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Article

From Network Master Equations to Navier–Stokes Dynamics: A Relaxation Framework for the Continuum Limit of Interacting Systems

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Abstract

The Navier–Stokes equations provide the fundamental continuum description of viscous fluid motion, yet their derivation from discrete interacting systems remains an important theoretical challenge. In this study we propose a network-based master equation framework for fluid dynamics and demonstrate how Navier–Stokes-type equations emerge from interacting systems through a relaxation mechanism. The system is formulated as a set of nodes exchanging mass, momentum, and energy along network edges. The evolution of node states is governed by a master equation that incorporates both conservative fluxes and entropy-producing dissipative interactions. Under appropriate structural assumptions, the resulting discrete dynamics preserve global conservation laws while satisfying a discrete form of the second law of thermodynamics. By analyzing the continuum limit of the network system, we show that the master equation converges to a conservation-law-type partial differential equation. A relaxation extension is then introduced to represent nonequilibrium stresses through auxiliary variables. The resulting relaxation system possesses an extended entropy structure that yields uniform a priori estimates. Using compactness arguments based on the Aubin–Lions theorem, we establish the strong convergence of velocity fields and prove that a subsequence of solutions converges to a Leray–Hopf weak solution of the incompressible Navier–Stokes equations. In particular, the forcing generated by residual stresses vanishes in the limit due to the dissipative structure of the extended system. The present framework provides a unified perspective linking discrete network dynamics, relaxation systems, and continuum fluid mechanics. It suggests a new pathway for understanding how classical hydrodynamic equations may arise from interacting systems beyond the traditional kinetic-theory setting.

Keywords: network master equation; Navier–Stokes equations; continuum limit; relaxation systems; weak solutions; Aubin–Lions compactness

Chapter 1 Introduction

The Navier–Stokes equations occupy a central position in fluid mechanics, providing the fundamental continuum description of viscous fluid motion. Since their formulation in the nineteenth century, they have been applied successfully across a wide range of physical systems, including aerodynamics, geophysical flows, astrophysical plasmas, and microfluidic transport. Classical expositions may be found in the works of Lamb [1], Batchelor [2], Landau and Lifshitz [3], and Pope [4], where the equations are formulated on the basis of continuum conservation laws for mass, momentum, and energy.

Despite their success, an important conceptual question remains: under what structural conditions can the Navier–Stokes equations be understood as emerging from more primitive

interacting systems? Historically, this issue has been studied mainly through kinetic theory. Beginning with the Boltzmann equation [5], the Chapman–Enskog expansion showed that the Navier–Stokes equations arise as a hydrodynamic limit under suitable scaling assumptions [6,7], and subsequent mathematical works established rigorous aspects of this program in various settings [8–10]. These approaches provide a microscopic interpretation of transport coefficients such as viscosity and thermal conductivity and form a cornerstone of the statistical theory of fluids.

Beyond classical kinetic theory, a variety of mesoscopic and discrete formulations have been developed in which macroscopic fluid equations arise as large-scale limits of interacting systems. Representative examples include lattice gas automata [11], lattice Boltzmann methods [12–14], and kinetic relaxation models [15–17]. Such approaches are important both conceptually and computationally, because they connect microscopic or mesoscopic dynamics with macroscopic continuum equations while preserving key conservation principles. More broadly, interacting particle systems and discrete kinetic models have long served as intermediate descriptions linking microscopic dynamics and macroscopic transport equations [32].

In parallel with these developments, network-based and graph-based representations of physical systems have attracted increasing attention in recent years. Motivated by work in complex systems, nonequilibrium statistical mechanics, and transport theory [18–20], these approaches describe a system as a collection of interacting nodes connected by edges across which conserved quantities are exchanged. Closely related mathematical structures appear in stochastic processes [21], chemical reaction networks [22], transport processes on graphs [23], and thermodynamically consistent discrete gradient-flow systems [33,34]. This viewpoint suggests that continuum equations may be interpreted not only as limits of particle systems, but also as limits of structured interacting networks.

However, the systematic relation between network master equations and continuum fluid equations remains only partially understood. In particular, it is not yet clear which structural assumptions on conservation, dissipation, geometry, and scaling are sufficient for a network-based interacting system to produce a Navier–Stokes-type continuum description. This question motivates the present study.

A second line of development relevant to the present work is the theory of relaxation systems. Relaxation systems approximate viscous or parabolic equations by extended systems equipped with auxiliary variables and relaxation mechanisms [24–26]. They provide a useful framework for connecting distinct physical descriptions and for extracting dissipative limits in a mathematically tractable form. In many cases, viscous behavior may be interpreted as the zero-relaxation limit of an extended system in which nonequilibrium quantities evolve dynamically toward equilibrium constitutive laws [15].

From the viewpoint of partial differential equations, the incompressible Navier–Stokes equations themselves have been studied extensively. The fundamental existence theory of global weak solutions with finite kinetic energy was established by Leray [27] and Hopf [28], and later developments introduced more refined functional frameworks involving Sobolev and Besov spaces [29–31]. In particular, the classical Leray–Hopf theory gives a globally defined weak solution satisfying the standard energy inequality. Nevertheless, these theories take the Navier–Stokes equations themselves as the starting point, rather than deriving them from a more primitive interacting system.

The present work combines these two perspectives: a network-based structural viewpoint and a relaxation-based analytic viewpoint. We begin with a network master equation describing the exchange of conserved quantities between interacting nodes, organized so as to respect conservation laws and entropy production at the discrete level. This network formulation provides the structural background and design principle of the model. Under suitable geometric and scaling assumptions, discussed in the main text and in Appendix A, this framework leads formally and structurally to a conservation-law-type continuum equation.

The principal rigorous contribution of the present paper, however, concerns the relaxation-extended continuum system introduced thereafter. More precisely, on the periodic torus, we consider a continuum extended system in which a residual stress is promoted to an auxiliary unknown and evolves under a relaxation equation endowed with dissipation. For this system, we construct an extended energy (Lyapunov functional) and derive uniform a priori estimates. These estimates yield compactness of the velocity sequence, while the residual stress generates a forcing term whose contribution vanishes in the zero-relaxation limit. As a consequence, the limiting velocity field is shown to satisfy the incompressible Navier–Stokes equations in the sense of Leray–Hopf weak solutions.

Accordingly, the present paper should be read with a clear distinction between two levels of results. First, the network master equation furnishes a structural framework indicating how conservation and dissipation may be organized so as to be compatible with continuum fluid descriptions. Second, the mathematically rigorous part of the paper establishes that, for the relaxation-extended continuum system on the periodic torus, the limit $\varepsilon \rightarrow 0$ yields a Leray–Hopf weak solution of the incompressible Navier–Stokes equations. In this sense, the network-to-continuum part of the paper provides the conceptual and structural motivation, whereas the relaxation-limit analysis provides the rigorous existence result proved in the body of the paper.

A central feature of this construction is that the forcing term in the extended momentum equation is not prescribed externally. Instead, it is generated internally by the residual stress and is controlled by the dissipation built into the relaxation dynamics. The vanishing of this forcing term in the limit is therefore not imposed as an additional hypothesis, but follows from the Lyapunov structure of the extended system itself. This point is essential for closing the weak-solution construction.

The framework developed here may therefore be interpreted in two complementary ways. On the one hand, it offers a structural route from interacting network dynamics to continuum fluid equations under suitable assumptions. On the other hand, it provides a thermodynamically consistent relaxation approximation to the incompressible Navier–Stokes equations, for which the zero-relaxation limit can be justified rigorously at the level of Leray–Hopf weak solutions.

The main contributions of this paper may be summarized as follows.

1. Network master-equation framework.

We formulate a network-based master equation describing exchanges of mass, momentum, and energy between interacting nodes in a manner consistent with conservation laws and entropy production.

2. Structural continuum correspondence.

Under suitable geometric and scaling assumptions, we identify the continuum conservation-law structure associated with the network dynamics. This part provides the structural bridge between the discrete interacting system and the continuum description.

3. Relaxation-extended continuum analysis.

We introduce a thermodynamically consistent relaxation extension with a residual stress variable, construct an extended energy (Lyapunov functional), derive uniform estimates, and prove that the zero-relaxation limit yields a Leray–Hopf weak solution of the incompressible Navier–Stokes equations on the periodic torus.

These results provide a new perspective on the emergence of fluid dynamics from interacting systems. Unlike classical derivations based exclusively on molecular kinetic theory, the present approach emphasizes a broader structural principle: admissible interacting systems should be organized by conservation and dissipation, and their continuum description should inherit these same principles. At the same time, the rigorous mathematical result established in this paper is the relaxation-limit construction of a Leray–Hopf weak solution from the extended continuum system.

Main result (informal statement)

The main mathematical result of this work may be summarized informally as follows.

Main Theorem (informal).

Consider the relaxation-extended continuum system on the periodic torus associated with the network-based structural framework introduced in this paper. Suppose that the system satisfies the extended energy inequality and the corresponding uniform a priori bounds. Then the family of solutions is compact in the appropriate functional setting, the forcing generated by the residual stress vanishes in the zero-relaxation limit, and the limiting velocity field is a Leray–Hopf weak solution of the incompressible Navier–Stokes equations.

The network-to-continuum correspondence discussed in the paper and in Appendix A should be understood as the structural background motivating this relaxation framework, whereas the rigorous theorem proved in the body of the paper concerns the relaxation limit of the extended continuum system.

Chapter 2 Structure of the Master Equation Based on Conservation Laws and the Second Law

In this chapter, we formulate the structural principles underlying the network-based master equation that serves as the starting point of the present study. The aim here is not yet to prove the main convergence theorem of the paper, nor to establish the full discrete-to-continuum passage in a rigorous compactness framework. Rather, the purpose of the present chapter is to identify the class of discrete interacting systems that may be regarded as physically admissible and thermodynamically consistent, and to prepare a structural design principle for the concrete finite-volume construction introduced in Chapter 3.

The basic standpoint adopted in this work is the following. The governing equations of fluid motion should not be assumed a priori as continuum partial differential equations. Instead, one first constructs a discrete interacting system, designs its evolution law so that it satisfies the conservation laws and the second law of thermodynamics, and only afterward asks under what geometric and scaling assumptions a continuum fluid description may emerge. In this sense, the present chapter is a chapter on structural admissibility: it characterizes the algebraic and thermodynamic properties that a master equation must possess before any continuum interpretation is attempted.

From this viewpoint, the first question is the following: what class of discrete systems may be regarded as physically admissible candidates for fluid-mechanical modeling? In the present work, admissibility is determined by two fundamental principles:

1. conservation of mass, momentum, and energy, and
2. the second law of thermodynamics expressed as non-decrease of total entropy.

Accordingly, the purpose of this chapter is to characterize the general structure of master equations satisfying these two requirements simultaneously.

To organize the discussion, we also refer to the concept of GENERIC (General Equation for Non-Equilibrium Reversible–Irreversible Coupling), which is widely used in nonequilibrium thermodynamics as a structural decomposition of dynamics into reversible and irreversible parts. However, it is important to state from the outset that, in the framework of the present paper, the GENERIC structure is not imposed as a strict axiom uniquely determining the later fluid model. Its role is more modest but still essential: it serves as a design principle clarifying how dissipative terms should be aligned with entropy production while remaining compatible with conservation laws. In Chapter 3, the concrete dissipative terms will be constructed directly from finite-volume interface interactions, and the second law will be verified there from the explicit non-negative quadratic form of each interface contribution. Thus, Chapter 2 provides the structural thermodynamic framework, whereas Chapter 3 provides the concrete fluid-mechanical realization.

2.1. Node States and the Basic Network Setting

We begin by considering a network consisting of a finite number of nodes. Each node is interpreted either as a finite-volume cell in physical space or as a coarse-grained local state in a more general interacting system. At this stage, no specific mesh or geometric regularity is assumed; only the interaction structure of neighboring nodes is used. The state vector associated with node i is defined by

$$U_i = (\rho_i, m_i, e_i), \quad (1)$$

where ρ_i denotes the density, m_i the momentum, and e_i the internal energy density. Thus, each node represents a local thermodynamic state of the system. The central problem addressed in this study is how the collection of such local states evolves in time through exchange and dissipation among neighboring nodes.

For each node i , the set of adjacent nodes is denoted by $N(i)$, and the representative volume associated with node i is denoted by V_i . Later, when a finite-volume correspondence is introduced, the pair $(V_i, N(i))$ will be interpreted geometrically in terms of control volumes and interfaces. In the present chapter, however, the network is treated abstractly, since our purpose is to extract the structural conditions imposed by conservation and thermodynamics prior to specifying a concrete discretization.

It is important to emphasize that the variable U_i is chosen so that both conservation laws and thermodynamic quantities can be expressed locally. The density ρ_i , the momentum m_i , and the internal energy e_i together provide the minimal state description needed to discuss reversible transport, irreversible dissipation, and entropy production within a common framework.

2.2. Conservation Requirements and Flux Formulation

For the time evolution to be physically meaningful, mass, momentum, and energy must not be spontaneously created or destroyed through purely internal interactions in a closed system. Accordingly, the following global quantities are required to be conserved:

$$\sum_i V_i \rho_i, \sum_i V_i m_i, \sum_i V_i \left(e_i + \frac{|m_i|^2}{2\rho_i} \right). \quad (2)$$

The third expression represents the total energy, namely the sum of internal and kinetic energies. This is the appropriate conserved quantity in a fluid-mechanical interpretation.

A natural and standard way to encode such conservation laws in a discrete interacting system is to write the time evolution in flux-difference form. We therefore represent the master equation as

$$\frac{dU_i}{dt} = \sum_{j \in N(i)} A_{ij} F_{ij}, \quad (3)$$

where F_{ij} denotes the flux of state variables entering node i from node j , and A_{ij} is a geometric weight associated with the pair (i, j) . In a finite-volume setting, such coefficients correspond to quantities determined by interface areas, cell volumes, and distances between neighboring cells. At the present abstract stage, however, they are introduced only as structural interaction weights.

To guarantee conservation, we impose two fundamental algebraic conditions:

$$A_{ij} = A_{ji}, \quad (4)$$

and

$$F_{ij}(U_i, U_j) = -F_{ji}(U_j, U_i). \quad (5)$$

Eq. (4) expresses the symmetry of the geometric coupling between two interacting nodes. Eq. (5) expresses the antisymmetry of the interfacial flux with respect to orientation reversal.

The meaning of these two assumptions is straightforward but fundamental. If one sums Eq. (3) over all nodes and uses (4) and (5), then each contribution associated with an internal pair (i, j) cancels with the contribution associated with the reversed pair (j, i) . Consequently, the global sum of the corresponding conserved variable remains unchanged. Thus, conservation is built into the very algebraic structure of the master equation rather than imposed afterward.

More precisely, summing (3) over i yields

$$\sum_i \frac{dU_i}{dt} = \sum_i \sum_{j \in N(i)} A_{ij} F_{ij}. \quad (6)$$

Reorganizing the double sum into unordered pairs $\{i, j\}$, one obtains

$$\sum_i \frac{dU_i}{dt} = \sum_{\{i,j\}} (A_{ij}F_{ij} + A_{ji}F_{ji}). \quad (7)$$

Using (4) and (5), the right-hand side becomes

$$\sum_{\{i,j\}} A_{ij} (F_{ij} - F_{ji}) = 0. \quad (8)$$

Therefore,

$$\sum_i \frac{dU_i}{dt} = 0, \quad (9)$$

which is the discrete conservation statement underlying the master equation. In later chapters, this structural cancellation will reappear in concrete finite-volume form.

The significance of this formulation is that it is completely independent of the particular constitutive choice of the flux. Conservation follows solely from the symmetry of the weights and the antisymmetry of the fluxes. This separation between algebraic conservation structure and constitutive flux modeling is essential for the later decomposition into reversible and dissipative parts.

2.3. Entropy and the Second Law

Conservation laws alone do not determine the direction of irreversible processes. They state only that certain totals remain constant, but they do not distinguish between reversible redistribution and dissipative evolution. To characterize physically admissible dynamics, we therefore supplement conservation by the second law of thermodynamics.

Let the specific entropy at each node be given by

$$s = s(\rho, e), \quad (10)$$

and define the total entropy of the system by

$$S = \sum_i V_i \rho_i s_i, \quad (11)$$

where $s_i = s(\rho_i, e_i)$. The quantity $\rho_i s_i$ is the entropy density per unit volume associated with node i . Summing over all nodes yields the total entropy of the closed system. The second law requires that, for an isolated system, total entropy must not decrease in time:

$$\frac{dS}{dt} \geq 0. \quad (12)$$

In the present work, this inequality is not treated merely as an a posteriori consequence of the dynamics. Rather, it is adopted as a design constraint that determines which master equations are physically admissible. Among all discrete evolution laws compatible with conservation, we select only those whose dissipative structure guarantees the entropy inequality (12). In this respect, the present viewpoint is closely related to entropy-based formulations of nonequilibrium dynamics and to gradient-flow ideas in metric and thermodynamic settings [35].

The role of entropy here is therefore twofold. First, it distinguishes physically meaningful dissipative dynamics from arbitrary conservative rearrangements. Second, it provides the guiding principle by which dissipative terms are later designed in the finite-volume framework of Chapter 3.

2.4. Decomposition into Reversible and Dissipative Parts

We now decompose the dynamics into reversible and irreversible components. This decomposition is standard in nonequilibrium thermodynamics and plays a central role in the GENERIC viewpoint. We write

$$\frac{dU_i}{dt} = R_i(U) + G_i(U), \quad (13)$$

where $R_i(U)$ denotes the reversible exchange term and $G_i(U)$ the irreversible dissipative term. The reversible part is understood as redistribution without entropy production. Accordingly, it must satisfy

$$\frac{dS}{dt} |_R = 0. \quad (14)$$

By contrast, the dissipative part represents irreversible processes and must therefore satisfy

$$\frac{dS}{dt} |_G \geq 0. \quad (15)$$

This decomposition has a clear physical interpretation. The reversible part corresponds, in later chapters, to Euler-type exchange fluxes that transport mass, momentum, and energy without

producing entropy. The dissipative part corresponds to viscous relaxation and heat conduction, which do produce entropy. The importance of separating the two at the abstract level is that it clarifies which part of the dynamics is constrained by conservation alone and which part must be designed to enforce the second law.

The decomposition (13) also clarifies a key methodological point of the paper. The objective is not to deduce a concrete fluid model immediately from an abstract thermodynamic formalism. Rather, the purpose is first to establish the structural template—reversible exchange plus irreversible dissipation—and then, in Chapter 3, to realize this template by explicit finite-volume interface fluxes.

2.5. GENERIC-Type Gradient Flow and Degeneracy Condition

To construct the dissipative term G in a way that automatically satisfies the second law, it is natural to refer to the GENERIC-type gradient-flow structure known in nonequilibrium thermodynamics. In such a formulation, dissipative evolution is directed along the entropy gradient. The generic form is

$$G = M(U)\nabla_U S, \quad (16)$$

where $M(U)$ is an operator representing the mobility or dissipation metric.

The essential assumption is that $M(U)$ is symmetric and positive semi-definite. Under this assumption, entropy production is non-negative, because the entropy gradient is paired with itself through a non-negative quadratic form. However, dissipative evolution should not violate the conserved quantities. To ensure compatibility with conservation laws, one imposes the degeneracy condition

$$\nabla C \cdot M(U) = 0, \quad (17)$$

for each conserved quantity C . This means that the dissipative directions are orthogonal to the gradients of the conserved quantities, so that dissipation redistributes the state toward higher entropy without changing the total conserved invariants.

At this point, it is crucial to state precisely the role of the GENERIC structure in the present work. In this paper, Eqs. (16) and (17) are not imposed as a strict axiom uniquely determining the formulas of the later finite-volume model. Rather, they are introduced as a structural guideline clarifying what any acceptable dissipative mechanism should achieve:

- it should produce entropy,
- it should be compatible with conservation laws, and
- it should admit a Lyapunov interpretation.

This distinction is important for the logical architecture of the paper. Chapter 2 is a chapter of thermodynamic design principles. The actual dissipative fluxes used later are not derived by postulating a global operator $M(U)$ and then reading off all coefficients from it. Instead, in Chapter 3 the dissipation will be constructed directly from interface-level quadratic forms. Entropy production will then be verified by direct calculation, not by appealing abstractly to Eq. (16).

Thus, the purpose of introducing GENERIC here is to make transparent the underlying philosophy of the construction, not to replace the concrete interface analysis that follows.

2.6. Abstract Master Equation

With the preceding preparation, we define the abstract master equation considered in this work as

$$\frac{dU_i}{dt} = \sum_{j \in N(i)} A_{ij} (F_{ij} + D_{ij}), \quad (18)$$

where F_{ij} denotes the reversible exchange flux and D_{ij} the dissipative flux. The former is responsible for conservative redistribution, while the latter represents irreversible processes giving rise to entropy production.

Eq. (18) summarizes the structural content of this chapter. The conservation structure is encoded by the symmetry $A_{ij} = A_{ji}$ together with the antisymmetry of the reversible flux. The dissipative

structure is encoded by the choice of D_{ij} , which must be designed so as to satisfy the second law while remaining compatible with the conserved quantities.

In Chapter 3, the abstract quantities in (18) will be specified concretely. The reversible flux F_{ij} will be realized as an Euler-type interfacial flux, while D_{ij} will be realized as an Onsager-type interface mechanism corresponding to viscosity and heat conduction. The key point is that the second law in the concrete fluid model will be proved directly from the explicit interface structure, rather than assumed abstractly through a global gradient-flow operator.

Theorem 2.1 (Entropy Production under GENERIC-Type Dissipation)

To clarify why the GENERIC-type construction is a useful design principle, we record the following structural theorem.

Theorem 2.1.

Suppose that the dissipative term has the GENERIC-type gradient-flow form

$$G = M(U)\nabla_U S, \quad (19)$$

and that $M(U)$ is symmetric and positive semi-definite. Then the total entropy satisfies

$$\frac{dS}{dt} \geq 0. \quad (20)$$

Proof. By the chain rule, the time derivative of the entropy along the dissipative dynamics is

$$\frac{dS}{dt} = (\nabla_U S)^T G. \quad (21)$$

Substituting (19), we obtain

$$\frac{dS}{dt} = (\nabla_U S)^T M(U) \nabla_U S. \quad (22)$$

Since $M(U)$ is symmetric and positive semi-definite, the quadratic form on the right-hand side is non-negative. Therefore,

$$\frac{dS}{dt} \geq 0, \quad (23)$$

which proves the claim.

The meaning of this theorem is simple but important. If the dissipative term is designed as a positive semi-definite operator acting in the direction of the entropy gradient, then the second law follows automatically. Thus, the GENERIC-type structure provides a robust thermodynamic design principle.

At the same time, the theorem should not be overinterpreted in the context of the present paper. It does not assert that the later finite-volume dissipation must literally be obtained by writing down a global operator $M(U)$ and applying it to $\nabla_U S$. Rather, it clarifies the thermodynamic structure that the later interface-level construction is designed to realize concretely.

Note on the Role of Eq. (19)

In the concrete fluid model constructed in Chapter 3, the dissipative term will not necessarily be written in the global abstract form

$$D = M(U)\nabla_U S. \quad (24)$$

Instead, viscous dissipation and heat conduction will be defined as symmetric quadratic forms built from finite-volume interface interactions. The entropy inequality will then be established directly from the non-negativity of each interface contribution. Thus, Theorem 2.1 is best interpreted as a statement of design principle, not as a formula that must be imposed verbatim on the later model.

Remark 2.1 (Role of the GENERIC Structure in This Study)

Theorem 2.1 confirms that GENERIC-type gradient flows automatically satisfy the second law. However, the central objective of the present work is not to develop the general theory of GENERIC itself. Rather, the aim is to construct a network master equation with a fluid-mechanically meaningful finite-volume flux structure and to connect that structure to continuum fluid equations under suitable assumptions, and then to study rigorously the relaxation-extended continuum system.

For this reason, the GENERIC framework is introduced in Chapter 2 primarily as a thermodynamic guide. Its function is to explain why dissipative terms should be aligned with entropy production and constrained by conservation. The actual proof of entropy production for the concrete fluid model is deferred to Chapter 3, where it is derived directly from interface quadratic forms. This division of roles is deliberate and is essential to the organization of the paper.

Remark 2.2 (Logical Connection to Chapter 3)

In Chapter 3, the abstract fluxes appearing in Eq. (18) will be specified explicitly. The reversible flux F_{ij} will be defined as a Euler-type conservative interfacial flux, whereas the dissipative flux D_{ij} will be constructed as an Onsager-type interface mechanism corresponding to viscosity and heat conduction.

In that formulation, the second law will not be established by assuming the abstract GENERIC form (19). Instead, it will be proved directly by showing that each interface contribution to entropy production is a non-negative quadratic form involving, in particular,

- the square of velocity differences, and
- the square of temperature differences, or equivalently inverse-temperature differences.

Thus, the logical structure of the paper may be summarized schematically as

$$\begin{aligned} \text{GENERIC structure (design principle)} &\rightarrow \text{finite-volume interface dissipation (explicit construction)} \\ &\rightarrow \text{direct proof of the entropy inequality} \end{aligned} \quad (25)$$

In this sense, Chapter 2 functions as a structural design chapter. It does not yet claim the complete rigorous discrete-to-continuum theorem. Rather, it identifies the admissible thermodynamic architecture of the master equation and prepares the framework within which the concrete interface model of Chapter 3 can be understood.

Chapter 3 Concrete Form of the Parent Equation: Reversible Fluxes and Dissipative Structure

In Chapter 2, we formulated the general structure of a network-type master equation on the basis of two fundamental principles: the conservation laws and the second law of thermodynamics. In particular, Chapter 2 introduced a GENERIC-type gradient-flow structure as a design principle for dissipative processes and showed that, at the abstract level, entropy production is automatic when the dissipative term is aligned with the entropy gradient through a positive semi-definite operator. At the same time, it was emphasized there that the purpose of the present study is not to impose the GENERIC formalism as a rigid axiom uniquely determining the later fluid model. Rather, its role is to clarify the thermodynamic architecture that an admissible dissipative mechanism should possess.

The purpose of the present chapter is therefore more concrete. Starting from the abstract master equation introduced in Chapter 2, we now specify the fluxes in a form that admits direct fluid-mechanical interpretation. More precisely, for the abstract master equation

$$\frac{du_i}{dt} = \sum_{j \in N(i)} A_{ij} (F_{ij} + D_{ij}), \quad (26)$$

we define F_{ij} as a reversible exchange flux corresponding to the Euler equations and construct D_{ij} as a dissipative flux corresponding to viscosity and heat conduction. The essential point is that the dissipative flux is not assumed a priori to arise from a global GENERIC-type operator. Instead, it is

constructed directly from finite-volume interface interactions. Entropy production is then verified by direct computation from the non-negative quadratic form of each interface contribution.

Thus, the role of the present chapter is twofold. First, it provides a concrete fluid-mechanical realization of the structural design principles introduced in Chapter 2. Second, it prepares the later continuum and relaxation analyses by identifying the precise conservative and dissipative interface mechanisms from which the extended continuum system is motivated.

It is important to state clearly what is, and is not, established in this chapter. The present chapter does not claim the full rigorous discrete-to-continuum compactness theorem. Rather, it constructs the concrete parent equation, proves that its interface structure is compatible with conservation and the second law, and explains how this structure leads naturally to a continuum fluid description and, after relaxation extension, to the framework analyzed rigorously in Chapters 4 and 5.

In this way, the present chapter shows that the master equation is consistent with the second law through the interplay of three structural ingredients:

- the conservation laws,
- the antisymmetric structure of interface fluxes, and
- the non-negative quadratic form of dissipation.

Furthermore, because this concrete formulation is fully compatible with the finite-volume viewpoint, it provides the structural bridge from discrete interacting systems to continuum fluid models under suitable geometric and scaling assumptions.

3.1. State Variables and Thermodynamic Closure (Pressure, Temperature, and Entropy)

For each node i , we define the state variables by

$$U_i = (\rho_i, m_i, e_i), \quad (27)$$

and the velocity u_i by

$$u_i := \frac{m_i}{\rho_i}. \quad (28)$$

Here ρ_i denotes the density, m_i the momentum, and e_i the internal energy density. The corresponding total energy density is defined by

$$E_i := \frac{|m_i|^2}{2\rho_i} + e_i. \quad (29)$$

This decomposition is standard in compressible fluid mechanics: the total energy consists of the kinetic part

$|m_i|^2/(2\rho_i)$ and the internal part e_i . Since the later conservative flux is modeled on the Euler equations, the use of E_i is natural already at the discrete level.

As the local thermodynamic closure, we prescribe the specific entropy $s(\rho, e)$, and define the temperature T and the pressure p through

$$\frac{1}{T} := \partial_e s(\rho, e), \quad \frac{p}{T} := \rho^2 \partial_\rho s(\rho, e), \quad (30)$$

which are the standard thermodynamic relations written in entropy variables. The total entropy is then defined by

$$S(U) := \sum_i \rho_i s(\rho_i, e_i) V_i, \quad (31)$$

where V_i denotes the representative volume associated with node i .

The importance of Eqs. (30) and (31) is the following. They ensure that the thermodynamic variables used in the interface construction are not introduced ad hoc, but are derived consistently from the entropy function itself. In particular, because the second law will be verified by computing the sign of the discrete entropy production, the use of the entropy-based thermodynamic closure is essential to the logic of the chapter.

3.2. Geometry (Finite-Volume Correspondence) and Antisymmetry ("Automatic" Implementation of the Conservation Laws)

We now introduce the geometric ingredients needed to interpret the network interactions in finite-volume form. Let S_{ij} denote the interface between node i and an adjacent node j , and let n_{ij} be the unit normal vector directed from j to i . We define the finite-volume-compatible weight

$$w_{ij} := \frac{S_{ij}}{V_i}, \quad (32)$$

and fix the interaction strength as

$$A_{ij} := w_{ij}. \quad (33)$$

This is the standard choice adopted in the present paper for the continuum correspondence, because it is exactly the scaling that converts an interfacial flux into a rate of change of a cell-averaged quantity.

Assuming bidirectional interfaces, we impose

$$S_{ij} = S_{ji}, n_{ji} = -n_{ij}. \quad (34)$$

Under this setting, if an arbitrary interface quantity Q_{ij} satisfies the antisymmetry property

$$Q_{ij} = -Q_{ji}, \quad (35)$$

then the global sum satisfies

$$\sum_i V_i \sum_{j \in N(i)} w_{ij} Q_{ij} = 0. \quad (36)$$

The proof is immediate but fundamental. Since $V_i w_{ij} = S_{ij}$, the left-hand side may be written as

$$\sum_i \sum_{j \in N(i)} S_{ij} Q_{ij}. \quad (37)$$

Grouping the terms associated with an unordered interface pair $\{i, j\}$, and using $S_{ij} = S_{ji}$ together with $Q_{ji} = -Q_{ij}$, one obtains

$$S_{ij} Q_{ij} + S_{ji} Q_{ji} = S_{ij} Q_{ij} - S_{ij} Q_{ij} = 0. \quad (38)$$

Hence the total contribution vanishes after summation over all internal interfaces, which proves (36).

This cancellation property is the discrete analogue of the divergence theorem in finite-volume form. Its significance is that global conservation follows purely from the interface antisymmetry and geometric symmetry, independently of the detailed constitutive formula for the flux. This is precisely the mechanism by which the conservative part of the master equation preserves mass, momentum, and total energy.

3.3. Reversible (Exchange) Flux F_{ij}

Implementation of the Euler-Type Structure

We next define the reversible flux. Let U_{ij} denote an interface state associated with the pair (i, j) , for example an arithmetic average or a consistent approximation obtained from a local Riemann solver. Using the Euler conservative flux

$$F(U) := \begin{pmatrix} \rho u \\ \rho u \otimes u + pI \\ (E + p)u \end{pmatrix}, \quad (39)$$

we define the reversible interface flux by

$$F_{ij}(U_i, U_j) := -F(U_{ij}) \cdot n_{ij}. \quad (40)$$

Here " $\cdot n_{ij}$ " denotes contraction of the flux tensor with the interface normal. This construction is the discrete analogue of the standard finite-volume Euler flux through an interface.

From Eqs. (34) and (40), and assuming the natural symmetry $U_{ji} = U_{ij}$ of the chosen interface state, we obtain

$$F_{ji}(U_j, U_i) = -F(U_{ji}) \cdot n_{ji} = +F(U_{ij}) \cdot n_{ij} = -F_{ij}(U_i, U_j). \quad (41)$$

Thus the antisymmetry condition (35) is satisfied. Consequently, by the cancellation argument of Section 3.2, the reversible part does not violate the global conservation laws.

This point deserves emphasis. The conservative structure of the parent equation does not arise from a postulated continuum PDE; rather, it is encoded directly at the discrete interface level through the Euler-type flux and the orientation reversal property. In this sense, the reversible part of the master equation is already fluid-mechanically meaningful before any continuum limit is taken.

3.4. Irreversible (Dissipative) Flux $\mathbf{G}_{ij}$: Onsager Form and Positive Semi-Definiteness

3.4.1. Basic Form of the Dissipative Flux

We now construct the dissipative part. In contrast to the reversible Euler-type flux, the dissipative flux is designed so as to encode viscosity and heat conduction while producing non-negative entropy. We define the dissipative flux by

$$G_{ij}^p = 0, \quad G_{ij}^m = \mu_{ij}(u_j - u_i), \quad G_{ij}^e = \kappa_{ij}(T_j - T_i), \quad (42)$$

where the coefficients satisfy

$$\mu_{ij} = \mu_{ji} \geq 0, \quad \kappa_{ij} = \kappa_{ji} \geq 0. \quad (43)$$

The first relation in Eq. (42) means that no dissipative mass flux is introduced. This is consistent with standard fluid mechanics, where viscosity and heat conduction do not create diffusive mass transfer in the basic single-component setting considered here.

Moreover, the dissipative contribution to the internal energy is understood more precisely as the sum of heat conduction and viscous heating:

$$G^e = G_{\text{cond}}^e + G_{\text{visc}}^e. \quad (44)$$

Eq. (42) gives the conductive part explicitly through temperature differences. The viscous-heating part is described in the next subsection. This separation is important, because the second law receives contributions from both mechanisms: heat conduction contributes directly through temperature differences, whereas viscosity contributes through conversion of kinetic energy into internal energy.

3.4.2. Energy Conversion Induced by Viscous Dissipation (Viscous Heating: Revised Form)

For the interface (i, j) , define the viscous dissipation power per unit area by

$$\Phi_{ij}^{\text{visc}} = \frac{1}{2} \mu_{ij} |u_j - u_i|^2 \geq 0. \quad (45)$$

This quantity measures the rate at which kinetic energy is dissipated at the interface due to velocity mismatch. The non-negativity is immediate from $\mu_{ij} \geq 0$.

The total kinetic energy dissipated per unit time through the interface is therefore $S_{ij} \Phi_{ij}^{\text{visc}}$. In order to preserve the total energy balance of the full system, this lost kinetic energy must reappear as an increase of internal energy. We implement this by distributing the dissipated power symmetrically between the two adjacent cells. Introducing the geometric weights

$$w_{ij} = \frac{S_{ij}}{V_i}, \quad w_{ji} = \frac{S_{ij}}{V_j}, \quad (46)$$

the viscous-heating contributions to the internal-energy densities become

$$\left(\frac{de_i}{dt}\right)_{\text{visc}} = \sum_{j \in N(i)} \frac{w_{ij}}{2} \mu_{ij} |u_j - u_i|^2, \quad (47)$$

and

$$\left(\frac{de_j}{dt}\right)_{\text{visc}} = \sum_{i \in N(j)} \frac{w_{ji}}{2} \mu_{ij} |u_j - u_i|^2. \quad (48)$$

This symmetrized allocation deserves a detailed explanation. The interface dissipation Φ_{ij}^{visc} is not associated with node i alone or node j alone; it is generated by the relative motion across the common interface. It is therefore natural, and fully consistent with the finite-volume viewpoint, to distribute the dissipated power equally between the adjacent cells. Under this choice, the kinetic-energy loss and internal-energy gain balance exactly at the level of each interface pair, so that the total energy of the whole discrete system remains consistent.

The role of Eqs. (47) and (48) is therefore not merely technical. They ensure that dissipation is represented not as destruction of energy, but as conversion from kinetic to internal energy. This is exactly the correct thermodynamic interpretation of viscosity.

Moreover, as will be stated explicitly in Section 3.7, the temperature field is later treated, in the low-Mach-number regime, as an auxiliary thermodynamic variable under an isothermal approximation at leading order. Consequently, the inclusion of viscous heating does not destroy the closure of the dynamical system. Rather, it preserves the correct thermodynamic bookkeeping needed for consistency with both the second law and the later continuum interpretation.

3.5. Dissipative Structure and the Second Law

We now verify directly that the dissipative interface structure constructed above is compatible with the second law. This is one of the most important points of the chapter. In accordance with the methodological distinction made in Chapter 2, the entropy inequality is not deduced here by postulating the abstract GENERIC formula. Instead, it is established directly from the explicit interface-level construction.

3.5.1. Basic Form of the Time Derivative of the Entropy

From Eq. (31), the time derivative of the total entropy is

$$\frac{ds}{dt} = \sum_i V_i (\partial_{\rho_i}(\rho_i s_i) \dot{\rho}_i + \partial_{e_i}(\rho_i s_i) \dot{e}_i), \quad (49)$$

where $s_i = s(\rho_i, e_i)$. Since e_i denotes the internal energy density per unit volume, the entropy density is $\rho_i s_i$. For the variables (ρ, e) , the Gibbs relation reads

$$\rho T ds = de - \frac{p}{\rho} d\rho. \quad (50)$$

Rewriting this relation in terms of the derivatives of ρs with respect to (ρ, e) , we obtain

$$\partial_e(\rho s) = \frac{1}{T}, \quad \partial_\rho(\rho s) = s - \frac{p}{\rho T}. \quad (51)$$

Now, for the dissipative part of the dynamics, we have $G_{ij}^p = 0$ by construction. Therefore the dissipative contribution to the mass equation vanishes, and the dissipative part of (49) reduces to

$$\left(\frac{ds}{dt}\right)_D = \sum_i V_i \frac{1}{T_i} \left(\frac{de_i}{dt}\right)_D. \quad (52)$$

This formula is crucial: it shows that, for the dissipative dynamics considered here, the sign of entropy production is determined entirely by the internal-energy update and its coupling with the reciprocal temperature.

3.5.2. Interface Representation of the Internal-Energy Update

The dissipative update of the internal energy density may be represented in terms of heat transfer across interfaces. Let q_{ij} denote the heat flux, or more generally the dissipative energy flux, per unit area through the interface (i, j) , with physical unit energy/(area×time). For the reversed orientation, we impose

$$q_{ji} = -q_{ij}. \quad (53)$$

Then the dissipative update of the internal energy density in cell i may be written as

$$\left(\frac{de_i}{dt}\right)_D = \sum_{j \in N(i)} w_{ij} q_{ij}, \quad (54)$$

because multiplying the area flux by the interface area gives power, and dividing by the cell volume gives the rate of change of an energy density. Substituting (54) into (52), we obtain

$$\left(\frac{ds}{dt}\right)_D = \sum_i V_i \frac{1}{T_i} \sum_{j \in N(i)} w_{ij} q_{ij}. \quad (55)$$

Since

$$V_i w_{ij} = S_{ij}, \quad (56)$$

Eq. (55) becomes

$$\left(\frac{ds}{dt}\right)_D = \sum_i \sum_{j \in N(i)} S_{ij} q_{ij} \frac{1}{T_i}. \quad (57)$$

Thus the entropy production is represented as a sum of interface contributions weighted by reciprocal temperatures.

3.5.3. Symmetrization over Bidirectional Interfaces

We now group together the two oriented contributions associated with the same geometric interface. Since Eq. (34) gives $S_{ij} = S_{ji}$, the two terms corresponding to (i, j) and (j, i) in (57) yield

$$S_{ij} q_{ij} \frac{1}{T_i} + S_{ji} q_{ji} \frac{1}{T_j}. \quad (58)$$

Because $q_{ji} = -q_{ij}$, this becomes

$$S_{ij} q_{ij} \left(\frac{1}{T_i} - \frac{1}{T_j} \right). \quad (59)$$

Therefore, after summing only once over each unordered interface pair, Eq. (57) may be written as

$$\left(\frac{dS}{dt}\right)_D = \sum_{i<j} S_{ij} q_{ij} \left(\frac{1}{T_i} - \frac{1}{T_j}\right). \quad (60)$$

We now choose the heat transfer according to the Onsager-type relation

$$q_{ij} = \kappa_{ij} \left(\frac{1}{T_j} - \frac{1}{T_i}\right), \kappa_{ij} = \kappa_{ji} \geq 0. \quad (61)$$

This is the standard linear-response form in nonequilibrium thermodynamics. It is written here in terms of reciprocal temperature, which is the entropy-conjugate thermodynamic variable and therefore the natural quantity for entropy production. Substituting (61) into (60), we obtain

$$q_{ij} \left(\frac{1}{T_i} - \frac{1}{T_j}\right) = \kappa_{ij} \left(\frac{1}{T_i} - \frac{1}{T_j}\right)^2, \quad (62)$$

and hence

$$\left(\frac{dS}{dt}\right)_D = \sum_{i<j} S_{ij} \kappa_{ij} \left(\frac{1}{T_i} - \frac{1}{T_j}\right)^2. \quad (63)$$

Since $S_{ij} > 0$ and $\kappa_{ij} \geq 0$, each interface contribution satisfies

$$S_{ij} \kappa_{ij} \left(\frac{1}{T_i} - \frac{1}{T_j}\right)^2 \geq 0, \quad (64)$$

and therefore

$$\left(\frac{dS}{dt}\right)_D \geq 0. \quad (65)$$

This proves the discrete entropy inequality for the conductive part of the dissipative mechanism.

The decisive point is that the non-negativity of entropy production has been obtained directly from the explicit interface construction. No abstract monotonicity hypothesis beyond $\kappa_{ij} \geq 0$ and the orientation symmetry has been required. In other words, the dissipative structure has been constructed in such a way that each interface contribution is itself a non-negative quadratic form.

For completeness, let us also note the role of the viscous contribution. By Eqs. (45), (47), and (48), the viscous heating contribution is likewise non-negative, because it is proportional to $|u_j - u_i|^2$ with coefficient $\mu_{ij} \geq 0$. Hence the viscous mechanism is consistent with the same thermodynamic directionality: kinetic energy is dissipated and reappears as internal energy, thereby contributing non-negatively to entropy production.

Accordingly, the full discrete dissipative structure constructed in this section is thermodynamically admissible.

3.6. Lifting to an Extended System (Regularization and Relaxation)

Up to this point, we have constructed a discrete parent equation that satisfies the correct conservation structure and a discrete entropy-production property. This provides the structural basis for a continuum fluid interpretation. At the same time, the later rigorous analysis of the paper is not carried out directly on the fully discrete network system. Instead, it is carried out on a continuum relaxation-extended system motivated by the present structure.

For this reason, we now explain how the discrete dissipative framework naturally suggests the introduction of an auxiliary stress variable in the continuum description. The idea is that nonequilibrium stresses, which are not yet fully relaxed to the Newtonian constitutive form, may be represented dynamically by an independent variable. The corresponding forcing term is then generated internally, not prescribed externally.

Accordingly, in the present work, we introduce a residual stress r^ε in the continuum system and define the associated forcing by

$$F^\varepsilon = \nabla \cdot r^\varepsilon. \quad (66)$$

Here F^ε is not an external source term chosen arbitrarily. It is an internally generated forcing arising from the fact that the stress variable has not yet completely relaxed to its viscous equilibrium form. This viewpoint is essential for the later analysis, because it allows the forcing to be controlled through the dissipation mechanism of the extended system itself.

At the structural level, the relevance of this definition is the following. The present parent equation identifies how dissipation should act at the level of interfaces—through velocity differences, temperature differences, and entropy production. In the continuum relaxation extension of Chapter 4, this same thermodynamic logic is encoded in the evolution of the residual stress variable r^ε . The forcing term F^ε is therefore not an extraneous modification of the model, but the continuum counterpart of the nonequilibrium dissipative correction suggested by the present interface-level structure.

A concrete analytic estimate of the forcing term, in particular the property

$$F^\varepsilon \in L^2(0, T; L^2(\Omega)), \quad (67)$$

together with the quantitative smallness and eventual disappearance of the residual stress in the relaxation limit, will be established in Chapter 4 from the extended energy inequality and then used in Chapter 5 in the compactness argument. To avoid overstating the present chapter's scope, we emphasize once more that equation (66) is introduced here as the structural passage from the parent equation to the continuum relaxation framework. The rigorous estimates justifying its properties belong to the subsequent chapters.

3.7. Incompressibilization by the Low-Mach-Number Limit and Internalization of the Forcing

In this section, we explain under what limiting procedure the network-based fluid model constructed above reduces formally to an incompressible system. The discussion is formal and is included for physical interpretation and structural motivation. It is not part of the rigorous compactness proof leading to the main theorem of the paper. That theorem is established later, in Chapters 4 and 5, from the extended energy inequality and the Aubin–Lions compactness argument applied to the continuum relaxation-extended system.

Thus, the purpose of the present section is not to prove the incompressible limit rigorously, but to show that the parent equation constructed in the preceding sections is naturally compatible with incompressible fluid mechanics in the low-Mach-number regime.

Throughout this chapter, the network equation has been formulated as a compressible thermodynamic system with state variable $U = (\rho, m, e)$. We now explain, through scaling arguments, how this system formally reduces to the incompressible Navier–Stokes equations under the low-Mach-number limit.

3.7.1. Nondimensionalization and the Low-Mach-Number Regime

Using a reference length L , a representative velocity U_0 , a reference density ρ_0 , and a reference temperature T_0 , we nondimensionalize the variables as

$$x' = \frac{x}{L}, t' = \frac{tU_0}{L}, u' = \frac{u}{U_0}. \quad (68)$$

The pressure is nondimensionalized using the sound speed c_s as

$$p' = \frac{p}{\rho_0 c_s^2}. \quad (69)$$

The Mach number is then defined by

$$Ma = \frac{U_0}{c_s}, \quad (70)$$

and is assumed to be small:

$$Ma \ll 1. \quad (71)$$

For well-prepared initial data, we assume the asymptotic expansions

$$p = p_0 + Ma^2 \pi, \quad \rho = 1 + Ma^2 \rho_1, \quad T = 1 + Ma^2 \theta, \quad (72)$$

where the primes are omitted from this point onward. We also impose the well-prepared condition

$$\nabla \cdot u_0 = O(Ma^2), \quad (73)$$

and adopt the dynamic-pressure scaling

$$\pi = O(1). \quad (74)$$

These assumptions are standard in low-Mach-number asymptotics. Their purpose is to suppress the leading-order acoustic oscillations that would otherwise prevent a clean incompressible limit.

The relevant time scale is the convective one:

$$t \sim \frac{L}{u_0}. \quad (75)$$

The well-prepared initial conditions play an especially important role. In general, when density and pressure fluctuations couple at the acoustic scale, rapid sound-wave oscillations appear and obstruct the incompressible regime. To eliminate this leading-order acoustic component, we assume more precisely that the initial data satisfy

$$\rho^\varepsilon = \rho_0 + O(Ma^2), \quad \nabla \cdot u^\varepsilon = O(Ma). \quad (76)$$

Then the density fluctuations remain negligible at leading order, and the mass conservation law reduces, as $Ma \rightarrow 0$, to

$$\nabla \cdot u = 0. \quad (77)$$

In the network representation adopted in this work, this corresponds to requiring that the initial node states

$$U_i = (\rho_i, m_i, e_i), \quad (78)$$

satisfy

$$\rho_i = \rho_0 + O(Ma^2). \quad (79)$$

This means that, at leading order, the density fluctuation is spatially uniform across the network, and hence the incompressibility condition emerges naturally in the continuum correspondence.

3.7.2. Asymptotic Expansion of the Density

In the low-Mach-number regime, the density satisfies

$$\rho = \rho_0 + O(Ma^2). \quad (80)$$

Hence, at leading order, one may regard

$$\rho \equiv \rho_0. \quad (81)$$

As a result, the mass conservation equation

$$\partial_t \rho + \nabla \cdot (\rho u) = 0, \quad (82)$$

reduces to

$$\nabla \cdot u = 0. \quad (83)$$

This is exactly the incompressibility condition.

3.7.3. Treatment of the Energy Equation

The internal-energy equation is generally written as

$$\partial_t e + \nabla \cdot (eu) = -p \nabla \cdot u + \Phi + \kappa \Delta T. \quad (84)$$

However, in the low-Mach-number limit one has

$$\nabla \cdot u = 0, \quad (85)$$

and therefore the compressional work term disappears. Moreover, when temperature variations are small, one may approximate

$$T \simeq T_0. \quad (86)$$

Since the main concern of the present paper is the dynamics of the velocity field and the relaxation of stresses, the temperature field is henceforth treated as an auxiliary thermodynamic variable under an isothermal approximation at leading order. This simplification is sufficient for the structural interpretation pursued here. The rigorous analysis later given in Chapters 4 and 5 is performed on the incompressible relaxation-extended system rather than on the full compressible thermodynamic system.

3.7.4. The Incompressible Navier–Stokes Equations

From this point onward, we return to dimensional variables. Let ρ_0 denote the reference density and μ the dynamic viscosity. Let Π denote the dimensional pressure corresponding to the nondimensional dynamic pressure π . Under the low-Mach-number limit $Ma \rightarrow 0$, the momentum equation becomes

$$\rho_0(\partial_t u + u \cdot \nabla u) = -\nabla \Pi + \mu \Delta u + f. \quad (87)$$

Introducing the redefined pressure

$$p := \frac{\pi}{\rho_0}, \quad (88)$$

and the kinematic viscosity

$$\nu := \frac{\mu}{\rho_0}, \quad (89)$$

we obtain

$$\partial_t u + u \cdot \nabla u = -\nabla p + \nu \Delta u + f, \nabla \cdot u = 0. \quad (90)$$

Thus, at the formal level, the incompressible Navier–Stokes equations are recovered.

The significance of this subsection is interpretive rather than theorem-proving. It shows that the conservative and dissipative structures introduced above are not arbitrary discrete devices, but are consistent with the standard incompressible fluid equations in the appropriate low-Mach-number regime.

3.8. Connection to Chapters 4 and 5

The discussion of this chapter yields the following structural conclusions:

First, by Eq. (40), the reversible flux possesses an antisymmetric interface structure, and therefore the global conservation laws hold automatically in a closed system by the cancellation mechanism explained in Section 3.2.

Second, by Eqs. (42)–(45) and the symmetrization argument leading to (63)–(65), the dissipative interface structure produces non-negative entropy. In other words, the second law is verified here directly from the finite-volume interface construction, exactly as announced in Chapter 2.

Third, the parent equation constructed in the present chapter motivates the continuum relaxation extension in which the internally generated forcing is represented by the divergence of a residual stress:

$$F^\varepsilon = \nabla \cdot r^\varepsilon. \quad (91)$$

The analytical properties of this forcing are not assumed here, but are established in Chapter 4 through the extended energy (Lyapunov) structure. In particular, Chapter 4 proves the dissipative estimates for the residual stress from which the integrability and eventual disappearance of the forcing follow.

More precisely, Chapter 4 will show that

$$F^\varepsilon = \nabla \cdot r^\varepsilon \in L^2(0, T; L^2(\Omega)), \quad (92)$$

and, more importantly, that the residual stress becomes small in the relaxation limit. Starting from those estimates, Chapter 5 will establish, in sequence,

- Lemma 5.1: global a priori bounds derived from the extended energy inequality,
- Lemma 5.2: compactness via the Aubin–Lions theorem, and
- Lemma 5.3: convergence, as $\varepsilon \rightarrow 0$, to a Leray–Hopf weak solution of the incompressible Navier–Stokes equations.

Thus, the conservation structure and entropy-production structure identified in the present chapter furnish the structural foundation of the later rigorous analysis. The direct proof of the second law obtained here is what justifies the later introduction of the relaxation-extended system as a thermodynamically consistent continuum approximation.

Chapter 4 Extended System (Regularization and Relaxation) and the Extended Energy Structure

The purpose of this chapter is to define rigorously, prior to proceeding to the relaxation-limit analysis leading to the incompressible Navier–Stokes equations, the continuum extended system that constitutes the analytic core of the present study, and to establish, in a self-contained manner and without reliance on external assumptions, the extended energy inequality (Lyapunov structure) satisfied by that system.

The role of the extended system introduced here is twofold. First, by promoting the nonequilibrium stress to an independent variable, the convergence toward the Newtonian viscous

stress can be represented as a dynamical relaxation process rather than as an instantaneous constitutive closure. Second, by incorporating higher-order dissipation into that relaxation process, one endows the coupled system with a coercive Lyapunov structure. From this structure, the ingredients needed in Chapter 5—namely, a priori bounds, the integrability of the forcing term, and the quantitative smallness of the residual stress—follow as intrinsic consequences of the model itself.

In the present chapter, the forcing term is not treated as an externally prescribed source. Instead, it is generated internally by the deviation of the stress variable from its viscous equilibrium. More precisely, after introducing the residual stress r^ε , we define

$$F^\varepsilon := \nabla \cdot r^\varepsilon. \quad (93)$$

By the end of this chapter, the estimates required later in Lemmas 5.1–5.3 will no longer remain as assumptions: they will have been established as theorems derived directly from the design of the extended system.

4.1. Extended Unknowns and the Definition of the Residual Stress

On the periodic domain $\Omega = \mathbb{T}^3$, we denote the unknowns of the extended system by $(u^\varepsilon, p^\varepsilon, \tau^\varepsilon)$.

Here $u^\varepsilon: \Omega \times [0, T] \rightarrow \mathbb{R}^3$ is the velocity field, p^ε is the pressure, and τ^ε is a symmetric second-order tensor-valued stress variable. The symmetry of τ^ε is imposed because the stress tensor in the fluid-mechanical interpretation is symmetric, and because the energy identity later relies on contraction against the symmetric part of ∇u^ε .

We define the deformation-rate tensor by

$$D(u) := \frac{1}{2}(\nabla u + (\nabla u)^\top). \quad (95)$$

This is the standard symmetric gradient. The correct use of $D(u)$ is essential, because the identity

$$\langle \nabla \cdot r, u \rangle = -\langle r: D(u) \rangle$$

holds only for symmetric r , and not with the incorrect quadratic expression $\frac{1}{2}\nabla u \nabla u^\top$.

We now introduce the residual, or deviation, stress by

$$r^\varepsilon := \tau^\varepsilon - 2\nu D(u^\varepsilon), \quad (96)$$

where $\nu > 0$ denotes the viscosity coefficient. Accordingly, the forcing term is defined by (93), namely,

$$F^\varepsilon = \nabla \cdot r^\varepsilon. \quad (97)$$

Under this definition, the force appearing in the extended momentum equation is not an externally prescribed input, but rather an internally generated residual caused by the fact that the stress variable has not yet completely relaxed to the Newtonian viscous stress $2\nu D(u^\varepsilon)$. The following observation will be used repeatedly.

Proposition 4.1 (Preservation of symmetry).

Assume that $r_0^\varepsilon \in L^2_{\text{sym}}(\Omega; \mathbb{R}^{3 \times 3})$. Then any weak solution r^ε of the relaxation equation introduced below remains symmetric for all $t \in [0, T]$.

Proof.

The relaxation equation acts componentwise on r^ε , and the operator $(-\Delta)^{1+\theta}$ is scalar, self-adjoint, and diagonal in Fourier space. Hence it preserves the symmetry of the tensor entries. Since the equation contains only linear combinations of symmetric tensors and componentwise operators, the antisymmetric part evolves independently with zero initial data and therefore remains zero. Thus $r^\varepsilon(t) \in L^2_{\text{sym}}$ for all t . This proposition resolves explicitly the symmetry issue on which the identity involving $D(u^\varepsilon)$ depends.

4.2. Extended Momentum Equation and Rewriting into Navier–Stokes Form

We write the momentum equation of the extended system as

$$\partial_t u^\varepsilon + (u^\varepsilon \cdot \nabla) u^\varepsilon + \nabla p^\varepsilon = \nabla \cdot \tau^\varepsilon, \nabla \cdot u^\varepsilon = 0. \quad (98)$$

Substituting the definition (96) of the residual stress, and using the incompressibility identity

$$\nabla \cdot (2\nu D(u^\varepsilon)) = \nu \Delta u^\varepsilon (\nabla \cdot u^\varepsilon = 0), \quad (99)$$

we obtain

$$\partial_t u^\varepsilon + (u^\varepsilon \cdot \nabla) u^\varepsilon - \nu \Delta u^\varepsilon + \nabla p^\varepsilon = \nabla \cdot r^\varepsilon = F^\varepsilon. \quad (100)$$

Thus, with respect to the left-hand side, the extended system has exactly the same structure as the incompressible Navier–Stokes equations, while on the right-hand side there appears an internally generated forcing term produced by the residual stress.

The analytic strategy of the paper is now transparent: we assign a dissipative relaxation equation to r^ε , derive quantitative bounds from its Lyapunov structure, and then show that the forcing F^ε disappears in the limit $\varepsilon \rightarrow 0$.

4.3. Relaxation Equation as a Regularizing Device

In the present framework, the nonequilibrium stress is assumed to relax toward zero on the residual level, while its high-frequency components are additionally suppressed by a fractional dissipation mechanism. We therefore postulate the closed relaxation equation

$$\varepsilon \partial_t r^\varepsilon + r^\varepsilon = -\delta(-\Delta)^{1+\theta} r^\varepsilon, \quad (101)$$

or equivalently,

$$\varepsilon \partial_t r^\varepsilon + r^\varepsilon + \delta(-\Delta)^{1+\theta} r^\varepsilon = 0, \quad (102)$$

where $\delta > 0$, $\theta \in (0, 1/2)$, and $\varepsilon > 0$.

The meaning of this equation is as follows. The term $\varepsilon \partial_t r^\varepsilon + r^\varepsilon$ represents a Maxwell-type first-order relaxation, so that r^ε decays on the time scale ε . The additional term $\delta(-\Delta)^{1+\theta} r^\varepsilon$ provides stronger damping of high-frequency modes than the standard Laplacian. This dissipation is essential both analytically and structurally: analytically, because it yields control of spatial derivatives of r^ε ; structurally, because it encodes the suppression of highly oscillatory nonequilibrium stresses.

The operator $(-\Delta)^{1+\theta}$ is defined on $\mathbb{T}^3$ through its Fourier symbol. If

$$r(x) = \sum_{k \in \mathbb{Z}^3} \hat{r}(k) e^{ik \cdot x}, \quad (103)$$

then

$$(-\Delta)^{1+\theta} r = \sum_{k \in \mathbb{Z}^3} |k|^{2(1+\theta)} \hat{r}(k) e^{ik \cdot x}. \quad (104)$$

Hence the operator is self-adjoint and nonnegative on $L^2(\mathbb{T}^3)$, and the corresponding quadratic form satisfies

$$\langle (-\Delta)^{1+\theta} r, r \rangle = \left\| (-\Delta)^{\frac{1+\theta}{2}} r \right\|_2^2. \quad (105)$$

This Fourier representation also clarifies why the range $\theta \in (0, 1/2)$ is analytically sufficient for the later forcing estimate: it yields control strictly stronger than one full derivative, which is enough to control ∇r^ε in L^2 on the torus. The higher-order dissipation introduced here may also be interpreted, at the level of the network motivation developed in earlier chapters, as the continuum manifestation of multi-stage or nonlocal interaction effects. However, for the rigorous analysis in the present chapter, its essential role is that it furnishes a coercive term in the extended energy inequality.

4.4. Definition of the Extended Energy (Lyapunov Functional)

We now define the energy functional associated with the extended system by

$$E^\varepsilon(t) := \frac{1}{2} \|u^\varepsilon(t)\|_{L^2}^2 + \frac{\lambda \varepsilon}{2} \|r^\varepsilon(t)\|_{L^2}^2, \quad (106)$$

where $\lambda > 0$ is a fixed weight constant.

The factor ε in front of $\|r^\varepsilon\|_2^2$ is essential. It is dictated directly by the relaxation equation. Indeed, multiplying (102) by r^ε and taking the L^2 -inner product gives

$$\frac{\varepsilon}{2} \frac{d}{dt} \|r^\varepsilon\|_2^2 + \|r^\varepsilon\|_2^2 + \delta \left\| (-\Delta)^{\frac{1+\theta}{2}} r^\varepsilon \right\|_2^2 = 0. \quad (107)$$

Thus the natural Lyapunov functional is one that contains $\varepsilon \|r^\varepsilon\|_2^2$, not $\varepsilon^2 \|r^\varepsilon\|_2^2$. With this correct scaling, one obtains the fundamental estimate

$$\int_0^T \|r^\varepsilon(t)\|_2^2 dt \leq C \varepsilon, \quad (108)$$

which is precisely the quantitative smallness needed later to show that the forcing disappears in the relaxation limit.

Henceforth, we refer to E^ε as the extended energy (Lyapunov functional) of the extended system.

4.5. Basic Energy Identity

We now derive the coupled energy identity for the velocity field and the residual stress.

Formal calculation

We begin by multiplying the momentum Eq. (100) by u^ε and integrating over Ω . Using the periodic boundary condition and $\nabla \cdot u^\varepsilon = 0$, the transport term and pressure term vanish in the standard way, and we obtain

$$\frac{1}{2} \frac{d}{dt} \|u^\varepsilon\|_2^2 + \nu \|\nabla u^\varepsilon\|_2^2 = \langle F^\varepsilon, u^\varepsilon \rangle = \langle \nabla \cdot r^\varepsilon, u^\varepsilon \rangle. \quad (109)$$

Since r^ε is symmetric by Proposition 4.1, integration by parts yields

$$\langle \nabla \cdot r^\varepsilon, u^\varepsilon \rangle = -\langle r^\varepsilon, \nabla u^\varepsilon \rangle = -\langle r^\varepsilon, D(u^\varepsilon) \rangle. \quad (110)$$

The last equality follows because the antisymmetric part of ∇u^ε has zero contraction with the symmetric tensor r^ε .

Next, we multiply the relaxation Eq. (102) by r^ε and take the L^2 -inner product. Since

$$\langle \partial_t r^\varepsilon, r^\varepsilon \rangle = \frac{1}{2} \frac{d}{dt} \|r^\varepsilon\|_2^2, \quad (111)$$

and since (105) holds, we obtain

$$\frac{\varepsilon}{2} \frac{d}{dt} \|r^\varepsilon\|_2^2 + \|r^\varepsilon\|_2^2 + \delta \|(-\Delta)^{\frac{1+\theta}{2}} r^\varepsilon\|_2^2 = 0. \quad (112)$$

Multiplying (112) by λ and adding the result to (109), we arrive at

$$\frac{d}{dt} E^\varepsilon(t) + \nu \|\nabla u^\varepsilon\|_2^2 + \lambda \|r^\varepsilon\|_2^2 + \lambda \delta \|(-\Delta)^{\frac{1+\theta}{2}} r^\varepsilon\|_2^2 = -\langle r^\varepsilon, D(u^\varepsilon) \rangle, \quad (113)$$

where

$$E^\varepsilon(t) = \frac{1}{2} \|u^\varepsilon(t)\|_2^2 + \frac{\lambda \varepsilon}{2} \|r^\varepsilon(t)\|_2^2. \quad (114)$$

Thus the only remaining task is to absorb the cross term

$$-\langle r^\varepsilon, D(u^\varepsilon) \rangle, \quad (115)$$

into the dissipative terms on the left-hand side.

4.6. Control of the Cross Term and the Dissipation Inequality

For any $\eta > 0$, the Cauchy–Schwarz and Young inequalities give

$$|\langle r^\varepsilon, D(u^\varepsilon) \rangle| \leq \|r^\varepsilon\|_2 \|D(u^\varepsilon)\|_2 \leq \eta \|r^\varepsilon\|_2^2 + \frac{1}{4\eta} \|D(u^\varepsilon)\|_2^2. \quad (116)$$

Moreover, because $D(u)$ is the symmetric part of ∇u ,

$$\|D(u^\varepsilon)\|_2^2 \leq \|\nabla u^\varepsilon\|_2^2. \quad (117)$$

Substituting (116) and (117) into (113), we obtain

$$\frac{d}{dt} E^\varepsilon(t) + (\nu - \frac{1}{4\eta}) \|\nabla u^\varepsilon\|_2^2 + (\lambda - \eta) \|r^\varepsilon\|_2^2 + \lambda \delta \|(-\Delta)^{\frac{1+\theta}{2}} r^\varepsilon\|_2^2 \leq 0. \quad (118)$$

If η is chosen so that

$$\frac{1}{4\nu} < \eta < \lambda, \quad (119)$$

then all coefficients in (118) are strictly positive. For example, one may first choose $\lambda > 1/(4\nu)$, and then select η in the interval $(1/(4\nu), \lambda)$. This proves that the extended system possesses a genuine Lyapunov structure.

Theorem 4.1 (Extended energy inequality: Lyapunov structure).

With appropriate choices of $\lambda > 0$ and $\eta > 0$, the extended energy E^ε satisfies

$$E^\varepsilon(t) + c_1 \int_0^t \|\nabla u^\varepsilon(s)\|_2^2 ds + c_2 \int_0^t \|r^\varepsilon(s)\|_2^2 ds + c_3 \int_0^t \|(-\Delta)^{\frac{1+\theta}{2}} r^\varepsilon(s)\|_2^2 ds \leq E^\varepsilon(0), \quad (120)$$

where $c_1, c_2, c_3 > 0$ are constants independent of ε .

Proof.

In Eq. (118), define

$$c_1 := \nu - \frac{1}{4\eta}, c_2 := \lambda - \eta, c_3 := \lambda\delta. \quad (121)$$

Under condition (119), all three constants are positive. Integrating (118) in time then yields (120).

This theorem is the core estimate of the chapter. It shows that the velocity dissipation, the residual-stress dissipation, and the higher-order residual-stress regularization are all controlled simultaneously by a single extended energy.

4.7. Existence of Weak Solutions to the Extended System (Faedo–Galerkin)

We now justify the above formal calculations rigorously by constructing weak solutions of the extended system.

The domain is the periodic torus $\Omega = \mathbb{T}^3$. The initial data are assumed to satisfy

$$u_0 \in L_\sigma^2(\Omega), r_0^\varepsilon \in L_{\text{sym}}^2(\Omega; \mathbb{R}^{3 \times 3}). \quad (122)$$

The symmetry assumption on r_0^ε is consistent with Proposition 4.1 and is needed for the identity (110).

Step 1. Galerkin approximation

Let $\{w_k\}_{k \geq 1} \subset H_\sigma^1(\Omega)$ be an orthogonal basis of the divergence-free space, and let $\{W_k\}_{k \geq 1} \subset H^{1+\theta}(\Omega; \mathbb{R}^{3 \times 3}_{\text{sym}})$ be an orthogonal basis of the symmetric tensor space. Define the finite-dimensional spaces

$$V_N = \text{span}\{w_1, \dots, w_N\}, S_N = \text{span}\{W_1, \dots, W_N\}. \quad (123)$$

We set the approximate solutions as

$$u_N(t) = \sum_{k=1}^N a_k(t) w_k, r_N(t) = \sum_{k=1}^N b_k(t) W_k. \quad (124)$$

Using the Galerkin projection P_N onto V_N , we define the approximate system by

$$\partial_t u_N + P_N[(u_N \cdot \nabla) u_N] - \nu \Delta u_N = P_N(\nabla \cdot r_N), \quad (125)$$

$$\varepsilon \partial_t r_N + r_N + \delta(-\Delta)^{1+\theta} r_N = 0. \quad (126)$$

This is a finite-dimensional system of ordinary differential equations, and hence standard ODE theory yields local existence.

Step 2. Energy inequality at the Galerkin level

Because the approximation is finite-dimensional and r_N remains symmetric by construction, the calculations of Sections 4.5–4.6 are rigorous. In particular, one has

$$\langle \nabla \cdot r_N, u_N \rangle = -\langle r_N, D(u_N) \rangle. \quad (127)$$

Therefore, the Galerkin approximate energy satisfies

$$E_\varepsilon^N(t) + \int_0^t (c_1 \|\nabla u_N(s)\|_2^2 + c_2 \|r_N(s)\|_2^2 + c_3 \|(-\Delta)^{\frac{1+\theta}{2}} r_N(s)\|_2^2) ds \leq E_\varepsilon^N(0), \quad (128)$$

where

$$E_\varepsilon^N(t) = \frac{1}{2} \|u_N(t)\|_2^2 + \frac{\lambda\varepsilon}{2} \|r_N(t)\|_2^2. \quad (129)$$

Since this estimate is independent of N , the local solution extends globally in time.

Step 3. Compactness

From the uniform estimate (128), we have

$$u_N \text{ bounded in } L^\infty(0, T; L^2) \cap L^2(0, T; H^1), \quad (130)$$

and

$$r_N \text{ bounded in } L^\infty(0, T; L^2) \cap L^2(0, T; H^{1+\theta}). \quad (131)$$

Moreover, from Eqs. (125)–(126) we infer

$$\partial_t u_N \in L^{4/3}(0, T; H^{-1}), \quad (132)$$

and

$$\partial_t r_N \in L^2(0, T; H^{-(1+\theta)}). \quad (133)$$

Indeed, the nonlinear term $(u_N \cdot \nabla)u_N$ is bounded in $L^{4/3}(0, T; H^{-1})$ by the standard three-dimensional estimate based on $u_N \in L_t^\infty L_x^2 \cap L_t^2 H_x^1$, while the right-hand side of (126) is bounded in $L^2(0, T; H^{-(1+\theta)})$ because $r_N \in L^2(0, T; H^{1+\theta})$.

$$H^1(\mathbb{T}^3) \hookrightarrow L^2(\mathbb{T}^3) \hookrightarrow H^{-1}(\mathbb{T}^3), \quad (134)$$

Therefore, by the Aubin–Lions lemma applied to the compact chain after extraction of a subsequence,

$$u_N \rightarrow u^\varepsilon \text{ strongly in } L^2(0, T; L^2), \quad (135)$$

and

$$r_N \rightharpoonup r^\varepsilon \text{ weakly in } L^2(0, T; H^{1+\theta}). \quad (136)$$

Step 4. Passage to the limit

Passing to the limit $N \rightarrow \infty$, we obtain a weak solution $(u^\varepsilon, r^\varepsilon)$ of the extended system (100), (102). By weak lower semicontinuity, the energy inequality (120) remains valid in the limit. Hence there exists a weak solution satisfying

$$u^\varepsilon \in L^2(0, T; H_\sigma^1), r^\varepsilon \in L^2(0, T; H_{\text{sym}}^{1+\theta}), \quad (137)$$

together with the extended energy inequality of Theorem 4.1. Thus the Lyapunov structure derived above is rigorously justified.

4.8. Internal Forcing, L^2 – Boundedness, and ε -Smallness

We now derive the two key consequences of Theorem 4.1:

1. $F^\varepsilon = \nabla \cdot r^\varepsilon$ is bounded in $L^2(0, T; L^2)$, and
2. $F^\varepsilon \rightarrow 0$ in $L^2(0, T; H^{-1})$ as $\varepsilon \rightarrow 0$.

From Theorem 4.1, one obtains in particular

$$\int_0^T \| (-\Delta)^{\frac{1+\theta}{2}} r^\varepsilon(s) \|_2^2 ds \leq C. \quad (138)$$

Since $\theta > 0$, the Fourier representation on the periodic torus implies the coercive estimate

$$\| (-\Delta)^{\frac{1+\theta}{2}} r \|_2 \geq c \| \nabla r \|_2, \quad (139)$$

for all tensor fields r with zero mean. For a general tensor field, the constant mode does not contribute to ∇r , so the same estimate is sufficient for the forcing analysis. Consequently,

$$\int_0^T \| \nabla r^\varepsilon(s) \|_2^2 ds \leq C. \quad (140)$$

Using the definition $F^\varepsilon = \nabla \cdot r^\varepsilon$ together with the componentwise inequality $|\nabla \cdot r|^2 \leq |\nabla r|^2$, we obtain

$$\int_0^T \| F^\varepsilon(s) \|_2^2 ds = \int_0^T \| \nabla \cdot r^\varepsilon(s) \|_2^2 ds \leq \int_0^T \| \nabla r^\varepsilon(s) \|_2^2 ds \leq C. \quad (141)$$

Hence

$$F^\varepsilon \in L^2(0, T; L^2(\Omega)), \| F^\varepsilon \|_{L_t^2 L_x^2} \leq C. \quad (142)$$

This proves the L^2 -integrability of the internally generated forcing. However, what is truly needed later is not only boundedness, but vanishing in the relaxation limit. This follows directly from the relaxation equation. Integrating (112) from 0 to t , we obtain

$$\frac{\varepsilon}{2} \|r^\varepsilon(t)\|_2^2 + \int_0^t \|r^\varepsilon(s)\|_2^2 ds + \delta \int_0^t \|(-\Delta)^{\frac{1+\theta}{2}} r^\varepsilon(s)\|_2^2 ds = \frac{\varepsilon}{2} \|r_0^\varepsilon\|_2^2. \quad (143)$$

In particular,

$$\int_0^T \|r^\varepsilon(s)\|_2^2 ds \leq \frac{\varepsilon}{2} \|r_0^\varepsilon\|_2^2. \quad (144)$$

If the initial data r_0^ε are uniformly bounded in L^2 , then

$$\|r^\varepsilon\|_{L^2(0,T;L^2)} \leq \sqrt{\frac{\varepsilon}{2}} \|r_0^\varepsilon\|_2 \leq C\varepsilon^{1/2} \rightarrow 0 (\varepsilon \rightarrow 0). \quad (145)$$

Now let $\phi \in L^2(0,T;H^1(\Omega;\mathbb{R}^3))$. Under the periodic boundary condition,

$$\int_0^T \int_\Omega F^\varepsilon \cdot \phi dx dt = - \int_0^T \int_\Omega r^\varepsilon : \nabla \phi dx dt. \quad (146)$$

Hence, by Cauchy–Schwarz,

$$|\int_0^T \int_\Omega F^\varepsilon \cdot \phi dx dt| \leq \|r^\varepsilon\|_{L^2(0,T;L^2)} \|\nabla \phi\|_{L^2(0,T;L^2)} \rightarrow 0. \quad (147)$$

Therefore,

$$F^\varepsilon \rightarrow 0 \text{ in } L^2(0,T;H^{-1}(\Omega)). \quad (148)$$

In particular,

$$\nabla \cdot r^\varepsilon \rightarrow 0 \text{ in } \mathcal{D}'((0,T) \times \Omega). \quad (149)$$

This result is decisive: the forcing term in the extended momentum equation is not an external input but an internally generated quantity whose magnitude vanishes automatically as $\varepsilon \rightarrow 0$.

Theorem 4.2 (ε -smallness of the residual stress and vanishing of the forcing).

For weak solutions of the extended system (100), (102), one has

$$\int_0^T \|r^\varepsilon(t)\|_2^2 dt \leq C\varepsilon. \quad (150)$$

Hence

$$r^\varepsilon \rightarrow 0 \text{ in } L^2(0,T;L^2(\Omega)). \quad (151)$$

Furthermore, defining the forcing by

$$F^\varepsilon = \nabla \cdot r^\varepsilon, \quad (152)$$

it follows that

$$F^\varepsilon \rightarrow 0 \text{ in } L^2(0,T;H^{-1}(\Omega)). \quad (153)$$

In particular,

$$\nabla \cdot r^\varepsilon \rightarrow 0 \text{ in } \mathcal{D}'((0,T) \times \Omega). \quad (154)$$

Proof.

Estimate (150) is exactly (144). Eq. (151) follows from (145). For the forcing, identity (146) and estimate (147) show that $F^\varepsilon \rightarrow 0$ in $L^2(0,T;H^{-1})$, which proves (153). The distributional convergence (154) is immediate.

4.9. Conclusion of This Chapter and Connection to Chapter 5

In this chapter, we have

- defined the unknowns of the extended system and the residual stress r^ε in (94)–(97),
- rewritten the extended momentum equation into Navier–Stokes form in (100),
- introduced the relaxation Eqs. (101)–(102) for the residual stress,

- constructed the extended energy E^ε as a Lyapunov functional in (106),
- proved the extended energy inequality (120), and
- established both the $L_t^2 L_x^2$ -boundedness of the forcing in (142) and the ε -smallness of the residual stress in (150).

Most importantly, Theorem 4.2 shows that

$$r^\varepsilon \rightarrow 0 \text{ in } L^2(0, T; L^2), \quad F^\varepsilon = \nabla \cdot r^\varepsilon \rightarrow 0 \text{ in } L^2(0, T; H^{-1}). \quad (155)$$

Thus, when we pass to the limit in the extended momentum equation in Chapter 5, the forcing term disappears without the introduction of any additional assumption.

Accordingly, Chapter 5 will use the estimates established here to prove, in sequence,

- Lemma 5.1: global a priori bounds,
- Lemma 5.2: compactness via the Aubin–Lions theorem, and
- Lemma 5.3: convergence, as $\varepsilon \rightarrow 0$, to a Leray–Hopf weak solution of the incompressible Navier–Stokes equations.

In this sense, the present chapter closes the analytic gap between the structural parent equation of Chapter 3 and the weak-solution construction carried out in Chapter 5. The forcing term has now been fully internalized, estimated, and shown to vanish in the relaxation limit as a direct consequence of the Lyapunov structure of the extended system.

Chapter 5 Global a priori Bounds, Compactness, and the Relaxation Limit for the Extended System — Construction of a Leray–Hopf Weak Solution to the Three-Dimensional Navier–Stokes Equations —

Moreover, the energy estimate obtained in the previous section implies that $u_\varepsilon \in L^2(0, T; H^1(\mathbb{T}^3))$. On the other hand, the momentum equation provides a bound on the time derivative in a negative Sobolev space. Indeed, the right-hand side of the equation consists of spatial derivatives of quantities that are bounded in L^2 , which yields the estimate $\partial_t u_\varepsilon \in L^2(0, T; H^{-1}(\mathbb{T}^3))$. Because the Sobolev embedding

$$H^1(\mathbb{T}^3) \hookrightarrow L^2(\mathbb{T}^3), \quad (156)$$

is compact, while $L^2(\mathbb{T}^3) \hookrightarrow H^{-1}(\mathbb{T}^3)$ is continuous, the Aubin–Lions compactness theorem applies. Consequently, the sequence $\{u_\varepsilon\}$ is relatively compact in $L^2(0, T; L^2(\mathbb{T}^3))$, and therefore $u_\varepsilon \rightarrow u$ strongly in $L^2(0, T; L^2(\mathbb{T}^3))$. This strong compactness allows us to pass to the limit in the nonlinear term and thereby complete the construction of the weak solution.

In this chapter, taking as the starting point the extended energy inequality established in Chapter 4, we construct, from the family of weak solutions $\{u^\varepsilon\}_{\varepsilon>0}$ of the extended system, a Leray–Hopf weak solution of the three-dimensional incompressible Navier–Stokes equations in the relaxation limit $\varepsilon \rightarrow 0$.

Conceptually, the logic of this chapter consists of three stages. First, from the Lyapunov structure established in Chapter 4, we derive ε -uniform energy bounds for the velocity field together with quantitative dissipative bounds for the residual stress. This is the role of Lemma 5.1. Next, using these uniform bounds, we estimate the time derivative in a dual space and thereby bring the family $\{u^\varepsilon\}$ into a framework to which the Aubin–Lions compactness theorem applies. This yields strong convergence of u^ε in $L^2(0, T; L^2)$. This is the content of Lemma 5.2. Finally, we pass to the limit in each term of the weak formulation of the extended system: the linear terms pass by weak or strong convergence, the nonlinear term is handled by strong convergence, and the forcing term vanishes automatically by virtue of the residual-stress estimate proved in Chapter 4. This yields a Leray–Hopf weak solution of the incompressible Navier–Stokes equations. This is the content of Lemma 5.3.

The decisive point is that we do not impose, as an external assumption, the statement that “the residual stress vanishes in the limit.” Instead, the vanishing of the residual stress r^ε , and hence of the forcing $F^\varepsilon = \nabla \cdot r^\varepsilon$, is derived as an internal consequence of the Lyapunov structure of the extended

system established in Chapter 4. Accordingly, the relaxation-limit argument forms a closed theory derived directly from the design of the extended system itself.

5.1. Setting and Reformulation of the Extended System

Let the domain be $\Omega = \mathbb{T}^3$, and fix the time interval $[0, T]$. For the extended unknowns $(u^\varepsilon, p^\varepsilon, \tau^\varepsilon)$, define the residual stress by

$$r^\varepsilon := \tau^\varepsilon - 2\nu D(u^\varepsilon), D(u) := \frac{1}{2}(\nabla u + (\nabla u)^\top), \quad (157)$$

and define the forcing term by

$$F^\varepsilon := \nabla \cdot r^\varepsilon. \quad (158)$$

Then the extended momentum equation, as established in Chapter 4, may be rewritten in Navier–Stokes form as

$$\partial_t u^\varepsilon + (u^\varepsilon \cdot \nabla) u^\varepsilon - \nu \Delta u^\varepsilon + \nabla p^\varepsilon = F^\varepsilon, \nabla \cdot u^\varepsilon = 0. \quad (159)$$

Furthermore, by Theorem 4.1, the extended system satisfies the extended energy inequality

$$E^\varepsilon(t) + c_1 \int_0^t \|\nabla u^\varepsilon(s)\|_2^2 ds + c_2 \int_0^t \|r^\varepsilon(s)\|_2^2 ds + c_3 \int_0^t \|(-\Delta)^{\frac{1+\theta}{2}} r^\varepsilon(s)\|_2^2 ds \leq E^\varepsilon(0), \quad (160)$$

where $\theta \in (0, 1/2)$, $c_1, c_2, c_3 > 0$ are independent of ε , and

$$E^\varepsilon(t) = \frac{1}{2} \|u^\varepsilon(t)\|_2^2 + \frac{\lambda\varepsilon}{2} \|r^\varepsilon(t)\|_2^2. \quad (161)$$

The initial condition is assumed to be

$$u^\varepsilon(0) = u_0 \in L_\sigma^2(\Omega), \nabla \cdot u_0 = 0. \quad (162)$$

For the construction of weak solutions in the present section, $u_0 \in L^2$ is sufficient; no higher regularity is required.

We also recall the decisive residual-decay estimate established in Theorem 4.2 of Chapter 4:

$$\int_0^T \|r^\varepsilon(t)\|_2^2 dt \leq C\varepsilon. \quad (163)$$

This estimate is much stronger than mere ε -uniform boundedness. It is the key fact allowing the forcing term to disappear in the limit. Indeed, since

$$F^\varepsilon = \nabla \cdot r^\varepsilon, \quad (164)$$

integration by parts under the periodic boundary condition gives, for every

$$\phi \in L^2(0, T; H^1(\Omega; \mathbb{R}^3)), \quad (165)$$

the identity

$$\int_0^T \int_\Omega F^\varepsilon \cdot \phi dx dt = - \int_0^T \int_\Omega r^\varepsilon : \nabla \phi dx dt. \quad (166)$$

Hence

$$| \int_0^T \int_\Omega F^\varepsilon \cdot \phi dx dt | \leq \|r^\varepsilon\|_{L^2(0, T; L^2)} \|\nabla \phi\|_{L^2(0, T; L^2)}. \quad (167)$$

By (163),

$$\|r^\varepsilon\|_{L^2(0, T; L^2)} \leq C\varepsilon^{1/2} \rightarrow 0, \quad (168)$$

and therefore

$$F^\varepsilon \rightarrow 0 \text{ in } L^2(0, T; H^{-1}(\Omega)). \quad (169)$$

Thus the forcing term vanishes in the weak sense as an automatic consequence of the residual-stress dissipation. This is precisely the point that closes the relaxation-limit argument without any external assumption.

Lemma 5.1 (ε -uniform a priori bounds of Leray type)

For any weak solution of the extended system satisfying Eq. (160), there exists a constant $C = C(T, u_0, E^\varepsilon(0))$, independent of ε , such that

$$\sup_{0 \leq t \leq T} \|u^\varepsilon(t)\|_2^2 \leq C, \quad (170)$$

$$\int_0^T \|\nabla u^\varepsilon(t)\|_2^2 dt \leq C, \quad (171)$$

$$\int_0^T \|r^\varepsilon(t)\|_2^2 dt \leq C\varepsilon, \quad (172)$$

and

$$\int_0^T \|(-\Delta)^{\frac{1+\theta}{2}} r^\varepsilon(t)\|_2^2 dt \leq C. \quad (173)$$

Moreover, the forcing term $F^\varepsilon = \nabla \cdot r^\varepsilon$ satisfies

$$F^\varepsilon \in L^2(0, T; L^2(\Omega)), \int_0^T \|F^\varepsilon(t)\|_2^2 dt \leq C. \quad (174)$$

In particular, Eq. (172) implies

$$\|r^\varepsilon\|_{L^2(0,T;L^2)} \leq C\varepsilon^{1/2}. \quad (175)$$

Proof.

From the extended energy inequality (160), one has

$$E^\varepsilon(t) + \int_0^t (c_1 \|\nabla u^\varepsilon(s)\|_2^2 + c_2 \|r^\varepsilon(s)\|_2^2 + c_3 \|(-\Delta)^{\frac{1+\theta}{2}} r^\varepsilon(s)\|_2^2) ds \leq E^\varepsilon(0). \quad (176)$$

Since $E^\varepsilon(t) \geq \frac{1}{2} \|u^\varepsilon(t)\|_2^2$, this immediately yields

$$\sup_{0 \leq t \leq T} \|u^\varepsilon(t)\|_2^2 \leq C, \quad (177)$$

and

$$\int_0^T \|\nabla u^\varepsilon(t)\|_2^2 dt \leq C. \quad (178)$$

The same inequality also gives

$$\int_0^T \|r^\varepsilon(t)\|_2^2 dt \leq C, \quad (179)$$

and

$$\int_0^T \|(-\Delta)^{\frac{1+\theta}{2}} r^\varepsilon(t)\|_2^2 dt \leq C. \quad (180)$$

However, by Theorem 4.2 we have the stronger estimate (163), namely

$$\int_0^T \|r^\varepsilon(t)\|_2^2 dt \leq C\varepsilon, \quad (181)$$

which is precisely statement (172).

To prove the forcing bound, recall from Chapter 4 that, since $\theta > 0$,

$$\|(-\Delta)^{\frac{1+\theta}{2}} r^\varepsilon\|_2 \geq c \|\nabla r^\varepsilon\|_2, \quad (182)$$

modulo the harmless constant mode, which does not contribute to ∇r^ε or $\nabla \cdot r^\varepsilon$. Using $|\nabla \cdot r|^2 \leq |\nabla r|^2$, we obtain

$$\int_0^T \|F^\varepsilon(t)\|_2^2 dt = \int_0^T \|\nabla \cdot r^\varepsilon(t)\|_2^2 dt \leq \int_0^T \|\nabla r^\varepsilon(t)\|_2^2 dt \leq C, \quad (183)$$

which proves (174). Finally, (175) follows immediately from (181).

5.2. Weak Formulation and Estimate of the Time Derivative

Preparation for Compactness

To apply the Aubin–Lions compactness theorem, it is necessary to obtain not only uniform spatial bounds but also a uniform estimate of the time derivative $\partial_t u^\varepsilon$ in a suitable dual space. For this purpose, we eliminate the pressure term from Eq. (159) by means of the Leray projection P , the L^2 -orthogonal projection onto divergence-free vector fields. Since $P\nabla p^\varepsilon = 0$, we may write

$$\partial_t u^\varepsilon = -P((u^\varepsilon \cdot \nabla)u^\varepsilon) + \nu \Delta u^\varepsilon + PF^\varepsilon. \quad (184)$$

The advantage of this representation is that each term on the right-hand side can be estimated naturally in $H^{-1}(\Omega)$.

To state the compactness framework explicitly, we recall the continuous and compact embeddings

$$H^1(\mathbb{T}^3) \hookrightarrow L^2(\mathbb{T}^3) \hookrightarrow H^{-1}(\mathbb{T}^3). \quad (185)$$

The first arrow is compact, and the second is continuous. Therefore, once we prove that

$$u^\varepsilon \text{ is bounded in } L^2(0, T; H^1(\Omega)), \quad (186)$$

and

$$\partial_t u^\varepsilon \text{ is bounded in } L^{4/3}(0, T; H^{-1}(\Omega)), \quad (187)$$

the Aubin–Lions theorem applies directly.

Lemma 5.2 (Strong convergence via Aubin–Lions)

Under the bounds of Lemma 5.1, there exists a subsequence, still denoted by the same symbol, such that

$$u^\varepsilon \rightharpoonup u \text{ in } L^2(0, T; H^1(\Omega)), \quad (188)$$

$$u^\varepsilon \rightharpoonup^* u \text{ in } L^\infty(0, T; L^2(\Omega)), \quad (189)$$

and

$$u^\varepsilon \rightarrow u \text{ strongly in } L^2(0, T; L^2(\Omega)). \quad (190)$$

Proof.

Step 1. Uniform spatial bounds.

By Lemma 5.1,

$$u^\varepsilon \text{ is bounded in } L^\infty(0, T; L^2(\Omega)) \cap L^2(0, T; H^1(\Omega)). \quad (191)$$

Therefore, by the Banach–Alaoglu theorem, after extraction of a subsequence we obtain the weak and weak- convergences (188) and (189).

Step 2. Estimate of the time derivative.

We estimate each term on the right-hand side of Eq. (184) in $L^{4/3}(0, T; H^{-1}(\Omega))$.

First, for the viscous term, $\Delta u^\varepsilon \in H^{-1}$ and

$$\|\Delta u^\varepsilon\|_{H^{-1}} \leq \|\nabla u^\varepsilon\|_2. \quad (192)$$

Hence

$$\nu \Delta u^\varepsilon \in L^2(0, T; H^{-1}(\Omega)) \subset L^{4/3}(0, T; H^{-1}(\Omega)). \quad (193)$$

Next, by Lemma 5.1, $F^\varepsilon \in L^2(0, T; L^2(\Omega))$. Since

$$L^2(\Omega) \hookrightarrow H^{-1}(\Omega), \quad (194)$$

continuously, we have

$$PF^\varepsilon \in L^2(0, T; H^{-1}(\Omega)) \subset L^{4/3}(0, T; H^{-1}(\Omega)). \quad (195)$$

Finally, we estimate the nonlinear term. On the three-dimensional torus, the Sobolev embedding $H^1(\Omega) \hookrightarrow L^6(\Omega)$ holds. Together with Hölder interpolation and the Gagliardo–Nirenberg inequality, this yields

$$\|u^\varepsilon\|_{L^4(0, T; L^3(\Omega))} \leq C \|u^\varepsilon\|_{L^\infty(0, T; L^2)}^{1/2} \|u^\varepsilon\|_{L^2(0, T; H^1)}^{1/2}. \quad (196)$$

Hence $u^\varepsilon \in L_t^4 L_x^3$ uniformly in ε . For any $\phi \in H^1(\Omega; \mathbb{R}^3)$ with $\nabla \cdot \phi = 0$,

$$\langle (u^\varepsilon \cdot \nabla) u^\varepsilon, \phi \rangle = - \int_\Omega (u^\varepsilon \otimes u^\varepsilon) : \nabla \phi \, dx, \quad (197)$$

and therefore

$$\|(u^\varepsilon \cdot \nabla) u^\varepsilon\|_{H^{-1}} \leq C \|u^\varepsilon \otimes u^\varepsilon\|_{L^{3/2}} \leq C \|u^\varepsilon\|_{L^3}^2. \quad (198)$$

Since $u^\varepsilon \in L_t^4 L_x^3$, it follows that

$$(u^\varepsilon \cdot \nabla) u^\varepsilon \in L^{4/3}(0, T; H^{-1}(\Omega)). \quad (199)$$

Collecting (193), (195), and (199), we conclude that

$$\partial_t u^\varepsilon \text{ is bounded in } L^{4/3}(0, T; H^{-1}(\Omega)). \quad (200)$$

Step 3. Application of Aubin–Lions.

Applying the Aubin–Lions theorem to the compact chain (185), together with the spatial bound (191) and the time-derivative bound (200), yields

$$u^\varepsilon \rightarrow u \text{ strongly in } L^2(0, T; L^2(\Omega)). \quad (201)$$

This is precisely statement (190).

The significance of the assumption $\theta \in (0, 1/2)$ should also be noted. This range is sufficient to ensure that $(-\Delta)^{(1+\theta)/2} r^\varepsilon \in L^2$ controls ∇r^ε in L^2 , which in turn guarantees that the forcing term belongs to the appropriate Sobolev space. On the other hand, the compactness argument for u^ε itself relies only on the standard chain $H^1 \hookrightarrow L^2 \hookrightarrow H^{-1}$, so there is no need to impose a stronger condition such as $\theta \geq 1/2$.

5.3. The Limit $\varepsilon \rightarrow 0$ and Construction of a Leray–Hopf Weak Solution

We now turn to the relaxation limit. The aim is to prove that the limit of the family u^ε is a Leray–Hopf weak solution of the standard three-dimensional incompressible Navier–Stokes equations without forcing. To do this, we must verify the following four points:

- (i) passage to the limit in the linear terms,
- (ii) passage to the limit in the nonlinear term,
- (iii) vanishing of the forcing term, and
- (iv) inheritance of the energy inequality.

Definition 5.1 (Leray–Hopf weak solution)

A function u is called a Leray–Hopf weak solution of the three-dimensional incompressible Navier–Stokes equations on $\Omega = \mathbb{T}^3$ with initial datum $u_0 \in L_\sigma^2(\Omega)$ if:

$$u \in L^\infty(0, T; L^2_\sigma(\Omega)) \cap L^2(0, T; H^1_\sigma(\Omega)), \quad (202)$$

and for every divergence-free test function

$$\phi \in C_c^\infty([0, T] \times \Omega; \mathbb{R}^3), \nabla \cdot \phi = 0, \quad (203)$$

one has

$$\int_0^T \int_\Omega (-u \cdot \partial_t \phi + (u \otimes u) : \nabla \phi + \nu \nabla u : \nabla \phi) dx dt = \int_\Omega u_0 \cdot \phi(0) dx, \quad (204)$$

and the energy inequality

$$\frac{1}{2} \|u(t)\|_2^2 + \nu \int_0^t \|\nabla u(s)\|_2^2 ds \leq \frac{1}{2} \|u_0\|_2^2, \quad (205)$$

holds for almost every $t \in [0, T]$. This is the standard Leray–Hopf notion adapted to the periodic setting.

Lemma 5.3 (Relaxation limit: existence of a Leray–Hopf weak solution)

Under the assumptions of Lemmas 5.1 and 5.2, there exists a subsequence, still denoted by u^ε , such that $u^\varepsilon \rightarrow u$ as $\varepsilon \rightarrow 0$, and the limit u is a Leray–Hopf weak solution of the three-dimensional incompressible Navier–Stokes equations in the sense of Definition 5.1.

Proof.

Step 1. Weak formulation of the extended system.

Let $\phi \in C_c^\infty([0, T] \times \Omega; \mathbb{R}^3)$ be divergence-free. Multiplying Eq. (159) by ϕ and integrating over $(0, T) \times \Omega$, then integrating by parts in time and space, we obtain

$$\int_0^T \int_\Omega (-u^\varepsilon \cdot \partial_t \phi + (u^\varepsilon \otimes u^\varepsilon) : \nabla \phi + \nu \nabla u^\varepsilon : \nabla \phi) dx dt = \int_\Omega u_0 \cdot \phi(0) dx + \int_0^T \int_\Omega F^\varepsilon \cdot \phi dx dt. \quad (206)$$

The pressure term disappears because $\nabla \cdot \phi = 0$.

Step 2. Limit of the linear terms.

By Lemma 5.2, $u^\varepsilon \rightarrow u$ strongly in $L^2(0, T; L^2)$ and $u^\varepsilon \rightharpoonup u$ weakly in $L^2(0, T; H^1)$. Since $\partial_t \phi$ and $\nabla \phi$ are smooth and bounded, it follows that

$$\int_0^T \int_\Omega u^\varepsilon \cdot \partial_t \phi dx dt \rightarrow \int_0^T \int_\Omega u \cdot \partial_t \phi dx dt, \quad (207)$$

and

$$\int_0^T \int_\Omega \nu \nabla u^\varepsilon : \nabla \phi dx dt \rightarrow \int_0^T \int_\Omega \nu \nabla u : \nabla \phi dx dt. \quad (208)$$

Step 3. Limit of the nonlinear term.

To pass to the limit in the quadratic term, strong convergence is essential. Using (190), we have

$$u^\varepsilon \otimes u^\varepsilon \rightarrow u \otimes u \text{ in } L^1(0, T; L^1(\Omega)). \quad (209)$$

Indeed,

$$u^\varepsilon \otimes u^\varepsilon - u \otimes u = (u^\varepsilon - u) \otimes u^\varepsilon + u \otimes (u^\varepsilon - u), \quad (210)$$

and therefore

$$\|u^\varepsilon \otimes u^\varepsilon - u \otimes u\|_{L^1_{t,x}} \leq \|u^\varepsilon - u\|_{L^2_{t,x}} (\|u^\varepsilon\|_{L^2_{t,x}} + \|u\|_{L^2_{t,x}}) \rightarrow 0. \quad (211)$$

Consequently,

$$\int_0^T \int_\Omega (u^\varepsilon \otimes u^\varepsilon) : \nabla \phi dx dt \rightarrow \int_0^T \int_\Omega (u \otimes u) : \nabla \phi dx dt. \quad (212)$$

Step 4. Limit of the forcing term.

By Eq. (169), we already know that

$$F^\varepsilon \rightarrow 0 \text{ in } L^2(0, T; H^{-1}(\Omega)). \quad (213)$$

Since $\phi \in L^2(0, T; H^1(\Omega))$, it follows immediately that

$$\int_0^T \int_\Omega F^\varepsilon \cdot \phi dx dt \rightarrow 0. \quad (214)$$

Equivalently, one may write, using (166),

$$\int_0^T \int_\Omega F^\varepsilon \cdot \phi dx dt = - \int_0^T \int_\Omega r^\varepsilon : \nabla \phi dx dt, \quad (215)$$

and the right-hand side tends to zero by (168). Thus the forcing term disappears in the limit without any additional assumption. Returning to the weak formulation (206) and letting $\varepsilon \rightarrow 0$, we obtain

$$\int_0^T \int_\Omega (-u \cdot \partial_t \phi + (u \otimes u) : \nabla \phi + \nu \nabla u : \nabla \phi) dx dt = \int_\Omega u_0 \cdot \phi(0) dx. \quad (216)$$

This is exactly the weak formulation (204) in Definition 5.1.

Step 5. Energy inequality.

It remains to show that the limit u satisfies the Leray–Hopf energy inequality. By Lemma 5.1 and weak lower semicontinuity, for almost every $t \in [0, T]$,

$$\|u(t)\|_2^2 \leq \liminf_{\varepsilon \rightarrow 0} \|u^\varepsilon(t)\|_2^2, \quad (217)$$

and

$$\int_0^t \|\nabla u(s)\|_2^2 ds \leq \liminf_{\varepsilon \rightarrow 0} \int_0^t \|\nabla u^\varepsilon(s)\|_2^2 ds. \quad (218)$$

From Chapter 4, the extended energy inequality gives

$$\frac{1}{2} \|u^\varepsilon(t)\|_2^2 + \nu \int_0^t \|\nabla u^\varepsilon(s)\|_2^2 ds \leq E^\varepsilon(0), \quad (219)$$

because all residual-stress terms are nonnegative. Since

$$E^\varepsilon(0) = \frac{1}{2} \|u_0\|_2^2 + \frac{\lambda\varepsilon}{2} \|r_0^\varepsilon\|_2^2 \rightarrow \frac{1}{2} \|u_0\|_2^2, \quad (220)$$

assuming r_0^ε is uniformly bounded in L^2 , we may pass to the limit in (219) using (217)–(218) to obtain

$$\frac{1}{2} \|u(t)\|_2^2 + \nu \int_0^t \|\nabla u(s)\|_2^2 ds \leq \frac{1}{2} \|u_0\|_2^2. \quad (221)$$

This is exactly the energy inequality (205). Therefore u is a Leray–Hopf weak solution.

5.4. Fourier Analysis of the Relaxation Equation and the Mechanism of Residual-Stress Decay

The previous section has already established the vanishing of the forcing term directly from the energy estimate

$$\int_0^T \|r^\varepsilon(t)\|_2^2 dt \leq C\varepsilon. \quad (222)$$

This estimate alone is sufficient for the weak-solution construction. Nevertheless, the extended system introduced in the present paper has been designed not merely as an analytic device, but as a model whose internal structure suppresses the residual stress itself. To make this point explicit, we now analyze the relaxation equation in Fourier space.

The purpose of this section is twofold. First, it clarifies how the higher-order dissipative operator $(-\Delta)^{1+\theta}$ damps the high-frequency components of the residual stress. Second, it shows, at the modal level, why the estimate $r^\varepsilon \rightarrow 0$ follows from the internal mechanism of the extended system rather than from any external assumption.

5.4.1. Coercivity of $(-\Delta)^{1+\theta}$

The residual stress satisfies the relaxation equation

$$\varepsilon \partial_t r^\varepsilon + r^\varepsilon + \delta(-\Delta)^{1+\theta} r^\varepsilon = 0. \quad (223)$$

Since the domain is $\Omega = \mathbb{T}^3$, we may consider the periodic Fourier expansion

$$r^\varepsilon(x, t) = \sum_{k \in \mathbb{Z}^3} \widehat{r}^\varepsilon(k, t) e^{ik \cdot x}. \quad (224)$$

Then the fractional Laplacian acts as

$$(-\Delta)^{1+\theta} \widehat{r}^\varepsilon(k, t) = |k|^{2(1+\theta)} \widehat{r}^\varepsilon(k, t). \quad (225)$$

Accordingly,

$$\langle (-\Delta)^{1+\theta} r^\varepsilon, r^\varepsilon \rangle = \sum_{k \in \mathbb{Z}^3} |k|^{2(1+\theta)} |\widehat{r}^\varepsilon(k, t)|^2 = \|(-\Delta)^{\frac{1+\theta}{2}} r^\varepsilon\|_2^2. \quad (226)$$

In particular, since $\theta > 0$,

$$\|(-\Delta)^{\frac{1+\theta}{2}} r\|_2^2 \geq c \|\nabla r\|_2^2, \quad (227)$$

again modulo the constant mode, which is irrelevant for the forcing $\nabla \cdot r$. Thus, the extended system possesses not merely L^2 -dissipation, but a genuinely high-frequency dissipative structure controlling the gradient of the residual stress.

5.4.2. Decay of individual Fourier modes

Writing Eq. (223) mode by mode in Fourier space gives

$$\varepsilon \partial_t \widehat{r}^\varepsilon(k, t) + \alpha_k \widehat{r}^\varepsilon(k, t) = 0, \quad (228)$$

where

$$\alpha_k := 1 + \delta |k|^{2(1+\theta)} \geq 1. \quad (229)$$

This first-order ODE is solved explicitly by

$$\widehat{r}^\varepsilon(k, t) = e^{-\alpha_k t / \varepsilon} \widehat{r}^\varepsilon(k, 0). \quad (230)$$

This formula makes the decay mechanism entirely explicit. Each Fourier mode decays exponentially on the time scale

$$t \sim \frac{\varepsilon}{\alpha_k}, \quad (231)$$

and for high frequencies,

$$\alpha_k \sim |k|^{2(1+\theta)}, \quad (232)$$

so high-wavenumber modes decay even more rapidly. Thus, the extended system preferentially removes highly oscillatory residual-stress components. There is an initial layer of width $O(\varepsilon)$ near $t = 0$, but once this layer is passed, each mode becomes rapidly small.

5.4.3. Consequence for the weak-solution construction

What is ultimately required for the weak-solution construction is

$$r^\varepsilon \rightarrow 0 \text{ strongly in } L^2(0, T; L^2(\Omega)). \quad (233)$$

This convergence is not an additional assumption. By Theorem 4.2, one already has

$$\int_0^T \|r^\varepsilon(t)\|_2^2 dt \leq C\varepsilon, \quad (234)$$

Hence

$$\|r^\varepsilon\|_{L^2(0, T; L^2)} \leq C\varepsilon^{1/2} \rightarrow 0. \quad (235)$$

Since the forcing is given by

$$F^\varepsilon = \nabla \cdot r^\varepsilon, \quad (236)$$

it follows for any test function ϕ that

$$\int_0^T \int_\Omega F^\varepsilon \cdot \phi dx dt = - \int_0^T \int_\Omega r^\varepsilon : \nabla \phi dx dt, \quad (237)$$

and therefore

$$| \int_0^T \int_\Omega r^\varepsilon : \nabla \phi dx dt | \leq \|r^\varepsilon\|_{L^2(0, T; L^2)} \|\nabla \phi\|_{L^2(0, T; L^2)} \rightarrow 0. \quad (238)$$

Hence

$$F^\varepsilon \rightarrow 0 \text{ in } \mathcal{D}'((0, T) \times \Omega), \quad (239)$$

consistently with the stronger H^{-1} -convergence already established in (213).

5.4.4. Position and significance of the present section

The foregoing shows that the limit treatment of the forcing term already closes completely through:

- Lemma 5.1: uniform bounds for the velocity field,
- Lemma 5.2: strong convergence via Aubin–Lions,
- Lemma 5.3: passage to the limit in the weak formulation, and
- Theorem 4.2 of Chapter 4.

The Fourier analysis presented here provides an alternative perspective on the same conclusion and makes explicit that the extended system possesses an intrinsic mechanism that dissipates the residual stress by itself.

Accordingly, the limit

$$\varepsilon \rightarrow 0, \quad (240)$$

leads to the appearance of a Leray–Hopf weak solution of the incompressible Navier–Stokes equations not as a consequence of an externally imposed assumption, but as a direct consequence of the internal Lyapunov structure of the extended system.

Chapter 6 Conclusions and Future Perspectives

6.1. Principal Results of the Present Study

In the present work, we have proposed a framework in which the governing equations of fluid motion are not introduced from the conventional standpoint that the continuum model is assumed a priori, but are instead approached from the viewpoint that continuum fluid equations should arise from structurally admissible interacting systems governed by conservation and dissipation. The point of departure is a network-based master equation in which the state at node i is represented by

$$U_i = (\rho_i, m_i, e_i), \quad (241)$$

and the exchange between neighboring nodes is described abstractly through interfacial fluxes of conservative and dissipative type.

The logical development of the paper has been organized into the following three stages.

(1) Structural formulation of the parent equation (Chapters 2–3).

In Chapters 2 and 3, we identified the admissible structure of a network master equation under two basic principles: conservation laws and the second law of thermodynamics. The reversible part was organized through antisymmetric interface fluxes consistent with Euler-type transport, while the dissipative part was constructed through non-negative interface quadratic forms corresponding to viscosity and heat conduction. In this way, the parent equation was endowed with a thermodynamically consistent discrete structure. Under suitable geometric and scaling assumptions, discussed in Appendix A, this framework is compatible with a conservation-law-type continuum description of the form

$$\partial_t U + \nabla \cdot F(U) = 0. \quad (242)$$

It is important, however, to state precisely the role of this step. In the present paper, the network-to-continuum passage is organized as a structural correspondence under explicit assumptions, not as the main compactness theorem proved in full detail in the body of the paper. Its function is to provide the structural background and motivation for the continuum relaxation framework analyzed rigorously in Chapters 4 and 5.

(2) Construction of the relaxation-extended continuum system (Chapter 4).

Rather than appending viscous effects and nonequilibrium stresses in an ad hoc manner, we introduced an extended system with an auxiliary variable, namely the residual stress r^ε , so as to preserve the design principles inherited from the parent equation—conservation together with dissipation. The residual stress is governed by the relaxation equation

$$\varepsilon \partial_t r^\varepsilon + r^\varepsilon + \delta(-\Delta)^{1+\theta} r^\varepsilon = 0, \quad \delta > 0, \theta \in (0, 1/2). \quad (243)$$

This equation contains two key mechanisms:

- Maxwell-type relaxation: $\varepsilon \partial_t r^\varepsilon + r^\varepsilon$,
- high-frequency stabilization: $\delta(-\Delta)^{1+\theta} r^\varepsilon$.

The crucial point is that Chapter 4 established an extended energy inequality (Lyapunov structure) for the coupled system. As a consequence, the residual stress was shown to satisfy the quantitative estimate

$$\int_0^T \|r^\varepsilon(t)\|_2^2 dt \leq C\varepsilon, \quad (244)$$

and therefore, to vanish in the relaxation limit. This is the key fact that closes the later limit argument without any external assumption.

(3) Relaxation limit $\varepsilon \rightarrow 0$ and weak-solution construction (Chapter 5).

Finally, for the solution family $\{u^\varepsilon\}$ of the extended continuum system, we combined

- ε -uniform a priori bounds of Leray type,
- strong convergence in $L^2_{t,x}$ via the Aubin–Lions compactness theorem, and
- vanishing of the internally generated forcing term $F^\varepsilon = \nabla \cdot r^\varepsilon$, to prove that the limit velocity field is a Leray–Hopf weak solution of the three-dimensional incompressible Navier–Stokes equations.

The most important feature here is that the disappearance of the forcing is not postulated as an additional hypothesis. Instead, Chapter 4 proves that

$$\|r^\varepsilon\|_{L^2(0,T;L^2(\Omega))} \leq C\varepsilon^{1/2}, \quad (245)$$

and hence

$$F^\varepsilon = \nabla \cdot r^\varepsilon \rightarrow 0 \text{ in } L^2(0,T;H^{-1}(\Omega)), \quad (246)$$

and in particular in the sense of distributions. As a result, the limiting equation takes the standard unforced incompressible form

$$\partial_t u + (u \cdot \nabla)u + \nabla p = \nu \Delta u, \quad \nabla \cdot u = 0. \quad (247)$$

Thus, the mathematically rigorous conclusion of the present paper is the construction of a Leray–Hopf weak solution of the incompressible Navier–Stokes equations as the relaxation limit of a thermodynamically consistent continuum extended system.

6.2. Main Theorem

Collecting the results proved in the body of the paper, we obtain the following principal conclusion.

Theorem 6.1 Relaxation limit on $\mathbb{T}^3$ yields a Leray–Hopf weak solution

Consider the relaxation-extended continuum system on the periodic torus $\Omega = \mathbb{T}^3$, consisting of the incompressible momentum equation coupled with the residual-stress relaxation equation introduced in Chapter 4. Assume that the initial data satisfy

$$u_0 \in L^2_\sigma(\Omega), \quad r_0^\varepsilon \in L^2_{\text{sym}}(\Omega; \mathbb{R}^{3 \times 3}), \quad (248)$$

with r_0^ε uniformly bounded in L^2 . Then the associated weak solutions $(u^\varepsilon, r^\varepsilon)$ satisfy the extended energy inequality, the residual-stress estimate

$$\int_0^T \|r^\varepsilon(t)\|_2^2 dt \leq C\varepsilon, \quad (249)$$

and, after extraction of a subsequence, the velocity field u^ε converges to a limit u such that

$$u \in L^\infty(0, T; L^2_\sigma(\Omega)) \cap L^2(0, T; H^1_\sigma(\Omega)), \quad (250)$$

and u is a Leray–Hopf weak solution of the incompressible Navier–Stokes equations

$$\partial_t u + (u \cdot \nabla)u + \nabla p = \nu \Delta u, \quad \nabla \cdot u = 0. \quad (251)$$

In this process, the forcing term is not assumed to vanish externally, but follows directly from the quantitative decay of the residual stress:

$$F^\varepsilon = \nabla \cdot r^\varepsilon \rightarrow 0 \text{ in } L^2(0, T; H^{-1}(\Omega)). \quad (252)$$

This theorem is the principal rigorous result established in the present paper.

To avoid any ambiguity, we emphasize once more that the theorem proven in the body of the paper concerns the relaxation limit of the continuum extended system. The network-to-continuum correspondence discussed in Chapters 2–3 and Appendix A provides the structural framework motivating this extended system, but is not claimed here as the same level of rigorously closed compactness theorem.

6.3. Mathematical Significance of the Present Study

The mathematical significance of the present work may be summarized in the following three points.

(1) Structural derivation beyond the a priori continuum hypothesis.

In conventional fluid mechanics, the continuum hypothesis and the constitutive assumption for the viscous stress are postulated first, and the governing equations are then written down. By contrast, the present study begins from a network-based interacting system organized by conservation and entropy production, and then identifies a continuum extended system whose relaxation limit reproduces the incompressible Navier–Stokes equations. In this sense, the Navier–Stokes equations are no longer treated solely as assumed continuum equations, but as limits compatible with an admissible interaction structure.

(2) Thermodynamic consistency and closure of the relaxation limit.

The extended system was designed so as to possess coercive dissipation both for the velocity field and for the residual stress. As a consequence, the classical Leray–Hopf energy inequality

$$\frac{1}{2} \|u(t)\|_2^2 + \nu \int_0^t \|\nabla u(s)\|_2^2 ds \leq \frac{1}{2} \|u_0\|_2^2, \quad (253)$$

is inherited in the limit from the extended energy inequality. Most importantly, the dissipation of the residual stress functions as the mechanism by which the forcing disappears in the limit, and

this disappearance is guaranteed quantitatively. Owing to this fact, the relaxation limit becomes a closed analysis independent of any external assumption of vanishing residuals.

(3) Natural construction of weak solutions through the extended system.

Since the following ingredients are obtained simultaneously in the course of the limit:

- ε -uniform a priori bounds,
- Aubin–Lions compactness, and
- vanishing of the internally generated forcing,

the existence of Leray–Hopf weak solutions is derived naturally as a limit of the extended system, while remaining fully consistent with Leray’s classical theory.

6.4. Relation to Navier–Stokes Theory

The derivational route presented in this paper occupies the following position relative to existing theories.

- Leray (1934) established the existence of weak solutions by taking the incompressible Navier–Stokes equations themselves as the starting point.
- Kinetic theory derives hydrodynamic equations from microscopic models such as the Boltzmann equation through scaling limits.
- The present work begins from a network-based structural viewpoint, constructs a thermodynamically consistent continuum relaxation extension, and proves that its relaxation limit yields a Leray–Hopf weak solution of the incompressible Navier–Stokes equations.

Accordingly, the derivational pathway relevant to the rigorous part of the present paper is

$$\begin{aligned} &\text{network-based structural framework} \rightarrow \text{continuum relaxation-extended system} \\ &\rightarrow \text{Leray–Hopf weak solution of Navier–Stokes.} \end{aligned} \quad (254)$$

This formulation is deliberately more precise than the overly compressed slogan “network dynamics $\rightarrow$ continuum PDE $\rightarrow$ Navier–Stokes.” The contribution of the present paper lies not in claiming that every step of that entire route has been proved at the same level of rigor, but in showing that the final relaxation-limit step can be closed rigorously by means of a Lyapunov structure, compactness, and internal vanishing of the forcing.

In this sense, the present work may be viewed as complementing both classical Leray theory and traditional hydrodynamic-limit approaches. It does not replace the kinetic derivation of transport coefficients, nor does it supersede the classical weak-existence theory. Rather, it provides a new structural route in which thermodynamic admissibility and nonequilibrium relaxation are placed at the center of the analysis.

6.5. Future Extensions

The framework developed in the present paper admits several natural directions of extension.

(1) Full discrete-to-continuum compactness theory.

A first and most direct extension is to upgrade the structural correspondence of Appendix A into a fully rigorous theorem for discrete solution families. This would require establishing discrete a priori estimates, time-regularity bounds, and compactness for the network system itself, together with a careful treatment of mesh geometry, coefficient scaling, and possibly time discretization. Such a development would complete, at the discrete level, the route that in the present paper is closed rigorously only after passage to the continuum extended system.

(2) Nonlocal fluid equations.

If the network coupling does not collapse to local nearest-neighbor structure in the limit and nonlocality persists, then the limiting equation may take the form of an integro-differential equation such as

$$\partial_t U(x) = \int_{\Omega} K(x, y) F(U(x), U(y)) dy. \quad (255)$$

In this direction, the structure of the kernel K , together with the conservation and dissipation constraints, becomes the central factor governing solvability and asymptotic behavior.

(3) Connection with turbulence models.

Formally, the residual stress r^ε plays a role analogous to a subgrid-scale stress in LES or a nonequilibrium stress in regularized turbulence models. The framework proposed here does not introduce this stress as a phenomenological closure assumption, but as an extended variable controlled by a thermodynamically consistent dissipative law. It may therefore provide a new design principle for turbulence closures grounded in entropy production and relaxation structure.

(4) Graph fluid dynamics and meshless CFD.

By interpreting the fluid equation on a discrete network as

$$\frac{dU_i}{dt} = \sum_j A_{ij} F(U_i, U_j), \quad (256)$$

one may expect applications to flows on complex graphs, meshless discretizations, and point-cloud-based CFD formulations. In particular, the geometric and scaling assumptions discussed in Appendix A may serve directly as structural conditions under which discrete and continuum models remain consistently linked.

6.6. General Summary

In the present study, starting from a network-based master-equation viewpoint for interacting systems, we developed a structural route toward incompressible fluid dynamics through the following three-stage program:

- formulation of a thermodynamically admissible parent equation through conservation and entropy production,
- construction of a continuum relaxation-extended system together with an extended energy inequality, and
- construction of a Leray–Hopf weak solution of the incompressible Navier–Stokes equations through the relaxation limit $\varepsilon \rightarrow 0$.

Most notably, the disappearance of the forcing term was not postulated as an external assumption. Instead, it was derived quantitatively from the dissipation of the residual stress:

$$\|r^\varepsilon\|_{L^2(0,T;L^2)} \leq C\varepsilon^{\frac{1}{2}}, \nabla \cdot r^\varepsilon \rightarrow 0 \text{ in } L^2(0,T;H^{-1}(\Omega)). \quad (257)$$

Because of this fact, the weak-solution construction is completed as a closed logical argument rather than as an assumption-dependent passage.

The principal rigorous conclusion of the paper is therefore the following: the relaxation-extended continuum system constructed from the network-based thermodynamic design admits a relaxation limit that yields a Leray–Hopf weak solution of the incompressible Navier–Stokes equations on the periodic torus. This provides a new perspective on the structural derivation of fluid equations—one in which admissible interaction architecture, dissipation, and nonequilibrium relaxation are treated as the organizing principles of the analysis.

Nomenclature

Roman symbols

A_{ij} — Coupling strength (adjacency matrix element) between nodes i and j

$D(u) = \frac{1}{2}(\nabla u + (\nabla u)^T)$ — Rate-of-deformation tensor

$D(U_i, U_j)$ — Dissipative interfacial flux between nodes

$E_i^{tot} = k_i + e_i$ — Total energy density at node i

$F(U_i, U_j)$ — Conservative exchange flux between nodes

$f = \nabla \cdot r$ — Forcing term generated by the residual stress

G — Interaction graph consisting of nodes and edges

I — Identity tensor

i, j — Node indices

$k_i = \frac{1}{2} \rho_i |u_i|^2$ — Kinetic energy density

$m_i = \rho_i u_i$ — Momentum density

$N(i)$ — Set of nodes adjacent to node i

n_{ij} — Unit normal vector at the interface between nodes i and j

p — Thermodynamic pressure

r — Residual stress tensor in the relaxation-extended system

$r_{ij} = |x_i - x_j|$ — Distance between nodes i and j

s — Specific entropy

S — Total entropy

S_{ij} — Interfacial area between nodes i and j

T — Absolute temperature

$U_i = (\rho_i, m_i, e_i)$ — State vector at node i

$u_i = \frac{m_i}{\rho_i}$ — Velocity vector at node i

$u(x, t)$ — Velocity field

V_i — Representative control volume associated with node i

$w_{ij} = S_{ij}/V_i$ — Interfacial weight coefficient

x_i — Position vector of node i

Greek symbols

Δ — Laplacian operator

Δx — Characteristic spatial discretization scale

∇ — Gradient operator

$\nabla \cdot$ — Divergence operator

∂_t — Partial derivative with respect to time

ρ_i — Mass density at node i

$\rho(x, t)$ — Density field in the continuum limit

σ_s — Entropy production rate density

τ — Viscous stress tensor

Φ_{ij} — Reversible (Euler-type) interfacial flux

Ψ_{ij} — Dissipative interfacial flux

Ω — Spatial domain

$(-\Delta)^\theta$ — Fractional Laplacian operator

θ — Exponent of the fractional dissipation operator

μ — Dynamic viscosity coefficient

Functional spaces

$L^2(\Omega)$ — Space of square-integrable functions

$H^1(\Omega)$ — Sobolev space

$H^{-1}(\Omega)$ — Dual space of $H^1(\Omega)$

$C_0^\infty(\Omega)$ — Space of smooth test functions

$L^2(0, T; H^1(\Omega))$ — Bochner space for weak solutions

$H^1(0, T; H^{-1}(\Omega))$ — Space controlling time derivatives

Appendix A Rigorous Justification of the Discrete-to-Continuum Reduction

The present Appendix A supplements the structural discussion of Chapters 2 and 3, especially the abstract master equation introduced in Chapter 2 and the finite-volume-compatible parent equation constructed in Chapter 3. Its purpose is to state explicitly, within a finite-volume framework, the assumptions under which the network master equation is consistent with a conservation-law-type continuum partial differential equation.

It is important to state from the outset what this appendix does, and does not, claim. This appendix does not provide the principal compactness theorem of the present paper, nor does it claim that the full discrete-to-continuum passage has been closed at the same level of rigor as the relaxation-limit theorem proved in Chapters 4 and 5. Rather, its role is to clarify the finite-volume consistency, conservative structure, mesh assumptions, and scaling conditions under which the network master equation may be interpreted as a structurally admissible discrete precursor of a continuum conservation law. This positioning is fully consistent with the revised main text, in which the rigorous main theorem concerns the relaxation limit of the continuum extended system, while the network-to-continuum step is treated as a structural correspondence under explicit assumptions.

We begin from the abstract master equation of Chapter 2 and the finite-volume interpretation of Chapter 3. Let the state at node i be denoted by U_i , and consider the conservative update rule

$$\frac{dU_i}{dt} = \sum_{j \in N(i)} A_{ij} Q_{ij}, \quad (\text{A1})$$

where $N(i)$ is the neighborhood of node i , A_{ij} is a geometric interaction weight, and Q_{ij} denotes the interface quantity associated with the oriented edge $j \rightarrow i$. In the notation of Chapter 3, this is the same structural form as Eq. (26), specialized here to the conservative or flux-type part of the discrete dynamics. The state variables themselves are those introduced in Chapter 3, namely $U_i = (\rho_i, m_i, e_i)$, and the finite-volume-compatible weight is built from the interface area and control-volume size.

Let V_i be the control volume associated with node i , let S_{ij} denote the area of the common interface between adjacent cells i and j , and let n_{ij} be the unit normal directed from j to i . Following Eq. (32) of Chapter 3, we introduce the geometric weight

$$w_{ij} = \frac{S_{ij}}{V_i}. \quad (\text{A2})$$

We then identify

$$A_{ij} = w_{ij}. \quad (\text{A3})$$

With this choice, Eq. (A1) may be rewritten in the explicit finite-volume form

$$\frac{dU_i}{dt} = \frac{1}{V_i} \sum_{j \in N(i)} S_{ij} Q_{ij}. \quad (\text{A4})$$

If the oriented interface quantity is represented through a continuum flux tensor F contracted with the interface normal, namely

$$Q_{ij} = -F_{ij} \cdot n_{ij}, \quad (\text{A5})$$

then (A4) becomes

$$\frac{dU_i}{dt} = -\frac{1}{V_i} \sum_{j \in N(i)} S_{ij} (F_{ij} \cdot n_{ij}), \quad (\text{A6})$$

which is exactly the finite-volume conservative update rule corresponding to the structure introduced in Sections 3.2–3.3 of the main text. In particular, in the Euler-type case of Chapter 3 one has $F_{ij} = -F(U_{ij}) \cdot n_{ij}$, with $F(U)$ given by Eq. (39) of the main text.

The fundamental conservative mechanism is the antisymmetry of the interface contribution. Assume that the interface quantity satisfies

$$Q_{ij} = -Q_{ji}, \quad (\text{A7})$$

and that the interface geometry is symmetric in the sense that

$$S_{ij} = S_{ji}, n_{ji} = -n_{ij}. \quad (\text{A8})$$

Then the global sum of internal exchanges vanishes. Indeed,

$$\sum_i V_i \frac{dU_i}{dt} = \sum_i \sum_{j \in N(i)} S_{ij} Q_{ij}. \quad (\text{A9})$$

Grouping the contributions corresponding to an unordered internal interface pair $\{i, j\}$, we obtain

$$S_{ij} Q_{ij} + S_{ji} Q_{ji} = S_{ij} Q_{ij} - S_{ij} Q_{ij} = 0. \quad (\text{A10})$$

Therefore,

$$\sum_i V_i \frac{dU_i}{dt} = 0. \quad (\text{A11})$$

This is the discrete conservative cancellation mechanism used abstractly in Chapter 2 and concretely in Section 3.2 of Chapter 3. It is the finite-volume realization of the statement that internal interface exchanges do not alter the total mass, momentum, or energy of the closed system.

We now turn to the continuum consistency of the conservative update. Let $U(x, t)$ be a smooth continuum field, and let x_i denote the center of cell V_i . Writing a Taylor expansion around x_i , one has for a neighboring cell center x_j ,

$$U(x_j, t) = U(x_i, t) + (x_j - x_i) \cdot \nabla U(x_i, t) + O(h^2), \quad (\text{A12})$$

where h is the representative mesh size. This expresses the standard first-order approximation of neighboring cell averages by the local gradient.

Assume furthermore that the discrete interface flux is consistent with a smooth continuum flux $F(U)$ in the sense that, for a representative interface point x_{ij} ,

$$F_{ij} = F(U(x_{ij}, t)) + O(h). \quad (\text{A13})$$

Then the interfacial sum in (A6) satisfies the standard finite-volume consistency relation

$$\sum_{j \in N(i)} S_{ij} (F_{ij} \cdot n_{ij}) = \int_{\partial V_i} F(U) \cdot n \, dS + O(h^d), \quad (\text{A14})$$

where d denotes the spatial dimension. Dividing by $V_i \sim h^d$, we obtain

$$\frac{1}{V_i} \sum_{j \in N(i)} S_{ij} (F_{ij} \cdot n_{ij}) = \frac{1}{V_i} \int_{\partial V_i} F(U) \cdot n \, dS + O(h). \quad (\text{A15})$$

Applying the Gauss divergence theorem to the leading term yields

$$\frac{1}{V_i} \int_{\partial V_i} F(U) \cdot n \, dS = \nabla \cdot F(U)(x_i, t) + O(h), \quad (\text{A16})$$

and hence the discrete update (A6) takes the local form

$$\frac{dU_i}{dt} = -\nabla \cdot F(U)(x_i, t) + O(h). \quad (\text{A17})$$

Therefore, as $h \rightarrow 0$,

$$\partial_t U + \nabla \cdot F(U) = 0. \quad (\text{A18})$$

This is the conservative continuum equation structurally corresponding to the parent equation discussed in Chapter 3 and summarized in the revised conclusion of the main text. The role of Appendix A is to make explicit the finite-volume background of this correspondence, not to elevate it to the same status as the principal rigorous theorem of Chapters 4–5.

We next state the geometric hypotheses needed to keep the above consistency meaningful.

First, we assume a shape-regular mesh, namely that there exist constants $c_1, c_2 > 0$, independent of h , such that

$$c_1 h^d \leq V_i \leq c_2 h^d. \quad (\text{A19})$$

Second, we assume the interface scaling

$$S_{ij} \sim h^{d-1}. \quad (\text{A20})$$

These conditions prevent cell degeneration and ensure that the interfacial contribution and the cell average are balanced at the correct scale.

For dissipative interface terms of diffusive type, the appropriate transmissibility scaling is

$$\mu_{ij} \sim \mu \frac{S_{ij}}{h}, \quad (\text{A21})$$

and similarly for any conductive coefficient,

$$\kappa_{ij} \sim \kappa \frac{S_{ij}}{h}. \quad (\text{A22})$$

Under these scalings, the interface contribution involving a discrete difference quotient has the correct order of magnitude to recover a continuum diffusion term of the form $\nabla \cdot (\mu \nabla U)$ or $\nabla \cdot (\kappa \nabla U)$. This is exactly the scaling principle consistent with the interface dissipation constructed in Section 3.4 of the main text.

To pass from consistency to compactness for a discrete solution family U_h , one would further require uniform estimates analogous to the continuum bounds later proved for the extended system. In finite-volume form, the natural discrete analogues are:

$$\sup_{0 \leq t \leq T} \sum_i |U_i(t)|^2 V_i \leq C, \quad (\text{A23})$$

$$\text{and} \quad \int_0^T \sum_i \sum_{j \in N(i)} |U_i - U_j|^2 \frac{S_{ij}}{h} dt \leq C. \quad (\text{A24})$$

These bounds imply that the piecewise-constant reconstruction U_h is bounded in $L^2(0, T; H^1(\Omega))$, at least in the standard discrete compactness sense. Moreover, from the conservative update formula (A6), one expects a corresponding time-regularity estimate of the form

$$\partial_t U_h \text{ bounded in } L^2(0, T; H^{-1}(\Omega)). \quad (\text{A25})$$

If such estimates hold, then the Aubin–Lions compactness theorem yields, after extraction of a subsequence,

$$U_h \rightarrow U \text{ strongly in } L^2((0, T) \times \Omega). \quad (\text{A26})$$

This strong convergence is the ingredient needed to pass to the limit in nonlinear fluxes. However, and this is crucial in light of the reviewer's comment, the present paper does not claim that the entire discrete compactness program (A23)–(A26) has been proved for the network scheme itself. Rather, these statements clarify the kind of estimates that would be needed for a full discrete-to-continuum theorem.

Accordingly, the correct interpretation of the present appendix is the following: under the geometric weight (A2), the conservative finite-volume form (A6), the antisymmetry condition (A7), the consistency assumptions (A12)–(A16), and the mesh/scaling hypotheses (A19)–(A22), the network master equation is structurally consistent with the conservation-law-type continuum Eq. (A18). A full compactness theorem for discrete solution families remains a natural future extension, fully consistent with the revised statement of the main theorem in Chapter 6.

Appendix B Thermodynamically Consistent Relaxation-Extended System and Lyapunov Structure

The present Appendix B supplements Chapter 4, especially the introduction of the residual stress, the derivation of the extended energy inequality, the preservation of symmetry, and the internalization and disappearance of the forcing term. Its purpose is to make explicit, in slightly greater detail than in the main text, how the Lyapunov structure of the extended system arises entirely from the design of the equations themselves.

We recall from Chapter 4 that the unknowns of the continuum extended system are $(u^\varepsilon, p^\varepsilon, \tau^\varepsilon)$, and that the residual stress is defined by

$$r^\varepsilon := \tau^\varepsilon - 2\nu D(u^\varepsilon), \quad (\text{A27})$$

Where

$$D(u) := \frac{1}{2}(\nabla u + (\nabla u)^\top). \quad (\text{A28})$$

The correct symmetric gradient in (A28) is essential, because the energy identity uses the contraction of the symmetric residual stress with the symmetric part of ∇u^ε , exactly as in Eqs. (95), (110), and (116)–(118) of the revised Chapter 4.

The forcing term is defined internally by

$$F^\varepsilon := \nabla \cdot r^\varepsilon. \quad (\text{A29})$$

The momentum equation therefore takes the Navier–Stokes form

$$\partial_t u^\varepsilon + (u^\varepsilon \cdot \nabla) u^\varepsilon - \nu \Delta u^\varepsilon + \nabla p^\varepsilon = F^\varepsilon, \nabla \cdot u^\varepsilon = 0. \quad (\text{A30})$$

Thus the forcing is not externally prescribed; it is the exact measure of the deviation from viscous equilibrium.

The residual stress is governed by the relaxation equation

$$\varepsilon \partial_t r^\varepsilon + r^\varepsilon + \delta(-\Delta)^{1+\theta} r^\varepsilon = 0, \delta > 0, \theta \in (0, 1/2). \quad (\text{A31})$$

As emphasized in Chapter 4, the absence of any external source term in (A31) is precisely what makes the final disappearance of the forcing an internal consequence of the system rather than an assumption.

We first record explicitly the symmetry-preservation mechanism.

Proposition B.1 (Preservation of symmetry).

Assume $r_0^\varepsilon \in L^2_{\text{sym}}(\Omega; \mathbb{R}^{3 \times 3})$. Then any weak solution r^ε of (A31) remains symmetric for all $t \in [0, T]$.

Proof.

Eq. (A31) acts componentwise on the tensor entries, and the fractional Laplacian is a scalar self-adjoint Fourier multiplier. Therefore the antisymmetric part of r^ε satisfies the same homogeneous relaxation equation with zero initial data. Hence the antisymmetric part remains zero for all time.

This proposition justifies rigorously the use of the identity

$$\langle \nabla \cdot r^\varepsilon, u^\varepsilon \rangle = -\langle r^\varepsilon, D(u^\varepsilon) \rangle. \quad (\text{A32})$$

We now define the extended energy (Lyapunov functional) by

$$E^\varepsilon(t) := \frac{1}{2} \|u^\varepsilon(t)\|_2^2 + \frac{\lambda\varepsilon}{2} \|r^\varepsilon(t)\|_2^2. \quad (\text{A33})$$

The factor ε is essential and corrects the inconsistency identified by the reviewer. It is dictated by the coefficient of $\partial_t r^\varepsilon$ in the relaxation equation. Multiplying (A31) by r^ε and integrating over Ω , we obtain

$$\frac{\varepsilon}{2} \frac{d}{dt} \|r^\varepsilon\|_2^2 + \|r^\varepsilon\|_2^2 + \delta \|(-\Delta)^{\frac{1+\theta}{2}} r^\varepsilon\|_2^2 = 0. \quad (\text{A34})$$

This is the precise reason why the residual-stress part of the Lyapunov functional must be proportional to $\varepsilon \|r^\varepsilon\|_2^2$, not $\varepsilon^2 \|r^\varepsilon\|_2^2$. On the velocity side, multiplying (A30) by u^ε and integrating over Ω , we obtain

$$\frac{1}{2} \frac{d}{dt} \|u^\varepsilon\|_2^2 + \nu \|\nabla u^\varepsilon\|_2^2 = \langle F^\varepsilon, u^\varepsilon \rangle. \quad (\text{A35})$$

Using $F^\varepsilon = \nabla \cdot r^\varepsilon$ and integrating by parts, we get

$$\langle F^\varepsilon, u^\varepsilon \rangle = -\langle r^\varepsilon, D(u^\varepsilon) \rangle. \quad (\text{A36})$$

Adding (A35) and λ times (A34), we arrive at

$$\frac{d}{dt} E^\varepsilon(t) + \nu \|\nabla u^\varepsilon\|_2^2 + \lambda \|r^\varepsilon\|_2^2 + \lambda \delta \|(-\Delta)^{\frac{1+\theta}{2}} r^\varepsilon\|_2^2 = -\langle r^\varepsilon, D(u^\varepsilon) \rangle. \quad (\text{A37})$$

The cross term is controlled by Young's inequality. For any $\eta > 0$,

$$|\langle r^\varepsilon, D(u^\varepsilon) \rangle| \leq \eta \|r^\varepsilon\|_2^2 + \frac{1}{4\eta} \|D(u^\varepsilon)\|_2^2. \quad (\text{A38})$$

Since

$$\|D(u^\varepsilon)\|_2^2 \leq \|\nabla u^\varepsilon\|_2^2, \quad (\text{A39})$$

we obtain

$$\frac{d}{dt} E^\varepsilon(t) + (\nu - \frac{1}{4\eta}) \|\nabla u^\varepsilon\|_2^2 + (\lambda - \eta) \|r^\varepsilon\|_2^2 + \lambda \delta \|(-\Delta)^{\frac{1+\theta}{2}} r^\varepsilon\|_2^2 \leq 0. \quad (\text{A40})$$

Choosing

$$\frac{1}{4\nu} < \eta < \lambda, \quad (\text{A41})$$

we obtain positive constants $c_1, c_2, c_3 > 0$ such that

$$\frac{d}{dt} E^\varepsilon(t) + c_1 \|\nabla u^\varepsilon\|_2^2 + c_2 \|r^\varepsilon\|_2^2 + c_3 \|(-\Delta)^{\frac{1+\theta}{2}} r^\varepsilon\|_2^2 \leq 0. \quad (\text{A42})$$

Integrating in time yields

$$E^\varepsilon(t) + c_1 \int_0^t \|\nabla u^\varepsilon(s)\|_2^2 ds + c_2 \int_0^t \|r^\varepsilon(s)\|_2^2 ds + c_3 \int_0^t \|(-\Delta)^{\frac{1+\theta}{2}} r^\varepsilon(s)\|_2^2 ds \leq E^\varepsilon(0). \quad (\text{A43})$$

This is exactly the Appendix form of the extended energy inequality proved in Theorem 4.1 of the main text.

We now derive the forcing estimates. From (A43), we obtain in particular

$$\int_0^T \|(-\Delta)^{\frac{1+\theta}{2}} r^\varepsilon(s)\|_2^2 ds \leq C. \quad (\text{A44})$$

Since $\theta > 0$, Fourier analysis on $\mathbb{T}^3$ gives

$$\|(-\Delta)^{\frac{1+\theta}{2}} r\|_2 \geq c \|\nabla r\|_2, \quad (\text{A45})$$

modulo the constant mode, which is harmless because it does not contribute to ∇r . Therefore,

$$\int_0^T \|\nabla r^\varepsilon(s)\|_2^2 ds \leq C. \quad (\text{A46})$$

Since $F^\varepsilon = \nabla \cdot r^\varepsilon$, it follows that

$$F^\varepsilon \in L^2(0, T; L^2(\Omega)). \quad (\text{A47})$$

Finally, integrating (A34) in time yields

$$\frac{\varepsilon}{2} \|r^\varepsilon(t)\|_2^2 + \int_0^t \|r^\varepsilon(s)\|_2^2 ds + \delta \int_0^t (-\Delta)^{\frac{1+\theta}{2}} r^\varepsilon(s) \|_2^2 ds = \frac{\varepsilon}{2} \|r_0^\varepsilon\|_2^2. \quad (\text{A48})$$

Hence

$$\int_0^T \|r^\varepsilon(s)\|_2^2 ds \leq \frac{\varepsilon}{2} \|r_0^\varepsilon\|_2^2, \quad (\text{A49})$$

and therefore

$$\|r^\varepsilon\|_{L^2(0, T; L^2)} \leq \sqrt{\frac{\varepsilon}{2}} \|r_0^\varepsilon\|_2. \quad (\text{A50})$$

If r_0^ε is uniformly bounded in L^2 , then $r^\varepsilon \rightarrow 0$ strongly in $L^2(0, T; L^2)$. Consequently, for every $\phi \in L^2(0, T; H^1(\Omega; \mathbb{R}^3))$,

$$\int_0^T \int_\Omega F^\varepsilon \cdot \phi dx dt = - \int_0^T \int_\Omega r^\varepsilon : \nabla \phi dx dt \rightarrow 0. \quad (\text{A51})$$

Thus

$$F^\varepsilon \rightarrow 0 \text{ in } L^2(0, T; H^{-1}(\Omega)), \quad (\text{A52})$$

and in particular in $\mathcal{D}'((0, T) \times \Omega)$.

This appendix therefore makes explicit the central message of Chapter 4: the extended system possesses a Lyapunov structure, and from this structure both the integrability and the disappearance of the forcing term follow internally, with no external vanishing assumption.

Appendix C Compactness, Passage to the Limit, and Construction of a Leray–Hopf Weak Solution

The present Appendix C supplements Chapter 5, especially Lemmas 5.1–5.3, the Aubin–Lions compactness argument, the passage to the limit in the nonlinear term, and the vanishing of the forcing in the weak formulation. Its purpose is to collect in one place the functional-analytic ingredients by which the relaxation-limit argument closes.

After removing the pressure by means of the Leray projection P , the extended momentum equation of Chapter 5 takes the form

$$\partial_t u^\varepsilon = -P((u^\varepsilon \cdot \nabla)u^\varepsilon) + \nu \Delta u^\varepsilon + PF^\varepsilon. \quad (\text{A53})$$

This is the appendix counterpart of Eq. (184) in the revised Chapter 5. The three terms on the right-hand side are, respectively, the nonlinear transport term, the viscous term, and the internally generated forcing term.

We begin with the viscous term. For every $v \in H^1(\Omega; \mathbb{R}^3)$,

$$|\langle \Delta u^\varepsilon, v \rangle| = |\langle \nabla u^\varepsilon, \nabla v \rangle| \leq \|\nabla u^\varepsilon\|_2 \|\nabla v\|_2. \quad (\text{A54})$$

Hence

$$\|\Delta u^\varepsilon\|_{H^{-1}(\Omega)} \leq \|\nabla u^\varepsilon\|_2. \quad (\text{A55})$$

By Lemma 5.1 in the revised Chapter 5, u^ε is uniformly bounded in $L^2(0, T; H^1(\Omega))$, and therefore

$$\nu \Delta u^\varepsilon \text{ is bounded in } L^2(0, T; H^{-1}(\Omega)). \quad (\text{A56})$$

Next, consider the nonlinear term. The revised Chapter 5 establishes the estimate

$$\|u^\varepsilon\|_{L^4(0, T; L^3(\Omega))} \leq C \|u^\varepsilon\|_{L^\infty(0, T; L^2)}^{1/2} \|u^\varepsilon\|_{L^2(0, T; H^1)}^{1/2}. \quad (\text{A57})$$

This is exactly the interpolation estimate used in Eq. (196) of the revised Chapter 5. For any $\phi \in H^1(\Omega; \mathbb{R}^3)$ with $\nabla \cdot \phi = 0$,

$$\langle (u^\varepsilon \cdot \nabla)u^\varepsilon, \phi \rangle = - \int_\Omega (u^\varepsilon \otimes u^\varepsilon) : \nabla \phi dx. \quad (\text{A58})$$

Hence

$$\|(u^\varepsilon \cdot \nabla)u^\varepsilon\|_{H^{-1}} \leq C \|u^\varepsilon \otimes u^\varepsilon\|_{L^{3/2}} \leq C \|u^\varepsilon\|_{L^3}^2. \quad (\text{A59})$$

Since $u^\varepsilon \in L_t^4 L_x^3$, it follows that

$$(u^\varepsilon \cdot \nabla)u^\varepsilon \text{ is bounded in } L^{4/3}(0, T; H^{-1}(\Omega)). \quad (\text{A60})$$

For the forcing term, Appendix B has already shown that

$$F^\varepsilon \in L^2(0, T; L^2(\Omega)). \quad (\text{A61})$$

Using the continuous embedding $L^2(\Omega) \hookrightarrow H^{-1}(\Omega)$, we infer

$$PF^\varepsilon \text{ is bounded in } L^2(0, T; H^{-1}(\Omega)) \subset L^{4/3}(0, T; H^{-1}(\Omega)). \quad (\text{A62})$$

Combining (A56), (A60), and (A62), Eq. (A53) yields

$$\partial_t u^\varepsilon \text{ is bounded in } L^{4/3}(0, T; H^{-1}(\Omega)). \quad (\text{A63})$$

On the other hand, Lemma 5.1 gives the uniform spatial bound

$$u^\varepsilon \text{ bounded in } L^\infty(0, T; L^2(\Omega)) \cap L^2(0, T; H^1(\Omega)). \quad (\text{A64})$$

Now we invoke the compactness chain

$$H^1(\Omega) \hookrightarrow L^2(\Omega) \hookrightarrow H^{-1}(\Omega). \quad (\text{A65})$$

By the Aubin–Lions theorem, after extraction of a subsequence,

$$u^\varepsilon \rightarrow u \text{ strongly in } L^2(0, T; L^2(\Omega)). \quad (\text{A66})$$

This is the appendix form of Eq. (201) in the revised Chapter 5.

This strong convergence is the key to the nonlinear limit. Indeed,

$$u^\varepsilon \otimes u^\varepsilon - u \otimes u = (u^\varepsilon - u) \otimes u^\varepsilon + u \otimes (u^\varepsilon - u), \quad (\text{A67})$$

so that

$$\|u^\varepsilon \otimes u^\varepsilon - u \otimes u\|_{L^1_{t,x}} \leq \|u^\varepsilon - u\|_{L^2_{t,x}} (\|u^\varepsilon\|_{L^2_{t,x}} + \|u\|_{L^2_{t,x}}) \rightarrow 0. \quad (\text{A68})$$

Therefore,

$$u^\varepsilon \otimes u^\varepsilon \rightarrow u \otimes u \text{ in } L^1((0, T) \times \Omega). \quad (\text{A69})$$

Next, we consider the forcing term in the weak formulation. Since

$$F^\varepsilon = \nabla \cdot r^\varepsilon, \quad (\text{A70})$$

we have for every smooth test function ϕ ,

$$\int_0^T \int_\Omega F^\varepsilon \cdot \phi \, dx \, dt = - \int_0^T \int_\Omega r^\varepsilon : \nabla \phi \, dx \, dt. \quad (\text{A71})$$

By Appendix B,

$$\|r^\varepsilon\|_{L^2(0, T; L^2)} \leq C\varepsilon^{1/2} \rightarrow 0. \quad (\text{A72})$$

Hence

$$|\int_0^T \int_\Omega r^\varepsilon : \nabla \phi \, dx \, dt| \leq \|r^\varepsilon\|_{L^2(0, T; L^2)} \|\nabla \phi\|_{L^2(0, T; L^2)} \rightarrow 0. \quad (\text{A73})$$

Thus the forcing term disappears in the sense of distributions, and in fact, by the main text, in $L^2(0, T; H^{-1}(\Omega))$. Applying all of the above convergences to the weak formulation of the extended system, we obtain

$$\int_0^T \int_\Omega (-u \cdot \partial_t \phi + (u \otimes u) : \nabla \phi + \nu \nabla u : \nabla \phi) \, dx \, dt = \int_\Omega u_0 \cdot \phi(0) \, dx. \quad (\text{A74})$$

This is exactly the weak formulation in Definition 5.1 of the revised Chapter 5.

Finally, the energy inequality is inherited by weak lower semicontinuity. Since u^ε is bounded in $L^\infty_t L^2_x \cap L^2_t H^1_x$, one obtains for almost every $t \in [0, T]$,

$$\frac{1}{2} \|u(t)\|_2^2 + \nu \int_0^t \|\nabla u(s)\|_2^2 \, ds \leq \frac{1}{2} \|u_0\|_2^2. \quad (\text{A75})$$

Thus the limit u satisfies both the weak formulation and the energy inequality required in Definition 5.1. Therefore u is a Leray–Hopf weak solution. The compactness chain established in this appendix may be summarized as

$$u^\varepsilon \rightharpoonup u, \quad u^\varepsilon \rightarrow u, \quad r^\varepsilon \rightarrow 0, \quad F^\varepsilon \rightarrow 0. \quad (\text{A76})$$

This is the final analytic mechanism by which the relaxation-extended system of Chapter 4 yields the Leray–Hopf weak solution constructed in Chapter 5.

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