Article

5D-Euler Equations for Rotating Bodies

Vladimir Kobelev *

Posted Date: 30 September 2025

doi: 10.20944/preprints202508.0136.v2

Keywords: Euler equations; multidimensional Euclidian spaces; tensor formulations

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 30 September 2025 doi:10.20944/preprints202508.0136.v2

Article

5D-Euler Equations for Rotating Bodies

Vladimir Kobelev

University of Siegen, Department of Natural Sciences, Paul-Bonatz-Str.9-11, 57078 Siegen, Germany;
vladimir.kobelev@uni-siegen.de or kobelev@imr.mb.uni-siegen.de

Abstract

The manuscript undertakes a study of the rotational behaviour of rigid bodies in spaces of higher dimensions. The primary objective of the manuscript is the derivation of the 5D Euler equations. The closed-form solutions of the 5D-Euler equations are presented. The visualization of the observable motions and its dependence upon the hypothetical parameters of the 5D-state are demonstrated in closed form. Within the paradigm of four-dimensional Euclidean spaces, the number of rotational degrees of freedom is six. In the case of a five-dimensional Euclidean space, the number of rotational degrees of freedom is increased to ten. The Euler equations are derived using the tensor representation of rotational velocities. The closed-form solutions were discovered for a specific relationship between the principal moments of inertia.

Keywords: Euler equations; multidimensional Euclidian spaces; tensor formulations

---

1. Rotation of Rigid Bodies

In Newtonian mechanics, a rigid body is defined as an extended solid piece of matter, whereby the relative position of any two of its points remains constant. It is evident from an intuitive standpoint that the motion of this body can be delineated in relation to an arbitrary "space-fixed" inertial reference frame with origin O. This can be achieved by first fixing an arbitrary point O' within the body and subsequently determining the orientation of an arbitrary body-fixed Cartesian basis system.

The motion of the rigid body can now be characterized as follows: The location of the body-fixed origin is specified by the position vector whilst the relative orientation of the body is determined by the rotation matrix, which establishes the body-fixed basis in relation to the space-fixed basis (Borisov & Mamaev, 2019).

The results of the mathematical investigation suggest a novel interpretation of the spin phenomenon. Spin angular momentum, also referred to as "spin," is a fundamental property inherent to all particles, irrespective of their status as elementary or composite entities. This property is intrinsic, meaning it is not dependent on spatial coordinates, and manifests as the particle's inherent angular momentum. The subsequent study will explore the quantization of the rotation motion of four-dimensional rigid bodies in order to analyze the symmetries and relations between the virtual moments of inertia in higher dimensions. It is possible to make these observations through experimentation using three-dimensional projections of four- or five-dimensional hypothetical spaces at the microscopic scale.

1.1. Tensor Description of the Rotations

In any given number of spatial dimensions, rotational dynamics can be characterized by mechanical properties embodied in bivectors—that is, antisymmetric rank-2 tensors. (Cayley, 1846), (Hollmeier, 2024), (Parker, 2024). The rotation velocity is represented by an antisymmetric tensor of rank 2:

$$(1) \quad \boldsymbol{\omega} = [\![\omega_{kl}]\!], \quad \omega_{kl} = -\omega_{lk}.$$

doi:10.20944/preprints202508.0136.v2

The elementary rotation of the body takes place along a specific axis. According to the established principles of rotational dynamics, the axis of rotation is determined to be normal to the plane of rotation. The plane of rotation is characterized by the two constituent vectors, which are designated as follows: l [1]. The elementary rotation ω_{kl} is thus characterized by the indices of both vectors k and l . The angular velocity (1) does not change if chooses a different origin for KS. The orientation of the body-fixed system relative to the IS is determined by the degrees of freedom. The non-zero elements of matrix (1) can be described as generalized coordinates. Their number is equal to:

$$(2) \qquad N_\omega = \frac{N\,(N-1)}{2}.$$

1.2. Inertia Tensor

It is evident that each constituent of the momentum is a linear function of the angular velocity component. In the context of vector notation, the angular velocity vectors are interconnected through a linear transformation, which is symbolized by the rank-2 inertia tensor. The moment of inertia is defined as the angular acceleration produced by an applied torque. Angular acceleration is contingent upon three factors. Firstly, the geometry and mass distribution of the body must be taken into consideration. Secondly, the orientation of the rotation axis is important. The concept of moment of inertia, analogous to the role of mass in translational motion, signifies the body's resistance to alterations in its state of motion. In an inhomogeneous body characterized by a continuous mass distribution, it has been demonstrated that the density is found to vary with the spatial coordinate.

In tensor notation, the rotational velocity is related to the angular momentum by the certain rank-4 inertia tensor $\boldsymbol{I} = [\![\boldsymbol{I}^{ijkl}]\!]$. In the rotation part we use the representation in the components of BS, with the inertia tensor (Parker, 2024) :

$$(3) \qquad \begin{cases} I^{ijkl} = -\int_\Omega 2\, r^{[i}\delta^{j][k}r^{l]}\; dm = \frac{1}{2}\int_\Omega \left(-r^i\delta^{jk}r^l + r^i\delta^{jl}r^k + r^j\delta^{ik}r^l - r^j\delta^{il}r^k\right) dm, \\ \qquad I^{ijkl} = -I^{jikl} = -I^{ijlk}, \quad I^{ijkl} = I^{klij}, I^{ijkl} + I^{iklj} + I^{iljk} = 0\,. \end{cases}$$

The expression (3) results if for a continuous mass density. It is essential that the quantities $r_i\,\delta_{jk}\,r_l$ are independent of the translational motion of the rigid body; since their calculation is carried out in BS.

1.3. Angular Momentum

Angular momentum, denoted by $\boldsymbol{L}$, is defined as the product of the moment of inertia, relative to the rotation axis, $\boldsymbol{I}$, and the angular velocity, $\boldsymbol{\omega}$:

$$(4) \qquad \boldsymbol{L} = \boldsymbol{I} \cdot\cdot\, \boldsymbol{\omega}, \quad L^{ij} = I^{ijkl}\omega_{kl}.$$

In accordance with the stipulations enumerated by the Einstein summation convention, in instances where an index variable is present in two separate terms within a singular expression yet remains unambiguously defined, it is conventionally interpreted as an implicit summation of the term in question across the entirety of possible values that the index might assume. For the sake of convenience and to facilitate summation, there is justification for the employment of both co- and contravariant indices. The components of tensors $\boldsymbol{I}$ and $\boldsymbol{\omega}$ are skew-symmetric. Therefore, it can be

---

[1] The Latin indices are running from 1 to N. The Einstein summation convention is employed in this context, stipulating that in any expression in which two equal indices emerge, the index pair must be summed over from 1 to N. These rules are as follows: 1. Repeated indices are implicitly summed over. 2. It is imperative to note that each index is permitted to appear no more than twice in any given term. 3. It is imperative that each term contains identical non-repeated indices.

concluded that their contraction product (6) is also skew-symmetric. The angular momentum is the antisymmetric tensor of rank 2:

$$(5) \quad \boldsymbol{L} = [\![L^{ij}]\!], \quad L^{ij} = -L^{ji}.$$

In the event that the body-fixed coordinate system BS is established at the center of mass, the kinetic energy of a rigid body is comprised of two distinct components (Krey & Owen, 2007). The initial segment pertains to the kinetic energy of translation of the center of mass with a specific velocity. The second component is the kinetic energy of rotational motion with angular velocity $\boldsymbol{\omega}$ about an axis that passes through the center of mass:

$$(6) \quad W = \frac{1}{2}\boldsymbol{\omega} \cdot\cdot\, \boldsymbol{I} \cdot\cdot\, \boldsymbol{\omega} \quad = \frac{1}{2}\omega_{ij} I^{ijkl} \omega_{kl}.$$

The mixed terms in (6) are eliminated here because we choose the center of gravity as the origin of KS. Consequently, the kinetic energy is dissociated into two distinct components: a translational component and a rotational component. In the event that the upper surface is found to contain a support point, the designation of said point as the origin results in the elimination of all but the rotational component. In the context of rotational motions, it can be posited that torque is equivalent to the change in angular momentum, $\boldsymbol{L}$ over a specified interval of time for a given external torque $\boldsymbol{m}$ (Hestenes, 2002):

$$(7) \quad d\boldsymbol{L}/dt = \boldsymbol{m}.$$

The external torque $\boldsymbol{m}$ is the antisymmetric tensor of rank 2. In the event of the applied external torque being set to zero, the angular momentum is shown to be an integral of motion. The second norm of angular momentum is an integral of motion as well. The coefficients of the characteristic polynomials of $\boldsymbol{L}$ are the integrals of motions in this case.

2. Euler's Equations in 5D

2.1. Rotational Transformations in 5D

For the derivation of Euler equation, we introduce ten Euler angles:

$$(8) \quad \alpha_{12}, \alpha_{13}, \alpha_{14}, \alpha_{15}, \alpha_{23}, \alpha_{24}, \alpha_{25}, \alpha_{34}, \alpha_{35}, \alpha_{45}.$$

Each Euler angle corresponds to a basic rotation of the body around a specific rotation plane. The body system BS axes, marked by indices i and j, remain in the rotation plane ij. The angle between the old vector and the new vector with the index i is α_{ij}. Correspondingly, the same angle exists between the new and old second vectors with the index j.

The first rotation matrix, which corresponds to the rotation in plane, marked by indices 1 and 2, reads:

$$(9) \quad \boldsymbol{T}^{(12)} = \begin{bmatrix} \cos(\delta\,\alpha_{12}) & -\sin(\delta\,\alpha_{12}) & & & \\ \sin(\delta\,\alpha_{12}) & \cos(\delta\,\alpha_{12}) & & & \\ & & 1 & & \\ & & & 1 & \\ & & & & 1 \end{bmatrix}.$$

The void components of the matrix are all zero. To make the derivation easier, we introduce an auxiliary parameter ε.

The second rotation matrix, which corresponds to the rotation in plane, marked by indices 1 and 3, reads:

$$(10)\quad \boldsymbol{T}^{(13)}=\begin{bmatrix} \cos(\delta\,\alpha_{13}) & & \sin(\delta\,\alpha_{13}) & & \\ & 1 & & & \\ -\sin(\delta\,\alpha_{13}) & & \cos(\delta\,\alpha_{13}) & & \\ & & & 1 & \\ & & & & 1 \end{bmatrix}.$$

The third rotation matrix, which corresponds to a rotation in the 1-4 plane, is given by:

$$(11)\quad \boldsymbol{T}^{(14)}=\begin{bmatrix} \cos(\delta\,\alpha_{14}) & & & -\sin(\delta\,\alpha_{14}) & \\ & 1 & & & \\ & & 1 & & \\ \sin(\delta\,\alpha_{14}) & & & \cos(\delta\,\alpha_{14}) & \\ & & & & 1 \end{bmatrix}.$$

The fourth rotation matrix, which corresponds to the rotation in plane marked by indices 1 and 5, reads:

$$(12)\quad \boldsymbol{T}^{(15)}=\begin{bmatrix} \cos(\delta\,\alpha_{15}) & & & & \sin(\delta\,\alpha_{15}) \\ & 1 & & & \\ & & 1 & & \\ & & & 1 & \\ -\sin(\delta\,\alpha_{15}) & & & & \cos(\delta\,\alpha_{15}) \end{bmatrix}.$$

The fifth rotation matrix, equivalent to the rotation in plane 2 and 3 as indicated by the indices, has the following form:

$$(13)\quad \boldsymbol{T}^{(23)}=\begin{bmatrix} 1 & & & & \\ & \cos(\delta\,\alpha_{23}) & -\sin(\delta\,\alpha_{23}) & & \\ & \sin(\delta\,\alpha_{23}) & \cos(\delta\,\alpha_{23}) & & \\ & & & 1 & \\ & & & & 1 \end{bmatrix}.$$

The sixth rotation matrix, which corresponds to the rotation in plane, marked by indices 2 and 4, reads:

$$(14)\quad \boldsymbol{T}^{(24)}=\begin{bmatrix} 1 & & & & \\ & \cos(\delta\,\alpha_{24}) & & \sin(\delta\,\alpha_{24}) & \\ & & 1 & & \\ & -\sin(\delta\,\alpha_{24}) & & \cos(\delta\,\alpha_{24}) & \\ & & & & 1 \end{bmatrix}.$$

The seventh rotation matrix, which corresponds to the rotation in plane, marked by indices 2 and 5, reads:

$$(15)\quad \boldsymbol{T}^{(25)}=\begin{bmatrix} 1 & & & & \\ & \cos(\delta\,\alpha_{25}) & & & -\sin(\delta\,\alpha_{25}) \\ & & 1 & & \\ & & & 1 & \\ & \sin(\delta\,\alpha_{25}) & & & \cos(\delta\,\alpha_{25}) \end{bmatrix}.$$

The eighth rotation matrix, which corresponds to the rotation in plane, marked by indices 3 and 4, reads:

$$(16)\quad \boldsymbol{T}^{(34)}=\begin{bmatrix} 1 & & & & \\ & 1 & & & \\ & & \cos(\delta\,\alpha_{34}) & -\sin(\delta\,\alpha_{34}) & \\ & & \sin(\delta\,\alpha_{34}) & \cos(\delta\,\alpha_{34}) & \\ & & & & 1 \end{bmatrix}.$$

The ninth rotation matrix, equivalent to the rotation in plane 3 and 5 as indicated by the indices, has the following form:

$$(17)\quad \boldsymbol{T}^{(35)} = \begin{bmatrix} 1 & & & & \\ & 1 & & & \\ & & \cos(\delta\,\alpha_{35}) & & \sin(\delta\,\alpha_{35}) \\ & & & 1 & \\ & & -\sin(\delta\,\alpha_{35}) & & \cos(\delta\,\alpha_{35}) \end{bmatrix}.$$

It is finally possible to present the tenth rotation matrix, equivalent to the rotation in plane 45 as indicated by its indices. The following form is to be assigned to this matrix:

$$(18)\quad \boldsymbol{T}^{(45)} = \begin{bmatrix} 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & \cos(\delta\,\alpha_{45}) & -\sin(\delta\,\alpha_{45}) \\ & & & \sin(\delta\,\alpha_{45}) & \cos(\delta\,\alpha_{35}) \end{bmatrix}.$$

The direction of rotation is not specified in any particular case. In the context of symmetry, the rotation is assumed to occur in a clockwise direction if the sum of the indices is odd. Conversely, in the event of the sum of the indices being even, the rotation will occur in the opposite direction.

In the instance of ten elementary rotations occurring simultaneously, the rotation matrix $\boldsymbol{T}$ of the body is equivalent to the product of all ten matrices:

$$(19)\quad \boldsymbol{T} = T^{(12)}T^{(13)}T^{(14)}T^{(15)}T^{(23)}T^{(24)}T^{(25)}T^{(34)}T^{(35)}T^{(45)}.$$

In order to calculate angular velocities, it is necessary to employ the Taylor expansion of each component of the matrix T with respect to the parameter δ. Subsequently, it is imperative to retain solely the terms that are linear with respect to the expansion parameter δ, thereby resulting in the final rotation matrix:

$$(20)\quad \tilde{\boldsymbol{T}} = \begin{bmatrix} 1 & -\delta\alpha_{12} & \delta\alpha_{13} & -\delta\alpha_{14} & \delta\alpha_{15} \\ \delta\alpha_{12} & 1 & -\delta\alpha_{23} & \delta\alpha_{24} & -\delta\alpha_{25} \\ -\ \delta\alpha_{13} & \delta\alpha_{23} & 1 & -\delta\alpha_{34} & \delta\alpha_{33} \\ \delta\alpha_{14} & -\delta\alpha_{24} & \delta\alpha_{34} & 1 & -\delta\alpha_{45} \\ -\delta\alpha_{15} & \delta\alpha_{25} & -\delta\alpha_{35} & \delta\alpha_{45} & 1 \end{bmatrix}.$$

Let the body-fixed system BS be specifically the principal axis system $N = 5$. From the angle variation we get immediately the angular velocity:

$$(21)\quad \boldsymbol{\omega} = \left[\left[\begin{matrix} 0 & -\omega_{12} & \omega_{13} & -\omega_{14} & \omega_{15} \\ \omega_{12} & 0 & -\omega_{23} & \omega_{24} & -\omega_{25} \\ -\omega_{13} & \omega_{23} & 0 & -\omega_{34} & \omega_{35} \\ \omega_{14} & -\omega_{24} & \omega_{34} & 0 & -\omega_{45} \\ -\omega_{15} & \omega_{25} & -\omega_{35} & \omega_{45} & 0 \end{matrix}\right]\right], \text{ with } \omega_{\mathrm{ij}} = \dot{\alpha}_{\mathrm{ij}}.$$

Let the body-fixed system BS be specifically the principal axis system $N = 5$. It is evident that, since the consideration is situated within the principal axes of the body, the coefficients of the matrix of angular momentum will be proportional to the coefficients of angular velocity:

$$(22)\quad \boldsymbol{L} = \left[\left[\begin{matrix} 0 & -L^{12} & L^{13} & -L^{14} & L^{15} \\ L^{12} & 0 & -L^{23} & L^{24} & -L^{25} \\ -L^{13} & L^{23} & 0 & -L^{34} & L^{35} \\ L^{14} & -L^{24} & L^{34} & 0 & -L^{45} \\ -L^{15} & L^{25} & -L^{35} & L^{45} & 0 \end{matrix}\right]\right].$$

In the rotated coordinate system, the first variation of the angular moment is equal to the commutator of the matrices $\tilde{\boldsymbol{T}}$ and $\boldsymbol{L}$, (Arnold, 1989), (Hestenes, 2002):

$$(23)\quad \boldsymbol{\delta L} = \tilde{\boldsymbol{T}}\boldsymbol{L} - \boldsymbol{L}\tilde{\boldsymbol{T}}.$$

Correspondingly, the time derivative of the angular moment is equal:

$$(24)\quad \dot{\boldsymbol{L}} = \boldsymbol{\omega L} - \boldsymbol{L\omega} \overset{\text{def}}{=} \boldsymbol{\mathcal{L}\omega}.$$

For the sake of brevity in the notation, the auxiliary antisymmetric matrix $\boldsymbol{\mathcal{L}}$ was introduced in Equation (24). The Components of the auxiliary matrix $\boldsymbol{\mathcal{L}}$ are displayed in the Table 1. It is evident that the enumeration of rows and columns is congruent with the enumeration stipulated in the Equation (8).

Table 1. Components of the auxiliary matrix $\boldsymbol{\mathcal{L}}$ in 5D.

0	$-\boldsymbol{L}^{23}$	$\boldsymbol{L}^{24}$	$-\boldsymbol{L}^{25}$	$-\boldsymbol{L}^{13}$	$\boldsymbol{L}^{14}$	$-\boldsymbol{L}^{15}$	0	0	0
$\boldsymbol{L}^{23}$	0	$\boldsymbol{L}^{34}$	$-\boldsymbol{L}^{35}$	$\boldsymbol{L}^{12}$	0	0	$-\boldsymbol{L}^{14}$	$\boldsymbol{L}^{15}$	0
$-\boldsymbol{L}^{24}$	$-\boldsymbol{L}^{34}$	0	$-\boldsymbol{L}^{45}$	0	$-\boldsymbol{L}^{12}$	0	$\boldsymbol{L}^{13}$	0	$-\boldsymbol{L}^{15}$
$\boldsymbol{L}^{25}$	$\boldsymbol{L}^{35}$	$\boldsymbol{L}^{45}$	0	0	0	$\boldsymbol{L}^{12}$	0	$-\boldsymbol{L}^{13}$	$\boldsymbol{L}^{14}$
$\boldsymbol{L}^{13}$	$-\boldsymbol{L}^{12}$	0	0	0	$-\boldsymbol{L}^{34}$	$\boldsymbol{L}^{35}$	$-\boldsymbol{L}^{24}$	$\boldsymbol{L}^{25}$	0
$-\boldsymbol{L}^{14}$	0	$\boldsymbol{L}^{12}$	0	$\boldsymbol{L}^{34}$	0	$\boldsymbol{L}^{45}$	$\boldsymbol{L}^{23}$	0	$-\boldsymbol{L}^{25}$
$\boldsymbol{L}^{15}$	0	0	$-\boldsymbol{L}^{12}$	$-\boldsymbol{L}^{35}$	$-\boldsymbol{L}^{45}$	0	0	$-\boldsymbol{L}^{23}$	$\boldsymbol{L}^{24}$
0	$\boldsymbol{L}^{14}$	$-\boldsymbol{L}^{13}$	0	$\boldsymbol{L}^{24}$	$-\boldsymbol{L}^{23}$	0	0	$-\boldsymbol{L}^{45}$	$-\boldsymbol{L}^{35}$
0	$-\boldsymbol{L}^{15}$	0	$\boldsymbol{L}^{13}$	$-\boldsymbol{L}^{25}$	0	$\boldsymbol{L}^{23}$	$\boldsymbol{L}^{45}$	0	$\boldsymbol{L}^{34}$
0	0	$\boldsymbol{L}^{15}$	$-\boldsymbol{L}^{14}$	0	$\boldsymbol{L}^{25}$	$-\boldsymbol{L}^{24}$	$\boldsymbol{L}^{35}$	$-\boldsymbol{L}^{34}$	0

The components of the product (24) reads:

$$(25)\quad \begin{cases}
\mathfrak{L}\omega^{12} = -L^{13}\omega^{23} - L^{14}\omega^{24} - L^{15}\omega^{25} + L^{23}\omega^{31} + L^{24}\omega^{14} + L^{25}\omega^{15} \\
\mathfrak{L}\omega^{13} = -L^{12}\omega^{23} + L^{14}\omega^{34} + L^{15}\omega^{35} + L^{23}\omega^{21} - L^{34}\omega^{14} - L^{35}\omega^{15} \\
\mathfrak{L}\omega^{14} = L^{12}\omega^{24} + L^{13}\omega^{34} - L^{15}\omega^{45} - L^{24}\omega^{21} - L^{34}\omega^{31} + L^{45}\omega^{15} \\
\mathfrak{L}\omega^{15} = -L^{12}\omega^{25} - L^{13}\omega^{35} - L^{14}\omega^{45} + L^{25}\omega^{21} + L^{35}\omega^{31} + L^{45}\omega^{14} \\
\mathfrak{L}\omega^{23} = -L^{12}\omega^{31} + L^{31}\omega^{21} - L^{24}\omega^{34} - L^{25}\omega^{35} + L^{34}\omega^{24} + L^{35}\omega^{25} \\
\mathfrak{L}\omega^{24} = L^{12}\omega^{14} - L^{14}\omega^{21} - L^{23}\omega^{34} + L^{25}\omega^{45} + L^{34}\omega^{23} - L^{45}\omega^{25} \\
\mathfrak{L}\omega^{25} = -L^{12}\omega^{15} + L^{35}\omega^{21} + L^{23}\omega^{35} + L^{24}\omega^{45} - L^{35}\omega^{23} - L^{45}\omega^{24} \\
\mathfrak{L}\omega^{34} = -L^{13}\omega^{14} + L^{14}\omega^{31} - L^{23}\omega^{24} + L^{24}\omega^{23} - L^{35}\omega^{45} + L^{45}\omega^{35} \\
\mathfrak{L}\omega^{35} = L^{13}\omega^{15} - L^{15}\omega^{31} + L^{23}\omega^{25} - L^{25}\omega^{23} - L^{34}\omega^{45} + L^{45}\omega^{34} \\
\mathfrak{L}\omega^{45} = -L^{13}\omega^{15} + L^{15}\omega^{14} - L^{24}\omega^{25} + L^{25}\omega^{24} - L^{34}\omega^{35} + L^{35}\omega^{34}
\end{cases}$$

With the expression (25) for the moments of inertia (4), the angular momentum and its time derivative are:

$$(26)\quad \boldsymbol{L} = \begin{bmatrix}
\boldsymbol{0} & -\boldsymbol{I}^{1212}\boldsymbol{\omega}_{12} & \boldsymbol{I}^{1313}\boldsymbol{\omega}_{13} & -\boldsymbol{I}^{1414}\boldsymbol{\omega}_{14} & \boldsymbol{I}^{1515}\boldsymbol{\omega}_{15} \\
\boldsymbol{I}^{1212}\boldsymbol{\omega}_{12} & \boldsymbol{0} & -\boldsymbol{I}^{2323}\boldsymbol{\omega}_{23} & \boldsymbol{I}^{2424}\boldsymbol{\omega}_{24} & -\boldsymbol{I}^{2525}\boldsymbol{\omega}_{25} \\
-\boldsymbol{I}^{1313}\boldsymbol{\omega}_{13} & \boldsymbol{I}^{2323}\boldsymbol{\omega}_{23} & \boldsymbol{0} & -\boldsymbol{I}^{3434}\boldsymbol{\omega}_{34} & \boldsymbol{I}^{3535}\boldsymbol{\omega}_{35} \\
\boldsymbol{I}^{1414}\boldsymbol{\omega}_{14} & -\boldsymbol{I}^{2424}\boldsymbol{\omega}_{24} & \boldsymbol{I}^{3434}\boldsymbol{\omega}_{34} & \boldsymbol{0} & -\boldsymbol{I}^{4545}\boldsymbol{\omega}_{45} \\
-\boldsymbol{I}^{1515}\boldsymbol{\omega}_{15} & \boldsymbol{I}^{2525}\boldsymbol{\omega}_{25} & -\boldsymbol{I}^{3535}\boldsymbol{\omega}_{35} & \boldsymbol{I}^{4545}\boldsymbol{\omega}_{45} & \boldsymbol{0}
\end{bmatrix}.$$

$$(27)\quad \dot{\boldsymbol{L}} = \begin{bmatrix}
\boldsymbol{0} & -\boldsymbol{I}^{1212}\dot{\boldsymbol{\omega}}_{12} & \boldsymbol{I}^{1313}\dot{\boldsymbol{\omega}}_{13} & -\boldsymbol{I}^{1414}\dot{\boldsymbol{\omega}}_{14} & \boldsymbol{I}^{1515}\dot{\boldsymbol{\omega}}_{15} \\
\boldsymbol{I}^{1212}\dot{\boldsymbol{\omega}}_{12} & \boldsymbol{0} & -\boldsymbol{I}^{2323}\dot{\boldsymbol{\omega}}_{23} & \boldsymbol{I}^{2424}\dot{\boldsymbol{\omega}}_{24} & -\boldsymbol{I}^{2525}\dot{\boldsymbol{\omega}}_{25} \\
-\boldsymbol{I}^{1313}\dot{\boldsymbol{\omega}}_{13} & \boldsymbol{I}^{2323}\dot{\boldsymbol{\omega}}_{23} & \boldsymbol{0} & -\boldsymbol{I}^{3434}\dot{\boldsymbol{\omega}}_{34} & \boldsymbol{I}^{3535}\dot{\boldsymbol{\omega}}_{35} \\
\boldsymbol{I}^{1414}\dot{\boldsymbol{\omega}}_{14} & -\boldsymbol{I}^{2424}\dot{\boldsymbol{\omega}}_{24} & \boldsymbol{I}^{3434}\dot{\boldsymbol{\omega}}_{34} & \boldsymbol{0} & -\boldsymbol{I}^{4545}\dot{\boldsymbol{\omega}}_{45} \\
-\boldsymbol{I}^{1515}\dot{\boldsymbol{\omega}}_{15} & \boldsymbol{I}^{2525}\dot{\boldsymbol{\omega}}_{25} & -\boldsymbol{I}^{3535}\dot{\boldsymbol{\omega}}_{35} & \boldsymbol{I}^{4545}\dot{\boldsymbol{\omega}}_{45} & \boldsymbol{0}
\end{bmatrix}.$$

The substitution of (26), (27) in the Equation (24) results in the set of Euler equations:

$$(28)\quad -I^{1212}\frac{d\omega_{12}}{dt} + \left(I^{1313} - I^{2323}\right)\omega_{13}\omega_{23} + \left(I^{1414} - I^{2424}\right)\omega_{14}\omega_{24} + \left(I^{1515} - I^{2525}\right)\omega_{15}\omega_{25} = \widetilde{m}^{12},$$

$$(29)\quad I^{1313}\frac{d\omega_{13}}{dt}+\left(I^{1212}-I^{2323}\right)\omega_{12}\omega_{23}+\left(I^{3434}-I^{1414}\right)\omega_{14}\omega_{34}+\left(I^{3535}-I^{1515}\right)\omega_{15}\omega_{35}=\widetilde{m}^{13}.$$

$$(30)\quad -I^{1414}\frac{d\omega_{14}}{dt}+\left(I^{2424}-I^{1212}\right)\omega_{12}\omega_{24}+\left(I^{3434}-I^{1313}\right)\omega_{13}\omega_{34}+\left(I^{1515}-I^{4545}\right)\omega_{15}\omega_{45}=\widetilde{m}^{14},$$

$$(31)\quad I^{1515}\frac{d\omega_{15}}{dt}+\left(I^{1212}-I^{2525}\right)\omega_{12}\omega_{25}+\left(I^{1313}-I^{3535}\right)\omega_{13}\omega_{35}+\left(I^{1414}-I^{4545}\right)\omega_{14}\omega_{45}=\widetilde{m}^{15}.$$

$$(32)\quad -I^{2323}\frac{d\omega_{23}}{dt}+\left(I^{1212}-I^{1313}\right)\omega_{12}\omega_{13}+\left(I^{2424}-I^{3434}\right)\omega_{24}\omega_{34}+\left(I^{2525}-I^{3535}\right)\omega_{25}\omega_{35}=\widetilde{m}^{23}.$$

$$(33)\quad I^{2424}\frac{d\omega_{24}}{dt}+\left(I^{1414}-I^{1212}\right)\omega_{12}\omega_{14}+\left(I^{2323}-I^{3434}\right)\omega_{23}\omega_{34}+\left(I^{4545}-I^{2525}\right)\omega_{25}\omega_{45}=\widetilde{m}^{24}.$$

$$(34)\quad -I^{2525}\frac{d\omega_{25}}{dt}+\left(I^{1212}-I^{1515}\right)\omega_{12}\omega_{15}+\left(I^{3535}-I^{2323}\right)\omega_{23}\omega_{35}+\left(I^{4545}-I^{2424}\right)\omega_{24}\omega_{45}=\widetilde{m}^{25}.$$

$$(35)\quad -I^{3434}\frac{d\omega_{34}}{dt}+\left(I^{1313}-I^{1414}\right)\omega_{13}\omega_{14}+\left(I^{2323}-I^{2424}\right)\omega_{23}\omega_{24}+\left(I^{3535}-I^{4545}\right)\omega_{35}\omega_{45}=\widetilde{m}^{34}.$$

$$(36)\quad I^{3535}\frac{d\omega_{35}}{dt}+\left(I^{1515}-I^{1313}\right)\omega_{13}\omega_{15}+\left(I^{2525}-I^{2323}\right)\omega_{23}\omega_{25}+\left(I^{3434}-I^{4545}\right)\omega_{34}\omega_{45}=\widetilde{m}^{35}.$$

$$(37)\quad -I^{4545}\frac{d\omega_{45}}{dt}+\left(I^{1414}-I^{1515}\right)\omega_{14}\omega_{15}+\left(I^{2424}-I^{2525}\right)\omega_{24}\omega_{25}+\left(I^{3434}-I^{3535}\right)\omega_{34}\omega_{35}=\widetilde{m}^{45}.$$

The characteristic polynomial of the tensor $\boldsymbol{L}$, Equation (26) is the following:

$$(38)\quad |\boldsymbol{L}-\lambda\,\boldsymbol{E}|=k_1\lambda+k_3\lambda^3,\quad \boldsymbol{E}=diag[\![1,1,1,1,1]\!].$$

The coefficients k_1 and k_3 of the characteristic polynomial have been shown to be defined as the constants of motion; furthermore, it has been demonstrated that they represent the conservation of the moment of impulse and energy, respectively. It is possible to demonstrate this through calculation of the characteristic polynomial, with the terms of the first and third exponents of the parameter λ then collected. The coefficients thus obtained correspond to the classical expressions for the moment of impulse and energy. As illustrated in the 3D case study, the efficacy of the proposed approach was substantiated in (Landau & Lifshitz, Mechanics: Volume 1, Course of Theoretical Physics, 1976).

The method of the derivation of the Euler equations was explained using the bivectors notation in Ch.2, (Doran & Lasenby, 2003):

$$(39)\quad \dot{\boldsymbol{L}}=\boldsymbol{\omega}\wedge\boldsymbol{L}.$$

We use an orthonormal, time-invariant basis (also known as the Cartesian basis) in the five-dimensional Euclidean vector space. In the 5D there are five basis vectors, which form a right-handed basis:

$$\left\{\boldsymbol{e^1},\boldsymbol{e^2},\boldsymbol{e^3},\boldsymbol{e^4},\boldsymbol{e^5}\right\}.$$

From the five basis vectors the ten independent bivectors could be generated:

$$\left\{\boldsymbol{e^{12}}=\boldsymbol{e^1}\wedge\boldsymbol{e^2},\boldsymbol{e^{13}}=\boldsymbol{e^1}\wedge\boldsymbol{e^3},\boldsymbol{e^{14}}=\boldsymbol{e^1}\wedge\boldsymbol{e^4},\boldsymbol{e^{15}}=\boldsymbol{e^1}\wedge\boldsymbol{e^5},\boldsymbol{e^{23}}=\boldsymbol{e^2}\wedge\boldsymbol{e^3},\boldsymbol{e^{24}}=\boldsymbol{e^2}\wedge\boldsymbol{e^4},\boldsymbol{e^{25}}=\boldsymbol{e^2}\wedge\boldsymbol{e^5},\boldsymbol{e^{34}}=\boldsymbol{e^3}\wedge\boldsymbol{e^4},\boldsymbol{e^{35}}=\boldsymbol{e^3}\wedge\boldsymbol{e^5},\boldsymbol{e^{45}}=\boldsymbol{e^4}\wedge\boldsymbol{e^5}\right\}.$$

The products of basis bivectors $\boldsymbol{e^{ij}}\wedge\boldsymbol{e^{kl}}$ follow from the expression $\boldsymbol{\omega}\wedge\boldsymbol{L}$ in 5D. The products are displayed in the **Table 2**. The sequence of rows and columns is to be regarded as equivalent to the numerical sequence presented in Equation (8). The Table 2 is antisymmetric because bivector products are antisymmetric: $\mathbf{e}^{\mathbf{ij}} \wedge \mathbf{e}^{\mathbf{kl}} = -\mathbf{e}^{\mathbf{kl}} \wedge \mathbf{e}^{\mathbf{ij}}$. The diagonal terms vanish because of the antisymmetric nature of the matrix. If all the indices in both terms are different, the corresponding product will also vanish. It should be noted that there exist alternative definitions of the matrices for the bivector products. The definition is unique in that it enables the reduction of spatial dimensions from five to four, thereby resulting in the common Euler equations in all twenty possible three-dimensional scenarios. The model demonstrated a propensity for positive rotation frequencies in instances of rotation body asymmetry. It is evident that the signs of the elements in Tables 1 and 2 are identical.

Table 2. Products of basis bivectors $\boldsymbol{e}^{ij} \wedge \boldsymbol{e}^{kl}$ in 5D.

0	$-\boldsymbol{e}^{23}$	$\boldsymbol{e}^{24}$	$-\boldsymbol{e}^{25}$	$-\boldsymbol{e}^{13}$	$\boldsymbol{e}^{14}$	$-\boldsymbol{e}^{15}$	0	0	0
$\boldsymbol{e}^{23}$	0	$\boldsymbol{e}^{34}$	$-\boldsymbol{e}^{35}$	$\boldsymbol{e}^{12}$	0	0	$-\boldsymbol{e}^{14}$	$\boldsymbol{e}^{15}$	0
$-\boldsymbol{e}^{24}$	$-\boldsymbol{e}^{34}$	0	$-\boldsymbol{e}^{45}$	0	$-\boldsymbol{e}^{12}$	0	$\boldsymbol{e}^{13}$	0	$-\boldsymbol{e}^{15}$
$\boldsymbol{e}^{25}$	$\boldsymbol{e}^{35}$	$\boldsymbol{e}^{45}$	0	0	0	$\boldsymbol{e}^{12}$	0	$-\boldsymbol{e}^{13}$	$\boldsymbol{e}^{14}$
$\boldsymbol{e}^{13}$	$-\boldsymbol{e}^{12}$	0	0	0	$\boldsymbol{e}^{34}$	$-\boldsymbol{e}^{35}$	$\boldsymbol{e}^{24}$	$\boldsymbol{e}^{25}$	0
$-\boldsymbol{e}^{14}$	0	$\boldsymbol{e}^{12}$	0	$\boldsymbol{e}^{34}$	0	$\boldsymbol{e}^{45}$	$\boldsymbol{e}^{23}$	0	$-\boldsymbol{e}^{25}$
$\boldsymbol{e}^{15}$	0	0	$-\boldsymbol{e}^{12}$	$-\boldsymbol{e}^{35}$	$-\boldsymbol{e}^{45}$	0	0	$-\boldsymbol{e}^{23}$	$\boldsymbol{e}^{24}$
0	$\boldsymbol{e}^{14}$	$-\boldsymbol{e}^{13}$	0	$\boldsymbol{e}^{24}$	$-\boldsymbol{e}^{23}$	0	0	$-\boldsymbol{e}^{45}$	$-\boldsymbol{e}^{35}$
0	$-\boldsymbol{e}^{15}$	0	$\boldsymbol{e}^{13}$	$-\boldsymbol{e}^{25}$	0	$\boldsymbol{e}^{23}$	$\boldsymbol{e}^{45}$	0	$\boldsymbol{e}^{34}$
0	0	$\boldsymbol{e}^{15}$	$-\boldsymbol{e}^{14}$	0	$\boldsymbol{e}^{25}$	$-\boldsymbol{e}^{24}$	$\boldsymbol{e}^{35}$	$-\boldsymbol{e}^{34}$	0

To derive the bivector product, we use the expressions of the bivectors of an angular velocity and an angular moment:

$$(40) \quad \boldsymbol{\omega} = \omega_{12}\,\boldsymbol{e}^{12} + \omega_{13}\,\boldsymbol{e}^{13} + \omega_{14}\,\boldsymbol{e}^{14} + \omega_{15}\,\boldsymbol{e}^{15} + \omega_{23}\,\boldsymbol{e}^{23} + \omega_{24}\,\boldsymbol{e}^{24} + \omega_{25}\,\boldsymbol{e}^{25} + \omega_{34}\,\boldsymbol{e}^{34} + \omega_{35}\,\boldsymbol{e}^{35} + \omega_{45}\,\boldsymbol{e}^{45},$$

$$(41) \quad \boldsymbol{m} = m_{12}\,\boldsymbol{e}^{12} + m_{13}\,\boldsymbol{e}^{13} + m_{14}\,\boldsymbol{e}^{14} + m_{15}\,\boldsymbol{e}^{15} + m_{23}\,\boldsymbol{e}^{23} + m_{24}\,\boldsymbol{e}^{24} + m_{25}\,\boldsymbol{e}^{25} + m_{34}\,\boldsymbol{e}^{34} + m_{35}\,\boldsymbol{e}^{35} + m_{45}\,\boldsymbol{e}^{45}$$

The calculation of product of bivectors (40) and (41) uses Table 2 for calculation of the products of bivectors. For the $N = 5,\ N_\omega = 10$, such that there are 10 components of Euler's equations in five-dimensional space. The projections of the products of bivectors (40), (41) together with the corresponding time derivatives of the moments deliver once again the Equations (28)–(37)

The are $N_\omega = 10$ coupled second-order differential equations for N_ω components of the tensor $\boldsymbol{\omega}$. The set of Euler's equations has the disadvantage that the torque components $\widetilde{m}^{ij}$ are related to the body-fixed BS. The components L_{ij} are therefore generally time-dependent, and this time dependence depends on the body's motion. In the following, we limit ourselves to the force-free case $\widetilde{m}^{ij} = 0$.

Ten Equations (28)–(37) reduce to the trio of 4D-equations, if the components of rotations in the certain 4D-projection of the 5D-space disappear.

2.2. Reduction to the 4D Rotations

2.2.1. Rotation Over Axis One Restricted

If the rotation with respect to axis 1 is locked, we get:

$$S_4 = \{\omega_{12} = 0, \omega_{13} = 0, \omega_{14} = 0, \omega_{15} = 0\}\,.$$

The equations with the locked rotations are satisfied identically, and the remaining six equations condense to:

$$(42) \quad -I^{2323}\frac{d\omega_{23}}{dt} + \left(I^{2424} - I^{3434}\right)\omega_{24}\omega_{34} + \left(I^{2525} - I^{3535}\right)\omega_{25}\omega_{35} = \widetilde{m}^{23}.$$

$$I^{2424}\frac{d\omega_{24}}{dt}+\left(I^{2323}-I^{3434}\right)\omega_{23}\omega_{34}+\left(I^{4545}-I^{2525}\right)\omega_{25}\omega_{45}=\widetilde{m}^{24}. \tag{43}$$

$$-I^{2525}\frac{d\omega_{25}}{dt}+\left(I^{3535}-I^{2323}\right)\omega_{23}\omega_{35}+\left(I^{4545}-I^{2424}\right)\omega_{24}\omega_{45}=\widetilde{m}^{25}. \tag{44}$$

$$-I^{3434}\frac{d\omega_{34}}{dt}+\left(I^{2323}-I^{2424}\right)\omega_{23}\omega_{24}+\left(I^{3535}-I^{4545}\right)\omega_{35}\omega_{45}=\widetilde{m}^{34}. \tag{45}$$

$$I^{3535}\frac{d\omega_{35}}{dt}+\left(I^{2525}-I^{2323}\right)\omega_{23}\omega_{25}+\left(I^{3434}-I^{4545}\right)\omega_{34}\omega_{45}=\widetilde{m}^{35}. \tag{46}$$

$$-I^{4545}\frac{d\omega_{45}}{dt}+\left(I^{2424}-I^{2525}\right)\omega_{24}\omega_{25}+\left(I^{3434}-I^{3535}\right)\omega_{34}\omega_{35}=\widetilde{m}^{45}. \tag{47}$$

These equations describe the rotation of the rigid in $2-3-4-5$ four-dimensional projection of the 4D space.

2.2.2. Rotation Over Axis Two Restricted

Accordingly, if the rotation with respect to the axis 2 is blocked:

$$S_4=\{\omega_{12}=0,\omega_{23}=0,\omega_{24}=0,\omega_{25}=0\}.$$

The Equations (28)(32)(33)(34) erased, and the remaining six equations condense to:

$$I^{1313}\frac{d\omega_{13}}{dt}+\left(I^{3434}-I^{1414}\right)\omega_{14}\omega_{34}+\left(I^{3535}-I^{1515}\right)\omega_{15}\omega_{35}=\widetilde{m}^{13}. \tag{48}$$

$$-I^{1414}\frac{d\omega_{14}}{dt}+\left(I^{3434}-I^{1313}\right)\omega_{13}\omega_{34}+\left(I^{1515}-I^{4545}\right)\omega_{15}\omega_{45}=\widetilde{m}^{14}, \tag{49}$$

$$I^{1515}\frac{d\omega_{15}}{dt}+\left(I^{1313}-I^{3535}\right)\omega_{13}\omega_{35}+\left(I^{1414}-I^{4545}\right)\omega_{14}\omega_{45}=\widetilde{m}^{15}. \tag{50}$$

$$-I^{3434}\frac{d\omega_{34}}{dt}+\left(I^{1313}-I^{1414}\right)\omega_{13}\omega_{14}+\left(I^{3535}-I^{4545}\right)\omega_{35}\omega_{45}=\widetilde{m}^{34}. \tag{51}$$

$$I^{3535}\frac{d\omega_{35}}{dt}+\left(I^{1515}-I^{1313}\right)\omega_{13}\omega_{15}+\left(I^{3434}-I^{4545}\right)\omega_{34}\omega_{45}=\widetilde{m}^{35}. \tag{52}$$

$$-I^{4545}\frac{d\omega_{45}}{dt}+\left(I^{1414}-I^{1515}\right)\omega_{14}\omega_{15}+\left(I^{3434}-I^{3535}\right)\omega_{34}\omega_{35}=\widetilde{m}^{45}. \tag{53}$$

These equations describe the rotation of the rigid in $1-3-4-5$ four -dimensional projection of the 4D space.

2.2.3. Rotation Over Axis Three Restricted

If rotation around axis 3 is jammed, the result is as follows:

$$S_4=\{\omega_{13}=0,\omega_{23}=0,\omega_{34}=0,\omega_{35}=0\}.$$

The Equations (29)(32)(35)(36) erased, and the remaining six equations condense to:

$$-I^{1212}\frac{d\omega_{12}}{dt}+\left(I^{1414}-I^{2424}\right)\omega_{14}\omega_{24}+\left(I^{1515}-I^{2525}\right)\omega_{15}\omega_{25}=\widetilde{m}^{12}, \tag{54}$$

$$-I^{1414}\frac{d\omega_{14}}{dt}+\left(I^{2424}-I^{1212}\right)\omega_{12}\omega_{24}+\left(I^{1515}-I^{4545}\right)\omega_{15}\omega_{45}=\widetilde{m}^{14}, \tag{55}$$

$$(56)\quad I^{1515}\frac{d\omega_{15}}{dt}+\left(I^{1212}-I^{2525}\right)\omega_{12}\omega_{25}+\left(I^{1414}-I^{4545}\right)\omega_{14}\omega_{45}=\widetilde{m}^{15}.$$

$$(57)\quad I^{2424}\frac{d\omega_{24}}{dt}+\left(I^{1414}-I^{1212}\right)\omega_{12}\omega_{14}+\left(I^{4545}-I^{2525}\right)\omega_{25}\omega_{45}=\widetilde{m}^{24}.$$

$$(58)\quad -I^{2525}\frac{d\omega_{25}}{dt}+\left(I^{1212}-I^{1515}\right)\omega_{12}\omega_{15}+\left(I^{4545}-I^{2424}\right)\omega_{24}\omega_{45}=\widetilde{m}^{25}.$$

$$(59)\quad -I^{4545}\frac{d\omega_{45}}{dt}+\left(I^{1414}-I^{1515}\right)\omega_{14}\omega_{15}+\left(I^{2424}-I^{2525}\right)\omega_{24}\omega_{25}=\widetilde{m}^{45}.$$

These equations describe the rotation of the rigid in $1-2-4-5$ four -dimensional projection of the 4D space.

2.2.4. Rotation Over Axis Four Restricted

If the rotation with respect to the axis 4 is frozen, we have:

$$S_4=\{\omega_{14}=0,\omega_{24}=0,\omega_{34}=0,\omega_{45}=0\}.$$

The Equations (31),(34),(36),(37) erased, and the remaining six equations condense to:

$$(60)\quad -I^{1212}\frac{d\omega_{12}}{dt}+\left(I^{1313}-I^{2323}\right)\omega_{13}\omega_{23}+\left(I^{1515}-I^{2525}\right)\omega_{15}\omega_{25}=\widetilde{m}^{12},$$

$$(61)\quad I^{1313}\frac{d\omega_{13}}{dt}+\left(I^{1212}-I^{2323}\right)\omega_{12}\omega_{23}+\left(I^{3535}-I^{1515}\right)\omega_{15}\omega_{35}=\widetilde{m}^{13}.$$

$$(62)\quad I^{1515}\frac{d\omega_{15}}{dt}+\left(I^{1212}-I^{2525}\right)\omega_{12}\omega_{25}+\left(I^{1313}-I^{3535}\right)\omega_{13}\omega_{35}=\widetilde{m}^{15}.$$

$$(63)\quad -I^{2323}\frac{d\omega_{23}}{dt}+\left(I^{1212}-I^{1313}\right)\omega_{12}\omega_{13}+\left(I^{2525}-I^{3535}\right)\omega_{25}\omega_{35}=\widetilde{m}^{23}.$$

$$(64)\quad -I^{2525}\frac{d\omega_{25}}{dt}+\left(I^{1212}-I^{1515}\right)\omega_{12}\omega_{15}+\left(I^{3535}-I^{2323}\right)\omega_{23}\omega_{35}=\widetilde{m}^{25}.$$

$$(65)\quad I^{3535}\frac{d\omega_{35}}{dt}+\left(I^{1515}-I^{1313}\right)\omega_{13}\omega_{15}+\left(I^{2525}-I^{2323}\right)\omega_{23}\omega_{25}=\widetilde{m}^{35}.$$

These equations describe the rotation of the rigid in $1-2-3-5$ four -dimensional projection of the 4D space.

2.2.5. Rotation Over Axis Five Restricted

Finally, if the rotation with respect to axis 5 is restricted, the following equations apply:

$$S_4=\{\omega_{15}=0,\omega_{25}=0,\omega_{35}=0,\omega_{45}=0\}.$$

The Equations (31),(34),(36),(37) erased, and the remaining six equations condense to:

$$(66)\quad -I^{1212}\frac{d\omega_{12}}{dt}+\left(I^{1313}-I^{2323}\right)\omega_{13}\omega_{23}+\left(I^{1414}-I^{2424}\right)\omega_{14}\omega_{24}=\widetilde{m}^{12},$$

$$(67)\quad I^{1313}\frac{d\omega_{13}}{dt}+\left(I^{1212}-I^{2323}\right)\omega_{12}\omega_{23}+\left(I^{3434}-I^{1414}\right)\omega_{14}\omega_{34}=\widetilde{m}^{13}.$$

$$(68)\quad -I^{1414}\frac{d\omega_{14}}{dt}+\left(I^{2424}-I^{1212}\right)\omega_{12}\omega_{24}+\left(I^{3434}-I^{1313}\right)\omega_{13}\omega_{34}=\widetilde{m}^{14},$$

$$(69)\quad -I^{2323}\frac{d\omega_{23}}{dt}+\left(I^{1212}-I^{1313}\right)\omega_{12}\omega_{13}+\left(I^{2424}-I^{3434}\right)\omega_{24}\omega_{34}=\widetilde{m}^{23}.$$

$$(70)\quad I^{2424}\frac{d\omega_{24}}{dt}+\left(I^{1414}-I^{1212}\right)\omega_{12}\omega_{14}+\left(I^{2323}-I^{3434}\right)\omega_{23}\omega_{34}=\widetilde{m}^{24}.$$

$$(71)\quad -I^{3434}\frac{d\omega_{34}}{dt}+\left(I^{1313}-I^{1414}\right)\omega_{13}\omega_{14}+\left(I^{2323}-I^{2424}\right)\omega_{23}\omega_{24}=\widetilde{m}^{34}.$$

These equations describe the rotation of the rigid in $1-2-3-4$ four -dimensional projection of the 4D space.

Finally, the sequence of equations was arranged into five sextets, with the purpose of describing the rotations in the five 4D subspaces of the 5D Euclidean space:

$$(72)\quad \{1-2-3-4\},\ \{2-3-4-5\}\ ,\ \{3-4-5-1\},\ \{4-5-1-2\},\{5-1-2-3\}.$$

Evidently, that each of 4D equations reduces further if an additional degree of freedom will be fixed. We show this process on the following examples.

2.3. Additional Rotation Over Axis Four Restricted

If the rotation with respect to the axis 4 is additionally jammed, we get the restricted seven rotations from ten possible in 5D:

$$S_3=\{\omega_{14}=0,\omega_{15}=0,\omega_{24}=0,\omega_{25}=0,\omega_{34}=0,\omega_{35}=0,\omega_{45}=0\}\,.$$

The equations, which involved the rotation about the axis 4 will be satisfied identically. The remaining three equations over the axes $1-2-3$ condense to the classical Euler equations in 3D (Borisov & Mamaev, 2019):

$$(73)\quad -I^{1212}\frac{d\omega_{12}}{dt}+\left(I^{1313}-I^{2323}\right)\omega_{13}\omega_{23}+\left(I^{1515}-I^{2525}\right)\omega_{15}\omega_{25}=\widetilde{m}^{12},$$

$$(74)\quad I^{1313}\frac{d\omega_{13}}{dt}+\left(I^{1212}-I^{2323}\right)\omega_{12}\omega_{23}+\left(I^{3535}-I^{1515}\right)\omega_{15}\omega_{35}=\widetilde{m}^{13}.$$

$$(75)\quad -I^{2323}\frac{d\omega_{23}}{dt}+\left(I^{1212}-I^{1313}\right)\omega_{12}\omega_{13}+\left(I^{2525}-I^{3535}\right)\omega_{25}\omega_{35}=\widetilde{m}^{23}..$$

2.4. Additional Rotation Over Axis Three Restricted

If the rotation with respect to the axis 3 is additionally fixed, we get the restricted seven rotations from ten possible in 5D. The equations for rotations about the axis 3 will be satisfied identically. The remaining three equations condense to the classical Euler equations in 3D over the axes over the axes $1-2-4$, (Borisov & Mamaev, 2019):

$$(76)\quad -I^{1212}\frac{d\omega_{12}}{dt}+\left(I^{1313}-I^{2323}\right)\omega_{13}\omega_{23}+\left(I^{1414}-I^{2424}\right)\omega_{14}\omega_{24}=\widetilde{m}^{12},$$

$$(77)\quad -I^{1414}\frac{d\omega_{14}}{dt}+\left(I^{2424}-I^{1212}\right)\omega_{12}\omega_{24}+\left(I^{3434}-I^{1313}\right)\omega_{13}\omega_{34}=\widetilde{m}^{14},$$

$$(78)\quad I^{2424}\frac{d\omega_{24}}{dt}+\left(I^{1414}-I^{1212}\right)\omega_{12}\omega_{14}+\left(I^{2323}-I^{3434}\right)\omega_{23}\omega_{34}=\widetilde{m}^{24}.$$

2.5. Additional Rotation Over Axis Four Restricted

If the rotation with respect to the axis 3 is additionally blocked. The corresponding equations will be satisfied in this case identically. The remaining three equations condense to the classical Euler equations in 3D (Borisov & Mamaev, 2019) over the axes $1-3-4$:

$$(79)\quad I^{1313}\frac{d\omega_{13}}{dt}+\left(I^{1212}-I^{2323}\right)\omega_{12}\omega_{23}+\left(I^{3434}-I^{1414}\right)\omega_{14}\omega_{34}=\tilde{m}^{13}.$$

$$(80)\quad -I^{1414}\frac{d\omega_{14}}{dt}+\left(I^{2424}-I^{1212}\right)\omega_{12}\omega_{24}+\left(I^{3434}-I^{1313}\right)\omega_{13}\omega_{34}=\tilde{m}^{14},$$

$$(81)\quad -I^{3434}\frac{d\omega_{34}}{dt}+\left(I^{1313}-I^{1414}\right)\omega_{13}\omega_{14}+\left(I^{2323}-I^{2424}\right)\omega_{23}\omega_{24}=\tilde{m}^{34}.$$

Consequently, the ten equations in 5D project into four equations in the comprised 4D spaces. Each of 4D spaces dissolves into three corresponding classical Euler equations in the enclosed 3D spaces.

3. Solutions of the Euler Equations in 5D Spaces

3.1. All Moments of Inertia Equal

The system of nonlinear ordinary differential Equations (28)–(37) is nonlinear and depend upon ten initial conditions and ten fixed components of inertia tensor. In some special cases the system (28)–(37) allows the closed form solutions. In the initial phase of this investigation, we shall establish the solution to the Euler equations in the case of an equal moment of inertia across all possible axes of rotation:

$$(82)\quad I^{1212}=I^{1313}=I^{1414}=I^{1515}=I^{2323}=I^{2424}=I^{2525}=I^{3434}=I^{3535}=I^{4545}.$$

The Platonic bodies and hyperspheres satisfy this condition. For such bodies all gyration radii are equal. In this case the second term in all left sides of the Equations (28)–(37) vanish. The time derivatives of the rotation components nullify in the absence of the external moments:

$$(83)\quad \frac{d\omega_{12}}{dt}=\frac{d\omega_{23}}{dt}=\frac{d\omega_{13}}{dt}=\frac{d\omega_{15}}{dt}=\frac{d\omega_{23}}{dt}=\frac{d\omega_{24}}{dt}=\frac{d\omega_{25}}{dt}=\frac{d\omega_{34}}{dt}=\frac{d\omega_{35}}{dt}=\frac{d\omega_{45}}{dt}=0.$$

Consequently, it is evident that all components of the rotation velocity are sustained over a given period of time. It can be posited that this behavior holds in all multidimensional Euclidean spaces.

3.2. The Equal Moments with One Quartet of Different Moments of Inertia

The second case corresponds to six equal principal moments of inertia with four distinct principal moment. In the absence of any restrictions, the equal principal moments of inertia are set to two, and the distinct moments are referred to as $1+\kappa^2$:

$$(84)\quad I^{1212}=I^{1313}=I^{1414}=I^{2323}=I^{2424}=I^{3434}=2,$$

$$(85)\quad I^{1515}=I^{2525}=I^{3535}=I^{4545}=1+\kappa^2.$$

The aforementioned Equations (28)–(37) can be simplified to six nullifying expressions and four linear ordinary differential equations with constant coefficients:

$$(86)\quad \frac{d\omega_{12}}{dt}=0,\quad \omega_{12}(t=0)=c_1,$$

$$(87)\quad \frac{d\omega_{13}}{dt}=0,\quad \omega_{13}(t=0)=c_2,$$

$$(88)\quad \frac{d\omega_{14}}{dt} = 0, \quad \omega_{14}(t=0) = c_3,$$

$$(89)\quad \left(1+\kappa^2\right)\frac{d\omega_{15}}{dt} + \left(1-\kappa^2\right)(\omega_{12}\omega_{25} + \omega_{13}\omega_{35} + \omega_{14}\omega_{45}) = 0.$$

$$(90)\quad \frac{d\omega_{23}}{dt} = 0, \quad \omega_{23}(t=0) = c_4,$$

$$(91)\quad \frac{d\omega_{24}}{dt} = 0, \quad \omega_{24}(t=0) = c_5,$$

$$(92)\quad \left(1+\kappa^2\right)\frac{d\omega_{25}}{dt} + \left(1-\kappa^2\right)(\omega_{12}\omega_{15} - \omega_{23}\omega_{35} - \omega_{24}\omega_{45}) = 0.$$

$$(93)\quad \frac{d\omega_{34}}{dt} = 0, \quad \omega_{34}(t=0) = c_6,$$

$$(94)\quad \left(1+\kappa^2\right)\frac{d\omega_{35}}{dt} - \left(1-\kappa^2\right)(\omega_{13}\omega_{15} + \omega_{23}\omega_{25} - \omega_{34}\omega_{45}) = 0.$$

$$(95)\quad \left(1+\kappa^2\right)\frac{d\omega_{45}}{dt} + \left(1-\kappa^2\right)(\omega_{14}\omega_{51} + \omega_{24}\omega_{25} + \omega_{34}\omega_{35}) = 0.$$

The resolution (86),(87),(88),(93),(90),(91) of six ordinary differential equations of a nullifying nature guarantees that the six rotation velocities remain constant. It is evident that the coefficients of four additional ordinary differential equations are contingent upon the initial six constant velocities $\omega_{12} \equiv c_1, \omega_{13} \equiv c_2, \omega_{14} \equiv c_3, \omega_{23} \equiv c_4, \omega_{24} \equiv c_5, \omega_{34} \equiv c_6$:

$$(96)\quad \frac{d\omega_{15}}{dt} - \Theta c_1\omega_{25} - \Theta c_2\omega_{35} - \Theta c_3\omega_{45} = 0,$$

$$(97)\quad \frac{d\omega_{25}}{dt} + \Theta c_1\omega_{15} - \Theta c_4\omega_{35} - \Theta c_5\omega_{45} = 0,$$

$$(98)\quad \frac{d\omega_{35}}{dt} + \Theta c_2\omega_{15} + \Theta c_4\omega_{25} - \Theta c_6\omega_{45} = 0,$$

$$(99)\quad \frac{d\omega_{45}}{dt} + \Theta c_3\omega_{15} + \Theta c_5\omega_{25} + \Theta c_6\omega_{35} = 0,$$

$$(100)\ \Theta = \frac{\kappa^2-1}{1+\kappa^2}, \kappa = \frac{\sqrt{1-\Theta^2}}{\Theta-1}, \ -1 < \Theta < 1.$$

The Equations (96), (97), (98) and (99) are the linear ordinary differential equations of the first order with the constant coefficients. The closed-form solutions of Equations (96), (97), (98) and (99) will be sought using the exponential ansatz:

$$(101)\ \omega_{15}(t) = C_{15}\,exp(i\Omega t)\,, \omega_{25}(t) = C_{25}\,exp(i\Omega t)\,, \omega_{35}(t) = C_{35}\,exp(i\Omega t)\,, \omega_{45}(t) = C_{45}\,exp(i\Omega t).$$

With the ansatz (101) the ordinary differential equations reduce to the system of linear algebraic equations with the eigenvalue parameter Ω:

$$(102)\ i\,\Omega\,C_{15} - \Theta c_1 C_{25} - \Theta c_2 C_{35} - \Theta c_3 C_{45} = 0,$$

$$(103)\ i\,\Omega\,C_{25} + \Theta c_1 C_{15} - \Theta c_4 C_{35} - \Theta c_5 \omega_{45} = 0,$$

$$(104)\ i\,\Omega\, C_{35} + \Theta c_2 C_{15} + \Theta c_4 C_{25} - \Theta c_6 \omega_{45} = 0,$$

$$(105)\ i\,\Omega\, C_{45} + \Theta c_3 C_{15} - \Theta c_5 C_{25} + \Theta c_6 \omega_{35} = 0.$$

The eigenvalue problem (102), (103),(104),(105) can be written in matrix form:

$$(106)\ \mathcal{M}\mathcal{Y} = 0, \quad \mathcal{Y} = \begin{bmatrix} C_{15} \\ C_{25} \\ C_{35} \\ C_{45} \end{bmatrix}, \quad \mathcal{M} = \begin{bmatrix} i\,\Omega & -\Theta c_1 & -\Theta c_2 & -\Theta c_3 \\ \Theta c_1 & i\,\Omega & -\Theta c_4 & \Theta c_5 \\ \Theta c_2 & \Theta c_4 & i\,\Omega & -\Theta c_6 \\ \Theta c_3 & \Theta c_5 & \Theta c_6 & i\,\Omega \end{bmatrix}.$$

It is notable that the matrix must be antisymmetric, as can be seen from the above equation.

It is well known that some boundary value problems have unique solutions, while others only have solutions if certain solvability conditions are met (Gantmacher, 1959). The nontrivial solvability condition of the eigenvalue problem follows from the Fredholm Alternative Theorem (Fredholm, 1903). In the above case, the determinant of the matrix must vanish for a nontrivial solution to exist:

$$(107)\ |\mathcal{M}| \equiv \Omega^4 + l_2\Omega^2 + l_0 = 0,$$

$$(108)\ l_2 = -\,\Theta^2 \cdot \left(c_1^2 + c_2^2 + c_3^2 + c_4^2 + c_5^2 + c_6^2\right) < 0, \quad l_0 = \Theta^4 \cdot (c_1c_6 - c_2c_5 + c_3c_4)^2 > 0,$$

$$(109)\ {l_2}^2 - 4l_0 = \Theta^4 \cdot [(c_5 - c_2)^2 + (c_3 + c_4)^2 + (c_1 + c_6)^2][(c_5 + c_2)^2 + (c_3 - c_4)^2 + (c_1 - c_6)^2],$$

$$(110)\ {l_2}^2 > {l_2}^2 - 4l_0 > 0.$$

From Equation (107),(108),(109) follows, that the fundamental frequency Ω of the oscillation solution (101) is the function of the initial six constant velocities:

$$\omega_{12} \equiv c_1, \omega_{13} \equiv c_2, \omega_{14} \equiv c_3, \omega_{23} \equiv c_4, \omega_{24} \equiv c_5, \omega_{34} \equiv c_6.$$

The rotation velocities $\omega_{15}(t), \omega_{25}(t), \omega_{35}(t), \omega_{45}(t)$ are harmonically oscillating functions. The oscillation periods of the third last velocities depend on the six constant velocities.

Evidently, that the equivalent results could be obtained in four other cases:

$$(111)\ I^{1212} = I^{1313} = I^{1515} = I^{2323} = I^{2525} = I^{3535} = 2,\ I^{1414} = I^{2424} = I^{3434} = I^{4545} = 1 + \kappa^2\,,$$

$$(112)\ I^{1212} = I^{1414} = I^{1515} = I^{2525} = I^{2424} = I^{4545} = 2,\ I^{1313} = I^{2323} = I^{3434} = I^{3535} = 1 + \kappa^2,$$

$$(113)\ I^{1313} = I^{1414} = I^{1515} = I^{3434} = I^{3535} = I^{4545} = 2,\ I^{1212} = I^{2323} = I^{2424} = I^{2525} = 1 + \kappa^2,$$

$$(114)\ I^{2323} = I^{2424} = I^{2525} = I^{3434} = I^{3535} = I^{4545} = 2,\ I^{1212} = I^{1313} = I^{1414} = I^{1515} = 1 + \kappa^2.$$

The constant velocities will be correspondingly equal to:

$$(115)\ \omega_{12} \equiv c_1, \omega_{13} \equiv c_2, \omega_{15} \equiv c_3, \omega_{23} \equiv c_4, \omega_{25} \equiv c_5, \omega_{35} \equiv c_6,$$

$$(116)\ \omega_{12} \equiv c_1, \omega_{14} \equiv c_2, \omega_{15} \equiv c_3, \omega_{24} \equiv c_4, \omega_{25} \equiv c_5, \omega_{45} \equiv c_6,$$

$$(117)\ \omega_{13} \equiv c_1, \omega_{14} \equiv c_2, \omega_{15} \equiv c_3, \omega_{34} \equiv c_4, \omega_{35} \equiv c_5, \omega_{45} \equiv c_6,$$

$$(118)\ \omega_{23} \equiv c_1, \omega_{24} \equiv c_2, \omega_{25} \equiv c_3, \omega_{34} \equiv c_4, \omega_{35} \equiv c_5, \omega_{45} \equiv c_6.$$

The oscillating frequencies of the rest degree of freedom will be given from the solution of Equation (107):

$$\Omega_{1,2} = \pm\frac{1}{2}\sqrt{2\sqrt{{l_2}^2 - 4l_0} - 2l_2}, \quad \Omega_{3,4} = \pm\frac{1}{2}\sqrt{-2\sqrt{{l_2}^2 - 4l_0} - 2l_2}.$$

4. Stability of the Rotations

Consider a body rotating about the one principal axis with the certain angular velocity Ω . Assume a small impulsive moment that initiates a small rotation about the other axes and thereafter the motion proceeds with no applied external moments. For this case, Euler's equations become (28)..(37) with the zero right terms. We examine the question of stability to small perturbations, rotations about the other axes of magnitude of a small parameter ε.

For example, we study the stability of the rotation ω_{12} . In the first example, we assume the solution in the following form:

$$(119)\ \omega_{12} = \Omega + \varepsilon\,\varpi_1(t), \omega_{13} = \varepsilon\,\varpi_2(t)\,, \omega_{14} = \varepsilon\,\varpi_3(t)\,, \omega_{15} = \varepsilon\,\varpi_4(t)\,, \omega_{23} = \varepsilon\,\varpi_5(t)\,,$$

$$(120)\ \omega_{24} = \varepsilon\,\varpi_6(t), \omega_{25} = \varepsilon\,\varpi_7(t), \omega_{34} = \varepsilon\,\varpi_8(t), \omega_{35} = \varepsilon\,\varpi_9(t), \omega_{45} = \varepsilon\,\varpi_{10}(t).$$

After substitution of the Equations (119), (120) in the Equations (28)–(37), we can find, that the four equations will be satisfied in linear terms of ε. The remaining six equations reduce to:

$$(121)\ -J_1\frac{d\varpi_1}{dt} = 0,$$

$$(122)\ J_2\frac{d\varpi_2}{dt} + \Omega(J_1 - J_4)\varpi_5 = 0,$$

$$(123)\ -J_3\frac{d\varpi_3}{dt} + \Omega(J_5 - J_1)\varpi_6 = 0,$$

$$(124)\ J_7\frac{d\varpi_4}{dt} + \Omega(J_1 - J_8)\varpi_7 = 0,$$

$$(125)\ -J_4\frac{d\varpi_5}{dt} + \Omega(J_1 - J_2)\varpi_2 = 0,$$

$$(126)\ J_5\frac{d\varpi_6}{dt} + \Omega(J_3 - J_1)\varpi_3 = 0,$$

$$(127)\ -J_8\frac{d\varpi_7}{dt} + \Omega(J_1 - J_7)\varpi_4 = 0,$$

$$(128)\ -J_6\frac{d\varpi_8}{dt} = 0,$$

$$(129)\ J_9\frac{d\varpi_9}{dt} = 0,$$

$$(130)\ -J_{10}\frac{d\varpi_{10}}{dt} = 0.$$

We used the following notations for the briefness:

$$(131)\ J_1 = I^{1212}, J_2 = I^{1313}, J_3 = I^{1414}, J_4 = I^{2323}, J_5 = I^{2424}, J_6 = I^{3434}, J_7 = I^{1515}, J_8 = I^{2525}, J_9 = I^{3535}, J_{10} = I^{4545}.$$

The Equations (121)(128)(129)(130) leads in the first approximation with respect to ε to the constant angular velocities:

$$(132)\ \varpi_1 = 0, \varpi_6 = 0, \varpi_9 = 0, \varpi_{10} = 0.$$

The remaining equations could be grouped into pairs. One group consists of Equations (122) and (125). The solution to the first group of equations is as follows:

$$(133)\ \varpi_2 = C_1 \sin(\Omega v_1 t) + C_2 \cos(\Omega v_1 t),$$

$$(134)\ \varpi 5 = \mathrm{J}2 v1 J4 - J1 \mathrm{C}1 \cos \Omega v1 t - C2 \sin \Omega v1 t\ ,$$

$$(135)\ v1 = \mathrm{J}1 - J4 \mathrm{J}1 - J2 J2 J4.$$

The second group consists of Equations (123) and (126). The second group of equations has a solution:

$$\varpi_3 = C_1 \sin(\Omega v_2 t) + C2 \cos \Omega v2 t,$$

$$\varpi 6 = \mathrm{J}3 v2 J1 - J5 \mathrm{C}1 \cos \Omega v2 t - C2 \sin \Omega v2 t\ ,$$

$$(136)\ v2 = \mathrm{J}1 - J5 \mathrm{J}1 - J3 J3 J5.$$

The third group comprises a series of Equations (124) and (127). Their solution reads as:

$$\varpi_4 = C_1 \sin(\Omega v_3 t) + C2 \cos \Omega v3 t,$$

$$(137)\ \varpi 7 = \mathrm{J}7 v3 J8 - J1 \mathrm{C}1 \cos \Omega v3 t - C2 \sin \Omega v3 t\ ,$$

$$(138)\ v3 = \mathrm{J}1 - J8 \mathrm{J}1 - J7 J7 J8.$$

Assume at first, that the principal rotation axis rotation circular frequency Ω the axis with the highest principal moment of inertia:

$$(139)\ J_1 \geq J_2 \geq J_3 \geq J_4 \geq J5 \geq J6 \geq J7 \geq J8 \geq J9 \geq J10.$$

It is evident that, in the event of the inequalities being satisfied, all three frequencies of the oscillations will be positive real values. It can thus be concluded that if rotation occurs over the axis with the highest principal moment of inertia, then the rotation is stable with oscillations of three groups of order epsilon.

It should be posited for a second that the principal rotation axis is circular in its frequency of rotation (Ω), and that it is the axis that possesses the lowest principal moment inertia.

$$(140)\ J_1 \leq J_2 \leq J_3 \leq J_4 \leq J_5 \leq J_6 \leq J_7 \leq J_8 \leq J_9 \leq J_{10.}$$

In the event of the inequalities being satisfied, it can be demonstrated that all three frequencies of the oscillations will be positive real values once again. It can thus be concluded that if rotation occurs over the axis with the lowest principal moment of inertia, then the rotation is stable with oscillations of three groups of order ε as well.

5. Quantum Angular Momentum

For the derivation of the expressions of the angular momentum in spaces fifth dimension we apply the common principles of quantum mechanics (Landau & Lifshitz, Quantum Mechanics. Non-relativistic Theory. Course of Theoretical Physics. Volume 3, 1965). It is proposed that the tensor of an infinitely small rotation in 5D space with the cartesian coordinates x, y, z, w, v . The particles involved in such a rotation are characterized by the following parameters (8). The following expression may be regarded as the angular momentum operator:

$$(141)\ \mathbb{L} = -i\hbar \begin{Vmatrix} 0 & -\mathbb{I}^{12} & \mathbb{I}^{13} & -\mathbb{I}^{14} & \mathbb{I}^{15} \\ \mathbb{I}^{12} & 0 & -\mathbb{I}^{23} & \mathbb{I}^{24} & -\mathbb{I}^{25} \\ -\mathbb{I}^{13} & \mathbb{I}^{23} & 0 & -\mathbb{I}^{34} & \mathbb{I}^{35} \\ \mathbb{I}^{14} & -\mathbb{I}^{24} & \mathbb{I}^{34} & 0 & -\mathbb{I}^{45} \\ -\mathbb{I}^{15} & \mathbb{I}^{25} & -\mathbb{I}^{35} & \mathbb{I}^{45} & 0 \end{Vmatrix}.$$

The components of the angular momentum operator (144) are the follows:

$$(142)\begin{cases}\mathbb{I}^{12}=x\dfrac{\partial}{\partial y}-y\dfrac{\partial}{\partial x},\mathbb{I}^{12}=x\dfrac{\partial}{\partial z}-z\dfrac{\partial}{\partial x}, \quad \mathbb{I}^{14}=x\dfrac{\partial}{\partial w}-w\dfrac{\partial}{\partial x}, \quad \mathbb{I}^{15}=x\dfrac{\partial}{\partial v}-v\dfrac{\partial}{\partial x},\\ \mathbb{I}^{23}=y\dfrac{\partial}{\partial z}-z\dfrac{\partial}{\partial y},\mathbb{I}^{24}=y\dfrac{\partial}{\partial w}-w\dfrac{\partial}{\partial y}, \quad \mathbb{I}^{25}=y\dfrac{\partial}{\partial v}-v\dfrac{\partial}{\partial y},\\ \mathbb{I}^{34}=z\dfrac{\partial}{\partial w}-w\dfrac{\partial}{\partial z},\mathbb{I}^{35}=z\dfrac{\partial}{\partial v}-v\dfrac{\partial}{\partial z}, \quad \mathbb{I}^{45}=w\dfrac{\partial}{\partial v}-v\dfrac{\partial}{\partial w}\ .\end{cases}$$

Consider now two arbitrary differential operators in matrix form:

$$(143)\ \mathbb{L}=\left[\!\!\left[\begin{matrix}0 & -\mathbb{L}^{12} & \mathbb{L}^{13} & -\mathbb{L}^{14} & \mathbb{L}^{15}\\ \mathbb{L}^{12} & 0 & -\mathbb{L}^{23} & \mathbb{L}^{24} & -\mathbb{L}^{25}\\ -\mathbb{L}^{13} & \mathbb{L}^{23} & 0 & -\mathbb{L}^{34} & \mathbb{L}^{35}\\ \mathbb{L}^{14} & -\mathbb{L}^{24} & \mathbb{L}^{34} & 0 & -\mathbb{L}^{45}\\ -\mathbb{L}^{15} & \mathbb{L}^{25} & -\mathbb{L}^{35} & \mathbb{L}^{45} & 0\end{matrix}\right]\!\!\right];\ \Omega=\left[\!\!\left[\begin{matrix}0 & -\Omega^{12} & \Omega^{13} & -\Omega^{14} & \Omega^{15}\\ \Omega^{12} & 0 & -\Omega^{23} & \Omega^{24} & -\Omega^{25}\\ -\Omega^{13} & \Omega^{23} & 0 & -\Omega^{34} & \Omega^{35}\\ \Omega^{14} & -\Omega^{24} & \Omega^{34} & 0 & -\Omega^{45}\\ -\Omega^{15} & \Omega^{25} & -\Omega^{35} & \Omega^{45} & 0\end{matrix}\right]\!\!\right].$$

The components of both differential operators are the linear partial differential operators of the first order with respect to spatial variables x, y, z, w, v.

The products of two arbitrary differential operators $\mathbb{L}$ and Ω from (146) could be calculated using the already known formula (25):

$$(144)\begin{cases}\mathbb{L}\Omega^{12}=-\mathbb{L}^{13}\Omega^{23}-\mathbb{L}^{14}\Omega^{24}-\mathbb{L}^{15}\Omega^{25}+\mathbb{L}^{23}\Omega^{31}+\mathbb{L}^{24}\Omega^{14}+\mathbb{L}^{25}\Omega^{15},\\ \mathbb{L}\Omega^{13}=-\mathbb{L}^{12}\Omega^{23}+\mathbb{L}^{14}\Omega^{34}+\mathbb{L}^{15}\Omega^{35}+\mathbb{L}^{23}\Omega^{21}-\mathbb{L}^{34}\Omega^{14}-\mathbb{L}^{35}\Omega^{15},\\ \mathbb{L}\Omega^{14}=\mathbb{L}^{12}\Omega^{24}+\mathbb{L}^{13}\Omega^{34}-\mathbb{L}^{15}\Omega^{45}-\mathbb{L}^{24}\Omega^{21}-\mathbb{L}^{34}\Omega^{31}+\mathbb{L}^{45}\Omega^{15},\\ \mathbb{L}\Omega^{15}=-\mathbb{L}^{12}\Omega^{25}-\mathbb{L}^{13}\Omega^{35}-\mathbb{L}^{14}\Omega^{45}+\mathbb{L}^{25}\Omega^{21}+\mathbb{L}^{35}\Omega^{31}+\mathbb{L}^{45}\Omega^{14},\\ \mathbb{L}\Omega^{23}=-\mathbb{L}^{12}\Omega^{31}+\mathbb{L}^{31}\Omega^{21}-\mathbb{L}^{24}\Omega^{34}-\mathbb{L}^{25}\Omega^{35}+\mathbb{L}^{34}\Omega^{24}+\mathbb{L}^{35}\Omega^{25},\\ \mathbb{L}\Omega^{24}=\mathbb{L}^{12}\Omega^{14}-\mathbb{L}^{14}\Omega^{21}-\mathbb{L}^{23}\Omega^{34}+\mathbb{L}^{25}\Omega^{45}+\mathbb{L}^{34}\Omega^{23}-\mathbb{L}^{45}\Omega^{25},\\ \mathbb{L}\Omega^{25}=-\mathbb{L}^{12}\Omega^{15}+\mathbb{L}^{35}\Omega^{21}+\mathbb{L}^{23}\Omega^{35}+\mathbb{L}^{24}\Omega^{45}-\mathbb{L}^{35}\Omega^{23}-\mathbb{L}^{45}\Omega^{24},\\ \mathbb{L}\Omega^{34}=-\mathbb{L}^{13}\Omega^{14}+\mathbb{L}^{14}\Omega^{31}-\mathbb{L}^{23}\Omega^{24}+\mathbb{L}^{24}\Omega^{23}-\mathbb{L}^{35}\Omega^{45}+\mathbb{L}^{45}\Omega^{35},\\ \mathbb{L}\Omega^{35}=\mathbb{L}^{13}\Omega^{15}-\mathbb{L}^{15}\Omega^{31}+\mathbb{L}^{23}\Omega^{25}-\mathbb{L}^{25}\Omega^{23}-\mathbb{L}^{34}\Omega^{45}+\mathbb{L}^{45}\Omega^{34},\\ \mathbb{L}\Omega^{45}=-\mathbb{L}^{13}\Omega^{15}+\mathbb{L}^{15}\Omega^{14}-\mathbb{L}^{24}\Omega^{25}+\mathbb{L}^{25}\Omega^{24}-\mathbb{L}^{34}\Omega^{35}+\mathbb{L}^{35}\Omega^{34}.\end{cases}$$

Evidently, that the products in (147) symbolize the linear partial differential operators of the order two with respect to spatial variables x, y, z, w, v

From Equation (147) we get the commutation relations for the operators. Remarkably, that if $\Omega = \mathbb{L}^T$,

$$(145)\ \mathbb{L}\Omega = -3\mathbb{L}.$$

The Equation (148) is valid in 5D Euclidean space. In this particular instance, the product of two operators results in the shortening of the second-order powers.

This behaviour has been observed to occur in analogous ways in lower dimensional spaces. The corresponding expression 4D space reads:

$$(146)\ \mathbb{L}\Omega = -2\mathbb{L}.$$

In three-dimensional space, the conventional formula is employed:

$$(147)\ \mathbb{L}\Omega = -\mathbb{L}.$$

In order to derive the law of conservation of momentum, it was necessary to make use of the homogeneity of space relative to a closed system of particles. In addition to its homogeneity, space exhibits the property of isotropy. In accordance with the aforementioned principle, all directions within the system are considered to be equivalent. Therefore, the Hamiltonian of It is an irrefutable fact that a closed system is incapable of undergoing change when the system as a whole undergoes rotation. The angle is arbitrary, as is the axis. It is imperative that this condition be fulfilled for an arbitrary infinitely small rotation.

6. Conclusions and Directions for Future Research

The present manuscript undertakes a study of the rotational behavior of rigid bodies in spaces of higher dimensions. It is demonstrated that two integrals of motion exist for an unperturbed rotational motion. In the context of Euclidean spaces of dimension four, the number of rotational degrees of freedom is six. In the case of a five-dimensional Euclidean space, the number of rotational degrees of freedom is increased to ten. The Euler equations are derived using the tensor representation of rotational velocities. The closed form solutions were discovered for a certain relation between the principal moments of inertia.

Spin angular momentum, also known as spin, is a fundamental property of all particles, regardless of their status as elementary or composite entities. It is an intrinsic property, independent of spatial coordinates, and manifests as the particle's inherent angular momentum. The succeeding study will explore the quantization of the rotation motion of four-dimensional rigid bodies in order to analyze the symmetries and relations between the virtual moments of inertia in higher dimensions. These can be experimentally observed on the verifiable three-dimensional projections of the four- or five-dimensional hypothetical spaces on the microscopic scale.

References

1. Arnold, V. (1989). Rigid bodies. In: Mathematical Methods of Classical Mechanics. Graduate Texts in Mathematics, vol 60. New York: Springer https://doi.org/10.1007/978-1-4757-2063-1_6.
2. Borisov, A., & Mamaev, I. (2019). *Rigid Body Dynamics.* Berlin/Boston.: Higher Education Press and Walter de Gruyter GmbH.
3. Cayley, A. (1846). Sur quelques propriétés des déterminants gauches. *Journal für die reine und angewandte Mathematik , 32*, 119–123.
4. Doran, C., & Lasenby, A. (2003). *Geometric algebra for physicists.* Cambridge: Cambridge University Press.
5. Fredholm, E. I. (1903). Sur une classe d'equations fonctionnelles. *Acta Math., 27*, 365-390.
6. Gantmacher, F. (1959). *The Theory of Matrices, Volume 1.* Rhode Island: AMS Chelsea Publishing: An Imprint of the American Mathematical Society.
7. Hestenes, D. (2002). *New Foundations for Classical Mechanics.* NEW YORK, BOSTON, DORDRECHT, LONDON, MOSCOW: KLUWER ACADEMIC PUBLISHERS.
8. Hollmeier, V. A. (2024). *Geometric Algebra for Physicists.* https://physi.uni-heidelberg.de/~vhollmeier/ga4p, Heidelberg: University of Heidelberg.
9. Krey, U., & Owen, A. (2007). *Basic Theoretical Physics. A Concise Review.* Berlin Heidelberg : Springer-Verlag.
10. Landau, L., & Lifshitz, E. M. (1965). Quantum Mechanics. Non-relativistic Theory. Course of Theoretical Physics. Volume 3. Oxford: Pergamon Press Ltd.
11. Landau, L., & Lifshitz, E. M. (1976). *Mechanics: Volume 1, Course of Theoretical Physics.* Oxford: Butterworth-Heinemann.
12. Parker, E. (2024). The Ricci decomposition of the inertia tensor for a rigid body in arbitrary spatial dimensions. *International Journal of Theoretical Physics, https://doi.org/10.1007/s10773-023-05505-7*.