + (\nabla \cdot S)^2 - DS^T : DS),
\end{aligned}
$$

and hence the conclusion.

3. Dynamical Deformations

We continue to suppose that $\Omega \subset \mathbf{R}^n$ is a bounded Lipschitz domain. Here we study the dynamical deformation of domains introduced by [7].

To this end, we take a Lipschitz continuous vector field defined on a neighbourhood of $\overline{\Omega}$, denoted by $v = v(x)$. Then the transformation $T_t : \Omega \to \Omega_t = T_t(\Omega)$ is made by $T_t x = X(t)$, where $X = X(t)$, $|t| \ll 1$, is the solution to

$$
\frac{dX}{dt} = v(X), \quad X|_{t=0} = x.
$$

Then we have the group property,

$$
T_t \circ T_s = T_{t+s}, \quad |t|, |s| \ll 1,
$$

and therefore, the formulae derived for $t = 0$ are shifted to $t = s$.

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If v is a $C^{1,1}$ vector field, furthermore, this $T_t : \Omega \to \Omega_t$ is twice continuously differentiable, and it holds that (33) with

$$S = v, \quad R = (v \cdot \nabla)v \tag{34}$$

because of

$$\frac{d^2X}{dt^2} = [(v \cdot \nabla)]v(X).$$

Then we obtain the following lemma.

Lemma 1. *Under the above assumption, it holds that*

$$\nabla \cdot R = DS^T : DS + (v \cdot \nabla)(\nabla \cdot v).$$

Proof: Writing $v = (v^j)$, we obtain

$$\begin{aligned} \nabla \cdot R &= \nabla \cdot ((v \cdot \nabla)v) = \sum_{i,j} \frac{\partial}{\partial x_i}(v^j \frac{\partial v^i}{\partial x_j}) \\ &= \sum_{i,j} (\frac{\partial v^j}{\partial x_i} \frac{\partial v^i}{\partial x_j} + v^j \frac{\partial}{\partial x_j} \frac{\partial v^i}{\partial x_i}) = DS^T : DS + (v \cdot \nabla)(\nabla \cdot v) \end{aligned}$$

and the proof is complete. □

Here we take two categories that the vector fields are solenoidal and gradient. In the first category, we assume $\nabla \cdot v = 0$ everywhere. Then it holds that

$$|\Omega_t| = |\Omega|, \quad |t| \ll 1. \tag{35}$$

Theorem 5. *If $\nabla \cdot v = 0$, it holds that $\dot{B} = \ddot{B} = 0$ and*

$$\begin{aligned} \dot{A} &= -\int_\Omega (DS^T + DS)[\nabla \phi, \nabla \phi]\, dx \\ \ddot{A} &= \int_\Omega (2(DS)^2 - DR)^T + (2(DS)^2 - DR) \\ &\quad +2(DS)(DS)^T)[\nabla \phi, \nabla \phi]\, dx. \end{aligned} \tag{36}$$

in Theorem 4.

Proof: We recall (34). By the assumption we obtain $\nabla \cdot S = 0$ and

$$\nabla \cdot R = DS^T : DS$$

by Lemma 1, which implies $\dot{B} = \ddot{B} = 0$, and (36) by Theorem 4. □

Remark 1. *For the moment we write*

$$a_i = \frac{\partial a}{\partial x_i}, \quad a_{ij} = \frac{\partial^2 a}{\partial x_i \partial x_j}$$

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in short. Then, by the proof of Lemma 1 we obtain

$$DR = ((v^i v_i^j)_k)_{jk} = (DS)^2 + (v^i v_{ik}^j)_{jk}$$

and

$$DR^T = (DS)^{2T} + (v^i v_{ij}^k)_{jk}$$

similarly. Hence it holds that

$$\ddot{A} = \int_\Omega ((DS)^{2T} + (DS)^2 + 2(DS)(DS)^T - G)[\nabla\phi, \nabla\phi]\, dx$$

for

$$G = (v^i(v_{ik}^j + v_{ij}^k))_{jk}.$$

In the second category that the vector field is a gradient of a scalar field we obtain the following theorem.

Theorem 6. *If $v = \nabla\mu$ for the $C^{1,1}$ scalar field μ, it holds that*

$$\begin{aligned}\dot{A} &= \int_\Omega (-2H + (tr\ H)E)[\nabla\phi, \nabla\phi]\, dx \\ \ddot{A} &= \int_\Omega (6H^2 - K - 4(tr\ H)H \\ &\quad + ((tr\ H)^2 + \frac{1}{2} tr\ K - H : H)E)[\nabla\phi, \nabla\phi]\, dx \end{aligned} \tag{37}$$

and

$$\dot{B} = \int_\Omega (tr\ H)\phi^2\, dx, \quad \ddot{B} = \int_\Omega ((tr\ H)^2 + \frac{1}{2} tr\ K - H : H)\phi^2\, dx, \tag{38}$$

where

$$H = \nabla^2\mu, \quad K = \nabla^2(|\nabla\mu|^2)$$

and

$$|H|^2 = H : H = \sum_{ij} h_{ij}^2 \quad for\ H = (h_{ij}).$$

Proof: In this case, we obtain $S = \nabla\mu$ and

$$R = ((\nabla\mu)\cdot\nabla)\nabla\mu = \left(\sum_k \mu_k \mu_{ik}\right)_i = \left((\frac{1}{2}\sum_k \mu_k^2)_i\right)_i = \frac{1}{2}\nabla|\nabla\mu|^2. \tag{39}$$

Then it holds that

$$\dot{B} = \int_\Omega \phi^2 \nabla\cdot S\, dx = \int_\Omega \phi^2 \Delta\mu\, dx = \int_\Omega (\mathrm{tr}\ H)\phi^2\, dx.$$

It also holds that

$$\begin{aligned}(v\cdot\nabla)(\nabla\cdot v) &= ((\nabla\mu)\cdot\nabla)(\Delta\mu) = \nabla\mu\cdot\Delta\nabla\mu \\ &= \frac{1}{2}\Delta(|\nabla\mu|^2) - |\nabla^2\mu|^2, \end{aligned} \tag{40}$$

because

$$\Delta|a|^2 = 2\Delta a\cdot a + \sum_i \frac{\partial a}{\partial x_i}\cdot\frac{\partial a}{\partial x_i}$$

is valid to the vector field a.

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Since

$$\ddot{B} = \int_\Omega ((\nabla \cdot S)^2 + (v \cdot \nabla)(\nabla \cdot v))\phi^2 \, dx$$

is valid by Lemma 1, and hence

$$\begin{aligned} \ddot{B} &= \int_\Omega ((\Delta\mu)^2 + \frac{1}{2}\Delta(|\nabla\mu|^2) - |\nabla^2\mu|^2)\phi^2 \, dx \\ &= \int_\Omega (\mathrm{tr}\, H)^2 + \frac{1}{2}\mathrm{tr}\, K - |H|^2)\phi^2 \, dx. \end{aligned}$$

Next, we obtain

$$DS^T + DS = 2H$$

and hence

$$\begin{aligned} \dot{A} &= \int_\Omega (-2H + (\Delta\mu)E)[\nabla\phi, \nabla\phi] \, dx \\ &= \int_\Omega (-2H + (\mathrm{tr}\, H)E)[\nabla\phi, \nabla\phi] \, dx. \end{aligned}$$

Finally, we divide $\ddot{A}$ as in

$$\ddot{A} = I + II + III$$

for

$$\begin{aligned} I &= \int_\Omega (2((DS)^{2T} + (DS)^2 + DS^T DS)) - (DR^T + DR))[\nabla\phi, \nabla\phi] \, dx \\ II &= \int_\Omega -2(DS^T + DS)[\nabla\phi, \nabla\phi]\nabla \cdot S \, dx \\ III &= \int_\Omega (\nabla \cdot R + (\nabla \cdot S)^2 - DS^T : DS)|\nabla\phi|^2 \, dx. \end{aligned}$$

Here we have

$$(DS)^{2T} + (DS)^2 + DS^T DS = 3H^2.$$

Equality (39) now implies

$$R^i_j + R^j_i = (|\nabla\mu|^2)_{ij}$$

for $R = (R^i)$, and hence

$$\nabla R^T + \nabla R = \nabla^2(|\nabla\mu|^2) = K.$$

Thus we obtain

$$I = \int_\Omega (6H^2 - K)[\nabla\phi, \nabla\phi] \, dx.$$

It holds also that

$$II = \int_\Omega -4H[\nabla\phi, \nabla\phi](\Delta\mu) \, dx = \int_\Omega -4(\mathrm{tr}\, H)H[\nabla\phi, \nabla\phi] \, dx.$$

Finally, we have

$$\begin{aligned} III &= \int_\Omega ((\nabla \cdot S)^2 + (v \cdot \nabla)v)|\nabla\phi|^2 \, dx \\ &= \int_\Omega ((\Delta\mu)^2 + \frac{1}{2}\Delta(|\nabla\mu|^2) - |\nabla^2\mu|^2)|\nabla\phi|^2 \, dx \\ &= \int_\Omega ((\mathrm{tr}\, H)^2 + \frac{1}{2}\mathrm{tr}\, K - |H|^2)|\nabla\phi|^2 \, dx. \end{aligned}$$

We thus end up with the result by combining these equalities. □

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Remark 2. *Theorem 5 and Theorem 6 are consistent if* $v = \nabla\mu$ *with*

$$\Delta\mu = 0.$$

In this case it holds that

$$tr\ H = 0 \tag{41}$$

and

$$\frac{1}{2}tr\ K - |H|^2 = 0 \tag{42}$$

by (40):

$$0 = (v\cdot\nabla)(\nabla\cdot v) = \frac{1}{2}\Delta(|\nabla\mu|^2) - |\nabla^2\mu|^2.$$

Then we can reproduce the result of Theorem 5 from (38), as in

$$\dot{B} = 0, \quad \ddot{B} = 0. \tag{43}$$

We have, furthermore,

$$\dot{A} = \int_\Omega -2H|\nabla\phi|^2\, dx, \quad \ddot{A} = \int_\Omega (6H^2 - K)|\nabla\phi|^2\, dx \tag{44}$$

by (37). By (13), (43) and (44), first,

$$\dot{\lambda} = \dot{A} - \lambda\dot{B} = \dot{A}$$

cannot have a definite sign because of (41). Second, if

$$6H^2 - K \leq 0 \tag{45}$$

we obtain $\ddot{A} \leq 0$ *by (13), which implies*

$$\ddot{\lambda} \leq \ddot{A} - \lambda\ddot{B} - 2\dot{\lambda}\dot{B} = \ddot{A} \leq 0$$

regardless of the form of ϕ*. For inequality (45) to hold, however, it is necessary that*

$$tr\ (6H^2 - K) \geq 0,$$

or

$$3tr\ (H^2) \leq |H|^2$$

by (42). This inequality implies $H = 0$ *because*

$$tr\ (H^2) = \sum_{ij} h_{ij}^2 = |H|^2$$

for $H = (h_{ij})$ *with* $h_{ji} = h_{ij}$*. In other words, several possibilities can arise to the monotonicity or convexity of the first eigenvalue under the dynamical perturbation, provided with the property (35).*

4. Conformal Deformations

Here we sayt that $T_t : \Omega \to \Omega_t$ is conformal if

$$(DT_t)^{-1}(DT_t)^{-1T} = \frac{1}{\alpha_t}E$$

holds with a scalar field $\alpha_t > 0$. It follows that $\alpha_t E = (DT_t)(DT_t)^T$, and therefore, the matrix $F_t = \alpha_t^{-1/2} DT_t$ is orthogonal: $F_t F_t^T = E$.

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Then we assume, furthermore, that

$$\det F_t = 1$$

for simplicity. It follows that

$$a_t = \det DT_t = \det(\alpha_t^{1/2} F_t) = \alpha_t^{n/2}, \tag{46}$$

and hence $\alpha_t = a_t^{2/n}$, which results in

$$(DT_t)^{-1}(DT_t)^{-1T} = a_t^{-2/n} E,$$

and hence

$$A_t(u,v) = \int_\Omega (\nabla u \cdot \nabla v) a_t^{1-2/n} dx, \quad u, v \in V$$

and

$$B_t(u,v) = \int_\Omega uva_t \, dx.$$

Then we can show the following theorem.

Theorem 7. *Let $n = 2$, $k = m = 1$, and $T_t : \Omega \to \Omega_t$ be conformal at t. Assume, furthermore, that*

$$\frac{\partial^2 a_t}{\partial t^2} \geq 0 \quad in\ \Omega.$$

Then it holds that

$$\frac{d^2}{dt^2} \frac{1}{\lambda_1(t)} \geq 0. \tag{47}$$

Proof: Since $n = 2$ we obtain

$$A_t(u,v) = \int_\Omega \nabla u \cdot \nabla v \, dx, \quad B_t(u,v) = \int_\Omega uva_t \, dx. \tag{48}$$

Then it holds that $\dot{A} = \ddot{A} = 0$ and

$$\ddot{B} = \int_\Omega \phi^2 \frac{\partial^2 a_t}{\partial t^2} \, dx$$

in Theorem 1 for $\phi = \phi_t$. Then the result follows from Theorem 2. □

A consequence of Theorem 7 is the following isometric inequality.

Theorem 8. *Let $\Omega \subset \mathbf{R}^2$ be a simply-connected bounded Lipschitz domain and $f = f(z) : D \to \Omega$ be a univalent bi-Lipschitz homeomorphism, where $D = \{z \in \mathbb{C} \mid |z| < 1\}$. Let, furthermore,*

$$f(z) = \sum_{n=0}^{\infty} a_n z^n, \quad a_1 \in \mathbb{R}.$$

Then it holds that

$$\lambda_1(D) \geq \lambda_1(\Omega) \left(2 \int_D \phi_1^2 Ref'(z) \, dx - 1 \right), \tag{49}$$

where $\lambda_1(\Omega)$ and $\lambda_1(D)$ are the first eigenvalue of (18):

$$-\Delta u = \lambda u \ in\ \Omega, \quad u = 0 \ on\ \gamma_0, \quad \frac{\partial u}{\partial \nu} = 0 \ on\ \gamma_1 \tag{50}$$

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and that for $\Omega = D$,

$$-\Delta u = \lambda u \text{ in } D, \quad u = 0 \text{ in } \gamma_0', \quad \frac{\partial u}{\partial \nu} = 0 \text{ on } \gamma_1' \tag{51}$$

respectively with $\gamma_i = f(\gamma_i')$, $i = 1, 2$, *and furthermore,* $\phi_1 = \phi_1(x)$ *is the first eigenfunction of (51) such that*

$$\int_D \phi_1^2 \, dx = 1.$$

Remark 3. *Either* $\gamma_0 = \emptyset$ *or* $\gamma_1 = \emptyset$ *occurs if* $\Omega \subset \mathbb{R}^2$ *is simply-connected and (17), and in the former case it follows that* $\lambda_1(\Omega) = 0$ *and the above theorem is trivial. This assumption (17), however, is used in [9] just to formulate (18) as in (3) with (20)-(21), and (22). Hence we can exclude this assumption if we formulate (50) as in (19) with (20)-(21) for* V *standing for the closure of* V_0 *in* $H^1(\Omega)$, *where* $v \in V_0$ *if and only if there is* $\tilde{v}$, *a smooth extension of* v *in a neighborhood of* $\overline{\Omega}$, *such that*

$$\tilde{v}|_{\gamma_0} = 0.$$

Then we re-formulate (51) as in (30) with (31) for $t = 1$, *using the bi-Lipshichits homeomorphism* $T_1 = f^{-1} : \Omega \to D$. *Under this agreement of (50) and (51), we exclude the assumption (17) in the above theorem.*

Proof of Theorem 8: Let

$$g_t(z) = (1-t)z + tf(z), \quad 0 \le t \le 1 \tag{52}$$

be the conformal mapping on D, and define A_t and B_t by (48) for (46) with $n = 2$. We obtain $a_t = |g_t'(z)|^2$ in (46) and hence it follows that

$$|g_t'(z)|^2 = |1 + t(f'(z) - 1)|^2 = 1 + 2t \,\mathrm{Re}\,(f'(z) - 1) + t^2 |f'(z) - 1|^2$$

and hence

$$\begin{aligned} \frac{\partial}{\partial t} |g_t'(z)|^2 &= 2\mathrm{Re}(f'(z) - 1) + 2t|f'(z) - 1|^2 \\ \frac{\partial^2}{\partial t^2} |g_t'(z)|^2 &= 2|f'(z) - 1|^2 \ge 0. \end{aligned} \tag{53}$$

Thus we obtain

$$\frac{d^2}{dt^2} \frac{1}{\lambda_1(t)} \ge 0, \quad 0 \le t \le 1$$

for $\lambda_1(t)$ defined by (24) with $j = 1$ and $A = A_t$ and $B = B_t$, which implies

$$\frac{1}{\lambda_1(\Omega)} \ge \frac{1}{\lambda_1(D)} + \left(\frac{1}{\lambda_1(t)} \right)' \Bigg|_{t=0}, \tag{54}$$

or

$$\frac{1}{\lambda_1(\Omega)} \ge \frac{1}{\lambda_1(D)} + \frac{1}{\lambda_1(D)} \int_D \varphi_1^2 \frac{\partial}{\partial t} |g_t'(z)|^2 \Big|_{t=0} dx$$

by

$$\left(\frac{1}{\lambda_1} \right)' = -\frac{\lambda_1'}{\lambda_1^2}$$

and (12) with (48). Then we obtain the result by (53). □

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Remark 4. *If $\gamma_1 = \emptyset$, inequality (49) is reduced to*

$$\lambda_1(D) \geq \lambda_1(\Omega)(2Re\, a_1 - 1), \tag{55}$$

Since $f : D \to \Omega$ is univalent, it holds that

$$|\Omega| = \int_D |f'(z)|^2 dx = \pi \sum_{n=1}^{\infty} n|a_n|^2,$$

and therefore, $|\Omega| = |D|$ if and only if

$$\sum_{n=1}^{\infty} n|a_n|^2 = 1.$$

This equality implies $|a_1| \leq 1$ and in particular,

$$2Re\, a_1 - 1 \leq 1$$

in (55).

Inequality (55) for $\gamma_1 = \emptyset$ is proven in Appendix B. We conclude this section with an analogous result to (49).

Theorem 9. *Under the assumption of the previous theorem it holds that*

$$\lambda_1(\Omega) \geq \lambda_1(D)\left(2\int_D \tilde{\phi}_1^2 Re f'(z)\, dx - 1\right),$$

where $\tilde{\phi}_1 = \hat{\phi}_1 \circ f$ for the first eigenfunction $\hat{\phi}_1 = \hat{\phi}_1(x)$ to (50) such that

$$\int_\Omega \hat{\phi}_1^2\, dx = 1.$$

Proof: Similarly to (54), we obtain

$$\begin{aligned} \frac{1}{\lambda_1(D)} &\geq \frac{1}{\lambda_1(\Omega)} - \left(\frac{1}{\lambda_1(t)}\right)'\Big|_{t=1} = \frac{1}{\lambda_1(\Omega)}\left(1 + \frac{\lambda_1'(1)}{\lambda_1(\Omega)^2}\right) \\ &= \frac{1}{\lambda_1(\Omega)}\left(1 - \int_D \tilde{\phi}_1^2 \frac{\partial}{\partial t}|g_1'(z)|^2\Big|_{t=1} dx\right), \end{aligned}$$

which implies the result by

$$\int_D \tilde{\phi}_1^2 |f'(z)|^2\, dx = \int_\Omega \hat{\phi}_1^2\, dx = 1$$

and

$$\frac{\partial}{\partial t}|g'(z)|^2\Big|_{t=1} = 2|f'(z)|^2 - 2\mathrm{Re} f'(z). \qquad \square$$

5. Proof of Theorem 3

Although using the Harnack inequality is standard (c.f. Theorem 8.38 of [4]), here we follow the argument by [5]. In fact, by (24) it holds that

$$\lambda_1 = \min_{v \in V \setminus \{0\}} R[v].$$

Since $V \hookrightarrow X$ is compact, there is $\phi_1 \in V$ in $B(\phi_1, \phi_1) = 1$ which attains λ_1. Then it holds that

$$A(\phi_1, v) = \lambda_1 B(\phi_1, v), \quad \forall v \in V. \tag{56}$$

We have

$$v \in H^1(\Omega) \Rightarrow [\max\{v, 0\}]_{x_i} = \begin{cases} v_{x_i}, & \{x \in \Omega \mid v(x) > 0\} \\ 0, & \{x \in \Omega \mid v(x) \le 0\} \end{cases} \tag{57}$$

for $1 \le i \le n$ (Definition 6.7 and Theorem A.1 of [2]), and therefore,

$$v \in H^1(\Omega) \Rightarrow |v| \in H^1(\Omega)$$

We thus obtain

$$v \in V \Rightarrow |v| \in V, \quad A(|v|, |v|) = A(v, v) \tag{58}$$

because Ω is a Lipschitz domain. Hence we may assume $\phi_1 \ge 0$ in Ω.

Equality (56) implies

$$-\Delta\phi_1 = \lambda_1 \phi_1 \quad \text{in } \Omega$$

in the sense of distributions, and therefore, we obtain $\phi_1 \in C^2(\Omega)$ by the elliptic regularity. Then the strong maximum principle implies $\phi_1 > 0$ in Ω.

Now we use the following lemma derived from (57).

Lemma 2. *Given $v, w \in V$, let*

$$m = \min\{v, w\}, \quad M = \max\{v, w\}.$$

Then it follows that $m, M \in V$ and

$$A(m) + A(M) = A(v) + A(w), \quad B(m) + B(M) = B(v) + B(w). \tag{59}$$

Proof: We obtain $m, M \in H^1(\Omega)$ by (57) and hence $m, M \in V$ because Ω is a Lipschitz domain. It is obvious that

$$\begin{aligned} B(m) + B(M) &= \left(\int_{v>w} + \int_{v=w} + \int_{v<w} \right) \left(m^2 + M^2 \right) dx \\ &= \left(\int_{v>w} + \int_{v=w} + \int_{v<w} \right) \left(v^2 + w^2 \right) dx \\ &= B(v) + B(w), \end{aligned}$$

while

$$\begin{aligned} A(m) + A(M) &= \left(\int_{v>w} + \int_{v=w} + \int_{v<w} \right) \left(|\nabla m|^2 + |\nabla M|^2 \right) dx \\ &= \left(\int_{v>w} + \int_{v=w} + \int_{v<w} \right) \left(|\nabla v|^2 + |\nabla w|^2 \right) dx \\ &= A(v) + A(w) \end{aligned}$$

follows from (57). □

We are ready to give the following proof.

Proof of Theorem 3: It sufficies to show

$$\nabla(z/\phi_1) = 0 \quad \text{in } \Omega.$$

for any first eigenfunction $z \in V$. To this end, we take $x_0 \in \Omega$ and put

$$m = \min\{z, t_0\phi_1\}, \quad M = \max\{z, t_0\phi_1\}$$

for $t_0 = z(x_0)/\phi_1(x_0)$. The desired equality $\nabla(z/\phi_1)(x_0) = 0$ is thus reduced to

$$\phi_1(x_0)\nabla z(x_0) = z(x_0)\nabla\phi_1(x_0),$$

or,

$$\nabla z(x_0) = t_0\nabla\phi_1(x_0). \tag{60}$$

Letting $C = A - \lambda_1 B$, we obtain

$$C(m) + C(M) = C(z) + C(t_0\phi_1) = 0$$

by Lemma 2, while C is non-negative definite on $V \times V$. Hence it holds that $C(m) = C(M) = 0$, and hence $m, M \in V$ are the other first eigenfunctions. We thus obtain $M \in C^2(\Omega)$, in particular.

Given $e = (0, \cdots, \overbrace{1}^{i}, \cdots, 0)^T$, we obtain

$$z(x_0 + he) - z(x_0) \le M(x_0 + he) - M(x_0), \quad |h| \ll 1$$

by $t_0\phi_1(x_0) = z(x_0) = M(x_0)$. Dividing both sides by h and making $h \to \pm 0$, we thus obtain

$$z(x_0)_{x_i} = M(x_0)_{x_i}, \quad 1 \le i \le n,$$

and hence $\nabla M(x_0) = \nabla z(x_0)$. It holds that $\nabla M(x_0) = t_0\nabla\phi_1(x_0)$, and hence (60). $\square$

6. Proof of (55) for $\gamma_1 = \emptyset$

If $\gamma_1 = 0$, we have $\phi = \phi(r)$, and the result is a direct consequence of the following theorem.

Lemma 3. *If $\varphi = \varphi(r)$ for $r = |z|$ and $h = h(z)$ is holomorphic in D, it holds that*

$$\int_D \varphi(r)^2 h(z)\, dx = h(0) \int_D \phi(r)^2\, dx.$$

Proof of Lemma 3: Writing $z = re^{\imath\theta}$, we get

$$\int_D \varphi(r)^2 h(z)\, dx = \int_0^1 \varphi(r)^2 r dr \cdot \int_0^{2\pi} h(re^{\imath\theta}) d\theta.$$

Since $dz = \imath re^{\imath\theta} d\theta = izd\theta$ on $|z| = 1$, it holds that

$$\int_0^{2\pi} h(re^{\imath\theta}) d\theta = \int_{|z|=r} \frac{h(z)}{\imath z}\, dz = 2\pi h(0)$$

because $h = h(z)$ is homeomorphic. Then we obatin

$$\int_D \phi(r)^2 h(z)\, dx = 2\pi h(0) \int_0^1 \varphi(r)^2 r\, dr = h(0) \int_D \varphi(r)^2\, dx. \quad \square$$

7. Numerical Experiments

In this section, we present numerical experiments to confirm the theoretical results obtained.

Let $n = 2$ and $D \subset \mathbb{R}^2 \cong \mathbb{C}$ be the unit disk. Following (52), we define $g_t : D \to \mathbb{C}$ by $g_t(z) = (1-t)z + t\cos z$ for $z \in \mathbb{C}$. In this case, the images of the transformation are as follows.

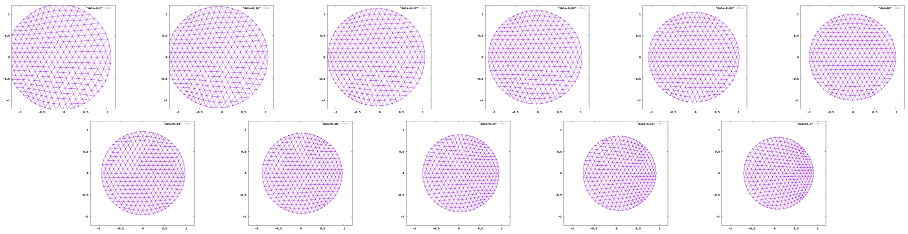

Figure 1. The images of the transformation of D. The value of t are from $t = -0.2$ (first row, leftmost) to $t = 0.2$ (second row, rightmost) with 0.4 increments.

Then, the eigenvalues of

$$-\Delta u = \lambda_k(t)u \quad \text{in } g_t(D), \qquad u = 0 \quad \text{on } \partial g_t(D) \tag{61}$$

are computed by the piecewise linear finite element method. By the calculus in the proof of Theorem 8 and Theorem 7, we have

$$\frac{d^2 a_t}{dt^2} = 2|\sin z - 1|^2 \geq 0 \qquad \text{and} \qquad \frac{d^2}{dt^2}\frac{1}{\lambda_1(t)} \geq 0,$$

although $\cos z$ is not univalent on D. The computed profiles of the first eigenvalue and its reciprocal given in Figure 2 are clearly consistent with Theorem 7.

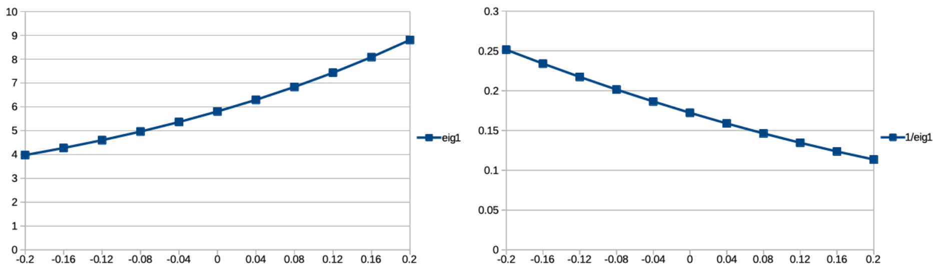

Figure 2. The profiles of the first eigen value (left) and its reciprocal (right).

Next, we define

$$g_t(z) = (1-t)z + te^z, \qquad z \in \mathbb{C}$$

to confirm the statement of Theorem 8. Note that e^z is univalent on D. Then, the images of $D = g_0(D)$ and $g_1(D)$ are given in Figure 3.

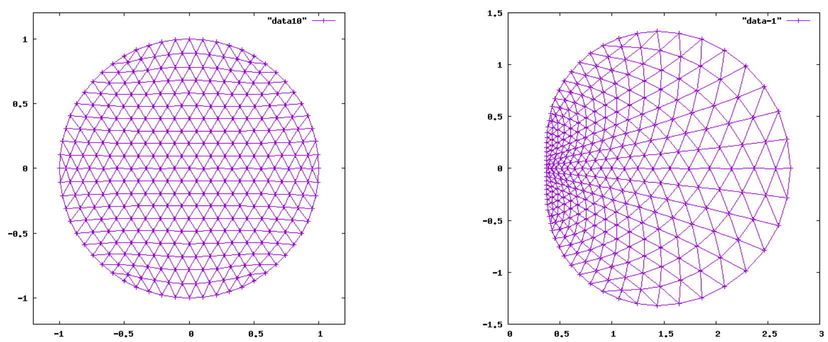

Figure 3. The unit disk and its image by g_1.

The first five eigenvalues of (61) on D and $g_1(D)$ are given in Table 1. Because $a_1 = 1$ in this case, the numbers on λ_1 are consistent with the inequality (55). Although Theorem 8 and Remark 4 claim only on the first eigenvalue λ_1, we observe similar inequalities hold for λ_k $(2 \le k \le 5)$.

Table 1. The first five eigenvalues of Laplacian on D and $g_1(D)$.

	λ_1	λ_2	λ_3	λ_4	λ_5
D	5.80728	14.8489	14.8489	26.9304	26.9304
$g_1(D)$	3.69736	8.96092	10.0331	16.8943	17.2069

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