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Article

On the Zeros of the Dirichlet Eta Function and the Riemann Hypothesis

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Abstract

This article establishes that no non-trivial zeros of the Dirichlet eta function exists outside the critical line, that is, where the real part is equal to $\frac{1}{2}$. As the eta function shares its non-trivial zeros with those of the Riemann zeta function, the Riemann Hypothesis is validated.

Keywords: Riemann; Riemann hypothesis; prime; prime distribution

1. Introduction

While the normal zeros of the Dirichlet eta function are well known, complex non-trivial zeros are less well known and retain some uncertainty about the value of their real part. All non-trivial zeros known to date have a real part equal to $\frac{1}{2}$.

Eta and zeta functions share there non-trivial zeros. Numerous studies have addressed the non-trivial zeros of the eta and zeta functions.

For example, for $s \in \mathbb{C}$, the prime number theorem is equivalent to the absence of zeros on the line $Re(s) = 1$ [1].

A more refined result [2], derived from an effective form of Vinogradov's mean-value theorem, asserts that $\zeta(a + it) \neq 0$

whenever $a \geq 1 - \frac{1}{57.54(\ln(|t|))^{\frac{2}{3}}(\ln(\ln(|t|)))^{\frac{1}{3}}}$ and $|t| \geq 3$.

In 2015, Mossinghoff and Trudgian demonstrated [3] that eta and zeta functions have no zeros in the region $a \geq 1 - \frac{1}{5.573412\ln(|t|)}$ for $|t| \geq 2$.

The ZetaGrid project, formerly the largest distributed computing initiative for exploring the non-trivial roots of the Riemann zeta function, checked more than one billion roots daily. Although the project concluded in November 2005, its results were never officially published.

This article presents a proof of the Riemann Hypothesis, but other approaches leading to the same conclusion certainly exist.

2. Riemann Zeta Function

The Riemann zeta function is basically defined as:

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \tag{2.1}$$

where $s \in \mathbb{C}$ with $Re(s) > 1$.

3. Riemann's Functional Equations

Moreover, in his seminal 1859 paper "On the Number of Primes Less Than a Given Magnitude" [4], Riemann established that the zeta function $\zeta(s)$ satisfies the following two functional identities

$$\zeta(s) = \frac{2}{(2\pi)^{1-s}} \sin\left(\frac{\pi s}{2}\right) \Gamma(1-s) \zeta(1-s) \quad (3.1)$$

and

$$\zeta(1-s) = \frac{2}{(2\pi)^s} \cos\left(\frac{\pi s}{2}\right) \Gamma(s) \zeta(s) \quad (3.2)$$

for $s \in \mathbb{C} \setminus \{0, 1\}$.

4. Analytic Continuation of the Zeta Function

This section is devoted to establishing the fact that the Dirichlet eta function

$$\eta(s) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^s} \quad (4.1)$$

with $s \in \mathbb{C}, 0 < \operatorname{Re}(s) < 1$ shares its non-trivial zeros with those of the analytic continuation of the Riemann zeta function.

The relationship between these functions is:

$$\eta(s) = (1 - 2^{1-s}) \zeta(s). \quad (4.2)$$

Proof. By adding the term $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^s}$ to both sides of the definition of zeta $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$, we obtain:

$$\zeta(s) + \sum_{n=1}^{\infty} \frac{(-1)^n}{n^s} = \sum_{n=1}^{\infty} \frac{1}{n^s} + \sum_{n=1}^{\infty} \frac{(-1)^n}{n^s}$$

It should be noted that only the even terms of the form $2n$ remain in the right-hand side:

$$\begin{aligned} \zeta(s) + \sum_{n=1}^{\infty} \frac{(-1)^n}{n^s} &= 2 \sum_{n=1}^{\infty} \frac{1}{(2n)^s} \\ \Leftrightarrow \zeta(s) + \sum_{n=1}^{\infty} \frac{(-1)^n}{n^s} &= 2^{1-s} \sum_{n=1}^{\infty} \frac{1}{n^s} \\ \Leftrightarrow (1 - 2^{1-s}) \zeta(s) + \sum_{n=1}^{\infty} \frac{(-1)^n}{n^s} &= 0 \\ \Leftrightarrow (1 - 2^{1-s}) \zeta(s) &= \eta(s). \end{aligned}$$

□

The coefficient $1 - 2^{1-s}$ never vanishes in the domain under consideration¹, so if $\hat{s}$ is a non-trivial zero of the zeta function $\zeta(\hat{s}) = 0$, then

$$\eta(\hat{s}) = 0 \quad (4.3)$$

as well.

Thus, the non-trivial zeros of the eta function are well the same of those of the zeta function.

¹ Indeed, for $1 - 2^{1-s}$ to be worth 0, it would be necessary, among other things, that $\operatorname{Re}(s) = 1$ but this value is outside the interval considered for the real part which is $]0, 1[$.

5. Normal Zeros of the Eta Function

The normal zeros of the eta function arise from the vanishing of $1 - 2^{1-s}$ in (4.2), yielding to the complex values $\hat{s} = 1 + i \frac{2n\pi}{\ln 2}$, $\forall n \in \mathbb{Z}$.

6. Non-Trivial Zeros of the Eta Function

We now show that the only valid value for the real part of the non-trivial zeros of the eta function is $\frac{1}{2}$.

6.1. On the Complex Non-Trivial Zeros of Zeta and Their Complex Conjugates

Lemma. If $\hat{s} \in \mathbb{C}$ is a non-trivial zero of the zeta function, then $1 - \hat{s}$, and the complex conjugates $\bar{\hat{s}}$ and $1 - \bar{\hat{s}}$ are also non-trivial zeros of the zeta function.

Proof. If, in the right-hand side of (3.2), $\zeta(\hat{s})$ equals zero, then the left-hand side $\zeta(1 - \hat{s})$ also equals zero, with the consequence that $1 - \hat{s}$ is a non-trivial zero as well. By taking the complex conjugate, we also obtain, through (4.1) and (4.2) and by the Schwartz reflection principle: $\overline{\zeta(\hat{s})} = \zeta(\bar{\hat{s}}) = 0$ and $\overline{\zeta(1 - \hat{s})} = \zeta(1 - \bar{\hat{s}}) = 0$.

Thus,

$$\zeta(\hat{s}) = \zeta(1 - \hat{s}) = \zeta(\bar{\hat{s}}) = \zeta(1 - \bar{\hat{s}}) = 0.$$

□

Corollary. We have in particular:

$$\zeta(\hat{s}) = \zeta(1 - \bar{\hat{s}}) = 0. \quad (6.1)$$

6.2. Analysis of the Non-Trivial Zeros of the Eta Function

Let the complex number $\hat{s} = a + i.t$ be a non-trivial zero of both the eta and zeta functions with $a \in]0, 1[$ and $t \in \mathbb{R} \setminus \{0\}$.

The relation (4.3) implies that we must then have

$$\eta(\hat{s}) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^{\hat{s}}} = 0 \quad (6.2)$$

and also by virtue of (6.1)

$$\eta(1 - \bar{\hat{s}}) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^{1-\bar{\hat{s}}}} = 0. \quad (6.3)$$

Making use of the real part a and the imaginary part t of $\hat{s}$, we obtain:

$$\eta(\hat{s}) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^a} e^{-i.t.\ln(n)} = 0 \quad (6.4)$$

and

$$\eta(1 - \bar{\hat{s}}) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^{1-a}} e^{-i.t.\ln(n)} = 0. \quad (6.5)$$

To satisfy (6.4) and (6.5), there are two possibilities:

- The series $\eta(\hat{s})$ and $\eta(1 - \bar{\hat{s}})$ are equal term by term
- Some terms of the series $\eta(\hat{s})$ and $\eta(1 - \bar{\hat{s}})$ are different, but both series converge to 0.

Now let us analyze these two cases.

6.2.1. The Series Are Equal Term by Term

This means that we must have $\forall n \in \mathbb{N} \setminus \{0\}$ and $a \in \mathbb{R}$ with $0 < a < 1$:

$$\begin{aligned} \frac{(-1)^{n-1}}{n^a} e^{-i.t.\ln(n)} &= \frac{(-1)^{n-1}}{n^{1-a}} e^{-i.t.\ln(n)} \\ \Leftrightarrow \frac{1}{n^a} &= \frac{1}{n^{1-a}} \\ \Leftrightarrow n^{1-2a} &= 1 \\ \Leftrightarrow 1 - 2a &= 0 \\ \Leftrightarrow a &= \frac{1}{2}. \end{aligned} \quad (6.6)$$

Thus the real part $a = \operatorname{Re}(\hat{s})$ of non-trivial zeros of the eta function, which are also those of the zeta function, must be $\frac{1}{2}$.

6.2.2. The Series Differ Term by Term But Both Converge to Zero

Our goal in this section is to prove that the convergence of the two series to the same limit zero is possible only when the real part of the non-trivial zeros is equal to $\frac{1}{2}$.

Obviously both the real and imaginary parts of (6.4) and (6.5) must be zero.

Let us consider the real parts of the relations (6.4) and (6.5) and let us define $s_1(n) = \frac{(-1)^{n-1}}{n^a} \cos(t.\ln(n))$, $s_2(n) = \frac{(-1)^{n-1}}{n^{1-a}} \cos(t.\ln(n))$. The real parts of (6.4) and (6.5) can then be written as:

$$\Re(\eta(\hat{s})) = \sum_{n=1}^{\infty} s_1(n) = 0 \quad (6.7)$$

and

$$\Re(\eta(1 - \bar{s})) = \sum_{n=1}^{\infty} s_2(n) = 0. \quad (6.8)$$

For the series (6.7) and (6.8) to converge to a zero value, either all the terms must be identically zero (which is mathematically impossible²), or $s_1(n)$ and $s_2(n)$ must take on both positive and negative values. If this were not the case, $\Re(\eta(\hat{s}))$ and $\Re(\eta(1 - \bar{s}))$ would be either strictly positive or strictly negative and could therefore never tend towards zero.

Corollary. $s_1(n)$ and $s_2(n)$ each have at least one positive value and one negative value.

Let us consider the ratio:

$$q = \frac{s_1(n)}{s_2(n)} = n^{1-2a} > 0.$$

Given $0 < a < 1$, we have the inequalities $-1 < 1 - 2a < 1$ and $\frac{1}{n} < q < n$. Since $n \in \mathbb{N} \setminus \{0\}$, we have $q \in]0, \infty[$ which implies that $s_1(n)$ and $s_2(n)$ always have the same sign for a given n .

Let us now distinguish two cases according to the value of $1 - 2a$:

1) $0 \leq 1 - 2a < 1$:

Then $s_1(n) = n^{1-2a} s_2(n) \geq s_2(n)$, from which we derive the following inequality:

$$s_2(n) - s_1(n) = s_2(n) (1 - n^{1-2a}) \leq 0. \quad (6.9)$$

As mentioned in the previous corollary, $s_2(n)$ presents at least one negative value. Therefore, if $s_2(n) < 0$, to satisfy (6.9), it is necessary that $1 - n^{1-2a} \geq 0$. However, $\forall n \in \mathbb{N} \setminus \{0\}$, the values of

² $s_1(n)$ as well as $s_2(n)$ can only tend towards zero when $n \rightarrow \infty$.

$1 - n^{1-2a} \in] - \infty, 0]$ and, therefore, the only acceptable value for $1 - n^{1-2a}$ reduces to zero, which leads us to:

$$a = \frac{1}{2}. \quad (6.10)$$

2) $-1 < 1 - 2a \leq 0$:

Then $s_1(n) = n^{1-2a}s_2(n) \leq s_2(n)$, from which we derive the following inequality:

$$s_1(n) - s_2(n) = s_2(n)(n^{1-2a} - 1) \leq 0. \quad (6.11)$$

As mentioned in the previous corollary, $s_2(n)$ presents at least one negative value. Therefore, if $s_2(n) < 0$, to satisfy (6.11), it is necessary that $n^{1-2a} - 1 \geq 0$. However, $\forall n \in \mathbb{N} \setminus \{0\}$, the values of $n^{1-2a} - 1 \in] - 1, 0]$ and, therefore, the only acceptable value for $n^{1-2a} - 1$ reduces to zero, which leads us to:

$$a = \frac{1}{2}. \quad (6.12)$$

Obviously, the reasoning we have held also applies to imaginary parts.

7. Proof of the Riemann Hypothesis

In sections 6.2.1. and 6.2.2., we analyzed the two possibilities to have $\eta(\hat{s}) = \eta(1 - \bar{\hat{s}}) = 0$ and we showed in (6.10) and (6.12) that the only possible real part $a = \text{Re}(\hat{s})$ for the non-trivial zeros is $\frac{1}{2}$.

Given the fact that the Riemann zeta function and the Dirichlet eta function share their non-trivial zeros, the result just mentioned also applies to the Riemann zeta function.

Riemann hypothesis is therefore confirmed.

8. Conclusion

By examining the non-trivial zeros of the Dirichlet eta function, conclusions are drawn that extend to the Riemann zeta function.

Although alternative proofs certainly exist, we hope that this article puts an end to doubts about the validity of the Riemann Hypothesis and will contribute to a better understanding of the distribution of non-trivial zeros and their connection with the distribution of primes.

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References

1. Harold G. Diamond, *Elementary methods in the study of the distribution of prime numbers*, Bulletin of the American Mathematical Society, **7** (1982), no. 3, 553–589, DOI 10.1090/S0273-0979-1982-15057-1. MR 0670132
2. K. Ford, *Vinogradov's integral and bounds for the Riemann zeta function*, Proc. London Math. Soc. **85** (2002), no. 3, 565–633, DOI 10.1112/S0024611502013655. S2CID 121144007
3. Michael J. Mossinghoff and Timothy S. Trudgian, *Nonnegative trigonometric polynomials and a zero-free region for the Riemann zeta-function*, J. Number Theory, **157** (2015), 329–349, DOI 10.1016/J.JNT.2015.05.010. S2CID 117968965
4. Bernhard Riemann, *Über die Anzahl der Primzahlen unter einer gegebenen GröÙe*, Monatsberichte der Preußischen Akademie der Wissenschaften, Berlin, (1859), 671ff

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