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Brief Report

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Abstract: In this note we discuss the relationships between the weak dictatorship axiom, as introduced by A. Mas-Colel and H. Sonnenschein, and the vetoer axiom, as given by P. C. Fishburn, in a wider context of traditional Social Choice Theory (SCT). Namely, we prove the equivalence between these two axioms. This note can be a good formal reasoning exercise for students in the field of preference logic. This note is written as an outline for the 'second' lesson in SCT for non-mathematicians, which contains elementary formal logical argumentation, after the basics of preference logic, and before the extensive and complicated treatment of impossibility theorems.

Introduction

Social Choice Theory (SCT) has been the subject of the Nobel Prize in Economics on several occasions over the past decades (e.g. K. J. Arrow (1921-2017), Nobel Prize in Economics 1972; J. M. Buchanan (1919-2013) Nobel Prize in Economics 1986; A. K. Sen (1933-), Nobel Prize in Economics 1998). This is one of the additional reasons why SCT, with modifications and simplifications, is permanently interesting as teaching content at all levels of economics studies. On the other hand, contrary to this attractiveness, the problem is the difficulty in understanding the basic concepts of SCT.

The central themes of traditional Arrow-Sen SCT are impossibilities and interdeducibilities of sets of axioms appearing in the theory (see Arrow (1963), Sen (1970), Fishburn (1973), Bori\v ci\'c (2009)). This short note is devoted to deductive interdependences between dictatorship, as originally defined by Arrow (see Arrow (1963), Sen (1970)), the weak dictatorship, as given by Mas--Colel and Sonnenschein (1972), and the vetoer axiom, as introduced by Fishburn (1973). In previous papers, e.g. Sre\'ckovi\'c (2017), and Bori\v ci\'c and Sre\'ckovi\'c (2024), the authors deal primarily with the cardinal status of vetoing, but also include a discussion regarding some ordinal aspects of simplified versions of the well-known axioms of Arrow-Sen SCT. The simplification of traditional SCT is the central point in Boričić (2023, 2024), and Boričić and Srećković (2024).

The high mathematization of fragments of contemporary economic theories often presents a barrier to a wide range of economics students. Our goal is to make the formalism we use in this note, as part of the widespread contemporary symbolic logic, accessible to non-mathematician students in order to enable a better understanding and connection between form and meaning.

In this note, we focus on traditionally defined (but not simplified) conditions, as entities of a higher-order language, and the facts that the dictatorship implies weak dictatorship, that the dictatorship implies vetoing, and, finally, that the quantifier-free parts of the axioms of the weak dictatorship **TWD** and vetoing **TV** are equivalent. We follow the definitions and notations given in Boričić and Srećković (2024), but we prefer a more descriptive rather than a formal style of argumentation. It may seem that these relationships are anticipated and almost trivial, but we believe that such considerations contribute to the popularization of the theory and its better understanding by the wider community, primarily students. Finally, this note can be considered as an outline for the

second lesson in SCT, after mastering the basic concepts of preference logic as given in Pefku (2025), but before the difficult lessons devoted to impossibility theorems.

Dictator, Weak Dictator, and Vetoer

The traditional definition is based on finite sets of individuals and alternatives V and X, respectively, P_i and P – individual and social strict preference relations on X, R – a social weak preference on X, and I – a social indifference relation on X. It is also supposed that P is asymmetric and transitive, R – linear and transitive, and I is reflexive, symmetric and transitive, as well as, that $R = P \cup I$, and $I = R \cup R^{-1}$ (v. Pefku (2025)).

Arrow's dictatorship condition **TD**, the Mas-Colell and Sonnenschein weak dictatorship **TWD**, and Fishburn's vetoer condition **TV** can be presented as follows:

TD: There exists an $i \in V$, such that for all n-tuples of individual preferences and each two alternatives $x, y \in X$,

$$xP_iy \to xPy.$$

TWD: There exists an $i \in V$, such that for all n-tuples of individual preferences and each two alternatives $x, y \in X$,

$$xP_iy \to xRy.$$

TV: There exists an $i \in V$, such that for all n-tuples of individual preferences and each two alternatives $x, y \in X$,

$$xP_iy \to \neg yPx.$$

The prefix "**T**" stands for the "traditional" versions of the axioms, as opposed to the "simplified" ones discussed in the papers Boričić (2023), Boričić and Srećković (2024), and Boričić (2024).

For negations **NTD** and **NTV** of **TD** and **TV**, respectively, Fishburn emphasised that **NTV** is stronger than **NTD**, "since a dictator is a vetoer, but not conversely" (see Fishburn (1973), p. 208). Here we provide an argument for contraposition, i.e. an equivalent statement for positive sentences.

Note that the quantification 'for all n-tuples of individual preferences' removes the above conditions from the first-order language.

Lemma 1. (i) **TD** *implies* **TV**; (ii) **TD** *implies* **TWD**.

Proof. (i) If we suppose **TD**, bearing in mind that P is asymmetric, $xPy \to \neg yPx$, for all $x, y \in X$, *a fortiori*, from $xP_iy \to xPy$, which is the quantifier-free part of **TD** we infer $xP_iy \to \neg yPx$, i.e. the quantifier-free part of **TV**.

(ii) Similarly, from $xP_iy \to xPy$, the quantifier-free part of **TD**, by weakening of the consequent, we infer $xP_iy \to xPy \lor xIy$, i.e. $xP_iy \to xRy$, the quantifier-free part of **TWD**. (Q.E.D.)

Lemma 2. Conditions **TV** and **TWD** are equivalent.

Proof. In order to prove that **TWD** and **TV** are equivalent, firstly, let us suppose that **TWD**, $xP_iy \to xPy \lor xIy$, for all $x, y \in X$, is satisfied. The in-difference relation I is defined as xIy iff $\neg xPy \land \neg yPx$, and the strict prefe-rence relation P is asymmetric, $xPy \to \neg yPx$, for all $x, y \in X$, where, from **TWD**, *a fortiori*, we obtain **TV**: $xP_iy \to \neg yPx$. Conversely, we start with the fact that the weak preference R is linear: $xRy \lor yRx$, for all $x, y \in X$, i.e. $xPy \lor xIy \lor yPx$, for all $x, y \in X$, which is logically equivalent to $\neg yPx \to xRy$, from which, through $xP_iy \to \neg yPx$, **TV**, we immediately have $xP_iy \to xRy$, for all $x, y \in X$, **TDW**. This means that a quantifier-free part of weak dictator axiom **TWD** is exactly a quantifier-free part of vetoer axiom **TV**, and vice versa.   (Q.E.D.)

Conclusion

If we denote Arrow's dictatorship, Mas-Colell's and Sonnenschein's weak dictatorship, and Fishburn's vetoer axiom by **TD**, **TWD** and **TV**, respectively, then we can prove that, for the quantifier-free parts of these axioms, we have: **TD** implies both **TWD** and **TV**, and that **TWD** and **TV** are mutually equivalent. These facts can be a good formal reasoning exercise for students when dealing with logic of preference or social choice theory.

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