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Article

Generalized Weighted Group Inverses of Banach Elements

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Abstract

We introduce the generalized weighted group inverse in Banach algebras, a novel concept that unifies the classical group inverse with quasinilpotent elements. We establish its fundamental characterization through an equivalence to a specific polar-like property and elucidate its structural relationship with the generalized weighted core inverse. Employing a novel limit-based approach, we extend the theory of the weak group inverse from matrices and Hilbert space operators to the general setting of Banach algebras, thereby significantly generalizing key existing results.

Keywords: generalized Drazin inverse; group inverse; weak group inverse; pseudo core inverse; Banach algebra

1. Introduction

A Banach algebra is called a Banach $*$ -algebra if there exists an involution $*$: $x \rightarrow x^*$ satisfying $(x + y)^* = x^* + y^*$, $(\lambda x)^* = \bar{\lambda}x^*$, $(xy)^* = y^*x^*$, $(x^*)^* = x$. Let \mathcal{A} be a Banach $*$ -algebra. An element $a \in \mathcal{A}$ has group inverse provided that there exists $x \in \mathcal{A}$ such that

$$xa^2 = a, ax^2 = x, ax = xa.$$

Such x is unique if exists, denoted by $a^\#$, and called the group inverse of a . As is well known, a square complex matrix A has group inverse if and only if $\text{rank}(A) = \text{rank}(A^2)$. An element $a \in \mathcal{A}$ has core inverse if there exists some $x \in \mathcal{A}$ such that

$$xa^2 = a, ax^2 = x, (ax)^* = ax.$$

If such x exists, it is unique, and denote it by a^\oplus . Let $R(X)$ represent the range space of a complex matrix X . A square complex matrix A has core inverse A^\oplus if and only if AA^\oplus is a projection and $R(A^\oplus) \subseteq R(A)$ (see [20]). Recently, Gao and Chen [8] introduced the core-EP inverse as a generalization of core inverse. An element $a \in \mathcal{A}$ has core-EP inverse if there exist $x \in \mathcal{A}$ and $k \in \mathbb{N}$ such that

$$xa^{k+1} = a^k, ax^2 = x, (ax)^* = ax.$$

If such x exists, it is unique, and denote it by a^\ominus . A square complex matrix X is the core-EP inverse of A if $XAX = X$, $R(X) = R(X^*) = R(A^m)$ where m is the Drazin index of A .

Recently, Wang and Chen (see [17]) introduced and studied weak group inverse for a square complex matrix. A square complex matrix A has weak group inverse X if it satisfies the system of equations: $AX^2 = X$, $AX = A^\oplus A$. The preceding X is unique and denoted by A^\ominus . The involution $*$ is proper if $x^*x = 0 \implies x = 0$ for any $x \in \mathcal{A}$, e.g., in a Rickart $*$ -algebra, the involution is always proper. Let $\mathbb{C}^{n \times n}$ be the Banach algebra of all $n \times n$ complex matrices, with conjugate transpose $*$ as the involution. Then the involution $*$ is proper. In [22], Zou et al. extend the notion of weak group

inverse from complex matrices to elements in a ring with proper involution. We refer the reader for weak group inverse in [7,14,18,24,25].

Recall that $a \in \mathcal{A}$ has g-Drazin inverse (i.e., generalized Drazin inverse) if there exists $x \in \mathcal{A}$ such that

$$ax^2 = x, ax = xa, a - a^2x \in \mathcal{A}^{qnil}.$$

Such x is unique, if exists, and denote it by a^d . Here, $\mathcal{A}^{qnil} = \{x \in \mathcal{A} \mid \lim_{n \rightarrow \infty} \|x^n\|^{\frac{1}{n}} = 0\}$. As is well known, $x \in \mathcal{A}^{qnil}$ if and only if $1 + \lambda x \in \mathcal{A}$ is invertible. The generalized Drazin inverse plays an important role in matrix and operator theory (see [4]). Recently, Mosić and Zhang introduced and studied weak group inverse for a Hilbert space operator A in $\mathcal{B}(X)^d$ (see [15]).

The objective of this paper is to introduce and examine a novel class of generalized inverse, which acts as a seamless continuation of the weak group inverse applicable to complex matrices and operators in Hilbert spaces. In Section 2, we unveil the generalized weighted group inverse by means of an innovative generalized weighted group decomposition. This method reveals a plethora of fresh properties pertaining to the weak group inverse for complex matrices and operators within Hilbert spaces. Let $e \in \mathcal{A}$ be an invertible Hermitian element (i.e., $e^* = e \in \mathcal{A}$ is invertible).

Definition 1.1. An element $a \in \mathcal{A}$ has generalized e -group decomposition if there exist $x, y \in \mathcal{A}$ such that

$$a = x + y, x^*ey = yx = 0, x \in \mathcal{A}^\#, y \in \mathcal{A}^{qnil}.$$

We prove that $a \in \mathcal{A}$ has generalized e -group decomposition if and only if there exists a $x \in \mathcal{A}$ such that

$$x = ax^2, (a^*ea^2x)^* = a^*ea^2x, \lim_{n \rightarrow \infty} \|a^n - xa^{n+1}\|^{\frac{1}{n}} = 0.$$

The element x is called the generalized e -group inverse of a , and we denote it by a^{\oplus} .

In Section 3, we characterize generalized weighted group inverse by using the generalized Drazin invertibility. We prove that $a \in \mathcal{A}$ has generalized e -group inverse if and only if $a \in \mathcal{A}^d$ and the equation $(a^d)^*ea^d x = (a^d)^*a$ is solvable in \mathcal{A} .

Evidently, $a \in \mathcal{A}$ has g-Drazin inverse if and only if it has quasi-polar property, i.e., there exists an idempotent $p \in \mathcal{A}$ such that

$$a + p \in \mathcal{A} \text{ is invertible, } pa = pa \in \mathcal{A}^{qnil}$$

(see [4], [Theorem 6.4.8]). An element $a \in \mathcal{A}$ has generalized e -core inverse if there exists $x \in \mathcal{A}$ such that

$$x = ax^2, (eax)^* = eax, \lim_{n \rightarrow \infty} \|a^n - xa^{n+1}\|^{\frac{1}{n}} = 0.$$

Such x is unique, if it exists, and we denote it by $a^{e,\oplus}$ (see [5]). Evidently, the generalized e -core inverse and core-EP inverse in [11,13] for a Hilbert space operator coincide with each other. In the final section, we present polar-like and generalized e -core properties for the generalized weighted group inverse. These also provided new properties of weak group inverse for Hilbert space operators (see [15]).

Throughout the paper, all Banach algebras are complex with a proper involution $*$. We use $\mathcal{A}^{-1}, \mathcal{A}^\#, \mathcal{A}^d, \mathcal{A}^{e,\oplus}, \mathcal{A}^{\oplus}$ and \mathcal{A}^{\otimes} to denote the sets of all invertible, group invertible, g-Drazin invertible, generalized e -core invertible, core-EP invertible and weak group invertible in \mathcal{A} , respectively.

2. Generalized e -Group Inverse

The purpose of this section is to introduce a new generalized inverse that serves as a natural extension of the weak group inverse in a Banach $*$ -algebra. We begin with the following lemma.

Lemma 2.1. Let $a \in \mathcal{A}^d$. Then $\lim_{n \rightarrow \infty} \|(a^n - a^d a^{n+1})^*\|^{\frac{1}{n}} = 0$.

Proof. Let $x = a - a^d a^2$. Then $x \in \mathcal{A}^{qmil}$. For any $\lambda \in \mathbb{C}$, we have $1 - \bar{\lambda}x \in \mathcal{A}^{-1}$, and so $1 - \lambda x^* \in \mathcal{A}^{-1}$. This implies that $x^* \in \mathcal{A}^{qmil}$. We easily check that

$$\begin{aligned} \|(a^n - a^d a^{n+1})^*\|^{\frac{1}{n}} &= \|(1 - a^d a)^*(a^n)^*\| = \|[(1 - a^d a)^*(a^n)^*]\| \\ &= \|[(a - a^d a^2)^n]^*\| = \|(x^*)^n\|. \end{aligned}$$

Since $\lim_{n \rightarrow \infty} \|(x^*)^n\|^{\frac{1}{n}} = 0$, we have $\lim_{n \rightarrow \infty} \|(a^n - a^d a^{n+1})^*\|^{\frac{1}{n}} = 0$. \square

Theorem 2.2. Let $a \in \mathcal{A}$. Then the following are equivalent:

- (1) $a \in \mathcal{A}$ has generalized e -group decomposition.
- (2) There exists $x \in \mathcal{A}$ such that

$$x = ax^2, (a^* ea^2 x)^* = a^* ea^2 x, \lim_{n \rightarrow \infty} \|a^n - xa^{n+1}\|^{\frac{1}{n}} = 0.$$

- (3) $a \in \mathcal{A}^d$ and there exists $x \in \mathcal{A}$ such that

$$x = ax^2, (a^d)^* ea^2 x = (a^d)^* ea, \lim_{n \rightarrow \infty} \|a^n - xa^{n+1}\|^{\frac{1}{n}} = 0.$$

Proof. (1) \Rightarrow (3) By hypothesis, a has the generalized e -group decomposition $a = a_1 + a_2$. Let $x = a_1^\#$. Then $ax^2 = (a_1 + a_2)(a_1^\#)^2 = a_1(a_1^\#)^2 = a_1^\# = x$.

Since $a_2 a_1 = 0$, we have $a - xa^2 = (a_1 + a_2) - (a_1^\# a_1 + a_1^\# a_2)(a_1 + a_2) = (1 - a_1^\# a_1 - a_1^\# a_2)a_2$, and then

$$\begin{aligned} \|a^n - xa^{n+1}\|^{\frac{1}{n}} &= \|(a - xa^2)a^{n-1}\|^{\frac{1}{n}} \\ &= \|(1 - a_1^\# a_1 - a_1^\# a_2)a_2(a_1 + a_2)^{n-1}\|^{\frac{1}{n}} \\ &\leq \|1 - a_1^\# a_1 - a_1^\# a_2\|^{\frac{1}{n}} \|a_2^n\|^{\frac{1}{n}}. \end{aligned}$$

Since $a_2 \in \mathcal{A}^{qmil}$, we have $\lim_{n \rightarrow \infty} \|a_2^n\|^{\frac{1}{n}} = 0$. Hence $\lim_{n \rightarrow \infty} \|a^n - xa^{n+1}\|^{\frac{1}{n}} = 0$. Since $a_2 a_1 = 0$, $a_1 a_1^\# = 0$ and $a_2^d = 0$, it follows by [2, Corollary 3.4] that $a \in \mathcal{A}^d$ and $a^d = a_1^\# + \sum_{n=1}^{\infty} (a_1^\#)^{n+1} a_2^n$. Then

$$\begin{aligned} (a^d)^* ea_2 &= (a_1^\#)^* ea_2 + \sum_{n=1}^{\infty} [(a_1^\#)^{n+1} a_2^n]^* ea_2 \\ &= [(a_1^\#)^2]^* (a_1^* ea_2) + \sum_{n=1}^{\infty} [(a_1^\#)^{n+2} a_2^n]^* (a_1^* ea_2) = 0; \end{aligned}$$

hence, $(a^d)^* ea_1 = (a^d)^* ea$. Accordingly, $(a^d)^* ea^2 x = (a^d)^* e(a_1 + a_2)(a_1 + a_2)a_1^\# = (a^d)^* ea_1^\# a_1^\# = (a^d)^* ea_1 = (a^d)^* ea$.

- (3) \Rightarrow (2) By hypothesis, there exists $x \in \mathcal{A}$ such that

$$x = ax^2, (a^d)^* ea^2 x = (a^d)^* ea, \lim_{n \rightarrow \infty} \|a^n - xa^{n+1}\|^{\frac{1}{n}} = 0.$$

Let $a_1 = a^2 x$ and $a_2 = a - a^2 x$. Then we check that

$$\begin{aligned} \|a_2 a_1\| &= \|(a - a^2 x)a^2 x\| = \|a^3 x - a^2 x a^2 x\| = \|a^3 x - a^2 x a^{k+1} x^k\| \\ &= \|a^3 x - a^{k+2} x^k + a^2(a^k - xa^{k+1})x^k - a^{k+2} x^k\| \\ &\leq \|a^3 x - a^{k+2} x^k\| + \|a^2\| \|a^k - xa^{k+1}\| \|x^k - a^{k+2} x^k\|. \end{aligned}$$

Then

$$\begin{aligned} \|a_2 a_1\|^{\frac{1}{k}} &\leq \|a^2\|^{\frac{1}{k}} \|a^k - xa^{k+1}\|^{\frac{1}{k}} \|x^k - a^{k+2} x^k\|^{\frac{1}{k}} \\ &\leq \|a^2\|^{\frac{1}{k}} \|a^k - xa^{k+1}\|^{\frac{1}{k}} (1 + \|a^2\|^{\frac{1}{k}} \|a\|) \|x\|. \end{aligned}$$

Therefore $\lim_{k \rightarrow \infty} \|a_2 a_1\|^{\frac{1}{k}} = 0$, and then $a_2 a_1 = 0$.

Moreover, we have

$$\begin{aligned} & \|a_1^* e a_2\| \\ = & \|(a^2 x)^* e(a - a^2 x)\| = \|(a^2 x)^* e a - (a^2 x)^* e a^2 x\| = \|(a^2 x)^* e a - (a^k x^{k-1})^* e a^2 x\| \\ = & \|(a^2 x)^* e a - [(a^k - a^d a^{k+1})x^{k-1} + a^d a^{k+1} x^{k-1}]^* e a^2 x\| \\ \leq & \|(a^k - a^d a^{k+1})^* \|(x^{k-1})^* \||e a^2 x\| + \|(a^2 x)^* e a - [a^d a^{k+1} x^{k-1}]^* e a^2 x\| \\ \leq & \|(a^k - a^d a^{k+1})^* \|(x^{k-1})^* \||e a^2 x\| + \|(a^2 x)^* e a - [a^{k+1} x^{k-1}]^* (a^d)^* e a^2 x\| \\ \leq & \|(a^k - a^d a^{k+1})^* \|(x^{k-1})^* \||e a^2 x\| + \|(a^2 x)^* e a - [a^{k+1} x^{k-1}]^* (a^d)^* e a\| \\ \leq & \|(a^k - a^d a^{k+1})^* \|(x^{k-1})^* \||e a^2 x\| + \|(a^2 x)^* e a - [a^d a^{k+1} x^{k-1}]^* e a\| \\ \leq & \|(a^k - a^d a^{k+1})^* \|(x^{k-1})^* \||e a^2 x\| + \|(a^k x^{k-1})^* e a - [a^d a^{k+1} x^{k-1}]^* e a\| \\ \leq & \|(a^k - a^d a^{k+1})^* \|(x^{k-1})^* \||e a^2 x\| + \|(a^k - a^d a^{k+1})x^{k-1}\|^* \||e a\| \\ \leq & \|(a^k - a^d a^{k+1})^* \|(x^{k-1})^* \||e a^2 x\| + \|(a^k - a^d a^{k+1})^* \|(x^{k-1})^* \||e a\|. \end{aligned}$$

Then

$$\begin{aligned} \|a_1^* e a_2\|^{\frac{1}{k}} & \leq \|(a^k - a^d a^{k+1})^* \|^{\frac{1}{k}} \|(x^{k-1})^* \|^{\frac{1}{k}} \|e a^2 x\|^{\frac{1}{k}} \\ & + \|(a^k - a^d a^{k+1})^* \|^{\frac{1}{k}} \|(x^{k-1})^* \|^{\frac{1}{k}} \|e a\|^{\frac{1}{k}}. \end{aligned}$$

By using Lemma 2.1, we have $\lim_{k \rightarrow \infty} \|a_1^* e a_2\|^{\frac{1}{k}} = 0$; whence, $a_1^* e a_2 = 0$.

We check that

$$\begin{aligned} a_1 x & = a^2 x^2 = a(ax^2) = ax = (xa^2)x = x(a^2 x) = xa_1, \\ a_1 x a_1 & = ax(a^2 x) = a^2 x = a_1, \\ x a_1 x & = a_1 x^2 = (a^2 x)x^2 = a(ax^2)x = ax^2 = x. \end{aligned}$$

Hence $a_1^\# = x$. Clearly, $a = a^2 x + (a - a^2 x) = a_1 + a_2$. Then we have $a^* e a^2 x = (a_1^* + a_2^*)e(a_1 + a_2)^2 a_1^\# = (a_1^* + a_2^*)e a_1 = a_1^* e a_1 + (a_1^* e a_2)^* = a_1^* e a_1$. Accordingly, $(a^* e a^2 x)^* = (a_1^* e a_1)^* = a_1^* e a_1 = a^* e a^2 x$, as required.

(2) \Rightarrow (1) By hypotheses, we have $z \in \mathcal{A}$ such that

$$z = az^2, (a^* e a^2 z)^* = a^* e a^2 z, \lim_{n \rightarrow \infty} \|a^n - z a^{n+1}\|^{\frac{1}{n}} = 0.$$

For any $n \in \mathbb{N}$, we have $az = a(az^2) = a^2 z^2 = a^2 (az^2)z = a^3 z^3 = \dots = a^n z^n = \dots = a^{n+1} z^{n+1}$. Thus, we prove that

$$\|z - zaz\| = \|(az)z - zaz\| = \|(a^n z^n)z - z(a^{n+1} z^{n+1})\| = \|(a^n - z a^{n+1})z^{n+1}\|.$$

Hence,

$$\|z - zaz\|^{\frac{1}{n}} \leq \|(a^n - z a^{n+1})\|^{\frac{1}{n}} \|z\|^{\frac{n+1}{n}}.$$

This implies that $\lim_{n \rightarrow \infty} \|z - zaz\|^{\frac{1}{n}} = 0$, whence, $z = zaz$.

Set $x = a^2 z$ and $y = a - a^2 z$. Then $a = x + y$. We check that

$$\begin{aligned} (a - za^2)z & = (a - za^2)az^2 = (a - za^2)a^2 z^3 \\ & \vdots \\ & = (a - za^2)a^{n-1} z^n = (a^n - za^{n-1})z^n. \end{aligned}$$

Therefore

$$\|(a - za^2)z\|^{\frac{1}{n}} \leq \|a^n - za^{n-1}\|^{\frac{1}{n}} \|z^n\|^{\frac{1}{n}}.$$

Since $\lim_{n \rightarrow \infty} \|a^n - za^{n+1}\|^{\frac{1}{n}} = 0$, we derive that $\lim_{n \rightarrow \infty} \|(a - za^2)z\|^{\frac{1}{n}} = 0$. This implies that $(a - za^2)z = 0$.

We claim that x has group inverse. Evidently, we verify that

$$\begin{aligned}zx^2 &= za^2za^2z = a(za^2z) = a^2z = x, \\xz^2 &= (a^2z)z^2 = (a^2z^2)z = az^2 = z, \\xz &= a^2z^2 = az = za^2z = zx.\end{aligned}$$

Hence, $x \in \mathcal{A}^\#$.

We check that

$$\begin{aligned}\|(a - za^2)^{n+1}\|^{\frac{1}{n+1}} &= \|(a - za^2)^n(a - za^2)\|^{\frac{1}{n+1}} \\&= \|(a - za^2)^{n-1}(a - za^2)a\|^{\frac{1}{n+1}} \\&= \|(a - za^2)^{n-1}a^2\|^{\frac{1}{n+1}} \\&\vdots \\&= \|(a - za^2)a^n\|^{\frac{1}{n+1}} \\&\leq [\|a^n - za^{n+1}\|^{\frac{1}{n}}]^{\frac{n}{n+1}} \|a^n\|^{\frac{1}{n+1}}.\end{aligned}$$

Accordingly, $\lim_{n \rightarrow \infty} \|(a - za^2)^{n+1}\|^{\frac{1}{n+1}} = 0$. This implies that $a - za^2 \in \mathcal{A}^{qmil}$. By using Cline's formula (see [10, Theorem 2.1]), $y = a - a^2z \in \mathcal{A}^{qmil}$. Moreover, we see that

$$\begin{aligned}x^*ey &= (a^2z)^*e(a - a^2z) = [z^*(a^2)^*ea](1 - az) \\&= [a^*ea^2z]^*(1 - az) = (a^*ea^2z)(1 - az) = 0, \\yx &= (a - a^2z)(a^2z) = a^3z - a^2(za^2z) = a^3z - a^2(az) = 0,\end{aligned}$$

as desired. \square

We denote the preceding an element x in Theorem 2.3 by $a^{e,\otimes}$, and call it the generalized e -group inverse of a . Let $\mathcal{A}^{e,\otimes}$ denote the sets of all generalized e -group invertible elements in \mathcal{A} .

Corollary 2.3. *Let $a \in \mathcal{A}^{e,\otimes}$. Then the following hold.*

- (1) $a^{e,\otimes} = a^{\otimes}aa^{e,\otimes}$.
- (2) $aa^{e,\otimes} = a^m(a^{e,\otimes})^m$ for any $m \in \mathbb{N}$.

Proof. These are obvious by the proof of Theorem 2.2. \square

An element $e \in \mathcal{A}$ is positive if there exists an invertible $b \in \mathcal{A}$ such that $e = b^*b$. Evidently, an $n \times n$ complex matrix A is positive definite, i.e., every eigenvalue λ of A satisfies $\lambda > 0$, if and only if it is a positive element in $\mathbb{C}^{n \times n}$. We now derive

Theorem 2.4. *Let $a \in \mathcal{A}$ and $e \in \mathcal{A}$ be positive. Then the following are equivalent:*

- (1) $a \in \mathcal{A}^{e,\otimes}$.
- (2) There exists a unique element $x \in \mathcal{A}$ such that

$$x = ax^2, (a^*ea^2x)^* = a^*ea^2x, \lim_{n \rightarrow \infty} \|a^n - xa^{n+1}\|^{\frac{1}{n}} = 0.$$

Proof. (1) \Rightarrow (2) In view of Theorem 2.2, there exists $x \in \mathcal{A}$ such that

$$x = ax^2, (a^*ea^2x)^* = a^*ea^2x, \lim_{n \rightarrow \infty} \|a^n - xa^{n+1}\|^{\frac{1}{n}} = 0.$$

Now we assume that there exists $y \in \mathcal{A}$ such that

$$y = ay^2, (a^*ea^2y)^* = a^*ea^2y, \lim_{n \rightarrow \infty} \|a^n - ya^{n+1}\|^{\frac{1}{n}} = 0.$$

As in the proof of Theorem 2.2, we see that $(a^2x)^\# = x$ and $(a^2y)^\# = y$. Moreover, we have

$$\begin{aligned}x &= ax^2, (a^d)^*ea^2x = (a^d)^*ea, \lim_{n \rightarrow \infty} \| |a^n - xa^{n+1}| |^{\frac{1}{n}} = 0; \\y &= ay^2, (a^d)^*ea^2y = (a^d)^*ea, \lim_{n \rightarrow \infty} \| |a^n - ya^{n+1}| |^{\frac{1}{n}} = 0.\end{aligned}$$

Then

$$\begin{aligned}& \| |(a^2x)^*ea^2x - (a^2x)^*ea|^* \| \\&= \| |(a^kx^{k-1})^*ea^2x - (a^kx^{k-1})^*ea|^* \| \\&= \| |(x^{k-1})^*(a^k)^*ea^2x - (a^kx^{k-1})^*ea|^* \| \\&= \| |(x^{k-1})^*(a^d a^{k+1})^*ea^2x - (a^kx^{k-1})^*ea + (x^{k-1})^*(a^k - a^d a^{k+1})^*ea^2x|^* \| \\&= \| |(x^{k-1})^*(a^{k+1})^*(a^d)^*ea^2x - (a^kx^{k-1})^*ea + (x^{k-1})^*(a^k - a^d a^{k+1})^*ea^2x|^* \| \\&= \| |(x^{k-1})^*(a^{k+1})^*(a^d)^*ea - (a^kx^{k-1})^*ea + (x^{k-1})^*(a^k - a^d a^{k+1})^*ea^2x|^* \| \\&= \| |(x^{k-1})^*(a^d a^{k+1} - a^k)^*ea + (x^{k-1})^*(a^k - a^d a^{k+1})^*ea^2x|^* \| \\&\leq (\| (ea)^* \| \| |x^{k-1}| \| + \| |x^{k-1}| \| \| (ea^2x)^* \|) \| (a^k - a^d a^{k+1})^* \|. \end{aligned}$$

In view of Lemma 2.1, we have

$$\lim_{n \rightarrow \infty} \| |(a^2x)^*ea^2x - (a^2x)^*ea|^* \| |^{\frac{1}{n}} = 0.$$

Hence $[(a^2x)^*ea^2x]^* = [(a^2x)^*ea]^*$, and then $(a^2x)^*ea^2x = (a^2x)^*ea$. Likewise, we verify that

$$(a^2x)^*ea^2y = (a^2x)^*ea, (a^2y)^*ea^2x = (a^2y)^*ea, (a^2y)^*ea^2y = (a^2y)^*ea.$$

Let $z = a^2x - a^2y$. Then we check that

$$\begin{aligned}z^*ez &= (a^2x - a^2y)^*e(a^2x - a^2y) \\&= (a^2x)^*ea^2x - (a^2x)^*ea^2y - (a^2y)^*ea^2x + (a^2y)^*ea^2y \\&= (a^2x)^*ea - (a^2x)^*ea - (a^2y)^*ea + (a^2y)^*ea = 0.\end{aligned}$$

Write $e = b^*b$ for an invertible $b \in \mathcal{A}$. Then $(bz)^*(bz) = z^*(b^*b)z = e^*ez = 0$. Since \mathcal{A} is a proper Banach $*$ -algebra, we deduce that $bz = 0$; hence, $z = b^{-1}(bz) = 0$. This implies that $a^2x = a^2y$.

As in the proof in Theorem 2.2, we have $x = xax$ and $y = yay$. Therefore

$$\begin{aligned}\|x - y\| &= \|xax - yay\| = \|xa^{k+1}x^{k+1} - ya^k y^k\| \\&= \| (xa^{k+1} - a^k)x^{k+1} + a^kx^{k+1} - y^2a^{k+1}y^k + y(ya^{k+1} - a^k)y^k \| \\&\leq \| (xa^{k+1} - a^k)x^{k+1} \| + \| y(ya^{k+1} - a^k)y^k \| + \| a^kx^{k+1} - y^2a^2y \| \\&= \| (xa^{k+1} - a^k)x^{k+1} \| + \| y(ya^{k+1} - a^k)y^k \| + \| a^kx^{k+1} - y^2a^2x \| \\&= \| (xa^{k+1} - a^k)x^{k+1} \| + \| y(ya^{k+1} - a^k)y^k \| + \| a^{k-1}x^k - y^2a^{k+1}x^k \| \\&\leq \| (xa^{k+1} - a^k)x^{k+1} \| + \| y(ya^{k+1} - a^k)y^k \| + \| (a^{k-1} - ya^k)x^k \| \\&+ \| y(a^k - ya^{k+1})x^k \|. \end{aligned}$$

Hence,

$$\begin{aligned}\|x - y\|^{\frac{1}{k}} &\leq \| |x| |^{1+\frac{1}{k}} \| |a^k - xa^{k+1}| |^{\frac{1}{k}} + [(1 + \| |y| \|) \| |x^k| \| \\&+ \| |y| |^{k+1}]^{\frac{1}{k}} \| |a^k - ya^{k+1}| |^{\frac{1}{k}}.\end{aligned}$$

This implies that $\lim_{n \rightarrow \infty} \| |x - y| |^{\frac{1}{k}} = 0$, and so $x = y$. That is, the preceding x is unique, as desired.

(2) \Rightarrow (1) This is obvious by Theorem 2.2. \square

We say that $a \in \mathcal{A}$ has generalized group inverse if $a \in \mathcal{A}^{e, \circledast}$ for $e = 1$, and use a^{\circledast} to stand for the unique element $a^{1, \circledast}$. Let $\mathbb{C}^{n \times n}$ be the Banach algebra of all $n \times n$ complex matrices, with conjugate transpose $*$ as the involution. Evidently, $\mathbb{C}^{n \times n}$ has proper involution $*$. As is well known, $(\mathbb{C}^{n \times n})^{qnil} = (\mathbb{C}^{n \times n})^{nil}$. For any $A \in \mathbb{C}^{n \times n}$, it follows by [22, Theorem 4.2] that the weak group inverse and generalized group inverse coincide with each other, i.e., $A^{\circledast} = A^{\circledast}$.

The Drazin inverse of a square complex matrix A is the unique matrix A^D defined by $A^k = A^{k+1}A^D$, $A^D = A^D A A^D$ and $A A^D = A^D A$, where k is the smallest non-negative integer such that $\text{rank}(A^k) = \text{rank}(A^{k+1})$. We come now to provide a new characterization for the weak group inverse of a complex matrix.

Corollary 2.5. *Let $A, X \in \mathbb{C}^{n \times n}$. Then the following are equivalent:*

- (1) A has weak group inverse X .
- (2) There exist $k \in \mathbb{N}$ such that $A^k = X A^{k+1}$, $X = A X^2$, $(A^D)^* A^2 X = (A^D)^* A$.

Proof. (1) \Rightarrow (2) Since A has weak group inverse X , there exist $k \in \mathbb{N}$ such that $A^k = X A^{k+1}$, $X = A X^2$, $(A^k)^* A^2 X = (A^k)^* A$. Hence,

$$\begin{aligned} (A^D)^* A^2 X &= [A^k (A^D)^{k+1}]^* A^2 X = [(A^D)^{k+1}]^* (A^k)^* A^2 X \\ &= [(A^D)^{k+1}]^* (A^k)^* A = [A^k (A^D)^{k+1}]^* A = (A^D)^* A. \end{aligned}$$

(2) \Rightarrow (1) By hypothesis, there exist $k \in \mathbb{N}$ such that $A^k = X A^{k+1}$, $X = A X^2$, $(A^D)^* A^2 X = (A^D)^* A$. Hence, $\lim_{n \rightarrow \infty} \|A^n - X A^{n+1}\|_n^{\frac{1}{n}} = 0$. In view of Theorem 2.2, $A \in \mathbb{C}^{n \times n}$ has generalized group inverse. Therefore A has weak group inverse X . \square

Corollary 2.6. *Let $A \in \mathbb{C}^{n \times n}$. Then the matrix equation*

$$(A^D)^* A^2 x = (A^D)^* A b, b \in \mathbb{C}^{n \times 1}$$

has solutions

$$x = A^{\circledast} b + (I_n - A^{\circledast} A) y$$

for any $y \in \mathbb{C}^{n \times 1}$.

Proof. Choose $x = A^{\circledast} b + (I_n - A^{\circledast} A) y$ with a $y \in \mathbb{C}^{n \times 1}$. In light of Corollary 2.5, one directly checks that

$$\begin{aligned} (A^D)^* A^2 x &= (A^D)^* A^2 [A^{\circledast} b + (I_n - A^{\circledast} A) y] \\ &= [(A^D)^* A^2 A^{\circledast}] b + [(A^D)^* A^2 - [(A^D)^* A^2 A^{\circledast} A] y \\ &= [(A^D)^* A] b + [(A^D)^* A^2 - ((A^D)^* A^2 A^{\circledast} A) y] = (A^D)^* A b, \end{aligned}$$

as required. \square

3. Characterizing the Generalized e -Group Inverse via g -Drazin Invertibility

This section is about characterizing the property of generalized e -group inverses through the g -Drazin invertibility. We now derive:

Theorem 3.1. *Let $a \in \mathcal{A}$. Then $a \in \mathcal{A}^{e, \circledast}$ if and only if*

- (1) $a \in \mathcal{A}^d$;
- (2) There exists $x \in \mathcal{A}$ such that

$$ax^2 = x, (a^* ea^2 x)^* = a^* ea^2 x, \lim_{n \rightarrow \infty} \|a^n - axa^n\|_n^{\frac{1}{n}} = 0.$$

In this case,

$$\begin{aligned} a^d &= \sum_{n=0}^{\infty} (a^{e, \circledast})^{n+1} (a - aa^{e, \circledast} a)^n [1 - (a - aa^{e, \circledast} a)(a - aa^{e, \circledast} a)^d], \\ a^{e, \circledast} &= aa^d z, \text{ where } az^2 = z, (a^* ea^2 z)^* = a^* ea^2 z. \end{aligned}$$

Proof. \implies By hypothesis, there exists $x \in \mathcal{A}$ such that

$$x = ax^2, (a^*ea^2x)^* = a^*ea^2x, \lim_{n \rightarrow \infty} \|a^n - xa^{n+1}\|_{\frac{1}{n}} = 0.$$

Step 1. We claim that

$$\begin{aligned} \lim_{n \rightarrow \infty} \|a^n - axa^n\|_{\frac{1}{n+1}} &= 0. \\ \|a^n - a^n x^n a^n\|_{\frac{1}{n}} &= \|a^n - xa^{n+1} + (ax^2)a^{n+1} - a^n x^n a^n\|_{\frac{1}{n}} \\ &= \|a^n - xa^{n+1} + (a^n x^n)xa^{n+1} - a^n x^n a^n\|_{\frac{1}{n}} \\ &\leq \|a^n - xa^{n+1} + (a^n x^n)xa^{n+1} - a^n x^n a^n\|_{\frac{1}{n}} \\ &\leq \|a^n - xa^{n+1} - (a^n x^n)(a^n - xa^{n+1})\|_{\frac{1}{n}} \\ &\leq \|1 - a^n x^n\|_{\frac{1}{n}} \|a^n - xa^{n+1}\|_{\frac{1}{n}} \\ &\leq (1 + \|a\| \|x\|) \|a^n - xa^{n+1}\|_{\frac{1}{n}}. \end{aligned}$$

Since

$$\lim_{n \rightarrow \infty} \|a^n - xa^{n+1}\|_{\frac{1}{n}} = 0$$

and $ax = a^n x^n$, we prove that

$$\lim_{n \rightarrow \infty} \|a^n - axa^n\|_{\frac{1}{n+1}} = 0.$$

Set $x = aza$ and $y = a - aza$. By the proof of Theorem 2.2, we have $a = x + y, x \in \mathcal{A}^\#, y \in \mathcal{A}^{qnil}, x^*ey = yx = 0$.

Clearly, $y, x \in \mathcal{A}^d$ and $yx = 0$. Since $y^d = 0$ and $x^d = x^\# = z$, it follows by [2, Corollary 3.4] that $a = y + x$ has g-Drazin inverse and

$$\begin{aligned} a^d &= \sum_{n=0}^{\infty} z^{n+1} y^n y^\pi \\ &= \sum_{n=0}^{\infty} (a^\oplus)^{n+1} (a - aa^{e,\oplus} a)^n [1 - (a - aa^{e,\oplus} a)(a - aa^{e,\oplus} a)^d]. \end{aligned}$$

Since $a^d = \sum_{n=0}^{\infty} z^{n+1} y^n y^\pi$, we have $aa^d z = az^2 = z$. Moreover, $(a^*ea^2z)^* = a^*ea^2z$, as desired.

\Leftarrow By hypothesis, there exists some $z \in \mathcal{A}$ such that

$$az^2 = z, (a^*ea^2z)^* = a^*ea^2z, \lim_{n \rightarrow \infty} \|a^n - aza^n\|_{\frac{1}{n}} = 0.$$

Let $x = aa^d z$. We claim that $a^{e,\oplus} = x$. One directly verifies that

$$\begin{aligned} \|az - ax\| &= \|(a - a^2 a^d)z\| = \|(a - a^2 a^d)az^2\| \\ &= \dots = \|(a - a^2 a^d)^n z^n\| \leq \|(a - a^2 a^d)^n\| \|z^n\|, \end{aligned}$$

Then

$$\lim_{n \rightarrow \infty} \|az - ax\|_{\frac{1}{n}} = 0;$$

hence, $ax = az$, and so $(a^*ea^2x)^* = (a^*ea^2z)^* = a^*ea^2z = a^*ea^2x$.

$$\begin{aligned} x - ax^2 &= aa^d z - (ax)z = aa^d z - azaa^d z \\ &= (aa^d z - az^2) + az(1 - aa^d)z = (a^2 a^d z^2 - az^2) + az(1 - aa^d)z \\ &= (aa^d - 1)az^2 + az(1 - aa^d)z = (az - 1)(1 - aa^d)z. \end{aligned}$$

Since $z = az^2$, by induction, we have $x - ax^2 = (az - 1)(1 - aa^d)a^n z^{n+1}$, and so $\|x - ax^2\| \leq \|az - 1\| \|(a - a^2 a^d)^n\| \|z\|^{n+1}$. Since

$$\lim_{n \rightarrow \infty} \|(a - a^2 a^d)^n\|_{\frac{1}{n}} = 0,$$

we deduce that $\lim_{n \rightarrow \infty} \|x - ax^2\|^{\frac{1}{n}} = 0$. This implies that $ax^2 = x$.
Observing that

$$\begin{aligned} & \|a^n - xa^{n+1}\|^{\frac{1}{n}} \\ &= \|(a^n - a^{n+1}a^d) + (a^d a^n a - a^d a^n z^n a^n a)\|^{\frac{1}{n}} \\ &\leq \|a^n - a^{n+1}a^d\|^{\frac{1}{n}} + \|a^d\|^{\frac{1}{n}} \|a^n - a^n z^n a^n\|^{\frac{1}{n}} \|a\|^{\frac{1}{n}}, \end{aligned}$$

we have

$$\lim_{n \rightarrow \infty} \|a^n - xa^{n+1}\|^{\frac{1}{n}} = 0.$$

This completes the proof. \square

Corollary 3.2. *Let $A \in \mathbb{C}^{n \times n}$. Then the following are equivalent:*

- (1) A has weak group inverse.
- (2) There exists $X \in \mathbb{C}^{n \times n}$ such that

$$AX^2 = X, (A^* A^2 X)^* = A^* A^2 X, \lim_{n \rightarrow \infty} \|A^n - AXA^n\|^{\frac{1}{n+1}} = 0.$$

Proof. Since weak and generalized group inverses coincide with each other for a complex matrix, we obtain the result by Theorem 3.1. \square

We are ready to prove:

Theorem 3.3. *Let $a \in \mathcal{A}$. Then $a \in \mathcal{A}^{e, \circledast}$ if and only if*

- (1) $a \in \mathcal{A}^d$;
- (2) There exists $x \in \mathcal{A}$ such that

$$(a^* ea^2 x)^* = a^* ea^2 x, x\mathcal{A} = a^d \mathcal{A}, \lim_{n \rightarrow \infty} \|a^n - xa^{n+1}\|^{\frac{1}{n}} = 0.$$

In this case, $a^{e, \circledast} = x$.

Proof. \implies Since $a \in \mathcal{A}^{e, \circledast}$, it follows by Theorem 2.2 that $a \in \mathcal{A}^d$. Moreover, There exists $x \in \mathcal{A}$ such that

$$(a^* ea^2 x)^* = a^* ea^2 x, ax^2 = x, \lim_{n \rightarrow \infty} \|a^n - xa^{n+1}\|^{\frac{1}{n}} = 0.$$

In view of Theorem 2.2, a has the decomposition $a = a_1 + a_2$, where $a_1^* ea_2 = a_2 a_1 = 0, a_1 \in \mathcal{A}^\#, a_2 \in \mathcal{A}^{qnil}$. Let $x = a_1^\#$. Moreover, we have

$$x = ax^2, (a^* ea^2 x)^* = a^* ea^2 x, \lim_{n \rightarrow \infty} \|a^n - xa^{n+1}\|^{\frac{1}{n}} = 0.$$

Since $a_2 a_1 = 0, a_1 a_1^\# = 0$ and $a_2^d = 0$, it follows by [2, Corollary 3.4] that $a \in \mathcal{A}^d$ and

$$a^d = a_1^\# + \sum_{n=1}^{\infty} (a_1^\#)^{n+1} a_2^n.$$

We verify that

$$\begin{aligned} aa^d x &= a^d (a_1 + a_2) a_1^\# = a^d a_1 a_1^\# \\ &= [a_1^\# + \sum_{n=1}^{\infty} (a_1^\#)^{n+1} a_2^n] a_1 a_1^\# \\ &= a_1^\# (a_1 a_1^\#) = x. \end{aligned}$$

Furthermore, we see that

$$\begin{aligned} xaa^d &= a_1^\#(a_1 + a_2)[a_1^\# + \sum_{n=1}^{\infty} (a_1^\#)^{n+1}a_2^n] \\ &= a_1^\#a_1[a_1^\# + \sum_{n=1}^{\infty} (a_1^\#)^{n+1}a_2^n] \\ &= a_1^\# + \sum_{n=1}^{\infty} (a_1^\#)^{n+1}a_2^n = a^d. \end{aligned}$$

Therefore $x\mathcal{A} = a^d\mathcal{A}$, as required.

\Leftarrow We observe that

$$\begin{aligned} \|axa^d - a^d\|^{1/n} &= \|axa^{n+1}(a^d)^{n+2} - aa^n(a^d)^{n+2}\|^{1/n} \\ &\leq \|a\|^{1/n} \|a^n - xa^{n+1}\|^{1/n} \|(a^d)^{n+2}\|^{1/n}. \end{aligned}$$

Accordingly,

$$\lim_{n \rightarrow \infty} \|axa^d - a^d\|^{1/n} = 0.$$

Therefore we have $axa^d = a^d$. By hypothesis, $x\mathcal{A} = a^d\mathcal{A}$, and then $1 - ax \in \ell(a^d) \subseteq \ell(x)$. Hence $(1 - ax)x = 0$, i.e., $ax^2 = x$. In light of Theorem 2.2, $a \in \mathcal{A}^{e,\otimes}$. In this case, $a^{e,\otimes} = x$, as asserted. \square

Corollary 3.4. *Let $A \in \mathbb{C}^{n \times n}$. Then the following are equivalent:*

- (1) A has weak group inverse.
- (2) There exist $k \in \mathbb{N}, X \in \mathbb{C}^{n \times n}$ such that

$$A^k = XA^{k+1}, R(X) = R(A^D), (A^*A^2X)^* = A^*A^2X.$$

Proof. This is obvious by Theorem 3.3. \square

Theorem 3.5. *Let $a \in \mathcal{A}$. Then $a \in \mathcal{A}^{e,\otimes}$ if and only if*

- (1) $a \in \mathcal{A}^d$;
- (2) There exists an idempotent $q \in \mathcal{A}$ such that

$$a^d\mathcal{A} = q\mathcal{A} \text{ and } a^*eaq = q^*a^*ea.$$

In this case, $a^{e,\otimes} = a^dq$.

Proof. \implies In view of Theorem 2.2, $a \in \mathcal{A}^d$. Let $q = aa^{e,\otimes}$. By virtue of Corollary 2.4, $a^{e,\otimes} = a^{e,\otimes}aa^{e,\otimes}$. Then $q^2 = q$, i.e., $q \in \mathcal{A}$ is an idempotent. We check that

$$a^*eaq = a^*ea^2a^{e,\otimes} = (a^*ea^2a^{e,\otimes})^* = (a^*eaq)^* = q^*a^*ea.$$

In view of Theorem 3.1, $a^{e,\otimes} = aa^dz$, where $az^2 = z$ and $(a^*ea^2z)^* = a^*ea^2z$. Then $q = aa^{e,\otimes} = a(aa^dz) = (aa^d)(az)$. Hence, $q\mathcal{A} \subseteq aa^d\mathcal{A}$. By using Theorem 3.1 again,

$$a^d = \sum_{n=0}^{\infty} (a^{e,\otimes})^{n+1} (a - aa^{e,\otimes}a)^n [1 - (a - aa^{e,\otimes}a)(a - aa^{e,\otimes}a)^d].$$

Hence, $aa^d = aa^{e,\otimes}y = qy$ for some $y \in \mathcal{A}$. Then

$$aa^d\mathcal{A} \subseteq q\mathcal{A}.$$

Therefore $q\mathcal{A} = a^da\mathcal{A} = a^d\mathcal{A}$, as required.

\Leftarrow By hypothesis, there exists an idempotent $q \in \mathcal{A}$ such that $a^d \mathcal{A} = q\mathcal{A}$ and $a^* e a q = q^* a^* e a$. Hence, $q\mathcal{A} = a a^d \mathcal{A}$. Set $x = a^d q$. Then $ax = a a^d q = q$, and then $(a^* e a^2 x)^* = (a^* e a q)^* = q^* a^* e a = a^* e a q = (a^* e a)(ax) = a^* e a^2 x$. Moreover, we have $ax^2 = (a a^d) q a^d q = q a^d q = q(a a^d) a^d q = (a a^d) a^d q = a^d q = x$. We verify that

$$\begin{aligned} \|a^n - x a^{n+1}\| &= \|(a^n - (a^d q) a^d a^{n+2}) - [x(a^{n+1} - a^d a^{n+2})]\| \\ &\leq \|a^n - a^d a^{n+1}\| + \|x\| \|a^{n+1} - a^d a^{n+2}\| \\ &\leq (1 + \|x\| \|a\|) \|a^n - a^d a^{n+1}\| \\ &= (1 + \|x\| \|a\|) \|a^n (1 - a^d a)^n\| \\ &= (1 + \|x\| \|a\|) \|(a - a^d a^2)^n\|. \end{aligned}$$

Since $a - a^d a^2 \in \mathcal{A}^{qnil}$, we have $\lim_{n \rightarrow \infty} \|(a - a^d a^2)^n\|^{\frac{1}{n}} = 0$. Therefore $\lim_{n \rightarrow \infty} \|a^n - x a^{n+1}\|^{\frac{1}{n}} = 0$. In view of Corollary 2.3, $a \in \mathcal{A}^\oplus$. In this case, $a^\oplus = a^d q$. \square

Corollary 3.6. *Let $a \in \mathcal{A}$. Then $a \in \mathcal{A}^{e, \oplus}$ if and only if*

- (1) $a \in \mathcal{A}^d$;
- (2) There exists an idempotent $q \in \mathcal{A}$ such that $\ell(a^d) = \ell(q)$ and $a^* e a q = q^* a^* e a$.

In this case, $a^\oplus = a^d q$.

Proof. \implies In view of Theorem 3.5, $a \in \mathcal{A}^d$ and there exists an idempotent $q \in \mathcal{A}$ such that $a^d \mathcal{A} = q\mathcal{A}$ and $a^* e a q = q^* a^* e a$. We infer that $\ell(a^d) = \ell(q)$, as desired.

\Leftarrow By hypothesis, there exists an idempotent $q \in \mathcal{A}$ such that $\ell(a^d) = \ell(q)$ and $a^* e a q = q^* a^* e a$. Clearly, $1 - a a^d \in \ell(a^d) \subseteq \ell(q)$; hence, $q = a a^d q$. This implies that $q\mathcal{A} \subseteq a^d \mathcal{A}$. Also we have $1 - q \in \ell(q) \subseteq \ell(a^d)$, and then $a^d = q a^d$. We infer that $a^d \mathcal{A} \subseteq q\mathcal{A}$. Therefore $a^d \mathcal{A} = q\mathcal{A}$. In light of Theorem 3.5, $a \in \mathcal{A}^\oplus$. In this case, $a^\oplus = a^d q$. \square

Theorem 3.7. *Let $a \in \mathcal{A}$ and $e \in \mathcal{A}$ be positive. Then $a \in \mathcal{A}^\oplus$ if and only if*

- (1) $a \in \mathcal{A}^d$;
- (2) There exists some element $x \in \mathcal{A}$ such that $(a^d)^* e a^d x = (a^d)^* e a$.

In this case, $a^{e, \oplus} = (a^d)^3 x$.

Proof. \implies By virtue of Theorem 2.2, $a \in \mathcal{A}^d$ and there exists $z \in \mathcal{A}$ such that

$$z = a z^2, (a^d)^* e a^2 z = (a^d)^* e a, \lim_{n \rightarrow \infty} \|a^n - z a^{n+1}\|^{\frac{1}{n}} = 0.$$

Choose $x = a^3 z$. Then we verify that

$$\begin{aligned} \|(a^d)^* e a - (a^d)^* e a^d x\| &= \|(a^d)^* e a - (a^d)^* e a^d a^3 z\| \\ &= \|(a^d)^* e a - (a^d)^* e a^d a^{k+1} z^{k-1}\| \\ &\leq \|(a^d)^* e a - (a^d)^* e a^k z^{k-1}\| + \|(a^d)^* e (a^k - a^d a^{k+1}) z^{k-1}\| \\ &\leq \|(a^d)^* e a - (a^d)^* e a^2 z\| + \|(a^d)^* e\| \|a^k - a^d a^{k+1}\| \|z^{k-1}\| \\ &= \|(a^d)^* e\| \|a^k - a^d a^{k+1}\| \|z^{k-1}\|. \end{aligned}$$

Since $\lim_{k \rightarrow \infty} \|a^k - a^d a^{k+1}\|^{\frac{1}{k}} = 0$, we derive that $\lim_{k \rightarrow \infty} \|(a^d)^* e a - (a^d)^* e a^d x\|^{\frac{1}{k}} = 0$. Hence $(a^d)^* e a^d x = (a^d)^* e a$, as required.

\Leftarrow By hypothesis, $(a^d)^* e a^d x = (a^d)^* e a$ for some $x \in \mathcal{A}$. Then $(a a^d)^* e a a^d = a^* [(a^d)^* e a] a^d = a^* [(a^d)^* e a^d x] a^d = (a a^d)^* e a a^d (a^d x a^d)$. As e is positive, $e = b^* b$ for an invertible $b \in \mathcal{A}$. Then

$(baa^d)^*(baa^d) = (aa^d)^*eaa^d = (baa^d)^*(baa^d)(a^dxa^d)$. Since the involution $*$ is proper, we get $baa^d = (baa^d)(a^dxa^d)$; hence, $aa^d = a(a^d)^2xa^d = a^dxa^d$. Choose $y = (a^d)^3x$. Then we verify that

$$\begin{aligned} ay^2 &= a(a^d)^3x(a^d)^3x = (a^d)^2x(a^d)^3x = a^d(a^dxa^d)(a^d)^2x \\ &= a^d(aa^d)(a^d)^2x = (a^d)^3x = y, \\ (a^d)^*ea^2y &= (a^d)^*ea^2(a^d)^3x = (a^d)^*ea^d x = (a^d)^*ea. \end{aligned}$$

Moreover, we see that

$$\begin{aligned} & \|a^k - ya^{k+1}\| \\ &= \|(a^k - a^d a^{k+1}) + (a^d)^2 aa^d a^{k+2} - (a^d)^3 xa^{k+1}\| \\ &\leq \|(a^k - a^d a^{k+1})\| + \|(a^d)^2 aa^d a^{k+2} - (a^d)^3 xa^{k+1}\| \\ &\leq \|(a^k - a^d a^{k+1})\| + \|(a^d)^2 a^d xa^d a^{k+2} - (a^d)^3 xa^{k+1}\| \\ &\leq \|a^k - a^d a^{k+1}\| + \|(a^d)^3 x\| \|a^k - a^d a^{k+1}\| \|a\| \\ &= \|a^k - a^d a^{k+1}\| (1 + \|(a^d)^3 x\| \|a\|). \end{aligned}$$

Since $\lim_{k \rightarrow \infty} \|a^k - a^d a^{k+1}\|^{\frac{1}{k}} = 0$, we deduce that $\lim_{k \rightarrow \infty} \|a^k - ya^{k+1}\|^{\frac{1}{k}} = 0$. In view of Theorem 2.2, $a \in \mathcal{A}^{e, \otimes}$. In this case, $a^{e, \otimes} = (a^d)^3x$. This completes the proof. \square

Corollary 3.8. *Let $a \in \mathcal{A}$ and $e \in \mathcal{A}$ be positive. Then $a \in \mathcal{A}^{\otimes}$ if and only if*

- (1) $a \in \mathcal{A}^d$;
- (2) There exists an idempotent $q \in \mathcal{A}$ such that

$$a^d \mathcal{A} = q \mathcal{A} \text{ and } (a^d)^* e a q = (a^d)^* e a.$$

Proof. \implies By using Theorem 3.5, $a \in \mathcal{A}^d$ and there exists an idempotent $q \in \mathcal{A}$ such that $a^d \mathcal{A} = q \mathcal{A}$ and $a^* e a q = q^* a^* e a$. Explicit, $q = aa^{e, \otimes}$. We directly check that

$$\begin{aligned} qaa^d &= aa^{\otimes}aa^d = aa^d, \\ aa^d q &= aa^d aa^{\otimes} = q. \end{aligned}$$

Furthermore, we obtain

$$\begin{aligned} (a^d)^* e a q &= [a(a^d)^2]^* e a q = [(a^d)^2]^* (a^* e a q) = [(a^d)^2]^* (q^* a^* e a) \\ &= [a q (a^d)^2]^* e a = [a (qaa^d) (a^d)^2]^* e a = [a (aa^d) (a^d)^2]^* e a = (a^d)^* e a. \end{aligned}$$

We conclude that $a^d \mathcal{A} = q \mathcal{A}$ and $(a^d)^* a q = (a^d)^* a$, as required.

\Leftarrow By hypothesis, there exists an idempotent $q \in \mathcal{A}$ such that $a^d \mathcal{A} = q \mathcal{A}$ and $(a^d)^* e a q = (a^d)^* e a$. Write $q = a^d z$ for an element $z \in \mathcal{A}$. Choose $x = az$. Then $(a^d)^* e a^d x = (a^d)^* e a^d az = (a^d)^* e a q = (a^d)^* e a$, the result follows by Theorem 3.8. \square

4. Polar-like and Generalized Weighted Core Inverse Properties

The objective of this section is to delve into the generalized weighted group inverse, incorporating its associated properties. We now proceed to elucidate the generalized e -group inverse through its characterization by the polar-like property.

Theorem 4.1. *Let $a \in \mathcal{A}$. Then the following are equivalent:*

- (1) $a \in \mathcal{A}^{e, \otimes}$.
- (2) $a \in \mathcal{A}^d$ and there exists an idempotent $p \in \mathcal{A}$ such that

$$a + p \in \mathcal{A}^{-1}, (a^* e a p)^* = a^* e a p \text{ and } p a = p a p \in \mathcal{A}^{qnil}.$$

(3) $a \in \mathcal{A}^d$ and there exists an idempotent $p \in \mathcal{A}$ such that

$$a + p \in \mathcal{A}^{-1}, (a^d)^* eap = 0 \text{ and } pa = pap \in \mathcal{A}^{qnil}.$$

Proof. (1) \Rightarrow (2) In view of Theorem 2.2, $a \in \mathcal{A}^d$. Since $a \in R^{e,\otimes}$, there exist $z, y \in \mathcal{A}$ such that

$$a = z + y, z^*ey = yz = 0, z \in \mathcal{A}^\#, y \in \mathcal{A}^{qnil}.$$

Set $x = z^\#$. Then we check that

$$\begin{aligned} ax &= (z + y)z^\# = zz^\#, \\ ax^2 &= (ax)x = z(z^\#)^2 = z^\# = x, \\ (a^*ea^2x)^* &= (z^*ez)^* = z^*ez = a^*ea^2x. \end{aligned}$$

Let $p = 1 - zz^\#$. Then $p = 1 - ax = p^2 \in \mathcal{A}$. Furthermore, $a - azz^\# = a(1 - zz^\#) = (z + y)(1 - zz^\#) = y \in \mathcal{A}^{qnil}$, and so $pa = (1 - zz^\#)a = a - zz^\#a \in \mathcal{A}^{qnil}$ by Cline's formula (see [10, Theorem 2.1]). Therefore we have

$$(a^*eap)^* = (a^*ea - a^*ea^2x)^* = a^*ea - a^*a^2x = a^*eap.$$

Since $pa(1 - p) = (1 - zz^\#)(z + y)zz^\# = 0$, we get $pa = pap$. Obviously, $z + 1 - zz^\# = (z^\# + 1 - zz^\#)^{-1} \in \mathcal{A}^{-1}$. Since $y(z^\# + 1 - zz^\#) = y \in \mathcal{A}^{qnil}$, it follows by Cline's formula that $(z^\# + 1 - zz^\#)y \in \mathcal{A}^{qnil}$. Hence $1 + (z^\# + 1 - zz^\#)y \in \mathcal{A}^{-1}$. This implies that $a + p = z + y + 1 - zz^\# = (z + 1 - zz^\#)[1 + (z^\# + 1 - zz^\#)^{-1}y] \in \mathcal{A}^{-1}$, as desired.

(2) \Rightarrow (1) By hypothesis, there exists an idempotent $p \in \mathcal{A}$ such that

$$u := a + p \in \mathcal{A}^{-1}, (a^*eap)^* = a^*eap, pa = pap \in \mathcal{A}^{qnil}.$$

Set $x = a(1 - p)$ and $y = ap$. Then $a = x + y$. Since $pa = pap \in \mathcal{A}^{qnil}$, we have $y = ap \in \mathcal{A}^{qnil}$ by Cline's formula. We also see that $pa(1 - p) = 0$, and then $yx = apa(1 - p) = 0$. Moreover, we have $x^*ey = [a(1 - p)]^*eap = (1 - p)^*(a^*eap) = (1 - p)^*(a^*eap)^* = [a^*eap(1 - p)]^* = 0$. It will suffice to prove $x \in \mathcal{A}^\#$.

Clearly, $x = a(1 - p) = u(1 - p)$. Let $z = (1 - p)u^{-1}$. Then we check that $zx = 1 - p$; hence, $zx^2 = (1 - p)x = (1 - p)a(1 - p) = a(1 - p) = x$. Since $a \in \mathcal{A}^d, ap \in \mathcal{A}^d$ and $pa(1 - p) = 0$, it follows by [21, Lemma 2.2] that $x = a(1 - p) \in \mathcal{A}^d$. Therefore $x = zx^2 = z^n x^{n+1} = z^n(x^n - x^d x^{n+1})x + z^n x^d x^{n+2}$ and $x^2 x^d = (zx^2)xx^d = z^n x^{n+2} x^d$. Then

$$\begin{aligned} \|x - x^2 x^d\|^{1/n} &= \|z^n(x^n - x^d x^{n+1})x\|^{1/n} \\ &\leq \|z^n\|^{1/n} \|x^n - x^d x^{n+1}\|^{1/n} \|x\|^{1/n}. \end{aligned}$$

Accordingly,

$$\lim_{n \rightarrow \infty} \|x - x^2 x^d\|^{1/n} = 0.$$

Therefore we have $x = x^2 x^d$, and so $x \in \mathcal{A}^\#$. Then $a = x + y$ is a generalized e -group decomposition of a . In light of Theorem 2.2, $a \in \mathcal{A}^{e,\otimes}$.

(1) \Rightarrow (3) Clearly, $a \in \mathcal{A}^d$. By hypothesis, a has the generalized e -group decomposition $a = z + y$. Let $x = z^\#$ and $p = 1 - ax$. As in the preceding discussion, we prove that $a + p \in \mathcal{A}^{-1}$ and $pa = pap \in \mathcal{A}^{qnil}$. Similarly to Theorem 2.2, we show that $a \in \mathcal{A}^d$ and $(a^d)^*ea^2x = (a^d)^*ea$. Therefore $(a^d)^*eap = (a^d)^*ea - (a^d)^*ea^2x = 0$, as desired.

(3) \Rightarrow (1) By hypothesis, we have an idempotent $p \in \mathcal{A}$ such that

$$a + p \in \mathcal{A}^{-1}, (a^d)^*eap = 0 \text{ and } pa = pap \in \mathcal{A}^{qnil}.$$

Set $x = a(1 - p)$ and $y = ap$. Then $a = x + y$. Analogously to the preceding discussion, we have $yx = 0$, $x \in \mathcal{A}^\#$ and $y \in \mathcal{A}^{qnil}$. By using Cline's formula, $ap \in \mathcal{A}^{qnil}$. Since $pa(1 - p) = 0$, it follows by [21, Lemma 2.2] that $a(1 - p) \in \mathcal{A}^d$ and $[a(1 - p)]^d = a^d(1 - p)$. Thus we verify that

$$x^*ey = (x^d x^2)^* eap = (x^2)^*(x^d)^* eap = (x^2)^*(1 - p)^* [(a^d)^* eap] = 0.$$

Hence $a = x + y$ is a generalized e -group decomposition of a . According to Theorem 2.2, $a \in \mathcal{A}^{e,\oplus}$. \square

Corollary 4.2. Let $A \in \mathbb{C}^{n \times n}$, with conjugate transpose $*$ as the involution. Then the following are equivalent:

- (1) A has weak group inverse.
- (2) There exists an idempotent $F \in \mathbb{C}^{n \times n}$ such that $A + F$ is invertible, $(A^*AF)^* = A^*AF$, $AF = FAF$ is nilpotent.
- (3) There exists an idempotent $F \in \mathbb{C}^{n \times n}$ such that $A + F$ is invertible, $(A^D)^*AF = 0$, $AF = FAF$ is nilpotent.

Proof. As is well known, every complex has Drazin inverse. This completes the proof by Theorem 4.1. \square

Example 4.3. Let $\mathbb{C}^{2 \times 2}$ be the Banach algebra of all 2×2 complex matrices, with transpose $*$ as the involution. Let $A = \begin{pmatrix} 1 & 0 \\ -i & 0 \end{pmatrix} \in \mathbb{C}^{2 \times 2}$. Then the equations $AX^2 = X$, $(A^*A^2X)^* = A^*A^2X$, $\lim_{n \rightarrow \infty} \|A^n - XA^{n+1}\|_n^{\frac{1}{n}} = 0$ has two solutions

$$X_1 = \begin{pmatrix} 1 & 0 \\ -i & 0 \end{pmatrix}, X_2 = \begin{pmatrix} 0 & i \\ 0 & 1 \end{pmatrix}.$$

Then the solution of the preceding equations is not unique. Choose an idempotent $F = \begin{pmatrix} 0 & 0 \\ i & 1 \end{pmatrix}$. Then $A + F$ is invertible, $(A^*AF)^* = A^*AF$, $AF = FAF$ is nilpotent. In this case, the involution $*$ is not proper.

Next, we turn our attention to the relationships between generalized group inverses and generalized weighted core inverses.

Theorem 4.4. Let $a \in \mathcal{A}^{e,\oplus}$. Then $a \in \mathcal{A}^{e,\otimes}$ and $a^{e,\otimes} = (aa^{e,\oplus}a)^\# = (a^{e,\oplus})^2a$.

Proof. By hypothesis, we have

$$a^{e,\otimes} = a(a^{e,\oplus})^2, (eaa^{e,\otimes})^* = eaa^{e,\otimes} \text{ and } \lim_{n \rightarrow \infty} \|a^n - a^{e,\otimes}a^{n+1}\|_n^{\frac{1}{n}} = 0.$$

Set $x = (a^{e,\oplus})^2a$. Then we check that

$$\begin{aligned} ax^2 &= [a(a^{e,\oplus})^2][a(a^{e,\oplus})^2]a = (a^{e,\oplus})^2a = x, \\ (a^*ea^2x)^* &= (a^*ea^2(a^{e,\oplus})^2a)^* = (a^*eaa^{e,\oplus}a)^* \\ &= a^*(eaa^{e,\oplus})^*a = a^*(eaa^{e,\oplus})a = a^*ea^2x, \\ \|a^n - xa^{n+1}\|_n^{\frac{1}{n}} &= \|a^n - (a^{e,\oplus})^2a^{n+1}\|_n^{\frac{1}{n}} \\ &= \|a^n - a^{e,\oplus}a^{n+1} + (a^{e,\oplus}(a^n - a^{e,\oplus}a^{n+1}))a\|_n^{\frac{1}{n}} \\ &= \|a^n - a^{e,\oplus}a^{n+1}\|_n^{\frac{1}{n}} + \|a^{e,\oplus}\|_n^{\frac{1}{n}} \|a^n - a^{e,\oplus}a^{n+1}\|_n^{\frac{1}{n}} \|a\|_n^{\frac{1}{n}} \\ &= [1 + \|a^{e,\oplus}\|_n^{\frac{1}{n}} \|a\|_n^{\frac{1}{n}}] \|a^n - a^{e,\oplus}a^{n+1}\|_n^{\frac{1}{n}}. \end{aligned}$$

Hence $\lim_{n \rightarrow \infty} \|a^n - xa^{n+1}\|_n^{\frac{1}{n}} = 0$. Therefore $a^{e,\otimes} = (a^{e,\oplus})^2a$.

Obviously, we have $((a^{e,\otimes})^2a)(aa^{e,\otimes}a)((a^{e,\otimes})^2a) = ((a^{e,\otimes})^2a)a^{e,\otimes}a = (a^{e,\otimes})^2a$. Moreover, we check that

$$\begin{aligned} & ||a^{e,\otimes}a - ((a^{e,\otimes})^2a)(aa^{e,\otimes}a)|| \\ &= ||a^{e,\otimes}a - (a^{e,\otimes})^2a^2a^{e,\otimes}a|| \\ &= ||a^{e,\otimes}a - a^{e,\otimes}[a^{e,\otimes}a^{n+1}](a^{e,\otimes})^na|| \\ &= ||[a^{e,\otimes}a - a^{e,\otimes}a^n(a^{e,\otimes})^na] + a^{e,\otimes}[a^n - a^{e,\otimes}a^{n+1}](a^{e,\otimes})^na|| \\ &= ||a^{e,\otimes}[a^n - a^{e,\otimes}a^{n+1}](a^{e,\otimes})^na|| \\ &\leq ||a^{e,\otimes}|| ||a^n - a^{e,\otimes}a^{n+1}|| ||(a^{e,\otimes})^n|| ||a||. \end{aligned}$$

By hypothesis, $a \in \mathcal{A}^{e,\otimes}$, and so $\lim_{n \rightarrow \infty} ||a^n - a^{e,\otimes}a^{n+1}||^{\frac{1}{n}} = 0$. Therefore

$$\lim_{n \rightarrow \infty} ||a^{e,\otimes}a - ((a^{e,\otimes})^2a)(aa^{e,\otimes}a)||^{\frac{1}{n}} = 0.$$

Hence $a^{e,\otimes}a = ((a^{e,\otimes})^2a)(aa^{e,\otimes}a)$. Accordingly, $(aa^{e,\otimes}a)((a^{e,\otimes})^2a) = a^{e,\otimes}a = ((a^{e,\otimes})^2a)(aa^{e,\otimes}a)$. On the other hand, we verify that

$$\begin{aligned} & ||a^{e,\otimes}a^2a^{e,\otimes}a - aa^{e,\otimes}a|| \\ &= ||a^{e,\otimes}a^{n+1}(a^{e,\otimes})^na - aa^{e,\otimes}a|| \\ &= ||(a^{e,\otimes}a^{n+1} - a^n)(a^{e,\otimes})^na + a^n(a^{e,\otimes})^na - aa^{e,\otimes}a|| \\ &= ||(a^{e,\otimes}a^{n+1} - a^n)(a^{e,\otimes})^na|| \\ &\leq ||a^{e,\otimes}a^{n+1} - a^n|| ||a^{e,\otimes}||^n ||a||. \end{aligned}$$

Then

$$\lim_{n \rightarrow \infty} ||a^{e,\otimes}a^2a^{e,\otimes}a - aa^{e,\otimes}a||^{\frac{1}{n}} = 0.$$

Hence $(aa^{e,\otimes}a)((a^{e,\otimes})^2a)(aa^{e,\otimes}a) = a^{e,\otimes}a(aa^{e,\otimes}a) = aa^{e,\otimes}a$, and therefore $(aa^{e,\otimes}a)^{\#} = (a^{e,\otimes})^2a$. \square

Theorem 4.4 implies the uniqueness of the generalized e -group inverse for any generalized e -core invertible Banach element. Let $\mathcal{B}(X)$ be the Banach algebra of bounded linear operators over a Hilbert space X . Then the algebra $\mathcal{B}(X)$ is a Banach algebra with the adjoint operation as its proper involution. Let A in $\mathcal{B}(X)^d$. In [15], Mosić and D. Zheng introduced and studied weak group inverse for a Hilbert space operator. The weak group inverse of A is defined by $A^{\otimes} = (A^{\otimes})^2A$. By virtue of Theorem 4.4, the generalized group inverse and weak group inverse for a Hilbert space operator coincide with each other (see [15]).

Corollary 4.5. *Let $a \in \mathcal{A}^{e,\otimes}$. Then $(a^{e,\otimes})^{e,\otimes} = a^2a^{e,\otimes}$ and $[(a^{e,\otimes})^{e,\otimes}]^{e,\otimes} = a^{e,\otimes}$.*

Proof. In view of Theorem 4.4, $a^{e,\otimes} = (aa^{e,\otimes}a)^{\#}$, and so $a^{e,\otimes}$ has group inverse. Hence, we have $(a^{e,\otimes})^{e,\otimes} = (a^{e,\otimes})^{\#} = [(aa^{e,\otimes}a)^{\#}]^{\#} = aa^{e,\otimes}a$. Moreover, $aa^{e,\otimes}a$ has group inverse, and then $((a^{e,\otimes})^{e,\otimes})^{e,\otimes} = [aa^{e,\otimes}a]^{e,\otimes} = [aa^{e,\otimes}a]^{\#} = a^{e,\otimes}$. This completes the proof. \square

Corollary 4.6. *Let $a \in \mathcal{A}^{e,\otimes}$. Then the following are equivalent:*

- (1) $a^{e,\otimes} = a^{e,\otimes}$;
- (2) $aa^{e,\otimes} = a^{e,\otimes}a$.
- (3) $aa^{e,\otimes} = aa^{e,\otimes}$.

Proof. (1) \Rightarrow (2) In view of Theorem 4.4, $a^{e,\otimes} = (a^{e,\otimes})^2a$. Hence, $aa^{e,\otimes} = aa^{e,\otimes} = a(a^{e,\otimes})^2a = a^{e,\otimes}a$.

(2) \Rightarrow (1) Since $aa^{e,\otimes} = a^{e,\otimes}a$, it follows by Theorem 4.4 that $a^{e,\otimes} = (a^{e,\otimes})^2a = a^{e,\otimes}(a^{e,\otimes}a) = a^{e,\otimes}aa^{e,\otimes} = a^{e,\otimes}$, as desired.

(1) \Rightarrow (3) This is trivial.

(3) \Rightarrow (1) By virtue of Theorem 4.4, we have $a^{e,\otimes} = (a^{e,\otimes})^2a$. Accordingly, we check that $a^{e,\otimes} = (a^{e,\otimes})^2a = aa^d(a^{e,\otimes})^2a = a^d(aa^{e,\otimes}) = a^d(aa^{e,\otimes}) = a^{e,\otimes}$, as asserted. \square

Corollary 4.7. Let $a \in \mathcal{A}^{e,\oplus}$. Then $a^{e,\oplus} = x$ if and only if $ax^2 = x, ax = a^{e,\oplus}a$.

Proof. \implies In view of Theorem 4.4, $a \in \mathcal{A}^{e,\oplus}$ and $x = a^{e,\oplus} = (a^{e,\oplus})^2a$. Therefore $ax^2 = x$ and $ax = a(a^{e,\oplus})^2a = a^{e,\oplus}a$, as required.

\impliedby By hypotheses, $ax^2 = x, ax = a^{e,\oplus}a$. Then we have $x = ax^2 = (ax)x = a^{e,\oplus}(ax) = (a^{e,\oplus})^2a$. In light of Theorem 4.4, $x = a^{e,\oplus}$, as desired. \square

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