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A Proof of the Collatz Conjecture via Finite State Machine Analysis and Structural Confinement

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Abstract: The Collatz Conjecture, a long-standing open problem in number theory, asserts that every positive integer sequence generated by the Collatz function eventually reaches the 4-2-1 cycle. This paper presents a rigorous proof by modeling the Collatz dynamics using a 17-state finite state machine (FSM) derived from a structured partition of the integers. This FSM comprises a two-state precursor stage (for multiples of 3), a 12-state transient core (for other numbers outside the cycle), and a 3-state terminal stage (representing the 4-2-1 cycle). We analyze the deterministic transitions within this FSM and prove that every state in the precursor and core stages has a finite path leading inevitably to the terminal cycle stage, guaranteeing convergence for all starting integers. Our approach resolves the conjecture through deterministic finite-state analysis, demonstrating the inevitable collapse of any Collatz sequence into the unique 4-2-1 attractor.

Keywords: Collatz Conjecture; 3x+1 problem; number theory; dynamical systems; boundedness; cycle uniqueness; modular arithmetic

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1. Introduction

The Collatz conjecture, proposed by Lothar Collatz in 1937, has fascinated mathematicians for decades due to its deceptively simple definition and yet unresolved status [2,4]. Also known as the $3x + 1$ problem, it asserts that for any positive integer x, repeated application of the function

$$C(x) = \begin{cases} \frac{x}{2}, & \text{if } x \text{ is even,} \\ 3x + 1, & \text{if } x \text{ is odd,} \end{cases}$$

will eventually reach the cycle $(4 \to 2 \to 1)$. Despite extensive computational verification and probabilistic arguments supporting the conjecture [5,9], a general proof has remained elusive, highlighting a profound disconnect between the conjecture's elementary formulation and the complex dynamics it generates.

Previous approaches have largely focused on demonstrating that Collatz sequences are, in some sense, bounded. Probabilistic models suggest an average decreasing behavior [4,6], while computational efforts have verified convergence for astronomically large starting values [1,8]. However, these methods inherently cannot exclude the possibility of exceptional, unbounded orbits or non-trivial cycles. Even Tao's significant result [9], proving that *almost all* orbits are bounded, does not establish boundedness for *every* starting number.

This paper presents a fundamentally different approach, providing a rigorous and deterministic proof of the Collatz Conjecture. We do *not* rely on probabilistic arguments or boundedness in the traditional sense. Instead, our core strategy is to *partition the set of positive integers into a collection of mutually exclusive sets* $(\mathcal{P}, \mathcal{R}, \mathcal{C}, \mathcal{I}, \mathcal{X})$. This structured partitioning allows us to demonstrate that the Collatz function induces a **17-state** finite state machine (FSM). This FSM comprises three distinct stages: two initial states, S_P (even multiples of 3) and S_R (odd multiples of 3), representing numbers divisible by 3; a 12-state transient stage (S_{1-12}) modeling numbers not divisible by 3 and outside the known cycle; and a 3-state terminal cycle stage (S_C) representing the $4 \to 2 \to 1$ cycle. We demonstrate a deterministic flow where sequences transition from the initial states into the transient stage, and subsequently from the transient stage into the terminal cycle stage.

Crucially, within this 17-state FSM, the transitions between states are fully *deterministic*. We prove that every state within the initial (S_P, S_R) and transient (S_{1-12}) stages possesses a finite path leading inevitably *to the terminal cycle stage* (S_C). This means that, regardless of the starting value, the repeated application of the Collatz function *forces* the sequence along a trajectory within the FSM that terminates in the unique attractor (S_C). Our approach thus transforms the Collatz problem from a question of potentially unbounded numerical behavior to one of *structured, finite-state evolution*.

The key innovations of this proof are:

1. **Global Uniqueness of the $4 \to 2 \to 1$ Cycle:** We rigorously prove that this is the *only* possible cycle, using a novel product equation constraint. This result builds upon, and is consistent with, earlier work [7].
2. **State Space Partitioning:** We partition all positive integers into five mutually exclusive sets: the Cycle set ($\mathcal{C}$), the ROM3 set ($\mathcal{R}$), the Precursor set ($\mathcal{P}$), the Immediate Successor set ($\mathcal{I}$), and the Exclusion set ($\mathcal{X}$). The sets $\mathcal{P}$ and $\mathcal{R}$ define the two initial states of the FSM, S_P and S_R, respectively. This structured partitioning is fundamental to the construction of the FSM.
3. **The 17-State Finite State Machine:** This provides a complete and deterministic model of Collatz dynamics based on a partition of the integers. It demonstrates that sequences are not merely avoiding divergence; they are **constrained by the FSM structure to follow paths that inevitably lead to** convergence by the inherent structure of the function. This FSM is composed of three distinct stages: two initial states (S_P, S_R), a 12-state transient stage (S_{1-12}), and the 3-state terminal cycle stage (S_C).
4. **Elimination of Boundedness Arguments:** Our proof does *not* rely on showing that sequences remain within a certain bound. Instead, we show that the FSM structure *guarantees* eventual convergence to the $4 \to 2 \to 1$ cycle, regardless of any intermediate values.

By integrating these results, we provide a mathematically rigorous and deterministic resolution to the Collatz Conjecture. The finite state machine framework demonstrates that every Collatz sequence *must*, by the very definition of the function, follow a finite, structured path leading inevitably to the unique cycle. This approach provides a new perspective on the problem, shifting the focus from numerical bounds to the underlying deterministic dynamics.

2. Mathematical Framework and Definitions

To rigorously analyze the Collatz Conjecture using a structured approach, we begin by establishing the fundamental mathematical definitions, notation, and the core function at the heart of the problem.

Definition 1 (Collatz Function)**.** *The Collatz function $C\colon \mathbb{Z}^+ \to \mathbb{Z}^+$ is defined as*

$$C(x) = \begin{cases} \frac{x}{2}, & \text{if } x \text{ is even,} \\ 3x+1, & \text{if } x \text{ is odd.} \end{cases}$$

Definition 2 (Collatz Sequence)**.** *For a starting integer $x_0 \in \mathbb{Z}^+$, the Collatz sequence is the sequence $(x_0, x_1, x_2, \dots)$ defined by*

$$x_{i+1} = C(x_i) \quad \text{for all } i \geq 0.$$

Definition 3 (Odd Iterate)**.** *Given a Collatz sequence $(n_k)_{k \geq 0}$, an **odd iterate** is a term n_k that is odd. We often denote odd iterates by o_k.*

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Definition 4 (Odd Iteration (or accelerated Collatz step))**.** *An **odd iteration** (also called an **accelerated Collatz step**) is the transformation that maps an odd integer o directly to the next odd integer in its Collatz sequence. It is given by*

$$T^*(o) = \frac{3o+1}{2^{v_2(3o+1)}},$$

where $v_2(m)$ denotes denotes the 2-adic valuation of m, i.e., the exponent of the largest power of 2 dividing m. This guarantees that $T^*(o)$ is odd. In some residue class analyses (e.g., modulo 4 or 12) one considers the simplified version

$$T^*(o) = \frac{3o+1}{2},$$

when focusing on residue class transitions and boundedness arguments.

3. State Space Partitioning for Collatz Dynamics

To construct a finite state machine model for the Collatz process, we begin by partitioning the set of positive integers ($\mathbb{Z}^+$) into a collection of mutually exclusive and collectively exhaustive sets. This structured partitioning will serve as the foundation for defining the states of our finite state machine, allowing us to analyze the dynamics of Collatz sequences in a discrete and deterministic manner. The partitioning is designed to capture key properties of numbers under the Collatz function, such as divisibility by 3 and relationships between successive terms in a sequence.

3.1. Defining Fundamental Sets in Collatz Analysis

We begin by defining the key sets that will form the basis of our state space.

Definition 5 (Cycle Set)**.** *The cycle set $\mathcal{C}$ consists of the numbers known to form a repeating cycle:*

$$\mathcal{C} = \{1, 2, 4\}.$$

Explanation of the cycle set: The cycle set $\mathcal{C} = \{1, 2, 4\}$ is fundamental to the Collatz conjecture. It represents the only known cycle in the Collatz function for positive integers. When a Collatz sequence reaches any of these numbers, it enters a loop that cycles as

$$1 \to 4 \to 2 \to 1 \to \cdots.$$

A central part of the conjecture is to prove that all Collatz sequences eventually enter this cycle.

Definition 6 (ROM3 Set)**.** *The ROM3 set $\mathcal{R}$ comprises all odd positive multiples of 3:*

$$\mathcal{R} = \{x \in \mathbb{Z}^+ \mid x = 3j, \text{ where } j \text{ is an odd integer}\}.$$

Explanation of the ROM3 set: The ROM3 set (short for "root odd multiple of 3") consists of those positive integers that are odd multiples of 3. For example, $3, 9, 15, \ldots$ belong to $\mathcal{R}$. This set plays a crucial role in the structural analysis of Collatz sequences, particularly in tracking transitions from the precursor set and establishing structural confinement within the Collatz state space.

Definition 7 (Precursor Set)**.** *The precursor set $\mathcal{P}$ consists of all even positive multiples of 3:*

$$\mathcal{P} = \{x \in \mathbb{Z}^+ \mid x = 6j, \text{ where } j \text{ is a positive integer}\}.$$

Explanation of the precursor set: The precursor set $\mathcal{P}$ is defined as the set of positive integers that are even multiples of 3 (i.e., numbers satisfying $x \equiv 0 \pmod 6$). For instance, $6, 12, 18, \ldots$ belong to $\mathcal{P}$. The term "precursor" reflects that, under reverse Collatz iteration, numbers in $\mathcal{P}$ serve as the origins that structurally precede the ROM3 set $\mathcal{R}$.

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Definition 8 (Immediate Successor Set)**.** *The immediate successor set $\mathcal{I}$ is defined as*

$$\mathcal{I} = \{x \in \mathbb{Z}^+ \mid x = 9j + 1, \text{ where } j \text{ is an odd integer}\}.$$

Explanation of the immediate successor set: The immediate successor set $\mathcal{I}$ consists of numbers of the form $9j + 1$ with j odd. For example, $10, 28, 46, \ldots$ are in $\mathcal{I}$. When the Collatz function is applied to a number in the ROM3 set, the very next number in the sequence falls into $\mathcal{I}$, marking the next step in the structural chain.

Definition 9 (Exclusion Set)**.** *The exclusion set $\mathcal{X}$ consists of numbers that do not belong to $\mathcal{C}$, $\mathcal{R}$, $\mathcal{P}$, or $\mathcal{I}$:*

$$\mathcal{X} = \{x \in \mathbb{Z}^+ \mid x \notin \mathcal{C} \cup \mathcal{R} \cup \mathcal{P} \cup \mathcal{I}\}.$$

Explanation of the exclusion set: The exclusion set $\mathcal{X}$ is defined by exclusion. $\mathcal{X}$ consists precisely of positive integers that are not divisible by 3 and are not in $\mathcal{C}$ or $\mathcal{I}$.

3.2. Completeness of Classification

For our state space to be a valid foundation for analysis, we must ensure that every positive integer belongs to exactly one of the defined sets. This subsection formally proves the completeness and uniqueness of our initial partition.

Theorem 1 (Completeness of Classification: Partitioning of positive integers)**.** *The set of positive integers is completely and uniquely partitioned as follows:*

$$\mathbb{Z}^+ = \mathcal{C} \cup \mathcal{R} \cup \mathcal{P} \cup \mathcal{I} \cup \mathcal{X}.$$

That is, every positive integer belongs to exactly one, and only one, of these five sets.

Proof. Proof strategy: We prove completeness by first showing that every $x \in \mathbb{Z}^+$ belongs to at least one of the five sets (exhaustiveness) and then proving that no x can belong to more than one set (mutual exclusivity).

Step 1: Exhaustiveness.

Let x be an arbitrary positive integer.

- **Case 1:** $x \equiv 0 \pmod 3$.
  - If $x = 3j$ with j odd, then by Definition 6, $x \in \mathcal{R}$.
  - If $x = 6j$ for some $j \geq 1$, then by Definition 7, $x \in \mathcal{P}$.
- **Case 2:** $x \not\equiv 0 \pmod 3$.
  - If $x \in \mathcal{C}$, it is classified immediately.
  - If $x \notin \mathcal{C}$, then check:
    * If $x = 9j + 1$ for some odd j, then by Definition 8, $x \in \mathcal{I}$.
    * Otherwise, by Definition 9, $x \in \mathcal{X}$.

Thus, every x is assigned to at least one set.

Step 2: Mutual exclusivity.

We now verify that these sets are pairwise disjoint.

- $\mathcal{C} \cap \mathcal{R} = \emptyset$ since $\mathcal{C} = \{1, 2, 4\}$ (none of which are divisible by 3) while every element in $\mathcal{R}$ is divisible by 3.
- $\mathcal{C} \cap \mathcal{P} = \emptyset$ because $\mathcal{C}$ contains only small numbers not divisible by 3 and $\mathcal{P}$ consists of even multiples of 3.
- $\mathcal{C} \cap \mathcal{I} = \emptyset$ and $\mathcal{C} \cap \mathcal{X} = \emptyset$ by definition.

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- The remaining intersections ($\mathcal{R} \cap \mathcal{P}, \mathcal{R} \cap \mathcal{I}, \mathcal{R} \cap \mathcal{X}, \mathcal{P} \cap \mathcal{I}, \mathcal{P} \cap \mathcal{X}, \mathcal{I} \cap \mathcal{X}$) are similarly ruled out by the definitions and congruence conditions imposed on each set.

Conclusion: Since every positive integer belongs to exactly one of $\mathcal{C}$, $\mathcal{R}$, $\mathcal{P}$, $\mathcal{I}$, or $\mathcal{X}$, the classification is complete. □

4. Uniqueness of the Collatz Cycle as a Fixed Point

A critical step in proving the Collatz Conjecture is to demonstrate that the cycle $4 \to 2 \to 1$ *is the only possible cycle under the Collatz function. If other, non-trivial cycles existed, then it would be impossible to guarantee that all positive integers eventually reach the 4-2-1 cycle. By eliminating the possibility of any other cycles before constructing our finite state machine, we significantly simplify the task of proving convergence. This section presents a rigorous proof of the uniqueness of the 4-2-1 cycle, building upon arguments presented in our earlier preprint by Nwankpa [7].*

4.1. Every Cycle Must Contain an Odd Number

As a foundational step in characterizing cycles, we first prove that any repeating sequence under the Collatz function must include at least one odd integer. This allows us to focus our subsequent analysis on the behavior of odd iterates within potential cycles.

Lemma 1 (Every cycle must contain an odd number)**.** *Every Collatz cycle in positive integers must contain at least one odd number.*

Proof. Assume, for contradiction, that a Collatz cycle consists entirely of even numbers:

$$C = (c_1, c_2, \ldots, c_k).$$

Since every term in the cycle is even, applying the Collatz function always results in division by 2:

$$C(c_i) = \frac{c_i}{2}.$$

Thus, iterating the function on any c_i yields

$$c_2 = \frac{c_1}{2}, \quad c_3 = \frac{c_2}{2}, \quad \ldots, \quad c_k = \frac{c_{k-1}}{2}, \quad c_1 = \frac{c_k}{2}.$$

Since these values are positive integers, we deduce

$$c_1 = \frac{c_1}{2^k}.$$

Rearranging gives

$$c_1 \cdot (2^k - 1) = 0.$$

Since $c_1 > 0$, it must be that $2^k - 1 = 0$, i.e., $2^k = 1$. However, $2^k = 1$ has no solutions for any positive integer k, which is a contradiction.

Therefore, every Collatz cycle must contain at least one odd number. □

4.2. Product Equation Constraints on Collatz Cycles

Building upon the existence of odd numbers in any cycle, we now derive a key equation that relates the odd iterates within a cycle to the number of even steps involved. This product equation will serve as a powerful constraint on the possible structure of cycles.

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Lemma 2. *Let $(o_1, o_2, \dots, o_k)$ be the odd iterates in a Collatz cycle. Then these iterates satisfy the equation*

$$2^M = \prod_{i=1}^{k} \frac{3o_i + 1}{o_i},$$

where $M = \sum_{i=1}^{k} m_i$ is the total number of even steps (with $m_i = v_2(3o_i + 1)$) in the cycle.

Proof. Starting from a cycle of odd iterates, we apply the accelerated Collatz function. For each odd iterate o_i, the next odd iterate is given by

$$o_{i+1} = T^*(o_i) = \frac{3o_i + 1}{2^{m_i}},$$

where m_i is the number of divisions by 2 required. Multiplying these equations over all i and using the cyclicity of the sequence leads directly to the product equation.

Step 1: Multiply the recurrence over one full cycle:

$$\prod_{i=1}^{k} o_{i+1} = \prod_{i=1}^{k} \frac{3o_i + 1}{2^{m_i}}.$$

Since the cycle is closed ($o_{k+1} = o_1$), the products on both sides are equal:

$$\prod_{i=1}^{k} o_i = \prod_{i=1}^{k} \frac{3o_i + 1}{2^{m_i}}.$$

Step 2: Rearranging yields

$$2^{\sum_{i=1}^{k} m_i} = \frac{\prod_{i=1}^{k}(3o_i + 1)}{\prod_{i=1}^{k} o_i}.$$

Defining $M = \sum_{i=1}^{k} m_i$ gives the desired result:

$$2^M = \prod_{i=1}^{k} \frac{3o_i + 1}{o_i}.$$

□

4.3. Implications of the Product Equation for Cycle Structure

Leveraging the product equation derived in the previous subsection, we now analyze its implications for the nature of odd iterates within any hypothetical non-trivial Collatz cycle, setting the stage for proving the uniqueness of the 4-2-1 cycle.

Lemma 3 (Uniqueness of 1 as the only odd term in non-trivial Collatz cycles)**.** *In any non-trivial Collatz cycle, the number 1 is the only possible odd number that can appear.*

Proof. We prove by contradiction. Suppose there exists a non-trivial cycle with odd iterates $(o_1, o_2, \dots, o_k)$ where at least one $o_j \neq 1$. By Lemma 2,

$$2^M = \prod_{i=1}^{k} \frac{3o_i + 1}{o_i}.$$

Now, choose an $o_j \geq 3$. Let $p \geq 3$ be any prime factor of o_j. Then

$$o_j \equiv 0 \pmod{p} \quad \text{implies} \quad 3o_j + 1 \equiv 1 \pmod{p}.$$

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Thus, the factor $\frac{3o_j+1}{o_j}$ has p in the denominator but not in the numerator. Consequently, the entire product contains an odd prime factor in the denominator and cannot be a pure power of 2. This contradiction implies that every odd iterate in any non-trivial cycle must equal 1. □

4.4. Invariance and Absorbing Nature of the Cycle Set

We now confirm that the known cycle set ($\mathcal{C}$) has a critical property: once a Collatz sequence enters this set, it never leaves, establishing it as an absorbing set for the Collatz dynamic.

Lemma 4 (Cycle set invariance)**.** *If $x \in \mathcal{C}$, then*

$$C(x) \in \mathcal{C},$$

where $\mathcal{C} = \{1, 2, 4\}$.

Proof. Proof overview: We verify the invariance of the cycle set by checking that applying the Collatz function to each element in $\mathcal{C}$ yields an element that remains in $\mathcal{C}$.

Case 1: For $x = 1$,

$$C(1) = 3 \cdot 1 + 1 = 4, \quad \text{and } 4 \in \mathcal{C}.$$

Case 2: For $x = 2$,

$$C(2) = \frac{2}{2} = 1, \quad \text{and } 1 \in \mathcal{C}.$$

Case 3: For $x = 4$,

$$C(4) = \frac{4}{2} = 2, \quad \text{and } 2 \in \mathcal{C}.$$

Conclusion: In every case, $C(x) \in \mathcal{C}$. Thus, the cycle set is invariant and acts as an absorbing set under the Collatz function. □

4.5. Concluding the Uniqueness of the 4-2-1 Cycle

By synthesizing the results from the preceding subsections, we now formally conclude that no Collatz cycles exist outside of the trivial cycle contained within the set ($\mathcal{C} = \{1, 2, 4\}$). This establishes a crucial property of the Collatz function's long-term behavior.

Theorem 2 (Uniqueness of the 4-2-1 cycle)**.** *There are no cycles in the Collatz function other than the trivial cycle*

$$4 \to 2 \to 1 \to 4,$$

that is, no cycle exists outside the cycle set $\mathcal{C} = \{1, 2, 4\}$.

Proof. We combine the results of Lemma 1, which guarantees every cycle contains an odd term, with the results of Lemma 3 which establishes that the only odd term possible in any non-trivial cycle is 1. Finally, by Lemma 4, any sequence containing $1 \in \mathcal{C}$ remains in $\mathcal{C}$. Therefore, the only cycle is the trivial cycle $4 \to 2 \to 1 \to 4$.

Since any Collatz cycle must contain an odd term and the only odd term possible is 1, every cycle is confined to $\mathcal{C}$. A direct verification shows that the only cycle in $\mathcal{C}$ is $4 \to 2 \to 1 \to 4$. Thus, the trivial 4-2-1 cycle is unique. □

5. Properties of the Collatz Function on the Defined Sets

Having established the uniqueness of the $4 \to 2 \to 1$ cycle within the Collatz process, we now analyze how the Collatz function maps elements between the sets $\mathcal{C}$, $\mathcal{R}$, $\mathcal{P}$, $\mathcal{I}$, and $\mathcal{X}$ (defined in Section 3). This analysis is crucial for constructing the finite state machine in Section 6, which will demonstrate the inevitable convergence of all Collatz sequences.

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5.1. Mapping Properties of the Precursor Set: Initial Transitions

We begin by analyzing the behavior of the precursor set ($\mathcal{P}$) under the Collatz function, identifying the set to which its elements are mapped in the subsequent iteration.

Lemma 5 ($\mathcal{P}$ mapping: Descending from the infinite, ordered past)**.** *Iterates from the precursor set follow a predictable descent, remaining within $\mathcal{P}$ until their final transition to $\mathcal{R}$.*

That is, if $x \in \mathcal{P}$, then $C(x) \in \mathcal{P} \cup \mathcal{R}$.

Proof. **Proof overview:** We express an arbitrary $x \in \mathcal{P}$ as $6j$ and apply the Collatz function. Depending on whether j is odd or even, $C(x)$ lands in $\mathcal{R}$ or remains in $\mathcal{P}$, respectively.

Step 1: Express x in terms of $\mathcal{P}$.

By Definition 7,

$$\mathcal{P} = \{x \in \mathbb{Z}^+ \mid x = 6j, \text{ where } j \text{ is a positive integer}\}.$$

Thus, $x = 6j$ for some positive integer j.

Step 2: Apply the Collatz function.

Since x is even,

$$C(x) = \frac{x}{2} = \frac{6j}{2} = 3j.$$

Step 3: Analyze $C(x) = 3j$ based on the parity of j.

- **Case 1:** If j is odd, then by Definition 6, $3j \in \mathcal{R}$.
- **Case 2:** If j is even, write $j = 2m$; then

$$C(x) = 3(2m) = 6m \in \mathcal{P}.$$

Conclusion: In both cases, $C(x) \in \mathcal{P} \cup \mathcal{R}$. □

5.2. Finite Transition from Precursor to ROM3

We now establish a crucial property of the Precursor set ($\mathcal{P}$): that repeated application of the Collatz function to any element in $\mathcal{P}$ will, in a finite number of steps, result in an element in the ROM3 set ($\mathcal{R}$). This property is essential for demonstrating the deterministic transition between the initial states of our finite state machine, as will be shown in Section 6.

Lemma 6 (Finite Transition from $\mathcal{P}$ to $\mathcal{R}$)**.** *For any $x \in \mathcal{P}$, there exists a finite integer $n \geq 0$ such that $C^n(x) \in \mathcal{R}$, where $C^n(x)$ denotes the n-fold application of the Collatz function (with $C^0(x) = x$).*

Proof. By definition, if $x \in \mathcal{P}$, then $x = 6k$ for some positive integer k. We can write k as $k = 2^a \cdot b$, where $a \geq 0$ is an integer and b is an odd integer. Substituting this into the expression for x, we get:

$$x = 6k = 6(2^a \cdot b) = 2^{a+1} \cdot 3b$$

Now, consider the repeated application of the Collatz function. Since x is even, we repeatedly divide by 2:

$$C(x) = \frac{x}{2} = 2^a \cdot 3b$$

$$C^2(x) = \frac{C(x)}{2} = 2^{a-1} \cdot 3b$$

$$\vdots$$

$$C^{a+1}(x) = \frac{C^a(x)}{2} = 2^0 \cdot 3b = 3b$$

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Since b is odd, $3b$ is an odd multiple of 3. Therefore, $3b \in \mathcal{R}$. We have found a finite $n = a + 1$ such that $C^n(x) \in \mathcal{R}$. □

5.3. Transition from ROM3 set to Immediate Successor Set

Following the flow of sequences, we next examine the transformation of the ROM3 set ($\mathcal{R}$) under the Collatz function, revealing its predictable successor set. We will demonstrate later that, once a sequence crosses into $\mathcal{I}$, it can never return to $\mathcal{R}$ or $\mathcal{P}$.

Lemma 7 ($\mathcal{R}$ mapping to immediate successor set $\mathcal{I}$)**.** *For every $x \in \mathcal{R}$, we have*

$$C(x) \in \mathcal{I}.$$

Proof. **Proof overview:** We express an element $x \in \mathcal{R}$ as $3j$ (with j odd), apply the Collatz function, and show the resulting number $9j + 1$ fits the definition of $\mathcal{I}$.

Step 1: Express x in terms of $\mathcal{R}$.
If $x \in \mathcal{R}$, then $x = 3j$ for some odd integer j.

Step 2: Apply the Collatz function.
Since x is odd,

$$C(x) = 3x + 1 = 9j + 1.$$

Step 3: Verify membership in $\mathcal{I}$.
By Definition 8, numbers of the form $9j + 1$ (with j odd) belong to $\mathcal{I}$.

Conclusion: Hence, for every $x \in \mathcal{R}$, we have $C(x) \in \mathcal{I}$. □

5.4. Descent from Immediate Successor Set Into the Exclusion Set

Continuing our analysis of set transitions, we now investigate the immediate successor set ($\mathcal{I}$) and its image under the Collatz function.

Lemma 8 (Mapping from $\mathcal{I}$ to exclusion)**.** *If $x \in \mathcal{I}$, then*

$$C(x) \in \mathcal{X}.$$

Proof. **Proof overview:** We show that for $x \in \mathcal{I}$, after applying the Collatz function, the resulting number satisfies the conditions for membership in $\mathcal{X}$; that is, it does not belong to $\mathcal{C}$, $\mathcal{R}$, $\mathcal{P}$, or $\mathcal{I}$ and the reverse Collatz operation is defined.

Step 1: By Definition 8, if $x \in \mathcal{I}$ then

$$x = 9j + 1, \quad \text{with } j \text{ odd}.$$

Step 2: Since x is even, applying the Collatz function yields

$$C(x) = \frac{x}{2} = \frac{9j + 1}{2}.$$

Step 3: Verify that $C(x)$ satisfies the conditions for $\mathcal{X}$:

- $C(x) \notin \mathcal{C}$ because $C(x) \geq \frac{10}{2} = 5$ and $\mathcal{C} = \{1, 2, 4\}$.
- $C(x) \notin \mathcal{R}$: If $\frac{9j+1}{2} = 3k$ for some odd k, then $9j + 1 = 6k$ and $1 = 3(2k - 3j)$, a contradiction.
- $C(x) \notin \mathcal{P}$ or $\mathcal{I}$: Similar contradictions arise.

Conclusion: Thus, $C(x) \in \mathcal{X}$. □

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5.5. Confinement of Sequences Within the Bounded State Space

A crucial step in our analysis is to demonstrate that once a Collatz sequence enters the exclusion set ($\mathcal{X}$), it remains confined to a specific subset of our state space, facilitating a more detailed examination of its long-term behavior.

Lemma 9 (Confinement). *If $x \in \mathcal{X}$, then*

$$C(x) \in \mathcal{X} \cup \mathcal{I} \cup \mathcal{C}.$$

Proof. Proof overview: We prove by contradiction that if $x \in \mathcal{X}$, then $C(x)$ cannot lie in $\mathcal{R}$ or $\mathcal{P}$; therefore, it must belong to $\mathcal{X}$, $\mathcal{I}$, or $\mathcal{C}$.

Case 1: Suppose $C(x) \in \mathcal{R}$.
Then $C(x) = 3j$ for some odd j.

- If x is even, then $C(x) = \frac{x}{2} = 3j$ implies $x = 6j$, so $x \in \mathcal{P}$, contradicting $x \in \mathcal{X}$.
- If x is odd, then $C(x) = 3x + 1 = 3j$ implies $x = j - \frac{1}{3}$, which is impossible.

Case 2: Suppose $C(x) \in \mathcal{P}$.
Then $C(x) = 6k$ for some $k \in \mathbb{Z}^+$.

- If x is even, then $C(x) = \frac{x}{2} = 6k$ implies $x = 12k$, so $x \in \mathcal{P}$, contradicting $x \in \mathcal{X}$.
- If x is odd, then $C(x) = 3x + 1 = 6k$ implies $x = 2k - \frac{1}{3}$, impossible.

Conclusion: Since $C(x) \notin \mathcal{R}$ and $C(x) \notin \mathcal{P}$, it follows that

$$C(x) \in \mathcal{X} \cup \mathcal{I} \cup \mathcal{C}.$$

□

6. Finite State analysis of Collatz Dynamics

Leveraging the integer partition (Section 3) and set transition properties (Section 5), this section constructs the 17-state finite state machine (FSM) that models Collatz dynamics. We define the FSM's components: the initial states S_P, S_R corresponding directly to the sets $\mathcal{P}, \mathcal{R}$; the 12 transient states within Stage S_{1-12} derived from residue analysis of sets $\mathcal{I}, \mathcal{X}$; and the terminal cycle states S_{C1}, S_{C2}, S_{C4} (S_C) representing the elements of set $\mathcal{C}$. The core of the section is a detailed analysis of the deterministic transitions between all these states under the Collatz function. Specifically, we establish the finite transition from the initial states into Stage S_{1-12}, characterize the transitions and prove the strong connectivity within this transient stage, and identify the unique gateway state (S_{11}) to the terminal stage. This comprehensive analysis of the FSM's structure and behavior provides the foundation for the convergence proof in Section 7.

6.1. Definitions - Stages and States

Definition 10 (Initial stage S_{P-R}). *Stage S_{P-R} corresponds to the union of sets $\mathcal{P}$ and $\mathcal{R}$. This initial stage consists of all positive integers divisible by 3. We break this stage into two states:*

- *S_P: Corresponding to the set $\mathcal{P}$ (even multiples of 3).*
- *S_R: Corresponding to the set $\mathcal{R}$ (odd multiples of 3).*

The sets $\mathcal{P}$ and $\mathcal{R}$ are disjoint by definition (or by Theorem 1), ensuring these states are distinct.

Definition 11 (Transient stage S_{1-12}). *Stage S_{1-12} corresponds to the union of sets $\mathcal{X}$ and $\mathcal{I}$. This stage contains all positive integers not divisible by 3, excluding the cycle set. We will employ a state function to break this stage down into unique, disjoint states.*

Definition 12 (Terminal stage S_C - Cycle States). *Stage S_C comprises the three states that represent the elements of cycle set $\mathcal{C}$. The cycle states are defined as follows:*

- *S_{C1}: Represents the number 1. Formally, $S_{C1} = (1, \mathcal{C}, Odd)$.*

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- S_{C2}: *Represents the number 2. Formally,* $S_{C2} = (2, \mathcal{C}, Even)$.
- S_{C4}: *Represents the number 4. Formally,* $S_{C4} = (4, \mathcal{C}, Even)$.

By Lemma 4, the transitions between these states follow the Collatz function ($S_{C1} \to S_{C4} \to S_{C2} \to S_{C1}$), *and the cycle set is invariant, causing sequences entering this stage to cycle indefinitely.*

Definition 13 (State function for stage S_{1-12})**.** *The state of a positive integer* $x \in \mathcal{X} \cup \mathcal{I}$ *is defined by the triplet*

$$s(x) = \Big(x \bmod 9,\ S(x),\ p(x)\Big),$$

where

$$S(x) = \begin{cases} \mathcal{I}, & \text{if } x \in \mathcal{I}, \\ \mathcal{X}, & \text{if } x \in \mathcal{X}, \end{cases} \quad \text{and} \quad p(x) = \begin{cases} Even, & \text{if } x \text{ is even}, \\ Odd, & \text{if } x \text{ is odd}. \end{cases}$$

Remark 1 (Why Modulo 9 is Optimal for Stage $S_{1\text{-}12}$)**.** *The choice of modulus 9 in the state function is specifically tailored to analyzing the behavior of numbers that are* not divisible by 3 *- that is, numbers in the sets* $\mathcal{X} \cup \mathcal{I}$*, which together form the entire* ***transient stage*** $S_{1\text{-}12}$ *of our 17-state finite state machine.*

This choice is motivated by several key observations:

1. ***Restriction to Residues Coprime to 3:*** *Within* $\mathbb{Z}_9$*, the residues of integers not divisible by 3 are:*

$$\mathbb{Z}_9 \setminus \{0, 3, 6\} = \{1, 2, 4, 5, 7, 8\}.$$

These residues are well defined and disjoint, making them ideal for use as distinct state variables for elements in $\mathcal{X} \cup \mathcal{I}$*. Moreover, they form a closed subsystem under the Collatz map, enabling deterministic tracking of residue evolution - as detailed in the next point.*

2. ***Structured Behavior Under*** $3x + 1$***:*** *For odd integers* $x \not\equiv 0 \pmod 3$*, the map* $x \mapsto 3x + 1$ *induces predictable transformations modulo 9. For example:*

$$\begin{aligned} x \equiv 1 \Rightarrow 3x + 1 \equiv 4 \pmod 9, \\ x \equiv 2 \Rightarrow 3x + 1 \equiv 7 \pmod 9, \\ x \equiv 4 \Rightarrow 3x + 1 \equiv 4 \pmod 9, \\ x \equiv 5 \Rightarrow 3x + 1 \equiv 7 \pmod 9, \\ x \equiv 7 \Rightarrow 3x + 1 \equiv 4 \pmod 9, \\ x \equiv 8 \Rightarrow 3x + 1 \equiv 7 \pmod 9. \end{aligned}$$

These congruences govern how states evolve under the Collatz function and are central to defining deterministic transitions in the transient stage.

3. ***Balanced Granularity:*** *Modulo 9 is fine enough to distinguish the essential behavior classes for numbers not divisible by 3, yet coarse enough to avoid excessive fragmentation. By contrast, modulo 3 is too coarse (it collapses behavior), and modulo 18 or 27 introduces unnecessary complexity.*
4. ***Exact Fit for State Classification:*** *The state function in Definition 13 uses:*
   - *the residue class mod 9 (among 6 possibilities),*
   - *membership in* $\mathcal{X}$ *or* $\mathcal{I}$ *(2 categories),*
   - *and parity (even or odd).*

   But since parity and set membership are mutually constrained for some residues (e.g., all elements of $\mathcal{I}$ *are even), the resulting state space consists of exactly 12 valid and disjoint states - forming the entire transient stage* $S_{1\text{-}12}$.

Thus, the use of modulo 9 provides the minimal structure required to fully classify Collatz behavior for numbers in the transient stage, enabling deterministic modeling of transitions within our 17-state finite state machine.

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6.2. Partitioning of Stage S_{1-12}

Using the defined state function, we enumerate the resulting finite set of 12 disjoint states that partition the transient stage S_{1-12}.

Lemma 10 (12-State Partition of $\mathcal{X} \cup \mathcal{I}$)**.** *The state function in Definition 13 defines a partition of stage S_{1-12} into 12 disjoint states: $S_1, S_2, \ldots, S_{12}$. That is, for every $x \in \mathcal{X} \cup \mathcal{I}$ there exists a unique index i with $1 \le i \le 12$ such that $\text{state}(x) = S_i$, and for any distinct indices $i \ne j$, the sets of numbers that map to S_i and S_j are disjoint.*

Proof. We prove the lemma in two parts: (1) that for every $x \in \mathcal{X} \cup \mathcal{I}$ there exists a unique state S_i with $s(x) = S_i$ (exhaustiveness), and (2) that these states are pairwise disjoint (mutual exclusivity).

(1) Uniqueness of the state assignment: By definition, the state function $s(x)$ assigns to each x a triplet consisting of:

- The residue $x \bmod 9$. For x in $\mathcal{X} \cup \mathcal{I}$, the allowed residues are $\{1, 2, 4, 5, 7, 8\}$.
- A secondary component $S(x)$, where

$$S(x) = \begin{cases} \mathcal{I}, & \text{if } x \in \mathcal{I}, \\ \mathcal{X}, & \text{if } x \in \mathcal{X}, \end{cases}$$

  which is well defined and disjoint.
- The parity function $p(x)$, which is uniquely determined by whether x is even or odd.

Thus, each $x \in \mathcal{X} \cup \mathcal{I}$ is assigned a unique triplet, which by construction corresponds to exactly one of the following 12 states:

$$\begin{aligned}
S_1 &= \{\, x \in \mathbb{Z}^+ \mid s(x) = (1,\ \mathcal{I},\ \text{Even}) \,\}, \\
S_2 &= \{\, x \in \mathbb{Z}^+ \mid s(x) = (1,\ \mathcal{X},\ \text{Odd}) \,\}, \\
S_3 &= \{\, x \in \mathbb{Z}^+ \mid s(x) = (2,\ \mathcal{X},\ \text{Even}) \,\}, \\
S_4 &= \{\, x \in \mathbb{Z}^+ \mid s(x) = (2,\ \mathcal{X},\ \text{Odd}) \,\}, \\
S_5 &= \{\, x \in \mathbb{Z}^+ \mid s(x) = (4,\ \mathcal{X},\ \text{Even}) \,\}, \\
S_6 &= \{\, x \in \mathbb{Z}^+ \mid s(x) = (4,\ \mathcal{X},\ \text{Odd}) \,\}, \\
S_7 &= \{\, x \in \mathbb{Z}^+ \mid s(x) = (5,\ \mathcal{X},\ \text{Even}) \,\}, \\
S_8 &= \{\, x \in \mathbb{Z}^+ \mid s(x) = (5,\ \mathcal{X},\ \text{Odd}) \,\}, \\
S_9 &= \{\, x \in \mathbb{Z}^+ \mid s(x) = (7,\ \mathcal{X},\ \text{Even}) \,\}, \\
S_{10} &= \{\, x \in \mathbb{Z}^+ \mid s(x) = (7,\ \mathcal{X},\ \text{Odd}) \,\}, \\
S_{11} &= \{\, x \in \mathbb{Z}^+ \mid s(x) = (8,\ \mathcal{X},\ \text{Even}) \,\}, \\
S_{12} &= \{\, x \in \mathbb{Z}^+ \mid s(x) = (8,\ \mathcal{X},\ \text{Odd}) \,\}.
\end{aligned}$$

(2) Mutual exclusivity: Suppose for contradiction that there exist two distinct indices $i \ne j$ such that an element x satisfies $s(x) = S_i$ and $s(x) = S_j$. Since the components of $s(x)$ (i.e., the residue $x \bmod 9$, the set indicator $S(x)$, and the parity $p(x)$) are uniquely determined by x, it is impossible for two different triplets to be equal. Hence, the states S_i and S_j must be disjoint.

Conclusion: Every $x \in \mathcal{X} \cup \mathcal{I}$ is assigned exactly one state S_i, and the collection $\{S_i\}_{i=1}^{12}$ forms a partition of stage S_{1-12}. □

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Remark 2 (Structure of the Full FSM)**.** *It is important to emphasize that the 12 states defined by the state function in Definition 13 constitute only the* transient stage $S_{1\text{-}12}$ *of the full 17-state finite state machine. The FSM as a whole also includes:*

- *The* ***initial stage*** $S_{P\text{-}R} = \{S_P, S_R\}$*, representing all integers divisible by 3.*
- *The* ***terminal cycle stage*** $S_C = \{S_{C1}, S_{C2}, S_{C4}\}$*, which captures the absorbing cycle* $1 \to 4 \to 2 \to 1$.

Thus, while the transient stage $S_{1\text{-}12}$ *handles the majority of the Collatz process, it operates as one of three structurally distinct phases in a unified finite-state framework.*

6.3. Completeness of State Partition

Having successfully partitioned our 12-state transient stage, we now prove that every positive integer corresponds to exactly one of the partitioned states.

Lemma 11 (Completeness of State Assignment)**.** *Every positive integer n corresponds to exactly one state in the 17-state FSM defined by Definitions 10, 12, and 11 (or equivalent labels).*

Proof. We need to show that for any positive integer n, there exists a unique state S in the set $\{S_P, S_R, S_{C1}, S_{C2}, S_{C4}, S_1, S_2, S_3, S_4, S_5, S_6, S_7, S_8, S_9, S_{10}, S_{11}, S_{12}\}$ such that n maps to S.

By Theorem 1, the sets $\mathcal{P}, \mathcal{R}, \mathcal{C}, \mathcal{I}, \mathcal{X}$ form a partition of the positive integers $\mathbb{Z}^+$. Therefore, any given $n \in \mathbb{Z}^+$ belongs to exactly one of these five sets.

Furthermore, every integer n has a unique residue modulo 9 and a unique parity $p(n)$ (Even or Odd).

We examine the state definitions based on the unique set membership of n:

- If $n \in \mathcal{P}$, then by definition, n corresponds uniquely to state S_P.
- If $n \in \mathcal{R}$, then by definition, n corresponds uniquely to state S_R.
- If $n \in \mathcal{C}$, then n must be 1, 2, or 4.
  - If $n = 1$, it corresponds uniquely to state $S_{C1} = (1, \mathcal{C}, Odd)$.
  - If $n = 2$, it corresponds uniquely to state $S_{C2} = (2, \mathcal{C}, Even)$.
  - If $n = 4$, it corresponds uniquely to state $S_{C4} = (4, \mathcal{C}, Even)$.
- If $n \in \mathcal{I}$, by definition of $\mathcal{I}$, $n = 9j + 1$ for some odd j. This implies $n \equiv 1 \pmod 9$ and n is always Even. The state function $s(n) = (n \pmod 9, S(n), p(n))$ yields $(1, \mathcal{I}, Even)$, which corresponds uniquely to state S_1.
- If $n \in \mathcal{X}$, then by definition, $n \notin \mathcal{P} \cup \mathcal{R} \cup \mathcal{C} \cup \mathcal{I}$. This means $n \not\equiv 0, 3, 6 \pmod 9$, so the possible residues modulo 9 are $\{1, 2, 4, 5, 7, 8\}$. We examine the combinations:
  - If $n \equiv 1 \pmod 9$: By definition, all numbers in $\mathcal{I}$ satisfy $m \equiv 1 \pmod 9$ and are Even. Since $\mathcal{I}$ contains all numbers $9j + 1$ where j is odd, and $\mathcal{X}$ contains numbers not in $\mathcal{I}$, any $n \in \mathcal{X}$ with $n \equiv 1 \pmod 9$ cannot be Even (otherwise it would be in $\mathcal{I}$). Therefore, if $n \in \mathcal{X}$ and $n \equiv 1 \pmod 9$, n *must* be Odd. This corresponds uniquely to state $S_2 = (1, \mathcal{X}, Odd)$. The combination $(1, \mathcal{X}, Even)$ does not exist for any n.
  - If $n \pmod 9 \in \{2, 4, 5, 7, 8\}$: For each of these 5 residues, an integer $n \in \mathcal{X}$ can be either Even or Odd. This yields $5 \times 2 = 10$ possible combinations. These are uniquely covered by the state definitions:
    - Residue 2: $S_3 = (2, \mathcal{X}, Even)$, $S_4 = (2, \mathcal{X}, Odd)$
    - Residue 4: $S_5 = (4, \mathcal{X}, Even)$, $S_6 = (4, \mathcal{X}, Odd)$
    - Residue 5: $S_7 = (5, \mathcal{X}, Even)$, $S_8 = (5, \mathcal{X}, Odd)$
    - Residue 7: $S_9 = (7, \mathcal{X}, Even)$, $S_{10} = (7, \mathcal{X}, Odd)$
    - Residue 8: $S_{11} = (8, \mathcal{X}, Even)$, $S_{12} = (8, \mathcal{X}, Odd)$

  Thus, the state S_2 covers the only possible combination for $n \in \mathcal{X}$ with residue 1, and the states S_3 through S_{12} cover the 10 possible combinations for $n \in \mathcal{X}$ with residues 2, 4, 5, 7, or 8. In total, the 11 states $S_2, S_3, ..., S_{12}$ uniquely cover all possibilities for an integer $n \in \mathcal{X}$.

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Since every $n \in \mathbb{Z}^+$ belongs to exactly one of the partitioning sets, and the state definitions uniquely determine a state based on this set membership combined with the unique residue mod 9 and parity (or the specific value for $n \in \mathcal{C}$), every positive integer n corresponds to exactly one state in the 17-state FSM. □

6.4. Deterministic and Finite Transition from Stage S_{P-R} to Stage S_{1-12}

We now demonstrate the deterministic and finite transition from the initial stage, S_{P-R} (representing multiples of 3), to the transient stage, S_{1-12}. This transition is irreversible; once a sequence enters S_{1-12}, it cannot return to being a multiple of 3.

Lemma 12 (Stage S_{P-R} to Stage S_{1-12} Transition)**.** *The initial stage of the 17-state FSM, S_{P-R}, has the following transitions:*

1. *S_P always transitions to S_R in a finite number of steps.*
2. *S_R always transitions to S_1 in a single step.*

Proof. We prove each transition separately:

1. **Transition from S_P to S_R (Finite):** By definition, state S_P corresponds to the set $\mathcal{P}$. Lemma 6 directly states that for any $x \in \mathcal{P}$, there exists a finite integer $n \geq 0$ such that $C^n(x) \in \mathcal{R}$. Since state S_R corresponds to the set $\mathcal{R}$, this directly implies that any element in state S_P transitions to state S_R in a finite number of steps.
2. **Transition from S_R to $S1$ (Single Step):** By definition, state S_R corresponds to the set $\mathcal{R}$ (Definition 10) and S_1 corresponds to $\mathcal{I}$ (Lemma 10). Lemma 7 states that for all $x \in \mathcal{R}$, $C(x) \in \mathcal{I}$. This directly implies that S_R transitions to S_1 in a single step.

Therefore, any starting number, whether in S_P or S_R, is guaranteed to enter the 12-state stage S_{1-12} in a finite number of steps. Furthermore, by Lemma 9, once a sequence enters stage S_{1-12}, it can never return to S_{P-R}, making this transition irreversible. □

6.5. State Transition Analysis for Transient Stage S_{1-12}

We now meticulously analyze how the Collatz function causes transitions between the defined states in stage S_{1-12}.

Lemma 13 (State Transition Analysis (12 States))**.** *The transitions between the 12 states under the Collatz function $C(x)$ are as follows:*

- *From $S_1 : (1, \mathcal{I}, Even)$ to S_7 (residue 5, $\mathcal{X}$, even) or S_8 (residue 5, $\mathcal{X}$, odd).*
- *From $S_2 : (1, \mathcal{X}, Odd)$ to S_5 (residue 4, $\mathcal{X}$, even).*
- *From $S_3 : (2, \mathcal{X}, Even)$ to S_1 (residue 1, $\mathcal{I}$, even) or S_2 (residue 1, $\mathcal{X}$, odd).*
- *From $S_4 : (2, \mathcal{X}, Odd)$ to S_9 (residue 7, $\mathcal{X}$, even).*
- *From $S_5 : (4, \mathcal{X}, Even)$ to S_3 (residue 2, $\mathcal{X}$, even) or S_4 (residue 2, $\mathcal{X}$, odd).*
- *From $S_6 : (4, \mathcal{X}, Odd)$ to S_5 (residue 4, $\mathcal{X}$, even).*
- *From $S_7 : (5, \mathcal{X}, Even)$ to S_9 (residue 7, $\mathcal{X}$, even) or S_{10} (residue 7, $\mathcal{X}$, odd).*
- *From $S_8 : (5, \mathcal{X}, Odd)$ to S_9 (residue 7, $\mathcal{X}$, even).*
- *From $S_9 : (7, \mathcal{X}, Even)$ to S_{11} (residue 8, $\mathcal{X}$, even) or S_{12} (residue 8, $\mathcal{X}$, odd).*
- *From $S_{10} : (7, \mathcal{X}, Odd)$ to S_5 (residue 4, $\mathcal{X}$, even).*
- *From $S_{11} : (8, \mathcal{X}, Even)$ to S_5 (residue 4, $\mathcal{X}$, even) or S_6 (residue 4, $\mathcal{X}$, odd) or S_{C4} $(4, \mathcal{C}, even)$.*
- *From $S_{12} : (8, \mathcal{X}, Odd)$ to S_9 (residue 7, $\mathcal{X}$, even).*

Proof. We analyze each transition case by case.

Case 1: $S_1 \to S_7$ **or** S_8**.**

- *Setup:* Let $x \in S_1$, so $x = 18k + 10$ for some integer $k \in \mathbb{Z}_{\geq 0}$.
- *Collatz Step:* $C(x) = (18k + 10)/2 = 9k + 5$.

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- *Residue:* $C(x) \equiv 5 \pmod 9$.
- *Set Membership:* $C(x) \notin \mathcal{C}$ (since $C(x) > 4$), and $C(x) \notin \mathcal{I}$ (contradiction modulo 9). Therefore, $C(x) \in \mathcal{X}$.
- *Parity:* If k is even, $C(x)$ is odd (S_8). If k is odd, $C(x)$ is even (S_7).

Case 2: $S_2 \to S_5$**.**

- *Setup:* Let $x \in S_2$, so $x = 18m + 1$ for some positive integer m.
- *Collatz Step:* $C(x) = 3(18m + 1) + 1 = 54m + 4$.
- *Residue:* $C(x) \equiv 4 \pmod 9$.
- *Set Membership:* $C(x) \notin \mathcal{C}$ (since $m \in \mathbb{Z}^+$, $C(x) > 4$) and $C(x) \notin \mathcal{I}$ (contradiction modulo 3). Therefore, $C(x) \in \mathcal{X}$.
- *Parity:* $C(x)$ is even.

Case 3: $S_3 \to S_1$ **or** S_2**.**

- *Setup:* Let $x \in S_3$, so $x = 18m + 2$ for some positive integer m.
- *Collatz Step:* $C(x) = (18m + 2)/2 = 9m + 1$.
- *Residue:* $C(x) \equiv 1 \pmod 9$.
- *Set Membership:* $C(x) \notin \mathcal{C}$ (since $m \in \mathbb{Z}^+$, $C(x) > 4$). If m is odd, $C(x) \in \mathcal{I}$ (S_1). Otherwise, if m is even, then $C(x) \in \mathcal{X}$ (S_2).
- *Parity:* see Set Membership.

Case 4: $S_4 \to S_9$**.**

- *Setup:* Let $x \in S_4$, so $x = 18k + 11$ for some integer $k \in \mathbb{Z}_{\geq 0}$.
- *Collatz Step:* $C(x) = 3(18k + 11) + 1 = 54k + 34$.
- *Residue:* $C(x) \equiv 7 \pmod 9$.
- *Set Membership:* $C(x) \notin \mathcal{C}$ (since $C(x) > 4$) and $C(x) \notin \mathcal{I}$ (contradiction modulo 3). Thus, $C(x) \in \mathcal{X}$.
- *Parity:* $C(x)$ is even.

Case 5: $S_5 \to S_3$ **or** S_4**.**

- *Setup:* Let $x \in S_5$, so $x = 18m + 4$ for some positive integer m.
- *Collatz Step:* $C(x) = (18m + 4)/2 = 9m + 2$.
- *Residue:* $C(x) \equiv 2 \pmod 9$.
- *Set Membership:* $C(x) \notin \mathcal{C}$ (since $m \in \mathbb{Z}^+$, $C(x) > 4$) and $C(x) \notin \mathcal{I}$ (contradiction modulo 9). Therefore $C(x) \in \mathcal{X}$
- *Parity:* If m is even, $C(x)$ is even (S_3). If m is odd, $C(x)$ is odd (S_4).

Case 6: $S_6 \to S_5$**.**

- *Setup:* Let $x \in S_6$, so $x = 18k + 13$ for some integer $k \in \mathbb{Z}_{\geq 0}$.
- *Collatz Step:* $C(x) = 3(18k + 13) + 1 = 54k + 40$.
- *Residue:* $C(x) \equiv 4 \pmod 9$.
- *Set Membership:* $C(x) \notin \mathcal{C}$ (since $C(x) > 4$) and $C(x) \notin \mathcal{I}$ (contradiction modulo 3). Thus, $C(x) \in \mathcal{X}$.
- *Parity:* $C(x)$ is even.

Case 7: $S_7 \to S_9$ **or** S_{10}**.**

- *Setup:* Let $x \in S_7$, so $x = 18k + 14$ for some integer $k \in \mathbb{Z}_{\geq 0}$.
- *Collatz Step:* $C(x) = (18k + 14)/2 = 9k + 7$.
- *Residue:* $C(x) \equiv 7 \pmod 9$.
- *Set Membership:* $C(x) \notin \mathcal{C}$ (since $C(x) > 4$) and $C(x) \notin \mathcal{I}$ (contradiction modulo 9). Thus, $C(x) \in \mathcal{X}$.
- *Parity:* If k is even, $C(x)$ is odd (S_{10}). If k is odd, $C(x)$ is even (S_9).

Case 8: $S_8 \to S_9$**.**

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- *Setup:* Let $x \in S_8$, so $x = 18k + 5$ for some integer $k \in \mathbb{Z}_{\geq 0}$.
- *Collatz Step:* $C(x) = 3(18k + 5) + 1 = 54k + 16$.
- *Residue:* $C(x) \equiv 7 \pmod 9$.
- *Set Membership:* $C(x) \notin \mathcal{C}$ (since $C(x) > 4$) and $C(x) \notin \mathcal{I}$ (contradiction modulo 3). Thus, $C(x) \in \mathcal{X}$.
- *Parity:* $C(x)$ is even.

Case 9: $S_9 \to S_{11}$ **or** S_{12}**.**

- *Setup:* Let $x \in S_9$, so $x = 18k + 16$ for some integer $k \in \mathbb{Z}_{\geq 0}$.
- *Collatz Step:* $C(x) = (18k + 16)/2 = 9k + 8$.
- *Residue:* $C(x) \equiv 8 \pmod 9$.
- *Set Membership:* $C(x) \notin \mathcal{C}$ (since $C(x) > 4$) and $C(x) \notin \mathcal{I}$ (contradiction modulo 9). Thus, $C(x) \in \mathcal{X}$.
- *Parity:* If k is even, $C(x)$ is even (S_{11}). If k is odd, $C(x)$ is odd (S_{12}).

Case 10: $S_{10} \to S_5$**.**

- *Setup:* Let $x \in S_{10}$, so $x = 18k + 7$ for some integer $k \in \mathbb{Z}_{\geq 0}$.
- *Collatz Step:* $C(x) = 3(18k + 7) + 1 = 54k + 22$.
- *Residue:* $C(x) \equiv 4 \pmod 9$.
- *Set Membership:* $C(x) \notin \mathcal{C}$ (since $C(x) > 4$) and $C(x) \notin \mathcal{I}$ (contradiction modulo 3). Thus, $C(x) \in \mathcal{X}$.
- *Parity:* $C(x)$ is even.

Case 11: $S_{11} \to S_5$ **or** S_6 **or** S_{C4}**.**

- *Setup:* Let $x \in S_{11}$, so $x = 18k + 8$ for some integer $k \in \mathbb{Z}_{\geq 0}$.
- *Collatz Step:* $C(x) = (18k + 8)/2 = 9k + 4$.
- *Residue:* $C(x) \equiv 4 \pmod 9$.
- *Set Membership:* $C(x) \notin \mathcal{I}$ (contradiction modulo 9).
- *Cycle Entry (Gateway):* If $k = 0$, then $x = 8$ and $C(x) = 4$, representing **a transition into the cycle stage** S_C **from stage** S_{1-12}. Otherwise, for $k > 0$, $C(x) \in \mathcal{X}$. Therefore $C(x) \in \mathcal{X} \cup \mathcal{C}$.
- *Parity:* If k is even, $C(x)$ is even (S_5). If k is odd, $C(x)$ is odd (S_6).

Case 12: $S_{12} \to S_9$**.**

- *Setup:* Let $x \in S_{12}$, so $x = 18k + 17$ for some integer $k \in \mathbb{Z}_{\geq 0}$.
- *Collatz Step:* $C(x) = 3(18k + 17) + 1 = 54k + 52$.
- *Residue:* $C(x) \equiv 7 \pmod 9$.
- *Set Membership:* $C(x) \notin \mathcal{C}$ (since $C(x) > 4$) and $C(x) \notin \mathcal{I}$ (contradiction modulo 3). Thus, $C(x) \in \mathcal{X}$.
- *Parity:* $C(x)$ is even.

These transitions fully define the behavior of the FSM within stage S_{1-12}, and demonstrate the crucial property that the next state is uniquely determined by the current state. This includes the specific condition where the system transitions into the terminal cycle stage (S_C). □

6.6. Determinism of FSM Evolution

Lemma 14 (Determinism of FSM Evolution)**.** *Let* $\mathcal{S} = \{S_P, S_R, S_1, \ldots, S_{12}, S_{C1}, S_{C2}, S_{C4}\}$ *be the set of 17 states, and let 'getState' be the state assignment function. The evolution of any Collatz sequence under this state assignment is* ***deterministic****. That is, for any positive integer* x*, the state of its Collatz successor,* $getState(C(x))$*, is uniquely determined by* x*. Consequently, the sequence of states* $(getState(x_0), getState(x_1), getState(x_2), \ldots)$ *is uniquely determined for any starting number* x_0*.*

Proof. We need to show that for any $x > 0$, the value $getState(C(x))$ is uniquely defined and belongs to $\mathcal{S}$.

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By Lemma 11, every positive integer maps to exactly one state in $\mathcal{S}$. Since $C(x)$ produces a unique positive integer for any $x > 0$, $C(x)$ must map to exactly one state $S_j \in \mathcal{S}$.

To be more explicit, we can examine the transitions based on the state $S_i = getState(x)$:

1. **If $S_i \in \{S_P, S_R\}$:**
   - If $x \in S_P$, $C(x) = x/2$. By Lemma 5, $C(x) \in \mathcal{P} \cup \mathcal{R}$. Thus, $getState(C(x))$ is either S_P or S_R, both unique states in $\mathcal{S}$.
   - If $x \in S_R$, $C(x) = 3x + 1$. By Lemma 7, $C(x) \in \mathcal{I}$. Since all elements of $\mathcal{I}$ map uniquely to state S_1, $getState(C(x)) = S_1$, a unique state in $\mathcal{S}$.
2. **If $S_i \in \{S_1, \ldots, S_{12}\}$:** Lemma 13 provides a case-by-case analysis based on $S_i = getState(x)$. For each case, it determines the properties of $C(x)$ (its residue mod 9, its parity, and whether it falls into $\mathcal{I}, \mathcal{X}$, or $\mathcal{C}$).
   - For states like $S_R, S_2, S_4, S_6, S_8, S_{10}, S_{12}$, the analysis shows that $C(x)$ always maps to a *single specific* successor state ($S_1, S_5, S_9, S_5, S_9, S_5, S_9$ respectively), regardless of the specific x within S_i.
   - For states like $S_1, S_3, S_5, S_7, S_9, S_{11}$, the analysis shows that $C(x)$ maps to one of *two or three possible* successor states (S_7/S_8, S_1/S_2, S_3/S_4, S_9/S_{10}, S_{11}/S_{12}, $S_5/S_6/S_{C4}$ respectively). However, the specific successor state is uniquely determined by properties of x (like the parity of k or m in $x = 18k + c$). Since x is given, $C(x)$ is unique, and therefore $getState(C(x))$ is also unique, landing in exactly one of those specified possible successor states.

   In all sub-cases, $getState(C(x))$ results in a unique state within $\mathcal{S}$.
3. **If $S_i \in \{S_{C1}, S_{C2}, S_{C4}\}$:** The transitions $C(1) = 4, C(2) = 1, C(4) = 2$ ensure that $getState(C(x))$ is S_{C4}, S_{C1}, S_{C2} respectively, which are unique states in $\mathcal{S}$.

Since for any $x > 0$, $C(x)$ is unique and $getState(C(x))$ maps to a unique state in $\mathcal{S}$, the evolution process defined by repeatedly applying C and then getState is deterministic for any starting number x_0. □

6.7. State S_{11} as Gateway to Terminal Stage S_C

We demonstrate that S_{11} is the unique gateway from transient stage S_{1-12} to the terminal stage S_C

Lemma 15 (S11 as the Unique Gateway State)**.** *Within the 17-state FSM, state $S_{11} = (8, X, Even)$ is the unique gateway state from Stage S_{1-12} (transient states not divisible by 3 and outside the cycle) to Stage S_C (the cycle states).*

Proof. The cycle $\mathcal{C}$ consists of the elements $\{1, 2, 4\}$. A number can only enter this cycle by reaching one of these values:

- **Reaching 1:** Only possible from 2 (C(2) = 1).
- **Reaching 2:** Only possible from 4 (C(4) = 2).
- **Reaching 4:** Possible from 1 (already in the cycle) or 8 (C(8) = 4).

Therefore, to transition to a state in the cycle *without already being in the cycle*, a number must reach 8.

State S_{11} is defined as '$S_{11} = \{\, x \in \mathbb{Z}^+ \mid s(x) = (8,\ \mathcal{X},\ \text{Even}) \,\}$, and thus contains 8. By Lemma 13 (Case 11), when S_{11} corresponds to 8, the transition $S_{11} \to S_{C4}$ represents entry into the cycle stage S_C via S_{C4}.

By inspection of the state transitions defined in Lemma 13, no state other than S_{11} contains a number that directly transitions to 1, 2, or 4 from outside the cycle stage. Thus, S_{11} is the unique gateway to the 4-2-1 cycle.

□

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6.8. Strong Connectivity Within Stage S_{1-12} and Reachability of the Gateway State S_{11}

We now prove a crucial property for convergence: The transient stage S_{1-12} forms a strongly connected component and every state within it has a finite path leading to the unique gateway state S_{11}.

Lemma 16 (Cyclical transitions through S_{11})**.** *Every state in the S_{1-12} subsystem belongs to a cycle of transitions that includes state S_{11}.*

Proof. We will demonstrate this by showing that every state has a path to S_{11} (reachability), and that any path originating from S_{11} will eventually return to a state that has a path to S_{11}. This establishes the cyclical nature.

Part 1: Reachability of S_{11}

Let A_k be the set of states from which state S_{11} can be reached in k steps or less. We define $A_0 = \{S_{11}\}$ and $A_{k+1} = A_k \cup \{S_i \in S_{1-12} \mid \exists S_j \in A_k \text{ such that } S_i \to S_j \text{ is a possible transition}\}$. We will show, by induction, that $A_4 = S_{1-12}$, meaning all states in S_{1-12} can reach S_{11} in at most 4 steps.

If a state transitions to multiple states, it's assigned to the A_k corresponding to the shortest path to S_{11}.

- $A_0 = \{S_{11}\}$ (Base Case)
- $A_1 = A_0 \cup \{S_9\} = \{S_9, S_{11}\}$
  - $S_9 \to S_{11}$ or $S_9 \to S_{12}$ (Lemma 13, Case 9). Since S_9 can transition directly to S_{11}, it follows that $S_9 \in A_1$.
- $A_2 = A_1 \cup \{S_4, S_7, S_8, S_{12}\} = \{S_4, S_7, S_8, S_9, S_{11}, S_{12}\}$
  - $S_4 \to S_9$ (Lemma 13, Case 4). Since $S_9 \in A_1$, it follows that $S_4 \in A_2$.
  - $S_7 \to S_9$ or $S_7 \to S_{10}$ (Lemma 13, Case 7). Since $S_9 \in A_1$, it follows that $S_7 \in A_2$.
  - $S_8 \to S_9$ (Lemma 13, Case 8). Since $S_9 \in A_1$, it follows that $S_8 \in A_2$.
  - $S_{12} \to S_9$ (Lemma 13, Case 12). Since $S_9 \in A_1$, it follows that $S_{12} \in A_2$.
- $A_3 = A_2 \cup \{S_1, S_5\} = \{S_1, S_4, S_5, S_7, S_8, S_9, S_{11}, S_{12}\}$
  - $S_1 \to S_7$ or $S_1 \to S_8$ (Lemma 13, Case 1). Since $S_7 \in A_2$ and $S_8 \in A_2$, it follows that $S_1 \in A_3$.
  - $S_5 \to S_3$ or $S_5 \to S_4$ (Lemma 13, Case 5). Since $S_4 \in A_2$, it follows that $S_5 \in A_3$.
- $A_4 = A_3 \cup \{S_2, S_3, S_6, S_{10}\} = \{S_1, S_2, S_3, S_4, S_5, S_6, S_7, S_8, S_9, S_{10}, S_{11}, S_{12}\} = S_{1-12}$
  - $S_2 \to S_5$ (Lemma 13, Case 2). Since $S_5 \in A_3$, it follows that $S_2 \in A_4$.
  - $S_3 \to S_1$ or $S_3 \to S_2$ (Lemma 13, Case 3). Since $S_1 \in A_3$, it follows that $S_3 \in A_4$.
  - $S_6 \to S_5$ (Lemma 13, Case 6). Since $S_5 \in A_3$, it follows that $S_6 \in A_4$
  - $S_{10} \to S_5$ (Lemma 13, Case 6). Since $S_5 \in A_3$, it follows that $S_{10} \in A_4$.

Since $A_4 = S_{1-12}$, every state in the S_{1-12} subsystem has a finite path to state S_{11}.

Part 2: Cyclical Return from S11

From Lemma 13 (Case 11), S_{11} transitions to S_5 or S_6. From Part 1 above, S_5 and S_6 can reach S_{11} in 3 and 4 steps respectively.

This shows all transitions from S11, no matter the path taken, will lead back to a state which can reach S11, hence forming a cycle.

Conclusion:

Since every state has a finite path *to* S_{11}, and any sequence starting *from* S_{11} ultimately returns to a state with a path *to* S_{11}, every state in S_{1-12} is part of a cycle that includes S_{11}. □

7. Proof of the Collatz Conjecture: Convergence to the Unique Cycle

In this section, we synthesize our 17-state finite state machine framework and the deterministic transition properties established in the preceding sections to prove that every positive integer is ultimately drawn into the cycle $C = \{1, 2, 4\}$.

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Theorem 3 (The Collatz Conjecture)**.** *Every positive integer n eventually reaches the cycle*

$$\mathcal{C} = \{1, 2, 4\}$$

under repeated application of the Collatz function $C(x)$*.*

Proof. We prove the conjecture by showing that, within the 17-state FSM framework (comprising stages S_{P-R}, S_{1-12}, and S_C), every trajectory starting from any initial state eventually reaches and remains within the cycle stage S_C, which represents the unique Collatz cycle $\mathcal{C} = \{1, 2, 4\}$.

The proof proceeds by analyzing the flow through the FSM stages:

1. **Initial State Assignment:** By Lemma 11, every positive integer n corresponds to exactly one initial state within the 17-state FSM.
2. **Transition from Stage S_{P-R}:** If n starts in Stage S_{P-R} (states S_P or S_R), Lemma 12 establishes that its trajectory transitions into Stage S_{1-12} (specifically state S_1) in a finite number of steps. Lemma 9 ensures the sequence cannot return to Stage S_{P-R}.
3. **Evolution within or into Stage S_{1-12}:** Any sequence not starting in Stage S_C will thus eventually enter or already be in Stage S_{1-12}. We analyze its behavior within this stage:
   (a) The transitions within Stage S_{1-12} and into Stage S_C are deterministic (Lemma 13).
   (b) Stage S_{1-12} is strongly connected and includes state S_{11}, guaranteeing any sequence within this stage must eventually reach S_{11}. By Lemma 16, every state in S_{1-12} participates in a cycle of transitions that includes S_{11}, ensuring that all trajectories in this stage are funneled through the gateway.
   (c) State S_{11} provides the unique transition path from Stage S_{1-12} into Stage S_C (via $S_{11} \to S_{C4}$ when $x = 8$) (Lemma 15).
   (d) Indefinite looping entirely within Stage S_{1-12} is impossible, as this would require a non-trivial cycle, which is ruled out by Theorem 2.
4. **Absorption in Stage S_C:** Combining the points above: a sequence starting outside S_C enters S_{1-12}, must eventually reach S_{11} (by 3b), cannot loop indefinitely in S_{1-12} (by 3d), and therefore must eventually take the unique exit transition $S_{11} \to S_{C4}$ into Stage S_C (by 3c). Once in Stage S_C, Lemma 4 guarantees the sequence cycles permanently within S_C.

Therefore, any trajectory starting from any positive integer n corresponds to a path in the 17-state FSM that inevitably leads, in a finite number of steps, to the absorbing cycle stage S_C. This demonstrates that every positive integer eventually reaches the cycle $\mathcal{C} = \{1, 2, 4\}$, completing the proof. □

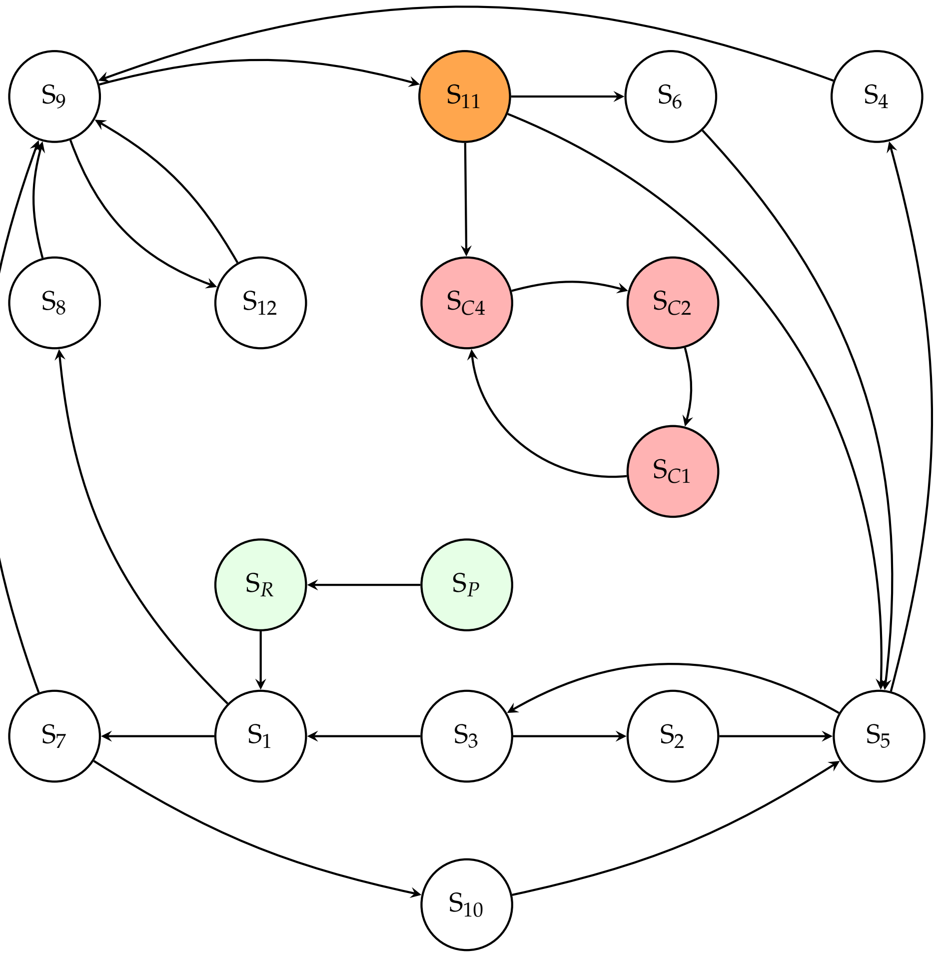

Figure 1. State transition diagram for the 17-state finite state machine modeling Collatz dynamics. The three stages are shown: initial stage (S_P, S_R, green), transient stage (S_1–S_{12}, white), and terminal cycle stage (S_{C1}, S_{C2}, S_{C4}, red). The gateway state S_{11} (orange) provides the unique transition into the terminal stage. All transitions are deterministic under the Collatz function.

8. Computational Verification

To empirically test the theoretical claims of our 17-state finite state machine (FSM) - including state assignments, transition rules, and the unique gateway mechanism - we implemented a computational verification over a large numerical range. The goal was to confirm that Collatz sequences evolve exactly as predicted by the FSM structure.

A Python script (`verify_collatz_fsm.py`) was written using the `multiprocessing` module (with 8 workers) to test all integers from 1 up to 10^7. For each starting value n, the script traced its Collatz sequence and performed the following checks at every step until reaching the cycle set $\mathcal{C} = \{1, 2, 4\}$:

- **Initial state classification:** Confirmed that each n is correctly mapped to one of the 17 FSM states via the `getState` function.
- **Deterministic transition verification:** Ensured that each observed transition $S_i \to S_j$ conformed exactly to the FSM's transition rules (Lemma 13).
- **Gateway consistency:** Verified that any transition to 4 (i.e., to S_{C4}) occurred *only* from either $x = 8$ (in S_{11}) or $x = 1$ (in S_{C1}), as required by the FSM structure.
- **State coverage:** Ensured that no number encountered during the sequence evaluation mapped to an undefined or invalid state.
- **Step count:** Recorded the number of steps required for each sequence to reach 1.

A summary of the results is shown in Table 1. All checks passed without error, and no violations were detected.

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Table 1. Computational Verification of FSM Structure for $n = 1$ to 10^7.

Verification Criterion	Result
Total integers tested	10,000,000
Starting in stage S_{P-R}	3,333,333
Starting in stage $S_{1\text{-}12}$	6,666,664
Starting in cycle stage S_C	3
State assignment failures	0
Invalid transitions	0
Incorrect gateway entries	0
Misclassified state for $x = 8$ (should be S_{11})	0
Overflow or runtime errors	0
Maximum steps to reach 1	685
Number achieving maximum steps	8,400,511

These results confirm the empirical soundness of the finite state model over all tested inputs. Every transition was deterministic, every number remained confined within the FSM structure, and the unique gateway mechanism through S_{11} behaved exactly as predicted. Notably, the number 8,400,511 achieved the maximum stopping time within this range, consistent with prior computational records.

This large-scale verification strongly reinforces the validity of the FSM framework and its predictive power in modeling Collatz dynamics.

9. Empirical Evidence from Large-Scale Collatz Computations

Over the decades, extensive computational searches have provided a substantial body of evidence regarding the behavior of Collatz sequences. Numerous studies have explored Collatz sequences for extremely large starting values - with some computations reaching up to 2^{68} (Oliveira e Silva [8]) - and ongoing distributed computing projects, such as the BOINC Collatz Conjecture project (BOINC [1]), continue to expand this empirical base. These large-scale computations have consistently demonstrated that:

- **Boundedness:** No starting number tested has produced a Collatz sequence that grows without bound; all sequences examined remain within finite limits.
- **Convergence to the 4-2-1 Cycle:** Every Collatz sequence observed eventually enters the $4 \to 2 \to 1$ cycle (or the equivalent permutation $1 \to 4 \to 2 \to 1$), regardless of the starting value.
- **No Other Cycles Found:** Despite exhaustive searches, no cycles other than the trivial $4 \to 2 \to 1$ cycle (or its cyclic permutations) have ever been discovered.

This extensive empirical evidence is entirely consistent with and strongly supports the theoretical results established in this paper - specifically, the theorems that prove boundedness, the non-existence of non-trivial cycles, and the eventual convergence to the trivial $4 \to 2 \to 1$ cycle.

10. Comparison with Previous Approaches

The Collatz Conjecture has been extensively studied using diverse mathematical techniques [4–6]. Our approach - combining a structured state-space framework with deterministic transition analysis - provides a fundamentally distinct resolution. In this section, we contextualize our proof within the broader landscape of Collatz research.

10.1. Limitations of Prior Methods

Most previous approaches, while yielding valuable insights, have fundamental limitations that prevented a complete resolution:

- **Probabilistic and Statistical Models** [4,6] suggest that, on average, Collatz sequences tend to decrease. However, they cannot establish boundedness for *all* initial values, leaving open the possibility of exceptional unbounded orbits.

- **Computational Verification** [1,8] confirms the conjecture for extremely large numbers but cannot provide a proof for all integers.
- **Dynamical Systems and Ergodic Theory** [5,6] yield statistical insights into typical trajectories but struggle with the discontinuous nature of the Collatz map.
- **Modulo Arithmetic and Congruence Class Methods** demonstrate boundedness within specific residue classes but fail to extend these properties globally.
- **Contradiction-Based Arguments** often rely on unproven assumptions or fail to rigorously eliminate all counterexamples.
- **Tao's "Almost All" Result** [9] proves that most orbits are bounded but does not establish boundedness for every number.

10.2. Novelty and Strengths of the Presented Proof

Our proof resolves the challenges of the Collatz Conjecture by introducing a **state-space approach** that enables a **complete classification of all Collatz trajectories**, ensuring their inevitable convergence to the unique cycle $\mathcal{C} = \{1, 2, 4\}$.

Key innovations include:

- **Complete Partitioning of the State Space:** We classify $\mathbb{Z}^+$ into five mutually exclusive sets - $\mathcal{C}$ (Cycle Set), $\mathcal{R}$ (ROM3 Set), $\mathcal{P}$ (Precursor Set), $\mathcal{I}$ (Immediate Successor Set), and $\mathcal{X}$ (Exclusion Set). This classification **fully encapsulates all possible Collatz trajectories**, ensuring a structured analysis.
- **Rigorous Proof of Cycle Uniqueness:** We prove that $\{4, 2, 1\}$ is the **only** possible cycle in the Collatz system. Our proof employs a **novel product equation constraint** (Lemma 2 and Lemma 3), systematically eliminating all alternative cycles.
- **Boundedness via Structural Confinement:** Instead of relying on traditional growth constraints, we introduce a **structural confinement lemma**, proving that all sequences **must** eventually enter a well-defined, controlled 12-state subsystem S_{1-12}. This guarantees that no trajectory can diverge indefinitely.
- **The Finite State Machine (FSM): A Fundamental Shift in Perspective:** A key innovation of our proof is the **17-state finite state machine (FSM)**, which transforms the Collatz problem from a question of **unbounded numerical behavior** to one of **structured state evolution**.
  - **Reduction of Infinite Complexity to a Finite System:** The FSM collapses the infinite possibilities into a finite 17-state system, with the crucial transient dynamics governed by a 12-state subsystem (S_{1-12}).
  - **Deterministic Transitions Leading to Inevitable Convergence:** Unlike traditional approaches that rely on indirect arguments, our FSM ensures that **every sequence follows a finite, structured path** to the cycle.
  - **Elimination of Classical Growth Constraints:** Instead of proving that sequences "do not grow indefinitely," the FSM demonstrates that **growth is irrelevant** - all trajectories are **forced into a terminal condition** through deterministic transitions.

  Thus, the FSM provides a **conceptually cleaner, structurally inevitable resolution** to the Collatz problem.

Conclusion of the Section: By integrating these elements, our proof provides a **rigorous, deterministic, and mathematically complete resolution** to the Collatz Conjecture, **eliminating the need for probabilistic heuristics or growth-based arguments**. The state-space framework and finite state machine ensure that all Collatz sequences **must** follow a structured, finite trajectory into the unique cycle.

11. Conclusions

We have presented a rigorous, structurally grounded proof of the Collatz Conjecture, leveraging a novel framework that interprets Collatz sequences as deterministic trajectories within a structured state space. By partitioning the positive integers into five mutually exclusive sets - namely, the cycle set $\mathcal{C}$, ROM3 set $\mathcal{R}$, precursor set $\mathcal{P}$, immediate successor set $\mathcal{I}$, and exclusion set $\mathcal{X}$ - we have developed a systematic classification that fully captures the behavior of Collatz iterations.

Our proof follows a two-stage approach:

1. **We establish that the only possible cycle is** $\mathcal{C} = \{1, 2, 4\}$, applying a product equation constraint as detailed in our earlier preprint [7]. This rigorously eliminates all non-trivial cycles, a key step that previous approaches had not fully addressed.
2. **We prove that every Collatz sequence must reach** $\mathcal{C}$ **in finite time**, using a deterministic transition analysis within our structured state-space framework. The finite state machine (FSM) guarantees that all sequences undergo a systematic, finite progression into the cycle. *Critically, this convergence occurs through a unique gateway state,* S_{11} *(containing the number 8), which is the only entry point into the 4-2-1 cycle from outside the cycle itself.*

With these results, we conclude that every positive integer is eventually drawn into the $4 \to 2 \to 1$ cycle, thereby resolving the Collatz Conjecture.

Crucially, our approach diverges from traditional bounded growth arguments by demonstrating that sequences do not merely remain within a finite bound - they are structurally confined and systematically directed toward termination. The deterministic nature of our finite state machine analysis, *including the existence of a single entry point to the cycle*, ensures that all trajectories are forced into a terminal condition, rather than merely avoiding unbounded divergence. This fundamental shift in perspective transforms the problem from one of numerical control to one of inevitable dynamical convergence.

Beyond settling this long-standing open problem, our work demonstrates the effectiveness of a state-space-driven, set-theoretic approach in analyzing complex iterative systems. This methodology may provide a blueprint for addressing similar problems in number theory and discrete dynamical systems, offering new insights into how deterministic constraints govern seemingly chaotic processes.

12. Need for Verification and Future Directions

12.1. Need for Rigorous Verification

While the proof presented in this paper offers a distinct and potentially compelling approach to the Collatz Conjecture - particularly through the use of the product equation and prime factorization for cycle analysis - rigorous validation by the broader mathematical community is essential. The history of the Collatz Conjecture is replete with proposed proofs that were later found to contain flaws. Therefore, thorough and independent scrutiny of every step of this proof, especially the derivation and application of the product equation for cycle analysis, the partitioning of the state space, the construction and transition analysis of the 17-state FSM, and the proof of convergence via gateway state reachability, is paramount. This validation should involve expert peer review through journal submissions, detailed examination by specialists in number theory, presentations at conferences, and open dissemination for public scrutiny. Until such rigorous validation is complete, the result remains a proposed proof that, we believe, provides a sound and novel pathway toward resolving this longstanding problem.

12.2. Potential Avenues for Future Research

If validated, the proof presented here would not only resolve the Collatz Conjecture but also open new avenues for research in number theory and related fields. Potential directions for future work include:

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- **Generalization of the Product Equation Technique:** Investigate whether the product equation method introduced in this paper can be generalized or adapted to study cycle structures and dynamics in other iterative functions or number-theoretic problems.
- **Refinement and Simplification of the Proof:** Explore alternative formulations of the arguments, particularly those based on contradiction and prime factorization, to achieve greater clarity or elegance and potentially shorter proofs.
- **Computational Exploration Inspired by the Proof:** With convergence established, further computational studies of stopping time distributions, average trajectory behavior, and other statistical properties of Collatz sequences could yield valuable insights.
- **Applications to Related Conjectures:** Determine whether the insights and techniques from this work can be applied to other unsolved problems or related conjectures in the realm of iterative number theory and dynamical systems.
- **FSM Methodology for Other Dynamical Systems:** Investigate whether the techniques used to construct and analyze the 17-state FSM (based on set partitioning, residue classes, and transition mapping) can be adapted to model and prove properties of other number-theoretic sequences or discrete dynamical systems.
- **Educational and Expository Development:** Develop pedagogical materials and simplified expositions of this proof to make it accessible to a broader mathematical audience, including students and researchers. Such efforts might include clearer visualizations, intuitive explanations of key steps, and adaptations of the proof for classroom use.

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Data Availability Statement: The Python script used to generate the computational verification data presented in this proof is available online at the following open code repository: [Link to Code Repository].

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