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Article

Langevin Equations Involving Two Fractional Orders with Infinite-Point Boundary Conditions

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Abstract: Recently, Li et al [1] investigated the nonlinear Langevin equations existence including two fractional orders with infinite-point boundary conditions. It has shown that the outcomes given it does count on the solution form for which it has boundary values. It means that their solution needs more to be in the final step. In current work, we will have discussion about the existence and uniqueness for the same boundary conditions by investigating the closely explicit solution form that has no boundary values. Numerical example is supported to have more vision to compare the between new and previous results. It turns out that our results are superior.

Keywords: fractional Langevin equations; fixed point theorem; existence and uniqueness

MSC: 26A33; 34A08; 34A12; 34B15

1. Introduction

In this paper, we discuss the following nonlinear Langevin equation of two fractional orders

$$ {}^cD^\gamma({}^cD^\alpha + \lambda)u(t) = f(t, u(t)), \qquad t \in [0,1] \tag{1.1}$$

supplemented with the infinite-point boundary conditions

$$ u(0) = 0, \quad {}^cD^\alpha u(0) = 0, \quad {}^cD^\alpha u(1) = \sum_{i=1}^{\infty} \beta_i \, {}^cD^\alpha u(\xi_i) \tag{1.2}$$

where ${}^cD^\alpha$ and ${}^cD^\gamma$ are the Caputo's fractional derivatives of orders $0 < \alpha \leq 1$ and $1 < \gamma \leq 2$, $\lambda, \beta_i \in \mathbb{R}$, $0 < \xi_1 < \xi_2 < \cdots < \xi_i < \cdots < 1, i \in \mathbb{N}$ and $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ is a continuously differentiable function.

It is worth pointing out that Li et al [1] have discussed the antecedent boundary problem and gave several new existence results of solutions by means of using Leray-Schauder's nonlinear alternative and Leray-Schauder degree theory. However, we note that their unique solution of the linear boundary value problem of fractional Langevin differential equation

$$ {}^cD^\gamma({}^cD^\alpha + \lambda)u(t) = h(t), \qquad t \in [0,1] \tag{1.3}$$

subject to the infinite-point boundary conditions (1.2) was given as

$$\begin{aligned} u(t) &= \int_0^t \frac{(t-s)^{\alpha+\gamma-1}}{\Gamma(\alpha+\gamma)} h(s)ds - \lambda \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} u(s)ds \\ &+ \frac{t^{\alpha+1}}{\Gamma(\alpha+2)\left(1 - \sum_{i=1}^{\infty} \beta_i \xi_i\right)} \left(\sum_{i=1}^{\infty} \beta_i \int_0^{\xi_i} \frac{(\xi_i - s)^{\gamma-1}}{\Gamma(\gamma)} h(s)ds \right. \\ &\left. - \int_0^1 \frac{(1-s)^{\gamma-1}}{\Gamma(\gamma)} h(s)ds + \lambda u(1) - \lambda \sum_{i=1}^{\infty} \beta_i u(\xi_i) \right) \end{aligned}$$

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with $\lambda > 0, \beta_i > 0, i \in \mathbb{N}$ and $1 - \sum_{i=1}^{\infty} \beta_i \xi_i > 0$, which contains the boundary values $u(1)$ and $u(\xi_i), i \in \mathbb{N}$ even though we can insert the values of these boundary values after obtaining the form of $u(t)$. This means that the solution above is not in the final form.

To be out of these criticisms, we resolve the boundary value problem (1.1)-(1.2) without the appearance of the boundary values $u(1)$ and $u(\xi_i), i \in \mathbb{N}$ in the unique solution $u(t)$. Also, we extend some restrictions on $\lambda, \beta_i, i \in \mathbb{N}$ and $\sum_{i=1}^{\infty} \beta_i \xi_i$.

It is worth mentioning that the nonlinear fractional Langevin equations have been developed by Mainardi and Pironi [2]. Recently, several contributions concerned with the existence and uniqueness of solutions for fractional Langevin equations, have been published, see [3–18] and the references given therein.

2. Preliminaries and Relevant Lemmas

In this section, we introduce some notations and definitions of fractional calculus and present preliminary results needed in our proofs later. We are indebted to the terminologies used in the books [19,20].

Definition 2.1. *The Riemann-Liouville fractional integral of order $\alpha > 0$ for a continuous function f is defined as*

$$I^{\alpha} f(t) = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s) ds$$

provided that the right-hand-side integral exists, where $\Gamma(\alpha)$ denotes the Gamma function is the Euler gamma function defined by

$$\Gamma(\alpha) = \int_0^{\infty} t^{\alpha-1} e^{-t} dt, \qquad \alpha > 0.$$

Definition 2.2. *Let $n \in \mathbb{N}$ be a positive integer and α be a positive real such that $n-1 < \alpha \leq n$, then the fractional derivative of a function f in the Caputo sense is defined as*

$$^{c}D^{\alpha} f(t) = \int_0^t \frac{(t-s)^{n-\alpha-1}}{\Gamma(n-\alpha)} f^{(n)}(s) ds$$

provided that the right-hand-side integral exists and is finite. We notice that the Caputo derivative of a constant is zero.

Lemma 2.1. *Let α and β be positive reals. If f is a continuous function, then we have*

$$I^{\alpha} I^{\beta} f(t) = I^{\alpha+\beta} f(t)$$

Lemma 2.2. *Let α be positive real. Then we have*

$$I^{\alpha} t^{\rho} = \frac{\Gamma(\rho+1)}{\Gamma(\rho+\alpha+1)} t^{\rho+\alpha}, \qquad \rho > -1.$$

Lemma 2.3. *Let $n \in \mathbb{N}$ and $n-1 < \alpha \leq n$. If u is a continuous function, then we have*

$$I^{\alpha}\, {}^{c}D^{\alpha} u(t) = u(t) + c_0 + c_1 t + \cdots + c_{n-1} t^{n-1}$$

Let us now consider the linear fractional Langevin differential equation (1.3) supplemented with the infinite-point boundary conditions (1.2), then we can state the following lemma:

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Lemma 2.4. *If $h \in C[0,1]$, then the unique solution of the boundary value problem (1.3) and (1.2) is given by*

$$\begin{aligned} u(t) &= \int_0^t \frac{(t-s)^{\alpha+\gamma-1}}{\Gamma(\alpha+\gamma)} h(s)ds - \lambda \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} u(s)ds \\ &- \frac{t^{\alpha+1}}{\Delta\Gamma(\alpha+2)} \sum_{i=1}^{\infty} \beta_i \left\{ \lambda \int_0^{\xi_i} \frac{(\xi_i-s)^{\alpha+\gamma-1}}{\Gamma(\alpha+\gamma)} h(s)ds - \lambda^2 \int_0^{\xi_i} \frac{(\xi_i-s)^{\alpha-1}}{\Gamma(\alpha)} u(s)ds \right. \\ &- \left. \int_0^{\xi_i} \frac{(\xi_i-s)^{\gamma-1}}{\Gamma(\gamma)} h(s)ds \right\} + \frac{t^{\alpha+1}}{\Delta\Gamma(\alpha+2)} \left\{ \lambda \int_0^1 \frac{(1-s)^{\alpha+\gamma-1}}{\Gamma(\alpha+\gamma)} h(s)ds \right. \\ &\left. -\lambda^2 \int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} u(s)ds - \int_0^1 \frac{(1-s)^{\gamma-1}}{\Gamma(\gamma)} h(s)ds \right\} \end{aligned}$$

where

$$\Delta = 1 - \sum_{i=1}^{\infty} \beta_i \xi_i - \frac{\lambda}{\Gamma(\alpha+2)} \left(1 - \sum_{i=1}^{\infty} \beta_i \xi_i^{\alpha+1} \right) \neq 0.$$

Proof. From Lemmas 2.1, 2.2 and 2.3 and the Definition 2.1, it follows that

$$^cD^{\alpha} u(t) = \int_0^t \frac{(t-s)^{\gamma-1}}{\Gamma(\gamma)} h(s)ds + c_0 + c_1 t - \lambda u(t) \tag{2.1}$$

and

$$\begin{aligned} u(t) = &\int_0^t \frac{(t-s)^{\alpha+\gamma-1}}{\Gamma(\alpha+\gamma)} h(s)ds + \frac{t^{\alpha}}{\Gamma(\alpha+1)} c_0 + \frac{t^{\alpha+1}}{\Gamma(\alpha+2)} c_1 \\ &- \lambda \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} u(s)ds + c_2. \end{aligned} \tag{2.2}$$

By inserting the boundary condition $u(0) = 0$ in (2.2) gives $c_2 = 0$ and also by inserting the boundary condition $^cDu(0) = 0$ in (2.1) gives $c_0 = 0$. Inserting (2.2) into (2.1) to obtain

$$\begin{aligned} D^{\alpha} u(t) = &-\lambda \left(\int_0^t \frac{(t-s)^{\alpha+\gamma-1}}{\Gamma(\alpha+\gamma)} h(s)ds - \lambda \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} u(s)ds \right) \\ &+ \int_0^t \frac{(t-s)^{\gamma-1}}{\Gamma(\gamma)} h(s)ds + \left(1 - \frac{\lambda t^{\alpha}}{\Gamma(\alpha+2)} \right) tc_1 \end{aligned}$$

Using the third boundary condition in (1.2) gives

$$\begin{aligned} c_1 = &\frac{1}{\Delta} \sum_{i=1}^{\infty} \beta_i \left\{ -\lambda \int_0^{\xi_i} \frac{(\xi_i-s)^{\alpha+\gamma-1}}{\Gamma(\alpha+\gamma)} h(s)ds + \lambda^2 \int_0^{\xi_i} \frac{(\xi_i-s)^{\alpha-1}}{\Gamma(\alpha)} u(s)ds \right. \\ &+ \left. \int_0^{\xi_i} \frac{(\xi_i-s)^{\gamma-1}}{\Gamma(\gamma)} h(s)ds \right\} + \frac{1}{\Delta} \left\{ \lambda \int_0^1 \frac{(1-s)^{\alpha+\gamma-1}}{\Gamma(\alpha+\gamma)} h(s)ds \right. \\ &\left. -\lambda^2 \int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} u(s)ds + \int_0^1 \frac{(1-s)^{\gamma-1}}{\Gamma(\gamma)} h(s)ds \right\} \end{aligned}$$

Substituting the above values in (2.2) to obtain the desired results. □

Remark 2.1. *If $\lambda = \Gamma(\alpha+2)$ and β_i are positive (negative) for all $i \in \mathbb{N}$, then we have*

$$\Delta = -\sum_{i=1}^{\infty} \beta_i \xi_i (1 - \xi_i^{\alpha}) \neq 0, \qquad 0 < \alpha \leq 1.$$

In the proofs of our main existence results for problem (1.1)-(1.2), we will use the Banach contraction mapping principle and nonlinear alternative Leray-Schauder theorem presented below:

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Lemma 2.5 ([22,23]). *Let $\mathbb{E}$ be a Banach space, C be a closed and convex subset of $\mathbb{E}$, U be an open subset of C and $0 \in U$. Suppose that the operator $\mathcal{T} : \overline{U} \to C$ is a continuous and compact map (that is, $\mathcal{T}(\overline{U})$ is a relatively compact subset of C). Then either*

(i) $\mathcal{T}$ has a fixed point in $x^* \in \overline{U}$, or
(ii) there is $x \in \partial U$ (the boundary of U in C) and $\delta \in (0,1)$ such that $\delta\mathcal{T}(x) = x$.

3. Main Results

Let $\mathbb{E} = C([0,1],\mathbb{R})$ be the Banach space of all continuous functions from $[0,1] \longrightarrow \mathbb{R}$ endowed the norm defined by

$$\|u\| = \sup\{|u(t)|, t \in [0,1]\}.$$

Before stating and proving the main results, we introduce the following hypotheses: Assume that

$(\mathcal{H}_1)$ The function $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ is a jointly continuous.
$(\mathcal{H}_2)$ The function f satisfies

$$|f(t,u) - f(t,v)| \leq \mathcal{L}|u - v|, \qquad \forall t \in [0,1], u,v \in \mathbb{R}$$

where $\mathcal{L}$ is the Lipschitz constant.
$(\mathcal{H}_3)$ There exists a positive function $\omega \in C([0,1],\mathbb{R}_+)$ and a nondecreasing function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$|f(t,u)| \leq \omega(t)\varphi(\|u\|), \qquad \forall (t,u) \in ([0,1],R).$$

$(\mathcal{H}_4)$ There exist two positive constants k and c such that

$$|f(t,u)| \leq \eta|u| + L, \qquad \forall (t,u) \in ([0,1],R).$$

For computational convenience, we set

$$\mathbf{A} = \frac{1}{\Gamma(\alpha+\gamma+1)} + |\lambda|S(\alpha+\gamma) + S(\gamma) + |\lambda|S_0(\alpha+\gamma) + S_0(\gamma) \tag{3.1}$$

$$\mathbf{B} = \frac{|\lambda|}{\Gamma(\alpha+1)} + \lambda^2 S(\alpha) + \lambda^2 S_0(\alpha) \tag{3.2}$$

where

$$S_0(p) = \frac{1}{|\Delta|\Gamma(\alpha+2)\Gamma(p+1)}, \qquad S(p) = \frac{1}{|\Delta|\Gamma(\alpha+2)} \sum_{i=1}^{\infty} \frac{|\beta_i|\xi_i^p}{\Gamma(p+1)}$$

In view of Lemma 2.4, we transform problem (1.1)-(1.2) as

$$u = T(u) \tag{3.3}$$

where the operator $T : \mathbb{E} \longrightarrow \mathbb{E}$ is defined by

$$(Tu)(t) = \int_0^t \frac{(t-s)^{\alpha+\gamma-1}}{\Gamma(\alpha+\gamma)} f(s,u(s))ds - \lambda \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} u(s)ds - \frac{t^{\alpha+1}}{\Delta\Gamma(\alpha+2)} U(u)$$

where

$$\begin{aligned} U(u) = \sum_{i=1}^{\infty} \beta_i \Big\{ &\lambda \int_0^{\xi_i} \frac{(\xi_i-s)^{\alpha+\gamma-1}}{\Gamma(\alpha+\gamma)} f(s,u(s))ds - \lambda^2 \int_0^{\xi_i} \frac{(\xi_i-s)^{\alpha-1}}{\Gamma(\alpha)} u(s)ds \\ &- \int_0^{\xi_i} \frac{(\xi_i-s)^{\gamma-1}}{\Gamma(\gamma)} f(s,u(s))ds \Big\} - \lambda \int_0^1 \frac{(1-s)^{\alpha+\gamma-1}}{\Gamma(\alpha+\gamma)} f(s,u(s))ds \\ &+ \lambda^2 \int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} u(s)ds + \int_0^1 \frac{(1-s)^{\gamma-1}}{\Gamma(\gamma)} f(s,u(s))ds. \end{aligned}$$

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The following theorem is devoted to provide the conditions that satisfy the assumptions of Banach contraction mapping principle to give a unique solution of the boundary value problem (1.1)-(1.2).

Theorem 3.1. *Assume that the assumptions $(\mathcal{H}_1)$ and $(\mathcal{H}_2)$ hold. Then the boundary value problem (1.1)-(1.2) has a unique solution if $\mathbf{Q} < 1$, where $\mathbf{Q} = \mathcal{L}\mathbf{A} + \mathbf{B}$ and $\mathbf{A}$ and $\mathbf{B}$ are given by (3.1) and (3.2), respectively.*

Proof. Let $\mathcal{B}_r = \{u \in \mathbb{E} : \|u\| \leq r\}$ be a closed ball with the radius $r \geq \mathcal{M}\mathbf{A}/(1-\mathbf{Q})$ where

$$\mathcal{M} = \sup_{t\in[0,1]} |f(t,0)|.$$

Then, for $u \in \mathcal{B}_r$, we have

$$\begin{aligned}\|f(t,u(t))\| &= \sup_{t\in[0,1]} |f(t,u(t)) - f(t,0) + f(t,0)| \\ &\leq \sup_{t\in[0,1]} |f(t,u(t)) - f(t,0)| + \sup_{t\in[0,1]} |f(t,0)| \leq \mathcal{L}\|u\| + \mathcal{M} \leq \mathcal{L}r + \mathcal{M}.\end{aligned}$$

From this, we obtain

$$\begin{aligned}\|U(u)\| \leq &\sum_{i=1}^{\infty} |\beta_i| \left\{ |\lambda|(\mathcal{L}r+\mathcal{M}) \int_0^{\xi_i} \frac{(\xi_i - s)^{\alpha+\gamma-1}}{\Gamma(\alpha+\gamma)} ds + \lambda^2 r \int_0^{\xi_i} \frac{(\xi_i - s)^{\alpha-1}}{\Gamma(\alpha)} ds \right. \\ &\left. +(\mathcal{L}r+\mathcal{M}) \int_0^{\xi_i} \frac{(\xi_i - s)^{\gamma-1}}{\Gamma(\gamma)} ds \right\} + |\lambda|(\mathcal{L}r+\mathcal{M}) \int_0^1 \frac{(1-s)^{\alpha+\gamma-1}}{\Gamma(\alpha+\gamma)} ds \\ &+ \lambda^2 r \int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} ds + (\mathcal{L}r+\mathcal{M}) \int_0^1 \frac{(1-s)^{\gamma-1}}{\Gamma(\gamma)} ds \\ &\leq \sum_{i=1}^{\infty} |\beta_i| \left\{ \frac{|\lambda|(\mathcal{L}r+\mathcal{M})\xi_i^{\alpha+\gamma}}{\Gamma(\alpha+\gamma+1)} + \frac{\lambda^2 r \xi_i^{\alpha}}{\Gamma(\alpha+1)} + \frac{(\mathcal{L}r+\mathcal{M})\xi_i^{\gamma}}{\Gamma(\gamma+1)} \right\} \\ &+ \frac{|\lambda|(\mathcal{L}r+\mathcal{M})}{\Gamma(\alpha+\gamma+1)} + \frac{\lambda^2 r}{\Gamma(\alpha+1)} + \frac{\mathcal{L}r+\mathcal{M}}{\Gamma(\gamma+1)}.\end{aligned}$$

Whence, we have

$$\begin{aligned}\|Tu(t)\| &= \sup_{t\in[0,1]} \left| \int_0^t \frac{(t-s)^{\alpha+\gamma-1}}{\Gamma(\alpha+\gamma)} f(s,u(s)) ds - \lambda \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} u(s) ds - \frac{t^{\alpha+1}}{\Delta\Gamma(\alpha+2)} U(u) \right| \\ &\leq \sup_{t\in[0,1]} \left[(\mathcal{L}r+\mathcal{M}) \int_0^t \frac{(t-s)^{\alpha+\gamma-1}}{\Gamma(\alpha+\gamma)} ds + |\lambda| r \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} ds \right] + \frac{\|U\|}{|\Delta|\Gamma(\alpha+2)} \\ &\leq \frac{(\mathcal{L}r+\mathcal{M})}{\Gamma(\alpha+\gamma+1)} + \frac{|\lambda| r}{\Gamma(\alpha+1)} \\ &+ \frac{1}{|\Delta|\Gamma(\alpha+2)} \sum_{i=1}^{\infty} |\beta_i| \left\{ \frac{|\lambda|(\mathcal{L}r+\mathcal{M})\xi_i^{\alpha+\gamma}}{\Gamma(\alpha+\gamma+1)} + \frac{\lambda^2 r \xi_i^{\alpha}}{\Gamma(\alpha+1)} + \frac{(\mathcal{L}r+\mathcal{M})\xi_i^{\gamma}}{\Gamma(\gamma+1)} \right\} \\ &+ \frac{1}{|\Delta|\Gamma(\alpha+2)} \left\{ \frac{|\lambda|(\mathcal{L}r+\mathcal{M})}{\Gamma(\alpha+\gamma+1)} + \frac{\lambda^2 r}{\Gamma(\alpha+1)} + \frac{\mathcal{L}r+\mathcal{M}}{\Gamma(\gamma+1)} \right\} \\ &= (\mathcal{L}\mathbf{A} + \mathbf{B})r + \mathcal{M}\mathbf{A} = \mathbf{Q}r + \mathcal{M}\mathbf{A} \leq r\end{aligned}$$

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which leads to $Tu \subset \mathcal{B}_r$. Now, let $u, v \in \mathcal{B}_r$, then we have

$$
\begin{aligned}
\|U(u) - U(v)\| \leq \sum_{i=1}^{\infty} |\beta_i| \Big\{ |\lambda| \int_0^{\xi_i} \frac{(\xi_i - s)^{\alpha+\gamma-1}}{\Gamma(\alpha+\gamma)} |f(s,u(s)) - f(s,v(s))| ds \\
&+ \lambda^2 \int_0^{\xi_i} \frac{(\xi_i - s)^{\alpha-1}}{\Gamma(\alpha)} |u(s) - v(s)| ds \\
&+ \int_0^{\xi_i} \frac{(\xi_i - s)^{\gamma-1}}{\Gamma(\gamma)} |f(s,u(s)) - f(s,v(s))| ds \Big\} \\
&+ |\lambda| \int_0^{1} \frac{(1 - s)^{\alpha+\gamma-1}}{\Gamma(\alpha+\gamma)} |f(s,u(s)) - f(s,v(s))| ds \\
&+ \lambda^2 \int_0^{1} \frac{(1 - s)^{\alpha-1}}{\Gamma(\alpha)} |u(s) - v(s)| ds \\
&+ \int_0^{1} \frac{(1 - s)^{\gamma-1}}{\Gamma(\gamma)} |f(s,u(s)) - f(s,v(s))| ds \\
&\leq \|u - v\| \sum_{i=1}^{\infty} |\beta_i| \Big\{ |\lambda| \mathcal{L} \int_0^{\xi_i} \frac{(\xi_i - s)^{\alpha+\gamma-1}}{\Gamma(\alpha+\gamma)} ds + \lambda^2 \int_0^{\xi_i} \frac{(\xi_i - s)^{\alpha-1}}{\Gamma(\alpha)} ds \\
&+ \mathcal{L} \int_0^{\xi_i} \frac{(\xi_i - s)^{\gamma-1}}{\Gamma(\gamma)} ds \Big\} + \|u - v\| \Big\{ |\lambda| \mathcal{L} \int_0^{1} \frac{(1 - s)^{\alpha+\gamma-1}}{\Gamma(\alpha+\gamma)} ds \\
&+ \lambda^2 \int_0^{1} \frac{(1 - s)^{\alpha-1}}{\Gamma(\alpha)} ds + \mathcal{L} \int_0^{1} \frac{(1 - s)^{\gamma-1}}{\Gamma(\gamma)} ds \Big\} \\
&\leq \|u - v\| \sum_{i=1}^{\infty} |\beta_i| \left\{ \frac{|\lambda| \mathcal{L} \xi_i^{\alpha+\gamma}}{\Gamma(\alpha+\gamma+1)} + \frac{\lambda^2 \xi_i^{\alpha}}{\Gamma(\alpha+1)} + \frac{\mathcal{L} \xi_i^{\gamma}}{\Gamma(\gamma+1)} \right\} \\
&+ \|u - v\| \left\{ \frac{|\lambda| \mathcal{L}}{\Gamma(\alpha+\gamma+1)} + \frac{\lambda^2}{\Gamma(\alpha+1)} + \frac{\mathcal{L}}{\Gamma(\gamma+1)} \right\}.
\end{aligned}
$$

Therefore, we have

$$
\begin{aligned}
\|(Tu)(t) - (Tv)(t)\| \leq \sup_{t \in [0,1]} \int_0^t \frac{(t - s)^{\alpha+\gamma-1}}{\Gamma(\alpha+\gamma)} |f(s,u(s)) - f(s,u(s))| ds \\
&+ \sup_{t \in [0,1]} |\lambda| \int_0^t \frac{(t - s)^{\alpha-1}}{\Gamma(\alpha)} |u(s) - v(s)| ds + \frac{\|U(u) - U(v)\|}{|\Delta| \Gamma(\alpha+2)} \\
&\leq \|u - v\| \sup_{t \in [0,1]} \left[\mathcal{L} \int_0^t \frac{(t - s)^{\alpha+\gamma-1}}{\Gamma(\alpha+\gamma)} ds + |\lambda| \int_0^t \frac{(t - s)^{\alpha-1}}{\Gamma(\alpha)} ds \right] + \frac{\|U(u) - U(v)\|}{|\Delta| \Gamma(\alpha+2)} \\
&\leq \frac{\mathcal{L} \|u - v\|)}{\Gamma(\alpha+\gamma+1)} + \frac{|\lambda| \|u - v\|}{\Gamma(\alpha+1)} + \frac{\|U(u) - U(v)\|}{|\Delta| \Gamma(\alpha+2)} \\
&= (\mathcal{L}\mathbf{A} + \mathbf{B}) \|u - v\| = \mathbf{Q} \|u - v\|.
\end{aligned}
$$

By the hypothesis $\mathbf{Q} < 1$, it follows that the operator T defined in (3.3) is a contraction. Therefore, with Banach contraction mapping principle, we deduce that the operator T has a fixed point, which equivalently implies that the boundary value problem (1.1)-(1.2) has a unique solution on $[0, 1]$. □

Theorem 3.2. *Assume that the assumptions $(\mathcal{H}_1)$ and $(\mathcal{H}_3)$ hold. Then the boundary value problem (1.1)-(1.2) has at least one solution if there exists a constant $M > 0$ such that $\mathbf{K} > 1$ where $\mathbf{K}$ is given*

$$
\mathbf{K} = \frac{M}{\|\omega\| \varphi(M) \mathbf{A} + M \mathbf{B}}.
$$

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Proof. The continuity of the function f implies that the operator $T : \mathbb{E} \to \mathbb{E}$ defined by (3.3) is continuous. Assume that $\mathcal{B}_r = \{u \in \mathbb{E} : \|u\| < r\}$ be an open subset of the Banach space $\mathbb{E}$ with radius $r > 0$. First, we are in a position to prove that the operator $T : \mathbb{E} \to \mathbb{E}$ is completely continuous. Assume that $u \in \mathcal{B}_r$. Then, we have

$$
\begin{aligned}
\|U(u)\| \leq & \sum_{i=1}^{\infty} |\beta_i| \left\{ |\lambda| \int_0^{\xi_i} \frac{(\xi_i - s)^{\alpha+\gamma-1}}{\Gamma(\alpha+\gamma)} |f(s,u(s))| ds + \lambda^2 \int_0^{\xi_i} \frac{(\xi_i - s)^{\alpha-1}}{\Gamma(\alpha)} |u(s)| ds \right. \\
& \left. + \int_0^{\xi_i} \frac{(\xi_i - s)^{\gamma-1}}{\Gamma(\gamma)} |f(s,u(s))| ds \right\} + |\lambda| \int_0^1 \frac{(1-s)^{\alpha+\gamma-1}}{\Gamma(\alpha+\gamma)} |f(s,u(s))| ds \\
& + \lambda^2 \int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} |u(s)| ds + \int_0^1 \frac{(1-s)^{\gamma-1}}{\Gamma(\gamma)} |f(s,u(s))| ds \\
\leq & \sum_{i=1}^{\infty} |\beta_i| \left\{ |\lambda| \|\omega\| \varphi(\|u\|) \int_0^{\xi_i} \frac{(\xi_i - s)^{\alpha+\gamma-1}}{\Gamma(\alpha+\gamma)} ds + \lambda^2 r \int_0^{\xi_i} \frac{(\xi_i - s)^{\alpha-1}}{\Gamma(\alpha)} ds \right. \\
& \left. + \|\omega\| \varphi(\|u\|) \int_0^{\xi_i} \frac{(\xi_i - s)^{\gamma-1}}{\Gamma(\gamma)} ds \right\} + |\lambda| \|\omega\| \varphi(\|u\|) \int_0^1 \frac{(1-s)^{\alpha+\gamma-1}}{\Gamma(\alpha+\gamma)} ds \\
& + \lambda^2 r \int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} ds + \|\omega\| \varphi(\|u\|) \int_0^1 \frac{(1-s)^{\gamma-1}}{\Gamma(\gamma)} ds \\
\leq & \sum_{i=1}^{\infty} |\beta_i| \left\{ \frac{|\lambda| \|\omega\| \varphi(r) \xi_i^{\alpha+\gamma}}{\Gamma(\alpha+\gamma+1)} + \frac{\lambda^2 r \xi_i^{\alpha}}{\Gamma(\alpha+1)} + \frac{\|\omega\| \varphi(r) \xi_i^{\gamma}}{\Gamma(\gamma+1)} \right\} \\
& + \frac{|\lambda| \|\omega\| \varphi(r)}{\Gamma(\alpha+\gamma+1)} + \frac{\lambda^2 r}{\Gamma(\alpha+1)} + \frac{\|\omega\| \varphi(r)}{\Gamma(\gamma+1)} \\
= & \|\omega\| \varphi(r) \mathbf{A} + r \mathbf{B}.
\end{aligned}
$$

It follows that

$$
\begin{aligned}
\|Tu(t)\| = & \sup_{t \in [0,1]} \left| \int_0^t \frac{(t-s)^{\alpha+\gamma-1}}{\Gamma(\alpha+\gamma)} f(s,u(s)) ds - \lambda \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} u(s) ds - \frac{t^{\alpha+1}}{\Delta \Gamma(\alpha+2)} U(u) \right| \\
\leq & \sup_{t \in [0,1]} \left[\|\omega\| \varphi(\|u\|) \int_0^t \frac{(t-s)^{\alpha+\gamma-1}}{\Gamma(\alpha+\gamma)} ds + |\lambda| r \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} ds \right] + \frac{\|U(u)\|}{|\Delta| \Gamma(\alpha+2)} \\
\leq & \frac{\|\omega\| \varphi(r)}{\Gamma(\alpha+\gamma+1)} + \frac{|\lambda| r}{\Gamma(\alpha+1)} + \frac{\|U(u)\|}{|\Delta| \Gamma(\alpha+2)} \\
= & \|\omega\| \varphi(r) \mathbf{A} + r \mathbf{B}
\end{aligned}
$$

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which concludes the boundedness of the operator T. Suppose that $t_1, t_2 \in [0,1]$ such that $t_1 < t_2$, it follows that

$$\begin{aligned}
\|Tu(t_2) - Tu(t_1)\| \leq & \left\| \int_0^{t_2} \frac{(t_2-s)^{\alpha+\gamma-1}}{\Gamma(\alpha+\gamma)} f(s,u(s))ds - \int_0^{t_1} \frac{(t_1-s)^{\alpha+\gamma-1}}{\Gamma(\alpha+\gamma)} f(s,u(s))ds \right\| \\
& + |\lambda| \left\| \int_0^{t_2} \frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)} u(s)ds - \int_0^{t_1} \frac{(t_1-s)^{\alpha-1}}{\Gamma(\alpha)} u(s)ds \right\| \\
& + \frac{\|U(u)\|}{|\Delta|\Gamma(\alpha+2)} |t_2^{\alpha+1} - t_1^{\alpha+1}| \\
& \leq \int_0^{t_1} \frac{(t_2-s)^{\alpha+\gamma-1} - (t_1-s)^{\alpha+\gamma-1}}{\Gamma(\alpha+\gamma)} |f(s,u(s))|ds \\
& + \int_{t_1}^{t_2} \frac{(t_2-s)^{\alpha+\gamma-1}}{\Gamma(\alpha+\gamma)} |f(s,u(s))|ds + |\lambda| \int_{t_1}^{t_2} \frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)} |u(s)|ds \\
& + |\lambda| \int_0^{t_1} \frac{(t_1-s)^{\alpha-1} - (t_2-s)^{\alpha-1}}{\Gamma(\alpha)} |u(s)|ds + \frac{\|U(u)\|}{|\Delta|\Gamma(\alpha+2)} |t_2^{\alpha+1} - t_1^{\alpha+1}| \\
& \leq \frac{|\lambda|\|\omega\|\varphi(r)}{\Gamma(\alpha+\gamma+1)} |t_2^{\alpha+\gamma} - t_1^{\alpha+\gamma}| + \frac{|\lambda| r}{\Gamma(\alpha+1)} |2(t_2-t_1)^{\alpha} - (t_2^{\alpha} - t_1^{\alpha})| \\
& + \frac{\|U(u)\|}{|\Delta|\Gamma(\alpha+2)} |t_2^{\alpha+1} - t_1^{\alpha+1}|.
\end{aligned}$$

It is clear that the right-hand side of the above inequality approaches zero independently of $u \in \mathbb{E}$ as $t_1 \to t_2$. Since the operator T satisfies the above assumptions, it follows by the Arzela-Ascoli theorem that $T : \mathbb{E} \to \mathbb{E}$ is completely continuous.

According to the Leray-Schauder nonlinear alternative Lemma 2.5, the result will follow once we prove the boundedness of the set of all solutions to equations $u = \delta Tu$ for some $\delta \in [0,1]$. Let u is a solution of the equation $u = \delta Tu$ for some $\delta \in [0,1]$, then for all $t \in [0,1]$, from the boundedness of the operator T, we have

$$\|u\| = \sup_{t\in[0,1]} |u(t)| = \sup_{t\in[0,1]} |\delta(Tu)(t)| \leq \|\omega\|\varphi(\|u\|)\mathbf{A} + \|u\|\mathbf{B}$$

which implies that

$$\frac{\|u\|}{\|\omega\|\varphi(\|u\|)\mathbf{A} + \|u\|\mathbf{B}} \leq 1.$$

By the assumption $\mathbf{K} > 1$, then there exists a constant $M > 0$ such that $\|u\| \neq M$. Setting the open set

$$\Omega = \{u \in \mathbb{E} : \|u\| < M\}.$$

Based on the form of Ω, there is no $u \in \partial\Omega$ such that $u = \delta T(u)$ for some $\delta \in (0,1)$. Since the operator $T : \overline{\Omega} \to \mathbb{E}$ is continuous and completely continuous, then by the nonlinear alternative of Leray-Schauder type Lemma 2.5, we deduce that T has a fixed point $u \in \overline{\Omega}$ which is a solution of problem (1.1)-(1.2). This ends the proof. □

Theorem 3.3. *Assume that the assumptions ($\mathcal{H}_1$) and ($\mathcal{H}_4$) hold. Then the boundary value problem (1.1)-(1.2) has at least one solution if*

$$0 < \eta < \frac{1-\mathbf{B}}{\mathbf{A}}$$

where $\mathbf{A}$ *and* $\mathbf{B}$ *are given by (3.1) and (3.2), respectively.*

Proof. Let us define the open ball $\mathcal{B}_r \subset \mathbb{E}$ with radius $r > 0$ as

$$\mathcal{B}_r = \{u \in \mathbb{E} : \|u\| < r\}$$

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where r will be determined later. It is adequate to prove that the operator $T : \overline{\mathcal{B}_r} \to \mathbb{E}$ satisfies

$$u \neq \lambda Tu, \quad \forall u \in \partial\mathcal{B}_r, \quad \sigma \in [0,1]. \tag{3.4}$$

To do this, assume that $u = \sigma Tu$ for some $\sigma \in [0,1]$. Then, as in the preceding results, we have

$$\|u\| \leq (\eta\mathbf{A} + \mathbf{B})\|u\| + L\mathbf{A}$$

which implies that

$$\|u\| \leq \frac{L\mathbf{A}}{1-(\eta\mathbf{A}+\mathbf{B})}$$

provided that $\eta\mathbf{A} + \mathbf{B} < 1$ which leads to $\eta < (1-\mathbf{B})/\mathbf{A}$. Now, suppose that there exists $\epsilon > 0$ such that

$$r = \frac{L\mathbf{A}}{1-(\eta\mathbf{A}+\mathbf{B})} + \epsilon.$$

By the analysis above, it follows that the relation (3.4) holds. Let us now define the continuous operator

$$h_\sigma(u) = u - \sigma Tu, \quad u \in \mathbb{E}, \quad \sigma \in [0,1].$$

In view of the results in Theorems above, it is clear that the operator $h_\sigma : \mathbb{E} \to \mathbb{E}$ is completely continuous. By the homotopy invariance of topological degree, it follows that

$$\deg(h_\sigma, \mathcal{B}_r, 0) = \deg(h_1, \mathcal{B}_r, 0) = \deg(h_0, \mathcal{B}_r, 0) = \deg(I, \mathcal{B}_r, 0) = 1 \neq 0$$

where I is the unit operator. By the nonzero property of the Leray-Schauder degree type, the equation $u = Tu$ has at least one solution in $\mathcal{B}_r$. That is, the boundary value problem (1.1)-(1.2) has at least one solution in $[0,1]$. □

4. Numerical Example

We will present the same example that is taken by [1] to illustrate our main results.

Example 4.1. *Consider the following boundary value problem for fractional Langevin equations:*

$$\begin{cases} {}^cD^\alpha({}^cD^\gamma + \lambda u(t) = f(t,u(t)), & 0 < t < 1 \\ u(0) = 0, \quad {}^cD^\alpha u(0) = 0, & {}^cD^\alpha u(1) = \sum_{i=1}^\infty \beta_i \, {}^cD^\alpha u(\xi_i) \end{cases} \tag{4.1}$$

Here we take $\alpha = 1/2, \gamma = 3/2, \lambda = 1/4, \beta_i = 1/2^i, \xi_i = 1/3^i$ *and*

$$f(t,u(t)) = \frac{u}{100(1+t^4)} + \frac{u}{10(1+t^{\frac{3}{2}})^5} + 1$$

$\mathbf{A} = 1.64833$ *and* $\mathbf{B} = 0.400111$ *and* $\eta < (1-\mathbf{B})/\mathbf{A} = 0.363938.$

It is easy to show that $f(t,u(t)) \leq 0.11\|u\| + 1 < \eta\|u\| + L$ for all $L \geq 1$ and $\eta < 0.363938$ which means that it satisfies our conditions in Theorem 3.3 and so the boundary value problem (4.1) has at least one solution on $[0,1]$.

By recalculate the calculations obtained by Li et al [1], we find that they are incorrect and the true value of η must be less than or equal 0.19315. This means that our result extend the domain of the restriction on η. To illustrate our results are better than their results, we provide the following tables that show the results obtained in this parer are superior.

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5. Conclusion

The existence and uniqueness of solutions for nonlinear Langevin equations involving two fractional orders with infinite-point boundary value problem (1.1)-(1.2) has been discussed. We apply the concepts of fractional calculus together with fixed point theorems to establish the existence and uniqueness results. To investigate our problem, we apply Banach contraction principle, nonlinear alternative Leray-Schauder theorem and Leray-Schauder degree theorem. Our approach is simple and is applicable to a variety of real world problems.

Numerical example is provided to introduce a comparison between our results and the results obtained by Li et al [1]. The comparison turns out that our results are superior.

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