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Regularity for quasi-linear p-Laplacian type non-homogeneous equations in the Heisenberg Group

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Abstract: When $2 - 1/Q < p \leq 2$, we establish the $C^{0,1}_{\text{loc}}$ and $C^{1,\alpha}_{\text{loc}}$-regularities of weak solutions to quasi-linear p-Laplacian type non-homogeneous equations in the Heisenberg group.

Keywords: p-Laplacian type; non-homogeneous equations; Heisenberg group; regularities; Riesz potentials.

1. Introduction

In this paper, we consider the equation

$$-\operatorname{div}_H a(x, Xu) = \mu \quad \text{in } \Omega \subset \mathbb{H}^n, \tag{1}$$

where Ω is a domain and μ is a Radon measure with $|\mu| < \infty$ and $\mu(\mathbb{H}^n \setminus \Omega) = 0$; hence the equation (1) can be considered as defined in all of $\mathbb{H}^n$. Here $Xu = (X_1 u, X_2 u, \ldots, X_{2n} u)$ is denoted as the horizontal gradient of a function $u : \Omega \to \mathbb{R}$, see Section 2 for more details, and the continuous function $a : \Omega \times \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ is assumed to be C^1 in the gradient variable and satisfies the following structural conditions for every $x, y \in \Omega$ and $z, \xi \in \mathbb{R}^{2n}$,

$$(|z|^2 + s^2)^{\frac{p-2}{2}} |\xi|^2 \leq \langle D_z a(x,z)\xi, \xi \rangle \leq L(|z|^2 + s^2)^{\frac{p-2}{2}} |\xi|^2; \tag{2}$$

$$|a(x,z) - a(y,z)| \leq L' |z| (|z|^2 + s^2)^{\frac{p-2}{2}} |x - y|^\alpha, \tag{3}$$

where $L, L' \geq 1, s \geq 0, \alpha \in (0,1]$ and $D_z a(x,z)$ is a symmetric matrix for every $x \in \Omega$.

A function $u \in HW^{1,p}_{\text{loc}}(\Omega)$ is called as a weak solution to (1) if

$$\int_\Omega \langle a(x, Xu), \varphi \rangle dx = \int_\Omega \varphi d\mu,$$

where $HW^{1,p}_{\text{loc}}(\Omega)$ is the first order p-th integrable horizontal local Sobolev space, namely, all functions $u \in L^p_{\text{loc}}(\Omega)$ with their distributional horizontal gradients $Xu \in L^p_{\text{loc}}(\Omega)$. Given the typical example $a(x,z) = (|z|^2 + s^2)^{\frac{p-2}{2}} z$, the equation (1) becomes the sub-elliptic non-degenerate p-Laplacian equation with measure data

$$-\operatorname{div}_H (|Xu|^2 + s^2)^{\frac{p-2}{2}} Xu = \mu \quad \text{if } s > 0,$$

and the sub-elliptic p-Laplacian equation with measure data

$$-\operatorname{div}_H |Xu|^{p-2} Xu = \mu \quad \text{if } s = 0. \tag{4}$$

When measure $\mu = 0$, the equation (4) becomes the sub-elliptic p-Laplacian equation

$$-\operatorname{div}_H |Xu|^{p-2} Xu = 0. \tag{5}$$

Particularly, we call weak solutions to the equation (5) as p-harmonic functions in $\Omega \subset \mathbb{H}^n$.

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For p-harmonic functions in Euclidean spaces $\mathbb{R}^n$, their $C^{1,\alpha}$-regualrity has been established by [6,9,15–17]. For p-harmonic functions in the Heisenberg group $\mathbb{H}^n$, their $C^{0,1}$ and $C^{1,\alpha}$-regularities have been established by [2,3,10,11,13,14,19]. It is therefore natural to consider the case of regularity for the corresponding inhomogeneous equation. In Euclidean spaces $\mathbb{R}^n$, when $2-1/n<p<\infty$, Duzaar-Mingione[4,5] built up the $C^{0,1}$-regularity of solutions to the equation (1) with measure $\mu\in L^1(\Omega)$. In the Heisenberg group $\mathbb{H}^n$, when $2\le p<\infty$, Mukherjee-Sire[12] built up the $C^{1,\gamma}$-regularity of solutions to the equation (1) with measure $\mu=f\in L^q(\Omega)$ for some $q>Q$ and some $\gamma\in(0,1)$. But when $1<p<2$, the $C^{0,1}$ and $C^{1,\gamma}$-regularities for the equation (1) in the Heisenberg group $\mathbb{H}^n$ are unknown. This paper aims to establish the $C^{0,1}$ and $C^{1,\gamma}$-regularities in the case $1<p<2$.

Before stating our main results, let us recall that truncated linear Riesz potentials are defined as

$$\mathbf{I}^{\mu}_{\beta}(x_0,2R):=\int_0^R\frac{\mu(B(x,\rho))}{\rho^{Q-\beta}}\frac{d\rho}{\rho},\quad \beta\in(0,Q].$$

Theorem 1. *Let $u\in HW^{1,p}(\Omega)$ be a weak solution to the euqation (1) with $\mu\in L^1_{\mathrm{loc}}(\Omega)$. If $2-1/Q<p\le 2$ and $a:\Omega\times\mathbb{R}^{2n}\to\mathbb{R}^{2n}$ satisfies the structural conditions (2) and (3), then there exist constants $c=c(n,p,L)>0$ and $\bar R=\bar R(n,p,L,L',\alpha,\mathrm{dist}(x_0,\partial\Omega))>0$, such that the pointwise estimate*

$$\begin{aligned}|Xu(x_0)|\le& c\fint_{B_{2R}}(|Xu|+s)dx+c\frac{|\mu|(B_{2R})}{R^{Q-1}}^{\frac{2}{p}}+c\frac{|\mu|(B_{2R})}{R^{Q-1}}^{\frac{3Q-Qp-2}{Q-p}}\\&+c[\mathbf{I}^{|\mu|}_1(x_0,2R)]^{\frac{2}{p}}+c[\mathbf{I}^{|\mu|}_1(x_0,2R)]^{\frac{3Q-Qp-2}{Q-p}}\end{aligned}\tag{6}$$

holds for any $x_0\in\mathbb{H}^n$, whenever $B_{2R}(x_0)\subset\Omega$ and $0<R\le\bar R$. Furthermore, if $a(x,z)$ is independent of x, then (6) holds for any $0<R<\frac{1}{2}\mathrm{dist}(x_0,\partial\Omega)$.

Theorem 2. *Let $u\in HW^{1,p}(\Omega)$ be a weak solution to the euqation (1). Assume that $2-1/Q<p\le 2$ and $a:\Omega\times\mathbb{R}^{2n}\to\mathbb{R}^{2n}$ satisfies the structural conditions (2) and (3). If we have $\mu=f\in L^q_{\mathrm{loc}}(\Omega)$ for some $q>Q$, then Xu is Hölder continuous and there exist constants $c=c(n,p,L)>0$ and $\bar R=\bar R(n,p,L,L',\alpha,\mathrm{dist}(x_0,\partial\Omega))>0$, such that for any $x_0\in\Omega$, $0<R\le\bar R$ and $x,y\in B_R(x_0)\subset\Omega$, the estimate*

$$\begin{aligned}|Xu(x)-Xu(y)|\le& cd(x,y)^{\gamma}\Big\{\fint_{B_R}(|Xu|+s)dx+\|f\|^{\frac{2}{p}}_{L^q(B_R)}+\|f\|^{\frac{3Q-Qp-2}{Q-p}}_{L^q(B_R)}\\&+[\mathbf{I}^{|\mu|}_1(x_0,2R)]^{\frac{2}{p}}+[\mathbf{I}^{|\mu|}_1(x_0,2R)]^{\frac{3Q-Qp-2}{Q-p}}\Big\}\end{aligned}\tag{7}$$

holds for some $\gamma=\gamma(n,p,L,\alpha,q)\in(0,1)$. In particular, if $a(x,z)$ is independent of x, then (7) holds for $\bar R=\bar R(n,p,L,L',\mathrm{dist}(x_0,\partial\Omega))>0$ and $\gamma=\gamma(n,p,L,q)\in(0,1)$.

1.1. Ideas of the proofs

We sketch the ideas to prove Theorems 1 and 2. The basic geometries and properties of the Heisenberg group used in this paper are stated in Section 2.

We will prove Theorem 1 in Section 4. The proof of Theorem 1 relies on novel techniques established by Duzaar-Mingione[4] based on sharp comparison estimates of homogeneous equations with frozen coefficients. In Section 3, we establish two comparison estimates, see Lemmas 1 and 2 for details. Basing on two comparison estimates, we establish the main estimate of the weak solution u to the equation (1), see Lemma 3 for details. Compared with the Euclidean setting, there exists the extra term $\sup\limits_{B_{\bar R}}|Xv|$ in (34), which comes from commutators of the horizontal vector fields, see Proposition 1 for details. We use Lemma 2 to estimate the extra term in Section 4. In Section 4, basing on Lemma 3, we use scientific induction to obtain Lemma 4. Finally, we use Lemma 4 to prove Theorem 1 in Section 4.

We will prove Theorem 2 in Section 5. The proof of Theorem 2 relies on a perturbation lemma established by Mukherjee-Sire[12], see Lemma 6 for details. In Section 5, we use Lemma 2 to establish the weaker integral decay estimate of the oscillation of the gradient of the weak solution u to the equation (1), see Lemma 5 for details. Basing on Lemmas 6 and 5, we obtain Proposition 2 in Section 5. Finally, we use Lemma 7 and Proposition 2 to prove Theorem 2 in Section 5. Lemma 7 follows from (13) and Lemma 2 in Section 5.

2. Preliminaries

2.1. Notations

In this paper, for $s \geq 0$, we denote

$$V(z) := (|z|^2 + s^2)^{\frac{p-2}{4}} z, \quad z \in \mathbb{R}^{2n}. \tag{8}$$

By [8, Lemma 2.1], the inequality

$$c^{-1}(|z_1|^2 + |z_2|^2 + s^2)^{\frac{p-2}{2}} \leq \frac{|V(z_2) - V(z_1)|^2}{|z_2 - z_1|^2} \leq c(|z_1|^2 + |z_2|^2 + s^2)^{\frac{p-2}{2}} \tag{9}$$

holds for any $z_1, z_2 \in \mathbb{R}^{2n}$ and any $s \geq 0$, where $c = c(n,p) > 0$ is independent of s, also see [4, (2.2)]. Inequality (9) and the structure condition (2) imply

$$c^{-1}|V(z_2) - V(z_1)|^2 \leq \langle a(x,z_2) - a(x,z_1), z_2 - z_1 \rangle. \tag{10}$$

2.2. The Heisenberg group

For an integer $n \geq 1$, we denote by $\mathbb{H}^n$ the Heisenberg group, which is identified with the Euclidean space $\mathbb{R}^{2n+1}$. The group multiplication on $\mathbb{H}^n$ is given by

$$x \circ y := \left(x_1 + y_1, \ldots, x_{2n} + y_{2n}, t + s + \frac{1}{2} \sum_{i=1}^{n} (x_i y_{n+i} - x_{n+i} y_i) \right)$$

for points $x = (x_1, \ldots, x_{2n}, t), y = (y_1, \ldots, y_{2n}, s) \in \mathbb{H}^n$. The left invariant vector fields corresponding to the canonical basis of the Lie algebra are

$$X_i = \partial_{x_i} - \frac{x_{n+i}}{2} \partial_t, \quad X_{n+i} = \partial_{x_{n+i}} + \frac{x_i}{2} \partial_t,$$

and the only non-trivial commutator $T = \partial_t$ for $1 \leq i \leq n$. For any $1 \leq i < j \leq 2n$, we have

$$[X_i, X_{n+i}] = T, \quad [X_i, X_j] = 0 \, \forall j \neq n+i.$$

We call $X_1, \ldots, X_{2n}$ as horizontal vector fields and T as the vertical vector field.

Let $\Omega \subset \mathbb{H}^n$ be any domain (open connected subset). For any scalar function $f \in C^1(\Omega)$, we denote $Xf = (X_1 f, \ldots, X_{2n} f)$ as the horizontal gradient; for any scalar function $f \in C^2(\Omega)$, we denote $XXf = (X_i X_j f)_{2n \times 2n}$ as the second order horizontal derivative and $\Delta_H f = \sum_{j=1}^{2n} X_j X_j f$ as the sub-Laplacian operator. We write lengths of Xf and XXf as

$$|Xu| = \left(\sum_{i=1}^{2n} |X_i u|^2 \right)^{1/2}, \quad |XXu| = \left(\sum_{i,j=1}^{2n} |X_i X_j u|^2 \right)^{1/2}.$$

For any vector valued function $F = (f_1, \ldots, f_{2n}) : \mathbb{H}^n \to \mathbb{R}^{2n}$, we denote $\mathrm{div}_H(F) = \sum_{i=1}^{2n} X_i f$ as the horizontal divergence. The Haar measure in $\mathbb{H}^n$ is the Lebesgue measure of $\mathbb{R}^{2n+1}$. We denote $|E|$ as the Lebesgue measure of a measurable set $E \subset \mathbb{H}^n$ and $⨍_E f dx = \frac{1}{|E|} \int_E f dx$ as the average of an integrable function f over set E.

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We denote d as the Carnot-Carathèodory metric (CC-metric) and $B_r(x) = B(x,r) := \{y \in \mathbb{H}^n : d(x,y) < r\}$ as the CC-metric balls with the center $x \in \mathbb{H}^n$ and the radius $r > 0$. Here the CC-metric d is defined as the length of the shortest horizontal curves connecting two points, see [1]. For any points $x, y \in \mathbb{H}^n$, the CC-metric $d(x,y)$ is equivalent to the homogeneous metric $d_{\mathbb{H}^n}(x,y) = \|y^{-1} \circ x\|_{\mathbb{H}^n}$. Here the homogeneous norm for $x = (x_1, \ldots, x_{2n}, t) \in \mathbb{H}^n$ is defined as $\|x\|_{\mathbb{H}^n} := \left(\sum_{i=1}^{2n} x_i^2 + |t|\right)^{1/2}$. Since these two metrics are equivalent, all the CC-metric balls $B_r(x)$ throughout this paper can be restated to the homogeneous metric balls $K_\rho(x) := \{y \in \mathbb{H}^n : d_{\mathbb{H}^n}(y,x) < \rho\}$.

The horizontal Sobolev space $HW^{1,p}(\Omega)$ with $1 \le p < \infty$ is the collection of all functions $u \in L^p(\Omega)$ with $Xu \in L^p(\Omega, \mathbb{R}^{2n})$. $HW^{1,p}(\Omega)$ is a Banach space equipped with the norm

$$\|u\|_{HW^{1,p}(\Omega)} = \|u\|_{L^p(\Omega)} + \|Xu\|_{L^p(\Omega,\mathbb{R}^{2n})}.$$

For any $m \ge 2$, the m-order horizontal Sobolev space $HW^{m,p}(\Omega)$ is the collection of all functions u with $Xu \in HW^{m-1,p}(\Omega)$, and its norm is defined in a similar way. For any $m \ge 1$, we denote $HW^{m,p}_{\text{loc}}(\Omega)$ as the collection of all functions $u : \Omega \to \mathbb{R}$ such that $u \in HW^{m,p}(U)$ for all $U \Subset \Omega$, and $HW^{m,p}_0(\Omega)$ as the completion of $C_c^\infty(\Omega)$ equipped with the $\|\cdot\|_{HW^{m,p}(\Omega)}$-norm.

In the rest of this section, we recall some regularities and apriori estimates of the homogeneous equation corresponding to the equation (1) with freezing of the coefficients. For any $x_0 \in \Omega$, we consider the equation

$$\operatorname{div}_H a(x_0, Xu) = 0 \quad \text{in } \Omega. \tag{11}$$

The following regularity theorem follows from [19, Theorem 1.1] and [13, Theorem 1.3], also see[12, Theorem 2.3].

Theorem 3. *Let $u \in HW^{1,p}(\Omega)$ be a weak solution to the equation (11). If $a(x_0,z)$ satisfies the condition (2) and $D_z a(x_0,z)$ is a symmetric matrix, then Xu is locally Hölder continuous. Moreover, there exist constants $c = c(n,p,L) > 0$ and $\beta = \beta(n,p,L) \in (0,1)$ such that the followings hold,*

$$\sup_{B_{R/2}} |Xu|^p \le c \fint_{B_R} (|Xu|^2 + s^2)^{\frac{p}{2}} dx; \tag{12}$$

$$\fint_{B_\rho} |Xu - (Xu)_{B_\rho}|^p dx \le c\left(\frac{\rho}{R}\right)^\beta \fint_{B_R} (|Xu|^2 + s^2)^{\frac{p}{2}} dx, \tag{13}$$

for every concentric $B_\rho \subset B_R \subset \Omega$ and $1 < p < \infty$.

Using Sobolev's inequality and Moser's iteration on the Caccioppoli type inequalities in [19], we have the following local estimate, for any $\sigma \in (0,1)$ and $q > 0$,

$$\sup_{B_{\sigma R}} |Xu| \le c(1-\sigma)^{-\frac{Q}{q}} \left(\fint_{B_R} (|Xu|^2 + s^2)^{\frac{q}{2}} dx\right)^{\frac{1}{q}} \tag{14}$$

for some $c = c(n,p,L,q) > 0$, also see [12, (2.14)], where $u \in C^{1,\beta}(\Omega)$ is a solution to the equation (11) for some $\beta \in (0,1)$. Using (14) with $\sigma = 1/2$ and $q = 1$, for all $0 < r \le R/2$, we have

$$\int_{B_r} |Xu| dx \le c\left(\frac{r}{R}\right)^Q \int_{B_R} (|Xu| + s) dx, \tag{15}$$

for some $c = c(n,p,L) > 0$, also see [12, (2.16)], where $u \in C^{1,\beta}(\Omega)$ is a solution to the equation (11) for some $\beta \in (0,1)$.

The next result has been proved for the case $p \ge 2$ in [12, Proposition 3.1]; the proof for the case $1 < p \le 2$ can be obtained with minor modifications. We omit the proof.

Proposition 1. *Let $B_{r_0} \subset \Omega$ and $u \in C^{1,\beta}(\Omega)$ be a solution to the equation (11), with $\beta = \beta(n,p,L) \in (0,1)$. Then there exists $c = c(n,p,L) > 0$ such that the inequality*

$$\fint_{B_\rho} |Xu - (Xu)_{B_\rho}| dx \le c\left(\frac{\rho}{r}\right)^\beta \left[\fint_{B_r} |Xu - (Xu)_{B_r}| dx + \chi r^\beta\right] \tag{16}$$

holds for all $0 < \rho < r < r_0$, where

$$\chi = \frac{1}{r_0^\beta}\left(s + \max_{1\le i\le 2n} \sup_{B_{r_0}} |X_i u|\right).$$

3. Comparison estimates

In this section, we fix $x_0 \in \Omega$ and denote $B_\rho = B(x_0, \rho)$ for every $\rho > 0$. For simplisity, we denote

$$M_\rho = \frac{|\mu|(B_\rho)}{\rho^{Q-1}}$$

for every $\rho > 0$. Fix $R > 0$ such that $B_{2R} \subset \Omega$. We consider the Dirichlet problem

$$\begin{cases} \operatorname{div}_H a(x, Xw) = 0 & \text{in } B_{2R}; \\ w - u \in HW_0^{1,p}(B_{2R}). \end{cases} \tag{17}$$

Now we give the first comparison lemma.

Lemma 1. *Let $u \in HW^{1,p}(\Omega)$ be a weak solution to the equation (1) and $2 - 1/Q < p \le 2$. Then the weak solution $w \in HW^{1,p}(B_{2R})$ to the equation (17) satisfies the inequality*

$$\begin{aligned} \fint_{B_{2R}} |Xu - Xw| dx \le & cM_{2R}^{\frac{2}{p}} + cM_{2R}^{\frac{3Q-Qp-2}{Q-p}} + cM_{2R}\left(\fint_{B_{2R}} (|Xu| + s) dx\right)^{\frac{2-p}{2}} \\ & + cM_{2R}\left(\fint_{B_{2R}} (|Xu| + s) dx\right)^{\frac{(Q-1)(2-p)}{3Q-Qp-2}}, \end{aligned} \tag{18}$$

where $c = c(n,p,L) > 0$.

Proof of Lemma 1. For any integer $k \ge 0$, $R > 0$ and $\gamma > 0$, we define the truncation operators

$$T_k(t) := \max\left\{-\frac{k}{R^\gamma}, \min\left\{\frac{k}{R^\gamma}, t\right\}\right\}, \quad \Phi_k(t) := T_1(t - T_k(t)), \quad t \in \mathbb{R}.$$

Denote

$$C_k := \left\{x \in B_{2R} : \frac{k}{R^\gamma} < \frac{|u(x) - w(x)|}{m} \le \frac{k+1}{R^\gamma}\right\},$$

where $m > 0$, we will choose constants $\gamma > 0$ and $m > 0$ in the following. Since $w - u \in HW_0^{1,p}(B_{2R})$, we use $\phi = \Phi_k(\frac{u-w}{m})$ to test equations (1) and (17), then we have

$$\int_{B_{2R}} \langle a(x, Xu) - a(x, Xw), X\phi \rangle dx = \int_{B_{2R}} \phi d\mu. \tag{19}$$

Note that

$$X_i\phi = \begin{cases} 0 & \text{in } B_{2R} \setminus C_k; \\ \frac{1}{m}(X_i u - X_i w) & \text{in } C_k. \end{cases}$$

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This, together with (10) and (19), yields

$$
\begin{aligned}
\int_{C_k} |V(Xu) - V(Xw)|^2 dx \leq & c \int_{C_k} \langle a(x, Xu) - a(x, Xw), Xu - Xw \rangle dx \\
= & cm \int_{B_{2R}} \langle a(x, Xu) - a(x, Xw), X\phi \rangle dx \\
= & cm \int_{B_{2R}} \Phi_k(\frac{u-w}{m}) d\mu \\
\leq & \frac{cm}{R^\gamma} |\mu|(B_{2R}).
\end{aligned}
$$

From this, by Hölder's inequality, we have

$$
\begin{aligned}
\int_{C_k} |V(Xu) - V(Xw)|^{\frac{2}{p}} dx \leq & c|C_k|^{\frac{p-1}{p}} \left(\int_{C_k} |V(Xu) - V(Xw)|^2 dx \right)^{\frac{1}{p}} \\
\leq & c|C_k|^{\frac{p-1}{p}} \left(\frac{m}{R^\gamma} \right)^{\frac{1}{p}} [|\mu|(B_{2R})]^{\frac{1}{p}} \\
= & c \left(\frac{m}{R^\gamma} \right)^{\frac{1}{p}} [|\mu|(B_{2R})]^{\frac{1}{p}} \left(\int_{C_k} 1 dx \right)^{\frac{p-1}{p}} \\
\leq & c \left(\frac{m}{R^\gamma} \right)^{\frac{1}{p}} [|\mu|(B_{2R})]^{\frac{1}{p}} \left[\frac{1}{\left(\frac{mk}{R^\gamma} \right)^{\frac{Q}{Q-1}}} \int_{C_k} |u-w|^{\frac{Q}{Q-1}} \right]^{\frac{p-1}{p}}. \qquad (20)
\end{aligned}
$$

Similarly, when $k = 0$, we have

$$
\begin{aligned}
\int_{C_0} |V(Xu) - V(Xw)|^{\frac{2}{p}} dx \leq & c|C_0|^{\frac{p-1}{p}} \left(\frac{m}{R^\gamma} \right)^{\frac{1}{p}} [|\mu|(B_{2R})]^{\frac{1}{p}} \\
\leq & c|B_{2R}|^{\frac{p-1}{p}} \left(\frac{m}{R^\gamma} \right)^{\frac{1}{p}} [|\mu|(B_{2R})]^{\frac{1}{p}}. \qquad (21)
\end{aligned}
$$

Combining (20) and (21), we have

$$
\begin{aligned}
& \int_{B_{2R}} |V(Xu) - V(Xw)|^{\frac{2}{p}} dx \\
= & \int_{C_0} |V(Xu) - V(Xw)|^{\frac{2}{p}} dx + \sum_{k=1}^{\infty} \int_{C_k} |V(Xu) - V(Xw)|^{\frac{2}{p}} dx \\
\leq & \ c|B_{2R}|^{\frac{p-1}{p}} \left(\frac{m}{R^\gamma} \right)^{\frac{1}{p}} [|\mu|(B_{2R})]^{\frac{1}{p}} \\
& + c \sum_{k=1}^{\infty} \left(\frac{m}{R^\gamma} \right)^{\frac{1}{p}} [|\mu|(B_{2R})]^{\frac{1}{p}} \left[\frac{1}{\left(\frac{mk}{R^\gamma} \right)^{\frac{Q}{Q-1}}} \int_{C_k} |u-w|^{\frac{Q}{Q-1}} dx \right]^{\frac{p-1}{p}}.
\end{aligned}
$$

Note that

$$
\begin{aligned}
& \sum_{k=1}^{\infty} \left[\frac{1}{k^{\frac{Q}{Q-1}}} \int_{C_k} |u-w|^{\frac{Q}{Q-1}} dx \right]^{\frac{p-1}{p}} \\
\leq & \left(\sum_{k=1}^{\infty} \left(\frac{1}{k} \right)^{\frac{Q(p-1)}{Q-1}} \right)^{\frac{1}{p}} \left(\sum_{k=1}^{\infty} \int_{C_k} |u-w|^{\frac{Q}{Q-1}} dx \right)^{\frac{p-1}{p}}.
\end{aligned}
$$

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Since $2-1/Q<p\le 2$ imlies $Q(p-1)/(Q-1)>1$, we have

$$\sum_{k=1}^{\infty}\left(\frac{1}{k}\right)^{\frac{Q(p-1)}{Q-1}}\le c.$$

Thus

$$\begin{aligned}&\int_{B_{2R}}|V(Xu)-V(Xw)|^{\frac{2}{p}}dx\\&\le c|B_{2R}|^{\frac{p-1}{p}}\left(\frac{m}{R^{\gamma}}\right)^{\frac{1}{p}}[|\mu|(B_{2R})]^{\frac{1}{p}}\\&\quad+c\left(\frac{m}{R^{\gamma}}\right)^{\frac{1}{p}-\frac{Q(p-1)}{(Q-1)p}}[|\mu|(B_{2R})]^{\frac{1}{p}}\left(\int_{B_{2R}}|u-w|^{\frac{Q}{Q-1}}dx\right)^{\frac{p-1}{p}}.\end{aligned}$$

By Sobolev inequality, we have

$$\begin{aligned}&\int_{B_{2R}}|V(Xu)-V(Xw)|^{\frac{2}{p}}dx\\&\le c|B_{2R}|^{\frac{p-1}{p}}\left(\frac{m}{R^{\gamma}}\right)^{\frac{1}{p}}[|\mu|(B_{2R})]^{\frac{1}{p}}\\&\quad+c\left(\frac{m}{R^{\gamma}}\right)^{\frac{1}{p}-\frac{Q(p-1)}{(Q-1)p}}[|\mu|(B_{2R})]^{\frac{1}{p}}\left(\int_{B_{2R}}|Xu-Xw|dx\right)^{\frac{Q(p-1)}{(Q-1)p}}.\end{aligned}\tag{22}$$

Noting that (9) implies

$$\begin{aligned}|Xu-Xw|=&\left[(|Xu|^2+|Xw|^2+s^2)^{\frac{p-2}{2}}|Xu-Xw|^2\right]^{\frac{1}{2}}(|Xu|^2+|Xw|^2+s^2)^{\frac{2-p}{4}}\\\le& c|V(Xu)-V(Xw)|(|Xu|^2+|Xw|^2+s^2)^{\frac{2-p}{4}}\\\le& c|V(Xu)-V(Xw)|[|Xu-Xw|^{\frac{2-p}{2}}+|Xu|^{\frac{2-p}{2}}+s^{\frac{2-p}{2}}].\end{aligned}\tag{23}$$

By Young's inequality, we have

$$|Xu-Xw|\le c|V(Xu)-V(Xw)|^{\frac{2}{p}}+\frac{1}{2}|Xu-Xw|+c|V(Xu)-V(Xw)|(|Xu|+s)^{\frac{2-p}{2}}.$$

By Hölder's inequality, we have

$$\begin{aligned}\int_{B_{2R}}|Xu-Xw|dx\le& c\int_{B_{2R}}|V(Xu)-V(Xw)|^{\frac{2}{p}}dx\\&+c\left(\int_{B_{2R}}|V(Xu)-V(Xw)|^{\frac{2}{p}}dx\right)^{\frac{p}{2}}\left(\int_{B_{2R}}(|Xu|+s)dx\right)^{\frac{2-p}{2}}.\end{aligned}\tag{24}$$

Let $m=|\mu|(B_{2R})$ and $\gamma=Q-2$. Then (22) becomes

$$\fint_{B_{2R}}|V(Xu)-V(Xw)|^{\frac{2}{p}}dx\le cM_{2R}^{\frac{2}{p}}+cM_{2R}^{\frac{3Q-Qp-2}{(Q-1)p}}\left(\fint_{B_{2R}}|Xu-Xw|dx\right)^{\frac{Q(p-1)}{(Q-1)p}},$$

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which, together with (24), yields

$$\begin{aligned}
&\fint_{B_{2R}} |Xu - Xw|dx \\
&\leq cM_{2R}^{\frac{2}{p}} + cM_{2R}^{\frac{3Q-Qp-2}{(Q-1)p}} \left(\fint_{B_{2R}} |Xu - Xw|dx \right)^{\frac{Q(p-1)}{(Q-1)p}} \\
&\quad + c\left[M_{2R} + M_{2R}^{\frac{3Q-Qp-2}{(Q-1)2}} \left(\fint_{B_{2R}} |Xu - Xw|dx \right)^{\frac{Q(p-1)}{(Q-1)2}} \right] \left(\fint_{B_{2R}} (|Xu| + s)dx \right)^{\frac{2-p}{2}}. \qquad (25)
\end{aligned}$$

Finally, using Young's inequality to estimate the second and last terms in the right hand side of (25), we conclude (18).
□

For the second comparison estimate, we require the Dirichlet problem with freezing of the coefficients. Let $w \in HW^{1,p}(B_{2R})$ be a weak solution to the equation (17). We consider the Dirichlet problem

$$\begin{cases} \mathrm{div}_H a(x_0, Xv) = 0 & \text{in } B_R; \\ v - w \in HW_0^{1,p}(B_R). \end{cases} \qquad (26)$$

Now we give the second comparison lemma.

Lemma 2. *Let $u \in HW^{1,p}(\Omega)$ be a weak solution to the equation (1) and let $w \in HW^{1,p}(B_{2R})$ be a weak solution to the equation (17). Assume that $2 - 1/Q < p \leq 2$. Then the weak solution $v \in HW^{1,p}(B_R)$ to the equation (26) satisfies*

$$\begin{aligned}
\fint_{B_R} |Xu - Xv|dx \leq & cM_{2R}^{\frac{2}{p}} + cM_{2R}^{\frac{3Q-Qp-2}{Q-p}} + cM_{2R}\left(\fint_{B_{2R}} (|Xu| + s)dx \right)^{\frac{2-p}{2}} \\
& + cM_{2R}\left(\fint_{B_{2R}} (|Xu| + s)dx \right)^{\frac{(Q-1)(2-p)}{3Q-Qp-2}} \\
& + cR^{\alpha} \fint_{B_{2R}} (|Xu| + s)dx, \qquad (27)
\end{aligned}$$

where $c = c(n, p, L, L') > 0$.

Proof of Lemma 2. By [7, Theorem 6.1] and the condition (2), we have

$$\int_{B_R} |Xv|^p dx \leq c_1 \int_{B_R} (|Xw| + s)^p dx, \qquad (28)$$

where $c_1 = c_1(n, p, L) \geq 1$. Here in the proof of [7, Theorem 6.1], only the condition (2) and Sobolev inequality are used, and therefore [7, Theorem 6.1] can also be used in the Heisenberg group.

Using sub-elliptic reverse Hölder's inequality and Gehring's lemma, see [18, Section 3], we have

$$\left(\fint_{B_R} (|Xw| + s)^p dx \right)^{\frac{1}{p}} \leq c\fint_{B_{2R}} (|Xw| + s)dx. \qquad (29)$$

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Using (9) and (10), the fact that both v and w are weak solutions and $v - w \in HW_0^{1,p}(B_R)$, we have

$$\begin{aligned}
&\int_{B_R} (|Xv|^2 + |Xw|^2 + s^2)^{\frac{p-2}{2}} |Xw - Xv|^2 dx \\
&\leq c \int_{B_R} |V(Xw) - V(Xv)|^2 dx \\
&\leq c \int_{B_R} \langle a(x_0, Xw) - a(x_0, Xv), Xw - Xv \rangle dx \\
&= c \int_{B_R} \langle a(x_0, Xw) - a(x, Xw), Xw - Xv \rangle dx,
\end{aligned}$$

which, together with condition (3), yields

$$\begin{aligned}
&\int_{B_R} (|Xv|^2 + |Xw|^2 + s^2)^{\frac{p-2}{2}} |Xw - Xv|^2 dx \\
&\leq cR^\alpha \int_{B_R} (|Xw|^2 + s^2)^{\frac{p-1}{2}} |Xw - Xv| dx \\
&\leq cR^\alpha \int_{B_R} (|Xv|^2 + |Xw|^2 + s^2)^{\frac{p-1}{2}} |Xw - Xv| dx.
\end{aligned}$$

By Young's inequality, we have

$$\begin{aligned}
&\int_{B_R} (|Xv|^2 + |Xw|^2 + s^2)^{\frac{p-2}{2}} |Xw - Xv|^2 dx \\
&\leq cR^{2\alpha} \int_{B_R} (|Xv|^2 + |Xw|^2 + s^2)^{\frac{p}{2}} |Xw - Xv|^2 dx.
\end{aligned}$$

This and (9) imply

$$\int_{B_R} |V(Xw) - V(Xv)|^2 dx \leq cR^{2\alpha} \int_{B_R} (|Xv|^2 + |Xw|^2 + s^2)^{\frac{p}{2}} dx.$$

Combining this and (28), we have

$$\int_{B_R} |V(Xw) - V(Xv)|^2 dx \leq cR^{2\alpha} \int_{B_R} (|Xw| + s)^p dx. \tag{30}$$

Similarly to (23), we have

$$|Xu - Xw|^p \leq c|V(Xw) - V(Xv)|^p (|Xv|^2 + |Xw|^2 + s^2)^{\frac{p(2-p)}{4}}.$$

From this, by Hölder's inequality, (28) and (30), we have

$$\begin{aligned}
&⨍_{B_R} |Xu - Xw|^p dx \\
&\leq c \left(⨍_{B_R} |V(Xw) - V(Xv)|^2 dx \right)^{\frac{p}{2}} \left(⨍_{B_R} (|Xv|^2 + |Xw|^2 + s^2)^{\frac{p}{2}} dx \right)^{\frac{2-p}{2}} \\
&\leq cR^{p\alpha} ⨍_{B_R} (|Xw| + s)^p dx. \qquad (31)
\end{aligned}$$

By Hölder's inequality, (31) and (29), we have

$$\begin{aligned}\fint_{B_R}|Xu-Xw|dx &\le c\left(\fint_{B_R}|Xu-Xw|^p dx\right)^{\frac{1}{p}}\\ &\le cR^\alpha\left(\fint_{B_R}(|Xw|+s)^p dx\right)^{\frac{1}{p}}\\ &\le cR^\alpha\fint_{B_{2R}}(|Xw|+s)dx. \end{aligned}\tag{32}$$

Using (18) in Lemma 1 and (32), we have

$$\begin{aligned}\fint_{B_R}|Xu-Xv|dx =&\fint_{B_R}|Xu-Xw|dx+\fint_{B_R}|Xw-Xv|dx\\ \le& cM_{2R}^{\frac{2}{p}}+cM_{2R}^{\frac{3Q-Qp-2}{Q-p}}+cM_{2R}\left(\fint_{B_{2R}}(|Xu|+s)dx\right)^{\frac{2-p}{2}}\\ &+cM_{2R}\left(\fint_{B_{2R}}(|Xu|+s)dx\right)^{\frac{(Q-1)(2-p)}{3Q-Qp-2}}+cR^\alpha\fint_{B_{2R}}(|Xw|+s)dx. \end{aligned}\tag{33}$$

Noting that

$$\fint_{B_{2R}}(|Xw|+s)dx=\fint_{B_{2R}}|Xw-Xu|dx+\fint_{B_{2R}}(|Xu|+s)dx,$$

then using (18) in Lemma 1 to estimate the last integral in the hand side of (33), we conclude (27). Here we can choose R small enough such that $R^\alpha\le 1$.
□

Now we give the main lemma.

Lemma 3. *Let $u\in HW^{1,p}(\Omega)$ be a weak solution to the equation (1) and $2-1/Q<p\le 2$ and let $v\in HW^{1,p}(B_{\tilde{R}})$ be a weak solution to the equation (26) with $B_{\tilde{R}}\subset\Omega$. Then there exist $\beta=\beta(n,p,L)\in(0,1)$ and $c=c(n,p,L,L')>0$ such that, for every $0<\rho<R<\tilde{R}$, we have*

$$\begin{aligned}&\fint_{B_\rho}|Xu-(Xu)_{B_\rho}|dx\\ &\le c\left(\frac{\rho}{R}\right)^\beta\fint_{B_{2R}}|Xu-(Xu)_{B_{2R}}|dx\\ &\quad+c\left(\frac{R}{\rho}\right)^Q\left[M_{2R}^{\frac{2}{p}}+M_{2R}^{\frac{3Q-Qp-2}{Q-p}}+M_{2R}\left(\fint_{B_{2R}}(|Xu|+s)dx\right)^{\frac{2-p}{2}}\right.\\ &\quad+M_{2R}\left(\fint_{B_{2R}}(|Xu|+s)dx\right)^{\frac{(Q-1)(2-p)}{3Q-Qp-2}}\\ &\quad\left.+R^\alpha\fint_{B_{2R}}(|Xu|+s)dx\right]+c\left(\frac{\rho}{\tilde{R}}\right)^\beta\sup_{B_{\tilde{R}}}|Xv|. \end{aligned}\tag{34}$$

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Proof of Lemma 3. By Proposition 1 with $r = R$ and $r_0 = \tilde{R}$, we have

$$\begin{aligned}
&\fint_{B_\rho} |Xv - (Xv)_{B_\rho}|dx \\
&\le c\left(\frac{\rho}{R}\right)^\beta \left[\fint_{B_R} |Xv - (Xv)_{B_R}|dx + \sup_{B_{5R/4}} |Xv|\right] \\
&\le c\left(\frac{\rho}{R}\right)^\beta \left[\fint_{B_R} |Xu - (Xu)_{B_R}|dx + 2\fint_{B_R} |Xu - Xv|dx\right] + c\left(\frac{\rho}{\tilde{R}}\right)^\beta \sup_{B_{\tilde{R}}} |Xv|.
\end{aligned}$$

Noting that

$$\fint_{B_\rho} |Xu - Xv|dx \le c\left(\frac{R}{\rho}\right)^Q \fint_{B_R} |Xu - Xv|dx,$$

we have

$$\begin{aligned}
\fint_{B_\rho} |Xu - (Xu)_{B_\rho}|dx \le& \fint_{B_\rho} |Xv - (Xv)_{B_\rho}|dx + 2\fint_{B_\rho} |Xu - Xv|dx \\
\le& c\left(\frac{\rho}{R}\right)^\beta \fint_{B_R} |Xu - (Xu)_{B_R}|dx + c\left(\frac{R}{\rho}\right)^Q \fint_{B_R} |Xu - Xv|dx \\
&+ c\left(\frac{\rho}{\tilde{R}}\right)^\beta \sup_{B_{\tilde{R}}} |Xv|.
\end{aligned}$$

Finally, using the inequality

$$\fint_{B_R} |Xu - (Xu)_{B_R}|dx \le 2^{Q+1}\fint_{B_{2R}} |Xu - (Xu)_{B_{2R}}|dx$$

and Lemma 2, we conclude (34).
□

4. Proof of Theorem 1

In this section, we prove Theorem 1. Fix $x_0 \in \mathbb{H}^n$ and denote $B_R := B(x_0, R)$. Assume that $0 < R < \tilde{R} \le \bar{R} = \bar{R}(n, p, L, L', \alpha, \text{dist}(x_0, \partial\Omega))$. For any $H > \tilde{H} > 1$ and $i \in \{0, 1, 2, \dots\}$, we denote $R_i = R/(2H)^i$, $\tilde{R}_i = 5R/[4(2\tilde{H})^i]$, $B_i := B_{R_i}$, $k_i := |(Xu)_{B_i}|$, $A_i := \fint_{B_i} |Xu - (Xu)_{B_i}|dx$ and $M_i := M_{R_i}$. Then

$$\begin{aligned}
k_{m+1} &= \sum_{i=0}^m (k_{i+1} - k_i) + k_0 \\
&\le \sum_{i=0}^m \fint_{B_{i+1}} |Xu - (Xu)_{B_i}|dx + k_0 \\
&\le (2H)^Q \sum_{i=0}^m A_i + k_0. \qquad (35)
\end{aligned}$$

Lemma 4. *Let $u \in HW^{1,p}(\Omega)$ be a weak solution to the equation (1) and $2 - 1/Q < p \le 2$, and let $v \in HW^{1,p}(B_R)$ be a weak solution to the equation (26). Assume that there exists an integer $\tilde{m} \in \mathbb{N} \cup \{\infty\}$ such that $\tilde{m} \ge 1$ and*

$$\fint_{B_i} |Xu|dx \le |Xu(x_0)| \qquad (36)$$

holds whenever $0 \le i \le \tilde{m} - 1$. Then for every $\epsilon \in (0, 1)$, there exists a constant $\tilde{c} = \tilde{c}(\epsilon) \ge 1$ such that

$$k_m \le 2c_4\mathcal{M} + 2c_3\epsilon|Xu(x_0)| \qquad (37)$$

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holds whenever $m \leq \tilde{m}+1$, where $c_3, c_4 \geq 1$ and

$$\mathcal{M} := \fint_{B_R} (|Xu|+s)dx + (1+c_3\tilde{c}(\epsilon))\left\{[\mathbf{I}_1^{|\mu|}(x_0,2R)]^{\frac{2}{p}} + [\mathbf{I}_1^{|\mu|}(x_0,2R)]^{\frac{3Q-Qp-2}{Q-p}}\right\} + \sup_{B_{5R/8}} |Xv|. \tag{38}$$

Proof of Lemma 4. By Lemma 3 with $0 < R/2H < R/2 < \tilde{R}/2$, we have

$$\begin{aligned} &\fint_{B_{R/2H}} |Xu-(Xu)_{B_{R/2H}}|dx \\ &\leq \frac{1}{4}\fint_{B_R} |Xu-(Xu)_{B_R}|dx \\ &\quad + cM_R^{\frac{2}{p}} + cM_R^{\frac{3Q-Qp-2}{Q-p}} + cM_R\left(\fint_{B_R}(|Xu|+s)dx\right)^{\frac{2-p}{2}} \\ &\quad + cM_R\left(\fint_{B_R}(|Xu|+s)dx\right)^{\frac{(Q-1)(2-p)}{3Q-Qp-2}} \\ &\quad + cR^\alpha\fint_{B_R}(|Xu|+s)dx + \frac{1}{4}\left(\frac{R}{\tilde{R}}\right)^\beta \sup_{B_{\tilde{R}/2}}|Xv|. \end{aligned} \tag{39}$$

Here we choose $H = H(n,p,L) > 1$ large enough such that $c/H^\beta \leq 1/4$. Noting that

$$\fint_{B_R} |Xu|dx = \fint_{B_R} |Xu| - (Xu)_{B_R}dx + (Xu)_{B_R}$$

and choosing $\bar{R}$ small enough such that $c\bar{R} \leq 1/4$, we write (39) as

$$\begin{aligned} &\fint_{B_{R/2H}} |Xu-(Xu)_{B_{R/2H}}|dx \\ &\leq \frac{1}{2}\fint_{B_R} |Xu-(Xu)_{B_R}|dx \\ &\quad + cM_R^{\frac{2}{p}} + cM_R^{\frac{3Q-Qp-2}{Q-p}} + cM_R\left(\fint_{B_R}(|Xu|+s)dx\right)^{\frac{2-p}{2}} \\ &\quad + cM_R\left(\fint_{B_R}(|Xu|+s)dx\right)^{\frac{(Q-1)(2-p)}{3Q-Qp-2}} \\ &\quad + cR^\alpha((Xu)_{B_R}+s) + \frac{1}{4}\left(\frac{R}{\tilde{R}}\right)^\beta \sup_{B_{\tilde{R}/2}}|Xv|. \end{aligned} \tag{40}$$

By (40) with $R = R_{i-1}$ and $\tilde{R} = \tilde{R}_{i-1}$, we have

$$\begin{aligned} A_i \leq &\frac{1}{2}A_{i-1} + cM_{i-1}^{\frac{2}{p}} + cM_{i-1}^{\frac{3Q-Qp-2}{Q-p}} + cM_{i-1}\left(\fint_{B_{i-1}}(|Xu|+s)dx\right)^{\frac{2-p}{2}} \\ &+ cM_{i-1}\left(\fint_{B_{i-1}}(|Xu|+s)dx\right)^{\frac{(Q-1)(2-p)}{3Q-Qp-2}} + cR_{i-1}^\alpha(k_{i-1}+s) \\ &+ \frac{1}{4}\left(\frac{\tilde{H}}{H}\right)^{\beta(i-1)} \sup_{B_{\tilde{R}_{i-1}/2}} |Xv|. \end{aligned}$$

Summing up over $i \in \{1, \dots, m\}$ the above inequality and letting $\tilde{H} = H/2^{1/\beta}$, and the fact

$$\sup_{B_{\tilde{R}_{i-1}/2}} |Xv| \leq \sup_{B_{5R/8}} |Xv|,$$

we have

$$\begin{aligned}\sum_{i=1}^{m} A_i \leq & \frac{1}{2}\sum_{i=0}^{m-1} A_i + c\sum_{i=0}^{m-1}\left[M_i^{\frac{2}{p}} + M_i^{\frac{3Q-Qp-2}{Q-p}} + M_i\left(\fint_{B_i}(|Xu|+s)dx\right)^{\frac{2-p}{2}}\right.\\ &\left. + M_i\left(\fint_{B_i}(|Xu|+s)dx\right)^{\frac{(Q-1)(2-p)}{3Q-Qp-2}}\right] + c\sum_{i=0}^{m-1} R_i^{\alpha}(k_i+s)\\ & + c\sup_{B_{5R/8}}|Xv|,\end{aligned}$$

and therefore

$$\begin{aligned}\sum_{i=1}^{m} A_i \leq & A_0 + 2c\sum_{i=0}^{m-1}\left[M_i^{\frac{2}{p}} + M_i^{\frac{3Q-Qp-2}{Q-p}} + M_i\left(\fint_{B_i}(|Xu|+s)dx\right)^{\frac{2-p}{2}}\right.\\ &\left. + M_i\left(\fint_{B_i}(|Xu|+s)dx\right)^{\frac{(Q-1)(2-p)}{3Q-Qp-2}}\right] + 2c\sum_{i=0}^{m-1} R_i^{\alpha}(k_i+s)\\ & + 2c\sup_{B_{5R/8}}|Xv|. \end{aligned} \tag{41}$$

Combining (35) and (41), we have

$$\begin{aligned}k_{m+1} \leq & cA_0 + k_0 + c\sum_{i=0}^{m-1}\left[M_i^{\frac{2}{p}} + M_i^{\frac{3Q-Qp-2}{Q-p}} + M_i\left(\fint_{B_i}(|Xu|+s)dx\right)^{\frac{2-p}{2}}\right.\\ &\left. + M_i\left(\fint_{B_i}(|Xu|+s)dx\right)^{\frac{(Q-1)(2-p)}{3Q-Qp-2}}\right] + c\sum_{i=0}^{m-1} R_i^{\alpha}(k_i+s)\\ & + c\sup_{B_{5R/8}}|Xv|. \end{aligned} \tag{42}$$

By (36) and (42), whenever $1 \leq m \leq \tilde{m}$, we have

$$\begin{aligned}k_{m+1} \leq & c\left(A_0 + k_0 + \sum_{i=0}^{m-1}\left[M_i^{\frac{2}{p}} + M_i^{\frac{3Q-Qp-2}{Q-p}}\right]\right)\\ & + c\left(|Xu(x_0)|^{\frac{2-p}{2}} + s^{\frac{2-p}{2}} + |Xu(x_0)|^{\frac{(Q-1)(2-p)}{3Q-Qp-2}} + s^{\frac{(Q-1)(2-p)}{3Q-Qp-2}}\right)\sum_{i=0}^{m-1} M_i\\ & + c\sum_{i=0}^{m-1} R_i^{\alpha}(k_i+s) + c\sup_{B_{5R/8}}|Xv|. \end{aligned} \tag{43}$$

Note that

$$\begin{aligned}\sum_{i=0}^{\infty} M_i \leq & \sum_{i=0}^{\infty}\frac{|\mu|(B_i)}{R_i^{Q-1}}\\ \leq & \frac{2^Q-1}{\log 2}\int_R^{2R}\frac{|\mu|(B(x_0,\rho))}{\rho^{Q-1}}\frac{d\rho}{\rho} + \frac{(2H)^{Q-1}}{\log 2H}\sum_{i=0}^{\infty}\int_{R_{i+1}}^{R_i}\frac{|\mu|(B(x_0,\rho))}{\rho^{Q-1}}\frac{d\rho}{\rho}\\ \leq & c(H)\mathbf{I}_1^{|\mu|}(x_0, 2R),\end{aligned}$$

the fact that $1 < p \leq 2$ implies $2/p \geq 1$ and $(3Q - Qp - 2)/(Q - p) \geq 1$, and

$$\sum_{i=0}^{\infty} R_i^{\alpha} = R^{\alpha} \sum_{i=0}^{\infty} \frac{1}{(2H)^{\alpha i}} \leq \frac{R^{\alpha}}{1 - 1/(2H)^{\alpha}} \leq \frac{R^{\alpha}}{1 - 1/2^{\alpha}} =: d(R).$$

For $1 \leq m \leq \tilde{m}$, we write (43) as

$$\begin{aligned} k_{m+1} \leq & c\left(A_0 + k_0 + [\mathbf{I}_1^{|\mu|}(x_0, 2R)]^{\frac{2}{p}} + [\mathbf{I}_1^{|\mu|}(x_0, 2R)]^{\frac{3Q-Qp-2}{Q-p}} \right) \\ & + c_3\left(|Xu(x_0)|^{\frac{2-p}{2}} + s^{\frac{2-p}{2}} + |Xu(x_0)|^{\frac{(Q-1)(2-p)}{3Q-Qp-2}} + s^{\frac{(Q-1)(2-p)}{3Q-Qp-2}} \right) \mathbf{I}_1^{|\mu|}(x_0, 2R) \\ & + c \sum_{i=0}^{m-1} R_i^{\alpha}(k_i + s) + c \sup_{B_{5R/8}} |Xv|. \end{aligned} \tag{44}$$

By Young's inequality, we have

$$\mathbf{I}_1^{|\mu|}(x_0, 2R)|Xu(x_0)|^{\frac{2-p}{2}} \leq \frac{\epsilon}{2}|Xu(x_0)| + \tilde{c}(\epsilon)[\mathbf{I}_1^{|\mu|}(x_0, 2R)]^{\frac{2}{p}}$$

and

$$\mathbf{I}_1^{|\mu|}(x_0, 2R)|Xu(x_0)|^{\frac{(Q-1)(2-p)}{3Q-Qp-2}} \leq \frac{\epsilon}{2}|Xu(x_0)| + \tilde{c}(\epsilon)[\mathbf{I}_1^{|\mu|}(x_0, 2R)]^{\frac{3Q-Qp-2}{Q-p}},$$

which, together with (44), yield

$$k_{m+1} \leq c_4\mathcal{M} + c_5 \sum_{i=0}^{m-1} R_i^{\alpha} k_i + c_3\epsilon|Xu(x_0)| \tag{45}$$

where $\mathcal{M}$ is as in (38). Here we choose $\bar{R}$ small enough such that $d(\bar{R}) \leq 1$.

Now we prove that the inequality

$$k_i \leq 2c_4\mathcal{M} + 2c_3\epsilon|Xu(x_0)| \tag{46}$$

holds for every $0 \leq i \leq \tilde{m} + 1$. When $i = 0$ and $i = 1$, we have

$$A_0 + k_0 + d(R)s \leq 3 \fint_{B_R} (|Xu| + s)dx$$

and

$$k_1 \leq 2^Q H^Q \fint_{B_R} |Xu|dx.$$

When $1 \leq i \leq \tilde{m} + 1$, we assume that (46) holds for every $i \leq m$ with $1 \leq m \leq \tilde{m}$, and prove it for $m + 1$. By using (45) and the assumption (46) for $i \leq m - 1$, we have

$$\begin{aligned} k_{m+1} \leq & c_4\mathcal{M} + c_5 \sum_{i=0}^{m-1} R_i^{\alpha}(2c_4\mathcal{M} + 2c_3\epsilon|Xu(x_0)|) + c_3\epsilon|Xu(x_0)| \\ = & [c_4 + 2c_4c_5d(R)]\mathcal{M} + [2c_3c_5d(R) + c_3]\epsilon|Xu(x_0)| \\ \leq & 2c_4\mathcal{M} + 2c_3\epsilon|Xu(x_0)|. \end{aligned}$$

Here we choose $\bar{R}$ small enough such that

$$d(\bar{R}) \leq \min\{1/(100c_3), 1/(100c_4), 1/(100c_5)\}.$$

We complete the proof.

□

Now we prove Theorem 1.

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Proof of Theorem 1. Define the set

$$\mathbb{S} := \left\{ i \in \mathbb{N} : |Xu(x_0)| \geq \fint_{B_i} |Xu| dx \right\},$$

and consider two cases: $\mathbb{S} = \mathbb{N}$ and $\mathbb{S} \neq \mathbb{N}$.

Case 1. When $\mathbb{S} = \mathbb{N}$, for every $i \in \mathbb{N}$, we have

$$\fint_{B_i} |Xu| dx \leq |Xu(x_0)|.$$

Using Lemma 4 with $\tilde{m} = \infty$, then letting $m \to \infty$, we have

$$|Xu(x_0)| = \lim_{m \to \infty} k_m \leq 2c_4 \mathcal{M} + 2c_3 \epsilon |Xu(x_0)|. \tag{47}$$

Choosing $\epsilon = 1/(4c_3)$, we have

$$|Xu(x_0)| \leq 4c_4 \mathcal{M}.$$

On the other hand, to estimate the last integral in $\mathcal{M}$, using (14) with $\sigma = 5/8$ and $q = 1$, we have

$$\begin{aligned} \sup_{B_{5R/8}} |Xv| \leq & c \fint_{B_R} (|Xv| + s) dx \\ \leq & c \fint_{B_R} (|Xu| + s) dx + c \fint_{B_R} |Xu - Xv| dx, \end{aligned}$$

from which, using Lemma 2 and Young's inequality, we have

$$\sup_{B_{5R/8}} |Xv| \leq c M_{2R}^{\frac{2}{p}} + c M_{2R}^{\frac{3Q - Qp - 2}{Q - p}} + c \fint_{B_{2R}} (|Xu| + s) dx. \tag{48}$$

Combining (47) and (48), we conclude (6) in the case.

Case 2. When $\mathbb{S} \neq \mathbb{N}$, we let $\tilde{m} := \min(\mathbb{N} \setminus \mathbb{S}) \geq 0$ and obtain

$$|Xu(x_0)| < \fint_{B_{\tilde{m}}} |Xu| dx, \tag{49}$$

and

$$\fint_{B_i} |Xu| dx < |Xu(x_0)| \tag{50}$$

for every $0 \leq i \leq \tilde{m} - 1$. When $\tilde{m} = 0$, we have $|Xu(x_0)| < (|Xu|)_{b_0}$, and therefore (6) holds true. When $\tilde{m} \geq 1$, the inequality (49) implies

$$|Xu(x_0)| < \fint_{B_{\tilde{m}}} |Xu| dx \leq \fint_{B_{\tilde{m}}} |Xu - (Xu)_{B_{\tilde{m}}}| dx + |(Xu)_{B_{\tilde{m}}}| = A_{\tilde{m}} + k_{\tilde{m}}. \tag{51}$$

Using (50) and Lemma 4, we have

$$k_{\tilde{m}} \leq 2c_4 \mathcal{M} + 2c_3 \epsilon |Xu(x_0)|. \tag{52}$$

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Since (50) satisfies the assumption (36), then combining (41) and (37), we have

$$
\begin{aligned}
A_{\tilde{m}} \leq & A_0 + 2c \sum_{i=0}^{\tilde{m}-1} \left[M_i^{\frac{2}{p}} + M_i^{\frac{3Q-Qp-2}{Q-p}} + M_i \left(\fint_{B_i} (|Xu| + s) dx \right)^{\frac{2-p}{2}} \right. \\
& \left. + M_i \left(\fint_{B_i} (|Xu| + s) dx \right)^{\frac{(Q-1)(2-p)}{3Q-Qp-2}} \right] + 2c \sum_{i=0}^{\tilde{m}-1} R_i^{\alpha} (2c_4 \mathcal{M} + 2c_3 \epsilon |Xu(x_0)| + s) \\
& + 2c \sup_{B_{5R/8}} |Xv|,
\end{aligned}
$$

from which, using (50) again, we have

$$
\begin{aligned}
A_{\tilde{m}} \leq & c \fint_{B_R} (|Xu| + s) dx + c \left[[\mathbf{I}_1^{|\mu|}(x_0, 2R)]^{\frac{2}{p}} + [\mathbf{I}_1^{|\mu|}(x_0, 2R)]^{\frac{3Q-Qp-2}{Q-p}} \right] \\
& + \mathbf{I}_1^{|\mu|}(x_0, 2R) \left[|Xu(x_0)|^{\frac{2-p}{2}} + s^{\frac{2-p}{2}} + |Xu(x_0)|^{\frac{(Q-1)(2-p)}{3Q-Qp-2}} + s^{\frac{(Q-1)(2-p)}{3Q-Qp-2}} \right] \\
& + c d(R)(2c_4 \mathcal{M} + 2c_3 \epsilon |Xu(x_0)| + s) + c \sup_{B_{5R/8}} |Xv|. \qquad (53)
\end{aligned}
$$

Estimating (53) as in (43)-(46) in the proof of Lemma 4, we have

$$
A_{\tilde{m}} \leq c\mathcal{M} + c\epsilon |Xu(x_0)|,
$$

which, together with (51), yields

$$
|Xu(x_0)| \leq c\mathcal{M} + c\epsilon |Xu(x_0)|.
$$

Choosing $\epsilon = 1/(2c)$, we have

$$
|Xu(x_0)| \leq 2c\mathcal{M}.
$$

Combining this and (48), we conclude (6) in the case.

Finally, we note that if $a(x,z)$ is independent of x then we can assume $L' = 0$ and therefore all items containing R^{α} disappear. Thus the proof holds for any $R > 0$ whenever $B_{2R} \subset \Omega$. We complete the proof.
□

5. Proof of Threorem 2

In this section, we prove Theorem 2. Fix $x_0 \in \mathbb{H}^n$ and denote $B_R := B(x_0, R)$. Assume that $0 < R < \bar{R} = \bar{R}(n, p, L, L', \alpha, \text{dist}(x_0, \partial\Omega))$. To prove Threorem 2, we need the following lemmas.

Lemma 5. *Let $u \in HW^{1,p}(\Omega)$ be a weak solution to the equation (1), $2 - 1/Q < p \leq 2$ and $B_{\bar{R}} \subset \Omega$. Then there exist $c = c(n, p, L, L') > 0$ such that, for every $0 < \rho \leq R \leq \bar{R}/2$, we have*

$$
\begin{aligned}
\fint_{B_\rho} (|Xu| + s) dx \leq & c \fint_{B_R} (|Xu| + s) dx \\
& + c \left(\frac{R}{\rho} \right)^Q \left[M_{2R}^{\frac{2}{p}} + M_{2R}^{\frac{3Q-Qp-2}{Q-p}} + M_{2R} \left(\fint_{B_{2R}} (|Xu| + s) dx \right)^{\frac{2-p}{2}} \right. \\
& \left. + M_{2R} \left(\fint_{B_{2R}} (|Xu| + s) dx \right)^{\frac{(Q-1)(2-p)}{3Q-Qp-2}} + R^{\alpha} \fint_{B_{2R}} (|Xu| + s) dx \right]. \qquad (54)
\end{aligned}
$$

Proof of Lemma 5. Letting $v \in HW^{1,p}(B_R)$ be a weak solution to the equation (26), we have

$$
\int_{B_\rho} (|Xu| + s) dx \leq \int_{B_\rho} (|Xv| + s) dx + \int_{B_\rho} |Xu - Xv| dx. \qquad (55)
$$

From (15), we have

$$\int_{B_\rho}(|Xv|+s)dx \le c\Big(\frac{\rho}{R}\Big)^Q\int_{B_R}(|Xv|+s)dx$$
$$\le c\Big(\frac{\rho}{R}\Big)^Q\int_{B_R}(|Xu|+s)dx + c\Big(\frac{\rho}{R}\Big)^Q\int_{B_R}|Xv-Xu|dx. \tag{56}$$

Combining (55) and (56), then using Lemma 2 and the inequality

$$\fint_{B_\rho}|Xu-Xv|dx \le c\Big(\frac{R}{\rho}\Big)^Q\fint_{B_R}|Xu-Xv|dx,$$

we conclude (54).
□

The following lemma is [12, Lemma 4.2].

Lemma 6. *Let $\phi:(0,\infty)\to[0,\infty)$ be a non-decreasing functions, $A>1$ and $\epsilon\ge 0$ be fixed constants. Let $\psi,\Phi:(0,\infty)\to[0,\infty)$ be functions such that $\sum_{j=0}^{\infty}\psi(t^j r)\le\Phi(r)$ for any $0<t<t_0<1$. Given any $a>0$, suppose that*

$$\phi(\rho)\le A\Big[\Big(\frac{\rho}{r}\Big)^a+\epsilon\Big]\phi(r)+r^a\psi(r) \tag{57}$$

holds for any $0<\rho<r\le R_0$, then there exists constants $\epsilon_0=\epsilon_0(A,a)>0$ and $c=c(A,a)>0$ such that if $\epsilon\le\epsilon_0$, then for all $0<\rho<r\le R_0$, we have

$$\phi(\rho)\le c\Big[\Big(\frac{\rho}{r}\Big)^{a-\bar\epsilon}\phi(r)+\rho^{a-\bar\epsilon}r^{\bar\epsilon}\Phi(r)\Big] \tag{58}$$

for any $0<\bar\epsilon<a$.

Based on Lemmas 5 and 6, we obtain the following proposition.

Proposition 2. *Let $u\in HW^{1,p}(\Omega)$ be a weak solution to the equation (1), $2-1/Q<p\le 2$ and $B_{\bar R}\subset\Omega$. Then there exist $c=c(n,p,L,L')>0$ such that, for any $0<\bar\epsilon<Q$ and $0<r<R\le\bar R$, we have*

$$\int_{B_r}(|Xu|+s)dx \le c\Big(\frac{r}{R}\Big)^{Q-\bar\epsilon}\Big[\int_{B_R}(|Xu|+s)dx$$
$$+R^Q\Big\{[\mathbf{I}_1^{|\mu|}(x_0,2R)]^{\frac{2}{p}}+[\mathbf{I}_1^{|\mu|}(x_0,2R)]^{\frac{3Q-Qp-2}{Q-p}}\Big\}\Big]. \tag{59}$$

Proof of Proposition 2. We fix $0<r<R\le\bar R$ and denote

$$\phi(r):=\int_{B_r}(|Xu|+s)dx.$$

By Lemma 5 with $\rho=r$ and $R\to R/2$, we have

$$\phi(r)\le c\Big(\frac{r}{R}\Big)^Q\phi(R)+cR^Q\Big[M_R^{\frac{2}{p}}+M_R^{\frac{3Q-Qp-2}{Q-p}}+M_R\Big(\fint_{B_R}(|Xu|+s)dx\Big)^{\frac{2-p}{2}}$$
$$+M_R\Big(\fint_{B_R}(|Xu|+s)dx\Big)^{\frac{(Q-1)(2-p)}{3Q-Qp-2}}\Big]+cR^\alpha\phi(R),$$

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which, together with Young's inequality, yields

$$\phi(r) \le c\left[\left(\frac{r}{R}\right)^Q + R^\alpha + \epsilon_1\right]\phi(R) + c\left(1+\frac{1}{\epsilon_1}\right)R^Q\left[M_R^{\frac{2}{p}} + M_R^{\frac{3Q-Qp-2}{Q-p}}\right].$$

Note that

$$\sum_{j=0}^{\infty} M_{t^j R} \le \mathbf{I}_1^{|\mu|}(x_0, 2R)$$

holds for any $t \in (0,1)$ and $R > 0$. Using Lemma 6 with $a = Q$, choosing $\bar{R}$ small enough such that $\bar{R}^\alpha < \epsilon_0(n,p,L)/2$ and letting $\epsilon_1 = \epsilon_0(n,p,L)/2$, we have

$$\phi(r) \le c\left[\left(\frac{r}{R}\right)^Q \phi(R) + r^{Q-\bar{\epsilon}}R^{\bar{\epsilon}}\left\{[\mathbf{I}_1^{|\mu|}(x_0,2R)]^{\frac{2}{p}} + [\mathbf{I}_1^{|\mu|}(x_0,2R)]^{\frac{3Q-Qp-2}{Q-p}}\right\}\right],$$

that is, (59). □

To obtain $C^{1,\gamma}_{\mathrm{loc}}$-regularity of u, we need the following lemma.

Lemma 7. *Let $u \in HW^{1,p}(\Omega)$ be a weak solution to the equation (1), $2 - 1/Q < p \le 2$ and $B_{\bar{R}} \subset \Omega$. Then there exist $\beta = \beta(n,p,L) \in (0,1)$ and $c = c(n,p,L,L') > 0$ such that, for every $0 < \rho < R < \bar{R}/2$, we have*

$$\begin{aligned}
&\fint_{B_\rho} |Xu - (Xu)_{B_\rho}| dx \\
&\le c\left(\frac{\rho}{R}\right)^\beta \fint_{B_R} (|Xu| + s)dx \\
&\quad + c\left(\frac{R}{\rho}\right)^Q\left[M_{2R}^{\frac{2}{p}} + M_{2R}^{\frac{3Q-Qp-2}{Q-p}} + M_{2R}\left(\fint_{B_{2R}}(|Xu|+s)dx\right)^{\frac{2-p}{2}}\right. \\
&\quad \left. + M_{2R}\left(\fint_{B_{2R}}(|Xu|+s)dx\right)^{\frac{(Q-1)(2-p)}{3Q-Qp-2}} + R^\alpha \fint_{B_{2R}}(|Xu|+s)dx\right]. \qquad (60)
\end{aligned}$$

Proof of Lemma 7. Letting $v \in HW^{1,p}(B_R)$ be a weak solution to the equation (26), we have

$$\begin{aligned}
\fint_{B_\rho} |Xu - (Xu)_{B_\rho}| dx &\le 2\fint_{B_\rho} |Xu - (Xv)_{B_\rho}| dx \\
&\le 2\fint_{B_\rho} |Xv - (Xv)_{B_\rho}| dx + 2\fint_{B_\rho} |Xu - Xv| dx.
\end{aligned}$$

By (13), we have

$$\begin{aligned}
\fint_{B_\rho} |Xv - (Xv)_{B_\rho}| dx &\le c\left(\frac{\rho}{R}\right)^\beta \fint_{B_R}(|Xv|+s)dx \\
&\le c\left(\frac{\rho}{R}\right)^\beta \fint_{B_R}(|Xu|+s)dx + c\left(\frac{\rho}{R}\right)^\beta \fint_{B_R}|Xv - Xu|dx.
\end{aligned}$$

Combining the above two inequalities, then using the inequality

$$\fint_{B_\rho} |Xu - Xv| dx \le c\left(\frac{R}{\rho}\right)^Q \fint_{B_R} |Xv - Xu| dx,$$

and Lemma 2, we conclude (60).
□

Now we prove Theorem 2.

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Proof of Theorem 2. Using Lemma 7 with $R = r/2$, we have

$$\begin{aligned}
&\int_{B_\rho} |Xu - (Xu)_{B_\rho}| dx \\
&\leq c\left(\frac{\rho}{r}\right)^{Q+\beta} \int_{B_r} (|Xu| + s) dx \\
&\quad + cr^Q \left[M_r^{\frac{2}{p}} + M_r^{\frac{3Q-Qp-2}{Q-p}} + M_r \left(\fint_{B_r} (|Xu| + s) dx \right)^{\frac{2-p}{2}} \right. \\
&\quad \left. + M_r \left(\fint_{B_r} (|Xu| + s) dx \right)^{\frac{(Q-1)(2-p)}{3Q-Qp-2}} \right] + r^\alpha \int_{B_r} (|Xu| + s) dx.
\end{aligned}$$

Using Young's inequality to estimate the second term in the hand side of the above inequality, then using Proposition 2, we have

$$\begin{aligned}
&\int_{B_\rho} |Xu - (Xu)_{B_\rho}| dx \\
&\leq c \frac{\rho^{Q+\beta} R^{\bar{\epsilon}}}{r^{\beta+\bar{\epsilon}} R^Q} \left[\int_{B_R} (|Xu| + s) dx + R^Q \left\{ [\mathbf{I}_1^{|\mu|}(x_0, 2R)]^{\frac{2}{p}} + [\mathbf{I}_1^{|\mu|}(x_0, 2R)]^{\frac{3Q-Qp-2}{Q-p}} \right\} \right] \\
&\quad + cr^Q \left[\left(1 + \frac{1}{\epsilon_2}\right) \left(M_r^{\frac{2}{p}} + M_r^{\frac{3Q-Qp-2}{Q-p}} \right) + (\epsilon_2 + r^\alpha) \fint_{B_r} (|Xu| + s) dx \right] \qquad (61)
\end{aligned}$$

for every $0 < \rho < r < R < \bar{R}$. Given $\mu = f \in L^q_{\text{loc}}(\Omega)$ for some $q > Q$, then by Hölder's inequality, we have

$$\frac{|\mu|(B_r)}{r^{Q-1}} = \frac{1}{r^{Q-1}} \int_{B_r} |f| dx \leq \frac{|B_r|^{1-1/q}}{r^{Q-1}} \left(\int_{B_r} |f|^q \right)^{1/q} \leq cr^{1-Q/q} \|f\|_{L^q(B_r)}$$

and therefore,

$$M_r^{\frac{2}{p}} \leq cr^{\left(1-\frac{Q}{q}\right)\frac{2}{p}} \|f\|_{L^q(B_r)}^{\frac{2}{p}}, \quad M_r^{\frac{3Q-Qp-2}{Q-p}} \leq cr^{\left(1-\frac{Q}{q}\right)\frac{3Q-Qp-2}{Q-p}} \|f\|_{L^q(B_r)}^{\frac{3Q-Qp-2}{Q-p}}.$$

Thus, by Proposition 2 and (61), we have

$$\begin{aligned}
&\int_{B_\rho} |Xu - (Xu)_{B_\rho}| dx \\
&\leq c \left[\frac{\rho^{Q+\beta} R^{\bar{\epsilon}}}{r^{\beta+\bar{\epsilon}}} + (r^\alpha + \epsilon_2) r^{Q-\bar{\epsilon}} R^{\bar{\epsilon}} \right] \\
&\quad \times \left[\fint_{B_R} (|Xu| + s) dx + [\mathbf{I}_1^{|\mu|}(x_0, 2R)]^{\frac{2}{p}} + [\mathbf{I}_1^{|\mu|}(x_0, 2R)]^{\frac{3Q-Qp-2}{Q-p}} \right] \\
&\quad + c\left(1 + \frac{1}{\epsilon_2}\right) r^{Q+\left(1-\frac{Q}{q}\right)\frac{2}{p}} \|f\|_{L^q(B_r)}^{\frac{2}{p}} + c\left(1 + \frac{1}{\epsilon_2}\right) r^{Q+\left(1-\frac{Q}{q}\right)\frac{3Q-Qp-2}{Q-p}} \|f\|_{L^q(B_r)}^{\frac{3Q-Qp-2}{Q-p}}
\end{aligned}$$

for every $0 < \rho < r < R < \bar{R}$. We choose $\delta, \bar{\epsilon}$ small enough such that

$$\delta + \bar{\epsilon} \leq \alpha, \quad 2\delta + \bar{\epsilon} \leq + \left(1 - \frac{Q}{q}\right)\frac{2}{p}, \quad Q\bar{\epsilon} < \beta\delta$$

and therefore,

$$\alpha + Q - \bar{\epsilon} \geq Q + \delta, \quad Q - \bar{\epsilon} - \delta + \left(1 - \frac{Q}{q}\right)\frac{2}{p} \geq Q + \delta, \quad \beta\delta - Q\bar{\epsilon} > 0.$$

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Here $1 < p \le 2$ implies $\frac{3Q-Qp-2}{Q-p} \ge \frac{2}{p}$. Thus, letting $\epsilon_2 = r^{\delta+\bar{\epsilon}}$, we have

$$\begin{aligned}
&\int_{B_\rho} |Xu - (Xu)_{B_\rho}| dx \\
&\le c\left[\frac{\rho^{Q+\beta}}{r^{\beta+\bar{\epsilon}}} + r^{Q+\delta}\right]\left[\fint_{B_R} (|Xu| + s) dx + [\mathbf{I}_1^{|\mu|}(x_0, 2R)]^{\frac{2}{p}} + [\mathbf{I}_1^{|\mu|}(x_0, 2R)]^{\frac{3Q-Qp-2}{Q-p}} \right.\\
&\quad \left. + \|f\|_{L^q(B_r)}^{\frac{2}{p}} + \|f\|_{L^q(B_r)}^{\frac{3Q-Qp-2}{Q-p}}\right]
\end{aligned}$$

for every $0 < \rho < r < R < \bar{R}$. Choosing $r = \rho^\kappa$ with some $\kappa \in (0,1)$, we rewrite the above inequality as

$$\begin{aligned}
&\int_{B_\rho} |Xu - (Xu)_{B_\rho}| dx \\
&\le c\left[\rho^{Q+(1-\kappa)\beta-\kappa\bar{\epsilon}} + \rho^{\kappa(Q+\delta)}\right] \\
&\quad \times \left[\fint_{B_R} (|Xu| + s) dx + [\mathbf{I}_1^{|\mu|}(x_0, 2R)]^{\frac{2}{p}} + [\mathbf{I}_1^{|\mu|}(x_0, 2R)]^{\frac{3Q-Qp-2}{Q-p}} \right.\\
&\quad \left. + \|f\|_{L^q(B_r)}^{\frac{2}{p}} + \|f\|_{L^q(B_r)}^{\frac{3Q-Qp-2}{Q-p}}\right] \\
&\le c\rho^{Q+\gamma}\left[\fint_{B_R} (|Xu| + s) dx + [\mathbf{I}_1^{|\mu|}(x_0, 2R)]^{\frac{2}{p}} + [\mathbf{I}_1^{|\mu|}(x_0, 2R)]^{\frac{3Q-Qp-2}{Q-p}} \right.\\
&\quad \left. + \|f\|_{L^q(B_r)}^{\frac{2}{p}} + \|f\|_{L^q(B_r)}^{\frac{3Q-Qp-2}{Q-p}}\right],
\end{aligned}$$

where the second inequality follows when $Q + \gamma \le \min\{Q + (1-\kappa)\beta - \kappa\bar{\epsilon}, \kappa(Q+\delta)\}$. Here we can make sure that this is true with the choice of $\kappa = \kappa(\gamma)$ such that

$$\frac{Q+\gamma}{Q+\delta} \le \kappa \le \frac{\beta-\gamma}{\beta+\bar{\epsilon}}$$

for any $0 < \gamma \le (\beta\delta - Q\bar{\epsilon})/(Q+\beta+\delta+\bar{\epsilon})$. Also, note that if $\gamma, \bar{\epsilon}$ are small enough, $\kappa = \kappa(\gamma)$ can be chosen close enough to 1 and we can make sure $\rho^\kappa < R$, whenever $0 < \rho < R$. Thus, we obtain

$$\begin{aligned}
&\fint_{B_\rho} |Xu - (Xu)_{B_\rho}| dx \\
&\le c\rho^{\gamma}\left[\fint_{B_R} (|Xu| + s) dx + [\mathbf{I}_1^{|\mu|}(x_0, 2R)]^{\frac{2}{p}} + [\mathbf{I}_1^{|\mu|}(x_0, 2R)]^{\frac{3Q-Qp-2}{Q-p}} \right.\\
&\quad \left. + \|f\|_{L^q(B_r)}^{\frac{2}{p}} + \|f\|_{L^q(B_r)}^{\frac{3Q-Qp-2}{Q-p}}\right]
\end{aligned}$$

for every $0 < \rho < R < \bar{R}$. We complete the proof.
□

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References

1. Capogna, L.; Danielli, D.; Pauls, S. D.; Tyson, J. T. An introduction to the Heisenberg group and the sub-Riemannian isoperimetric problem. In *Progress in Mathematics*; Publisher: Birkhäuser Verlag, Basel, 2007; pp. xvi+223.
2. Domokos, A.; Manfredi, J.J. $C^{1,\alpha}$-regularity for p-harmonic functions in the Heisenberg group for p near 2. *Contemp. Math.* **2005**, *370*, 17–23.
3. Domokos, A.; Manfredi, J.J. Subelliptic cordes estimates. *P. Am. Math. Soc.* **2005**, *133*, 1047–1056.
4. Duzaar, F.; Mingione, G. Gradient estimates via linear and nonlinear potentials. *J. Funct. Anal.* **2010**, *259*, 0022–1236.
5. Duzaar, F.; Mingione, G. Gradient estimates via non-linear potentials. *Amer. J. Math.* **2011**, *133*, 1093–1149.
6. Evans, L.C. A new proof of local $C^{1,\alpha}$-regularity for solutions of certain degenerate elliptic p.d.e. *J. Differ. Equations* **1982**, *45*, 356–373.
7. Giusti, E. Direct methods in the calculus of variations. In *World Scientific Publishing Co., Inc., River Edge*, 2003.
8. Hamburger, C. Regularity of differential forms minimizing degenerate elliptic functionals. *J. Reine Angew. Math.* **1992**, *431*, 7–64.
9. Lewis, J.L. Regularity of the derivatives of solutions to certain degenerate elliptic equations. *Indiana Univ. Math. J.* **1983**, *32*, 849–858.
10. Manfredi, J.J.; Mingione, G. Regularity results for quasilinear elliptic equations in the Heisenberg group. *Math. Ann.* **2007**, *339*, 485–544.
11. Mingione, G.; Zatorska-Goldstein, A.; Zhong, X. Gradient regularity for elliptic equations in the Heisenberg group. *Adv. Math.* **2009**, *222*, 62–129.
12. Mukherjee, S.; Sire, Y. Regularity of inhomogeneous quasi-linear equations on the Heisenberg group. *Anal. Theory Appl.* **2021**, *37*, 1672–4070.
13. Mukherjee, S.; Zhong, X. $C^{1,\alpha}$-regularity for variational problems in the Heisenberg group. *Anal. PDE* **2021**, *14*, 567–594.
14. Ricciotti, D. p-Laplace equation in the Heisenberg group. In *Springer International Publishing*; https://doi.org/10.1007/978-3-319-23790-9; 2015.
15. Tolksdorf, P. Regularity for a more general class of quasilinear elliptic equations. *J. Differ. Equations* **1984**, *51*, 126–150.
16. Uhlenbeck, K.K. Regularity for a class of non-linear elliptic systems. *Acta Math.* **1977**, *138*, 219–240.
17. Ural'ceva, N.N. Degenerate quasilinear elliptic systems. *Zap. Naučn. Sem. Leningrad. Otdel. Mat. Inst. Steklov.* **1968**, *7*, 184–222.
18. Zatorska-Goldstein, A. Very weak solutions of nonlinear subelliptic equations. *Ann. Acad. Sci. Fenn. Math.* **2005**, *30*, 407–436.
19. Zhong, X. Regularity for variational problems in the Heisenberg group. *Arxiv preprint: https://arxiv.org/abs/1711.03284* 2017.