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Article

The Born Rule as a Geometric Measure on Projective State Space—And an Octonionic Outlook

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Abstract

Presented is a geometric reformulation of the Born rule for finite-dimensional quantum systems ($n \geq 3$). The state space is identified with complex projective space $\mathbb{CP}^{n-1}$ equipped with its canonical Fubini–Study geometry. Three structural axioms — locality in projective distance, invariance under projective unitaries, and additivity/non-contextuality for orthogonal decompositions — are shown to reduce the problem of transition probabilities to the framework of Gleason’s theorem, thereby uniquely determining the transition probability $P([\psi] \rightarrow [\phi]) = |\langle \phi | \psi \rangle|^2$. The argument provides a transparent geometric reformulation and interpretation of Gleason’s theorem. I then show that any relativistic Dirac spinor theory automatically realizes this geometry locally, so the Born rule is inherited without further assumption. Finally, I formulate a precise conjecture for an octonionic analogue involving the exceptional group G_2 (or F_4/E_8), then illustrate the idea with a simple finite toy model derived from the seven imaginary octonions and the Fano plane.

Keywords: Born rule; Gleason theorem; projective geometry; Fubini–Study metric; quantum state space; octonions; exceptional lie groups; quantum foundations

Introduction

The Born rule is traditionally introduced as an independent probabilistic postulate linking the Hilbert-space description of quantum states to measurement statistics. A long-standing program in the foundations of quantum theory has been to derive it from more primitive physical or mathematical principles. The breakthrough result in this direction is Gleason’s 1957 theorem [1], which shows that any measure on the lattice of orthogonal projections in dimension $n \geq 3$ that is additive over orthogonal decompositions must be of the standard form $\mu(P) = \text{Tr}(\rho P)$. Numerous streamlined proofs and extensions have since appeared, including elementary proofs and extensions to generalized measurements [2,3].

Virtually all such derivations start from the linear and lattice structure of Hilbert space. In contrast, a geometric viewpoint takes the projective state space as primary: pure states form complex projective space $\mathbb{CP}^{n-1}$ equipped with its canonical Fubini–Study metric. This differential-geometric formulation goes back at least to Kibble and to Provost–Vallée and has been systematized in later expositions [4–7]. In this setting the projective-unitary symmetry acts isometrically, so any unitary-invariant transition probability between pure states can depend only on the Fubini–Study distance (equivalently the ray overlap). The standard Born rule is precisely the choice $P([\psi] \rightarrow [\phi]) = |\langle \phi | \psi \rangle|^2 = \cos^2 d_{\text{FS}}([\psi], [\phi])$.

The present work completes this programme by exhibiting a minimal set of purely geometric axioms — locality in the Fubini–Study metric, invariance under the full projective unitary group $\text{PU}(n)$, and additivity/non-contextuality over orthogonal decompositions — that uniquely determine the Born rule on $\mathbb{CP}^{n-1}$ (Theorem 1). The argument proceeds by showing that these axioms constrain any admissible probability measure to be a frame function on the lattice of projections, to which Gleason’s theorem applies. This yields a transparent geometric reformulation of Gleason’s theorem in which Hilbert-space linearity emerges from projective geometry and symmetry.

We then observe that any relativistic quantum field theory built on Dirac, Weyl, or Majorana spinors automatically realizes this geometry fibre-wise: at each spacetime point the local spinor fiber is canonically $\mathbb{CP}^3$ equipped with the Fubini–Study metric. Consequently, the geometric Born rule is inherited locally without further assumption, yielding a clean modular separation between an immutable kinematic probability geometry and arbitrary spinor dynamics (Corollary in Section).

Finally, we formulate a precise octonionic extension of the problem. Replacing complex projective space by an appropriate exceptional geometry with symmetry G_2 (or F_4/E_8) and imposing octonionic analogues of the same three geometric axioms leads naturally to a conjectural quadratic probability rule on the seven imaginary octonions that reduces to the standard Born rule on complex subalgebras (Conjecture 1). A simple seven-state toy model derived from the Fano plane and the finite subgroup $\Gamma \cong 2^3 \rtimes \text{GL}(3, 2)$ rigorously realizes this idea in a discrete setting (Appendix [A Finite Toy Model from the Seven Imaginary Octonions](#)).

Taken together, these results recast the origin of the Born rule as a geometric inevitability of highly symmetric state spaces, clarify its status in relativistic spinor theories, and open a concrete pathway toward exceptional-algebraic generalizations relevant to unification and quantum gravity.

Projective State Space as a Kähler Manifold

Let $\mathcal{H} \cong \mathbb{C}^n$ ($n \geq 3$) be a finite-dimensional complex Hilbert space with the standard Hermitian inner product $\langle \cdot | \cdot \rangle$. Pure quantum states are rays

$$[\psi] = \{e^{i\theta}\psi : \theta \in [0, 2\pi)\}, \quad \psi \in \mathcal{H} \setminus \{0\}.$$

The space of all rays is the complex projective space

$$\mathcal{S} = \mathbb{P}(\mathcal{H}) \cong \mathbb{CP}^{n-1}.$$

On $\mathcal{S}$ there exists a unique (up to scale) Kähler metric — the *Fubini–Study metric* — whose geodesic distance between rays (with unit-norm representatives) is

$$d_{\text{FS}}([\psi], [\phi]) = \arccos(|\langle \phi | \psi \rangle|) \in [0, \pi/2]. \quad (1)$$

The projective unitary group $\text{PU}(n) = U(n)/U(1)$ acts transitively and isometrically on $(\mathcal{S}, d_{\text{FS}})$, making it a homogeneous, isotropic Kähler manifold. The Fubini–Study metric determines a canonical volume form $d\mu_{\text{FS}}$ that is invariant under $\text{PU}(n)$; such an invariant measure is unique up to normalization.

Geometric Axioms for Transition Probabilities

We impose three structural axioms on transition probabilities $P : \mathcal{S} \times \mathcal{S} \rightarrow [0, 1]$.

Assumption 1 (Locality in projective geometry). *There exists a continuous function $f : [0, \pi/2] \rightarrow [0, 1]$ with $f(0) = 1$ and $f(\pi/2) = 0$ such that*

$$P([\psi] \rightarrow [\phi]) = f(d_{\text{FS}}([\psi], [\phi])). \quad (2)$$

Thus the transition probability depends only on the Fubini–Study distance (the projective angle) between the two rays and on no additional geometric data (e.g. relative phase).

Assumption 2 (Symmetry invariance).

$$P([\psi] \rightarrow [\phi]) = P(U[\psi] \rightarrow U[\phi]) \quad \forall U \in \text{PU}(n). \quad (3)$$

Assumption 3 (Additivity and non-contextuality). For any orthonormal basis $\{|\phi_i\rangle\}_{i=1}^n$ of $\mathcal{H}$,

$$\sum_{i=1}^n P([\psi] \rightarrow [\phi_i]) = 1 \quad \forall [\psi] \in \mathcal{S}, \quad (4)$$

and the probability assigned to each rank-one subspace depends only on that subspace itself (non-contextuality).

Geometric Uniqueness of the Born Rule

Theorem 1. Any assignment of transition probabilities on $\mathcal{S} = \mathbb{CP}^{n-1}$ ($n \geq 3$) satisfying Assumptions 1–3 must be of the form

$$P([\psi] \rightarrow [\phi]) = |\langle \phi | \psi \rangle|^2, \quad (5)$$

where $|\psi\rangle, |\phi\rangle$ are arbitrary unit-norm representatives.

Proof. Assumption 3 implies that the map sending a rank-one projector $P_i = |\phi_i\rangle\langle\phi_i|$ to $P([\psi] \rightarrow [\phi_i])$ extends to a frame function on the lattice of orthogonal projections. By Gleason's theorem [1], every such frame function is of the form $\text{Tr}(W_\psi P)$ for a unique positive trace-one operator W_ψ .

Assumption 2 implies that if $U \in \text{PU}(n)$ stabilizes $[\psi]$ (i.e. $U|\psi\rangle = e^{i\theta}|\psi\rangle$), then W_ψ must commute with the full stabilizer subgroup $U(1) \times U(n-1)$. The only positive trace-one operator with this property is the rank-one projector $W_\psi = |\psi\rangle\langle\psi|$.

Locality (A1) and the explicit form of the Fubini–Study distance (1) then force the functional dependence to be quadratic: $f(t) = \cos^2 t$ (see [3] for details). Normalization (A3) fixes the constant, yielding (5). $\square$

Spinor Realization of the Geometric State Space

In relativistic quantum field theory, the basic microscopic degree of freedom is a Dirac spinor field $\Psi(x)$, a section of the spinor bundle over spacetime. At each fixed spacetime point x , the value $\Psi(x)$ belongs to a 4-complex-dimensional fibre $\mathcal{H}_x \cong \mathbb{C}^4$ (8 real dimensions in a Majorana basis).

The physical Hilbert space consists of positive-frequency solutions (or, equivalently, of modes selected by a future-directed timelike vector field that defines a positive-definite inner product). With this standard restriction the Dirac inner product

$$\langle \Psi | \Phi \rangle_x = \bar{\Psi}(x) \Phi(x) = \Psi^\dagger(x) \gamma^0 \Phi(x)$$

is positive-definite on the relevant subspace of the local fibre.

Lemma 1. For any fixed x and with the positive-frequency restriction, the map

$$\pi_x : \Psi(x) \longmapsto [\Psi(x)] \in \mathbb{CP}^3 \quad (6)$$

is well-defined (independent of global phase) and equips the space of local spinor values with the structure of complex projective space $\mathbb{CP}^3$ endowed with the canonical Fubini–Study metric.

Proof. The inner product on the positive-frequency subspace of $\mathcal{H}_x$ is positive-definite and Hermitian. Rays are global phase classes, and the Fubini–Study distance between $[\Psi(x)]$ and $[\Phi(x)]$ is

$$d_{\text{FS}}([\Psi(x)], [\Phi(x)]) = \arccos(|\langle \Phi(x) | \Psi(x) \rangle|). \quad (7)$$

$\square$

Thus Theorem 1 applies directly to local spinor states. The unique transition-probability law compatible with the projective geometry at each spacetime point is

$$P([\Psi(x)] \rightarrow [\Phi(x)]) = \frac{|\langle \Phi(x) | \Psi(x) \rangle|^2}{\langle \Psi(x) | \Psi(x) \rangle \langle \Phi(x) | \Phi(x) \rangle} = \frac{|\langle \Phi | \Psi \rangle|^2}{|\Psi|^2 |\Phi|^2} \Big|_x. \quad (8)$$

When Ψ and Φ are normalized positive-frequency solutions, this coincides with the usual Born rule.

Corollary 1. *Any relativistic quantum field theory whose microscopic degrees of freedom are Dirac (or Weyl/Majorana) spinors automatically inherits the geometric Born rule of Theorem 1 as the unique probability assignment on the local projective state space $\mathbb{CP}^3$.*

The dynamics — be it the standard Dirac equation, Einstein–Cartan theory with torsion, or any other generally covariant spinor dynamics — determines how the local rays $[\Psi(x)]$ evolve and which global states are physically realized, but it not alter the kinematic probability rule on the local state space. In particular:

- The conserved Dirac current $j^\mu = \bar{\Psi} \gamma^\mu \Psi$ is a derived object; the geometric Born rule singles out the canonically normalized scalar density $|\Psi|^2 = \bar{\Psi} \Psi$.
- Modifications such as contorsion or torsion affect the Dirac Hamiltonian and therefore the time evolution of $[\Psi(x)]$, but the probability interpretation at each instant remains fixed by Theorem 1.

This clean separation between *kinematic probability geometry* (fixed once and for all by Theorem 1) and *relativistic spinor dynamics* is the modular structure exploited throughout the remainder of the paper.

Beyond Complex Numbers: An Octonionic Outlook

The complex projective construction of Sections and relies crucially on the field of complex numbers. A natural question is whether a similar geometric uniqueness result holds when the underlying division algebra is enlarged to the quaternions $\mathbb{H}$ or octonions $\mathbb{O}$.

For quaternions, the analogue of $\mathbb{CP}^{n-1}$ is quaternionic projective space $\mathbb{HP}^{n-1}$, with an isometry group of the form $\mathrm{Sp}(n)\mathrm{Sp}(1)/\mathbb{Z}_2$. Gleason-type theorems have been extended to this setting [8–10], and the standard quadratic Born rule survives, albeit with right-linear (quaternionic) Hilbert spaces.

For octonions the situation is far more subtle because of non-associativity. Nevertheless, one can still define octonionic projective spaces — notably the Cayley projective plane $\mathbb{OP}^2$, a 16-real-dimensional manifold with exceptional symmetry group F_4 — and consider G_2 -invariant geometric structures on the unit sphere of imaginary octonions. We close with a precise conjecture that extends Theorem 1 to this non-associative setting.

Conjecture 1 (Octonionic geometric Born rule). *Let $\mathcal{S}$ be an appropriate octonionic analogue of projective Hilbert space (for example, a space of normed octonionic spinors modulo the action of G_2 , or a suitable discretization via a finite subgroup such as $\Gamma \simeq 2^3 \rtimes \mathrm{GL}(3, 2)$). Assume octonionic versions of Axioms (A1)–(A3):*

- (O1) *Locality with respect to a G_2 -invariant distance on $\mathcal{S}$;*
- (O2) *Invariance under the full automorphism group G_2 (or an extended exceptional group such as E_8 in a larger construction);*
- (O3) *Additivity/non-contextuality over orthogonal decompositions defined by associative subalgebras $\mathbb{O} \supset \mathbb{H} \supset \mathbb{C}$.*

Then the natural candidate for a probability measure on $\mathcal{S}$ is quadratic in the octonionic norm:

$$P(o_1 \rightarrow o_2) \propto |o_1 \bar{o}_2|^2, \quad (9)$$

or, more generally, a suitable G_2 -invariant average thereof. When restricted to complex subalgebras, this measure reduces to the standard complex Born rule.

We do not attempt a rigorous proof of Conjecture 1 here. The plausibility rests on three observations:

- The only G_2 -invariant norm on $\mathbb{O}$ is quadratic.
- Restriction to complex slices reproduces ordinary quantum mechanics and the geometric Born rule of Theorem 1.
- Finite-orbit discretizations under subgroups such as $\Gamma \simeq 2^3 \rtimes \text{GL}(3, 2)$ yield a discrete, exactly computable setting in which one may hope to prove a Gleason-type uniqueness theorem using uniform measures weighted by quadratic norms (see Appendix A [Finite Toy Model from the Seven Imaginary Octonions](#) for a simple illustration).

An octonionic–spinor state space of this type would provide a single geometric arena in which both the standard Born rule (on complex subalgebras) and potential “beyond-complex” quantum phenomena (non-associative interference, symmetry-breaking collapse mechanisms) could naturally coexist.

Discussion

The starting question of this work was whether the Born rule can be understood as a *geometric inevitability* of the quantum state space, rather than an extra probabilistic postulate. Our main result (Theorem 1) is a reformulation of Gleason’s theorem in which the only primitive structure is the Kähler geometry of complex projective space: the Fubini–Study metric and its projective unitary isometries. On $\mathbb{CP}^{n-1}$ imposed are three explicitly geometric axioms — local dependence on the Fubini–Study distance, invariance under the full projective unitary group, and additivity/non-contextuality over orthogonal decompositions — and show that together they single out the Born rule as the unique probability assignment.

In contrast to the standard Hilbert-space presentation of Gleason’s theorem, where one starts from inner products and frame functions, our derivation treats the Fubini–Study geometry and $\text{PU}(n)$ symmetry as primary. The Hilbert space, inner product and density operators re-enter only at the end as a convenient representation of the unique $\text{PU}(n)$ -invariant, additive measure on $\mathbb{CP}^{n-1}$. This repackaging is not a replacement for Gleason’s original theorem, but a complementary viewpoint that is well-suited to settings where geometry and symmetry are more fundamental than linear structure, such as spinor bundles, curved backgrounds, or division-algebraic generalizations.

Section shows that this geometric Born rule is inherited automatically by relativistic spinor theories. At each spacetime point, the fibre of a Dirac spinor bundle is a copy of $\mathbb{C}^4$, and the associated local space of rays is canonically $\mathbb{CP}^3$ with its Fubini–Study metric. Theorem 1 therefore applies directly to local spinor values: any theory whose microscopic degrees of freedom are Dirac/Weyl/Majorana spinors comes equipped with a unique kinematic probability rule on this local projective state space. The Dirac or Einstein–Cartan dynamics then governs how these rays evolve in time, but does not modify the underlying geometric probability structure. This clean separation between *probability geometry* and *field dynamics* is, I believe, conceptually clarifying for relativistic quantum theories.

In Section we take a first step beyond complex numbers. For quaternionic quantum theory the projective geometry extends to $\mathbb{HP}^{n-1}$, where Gleason-type results are known. For octonions, however, non-associativity obstructs a straightforward generalization. We therefore formulate Conjecture 1: an octonionic analogue of the geometric Born problem in which $\mathbb{CP}^{n-1}$ is replaced by an appropriate octonionic projective manifold, $\text{PU}(n)$ by an exceptional group such as G_2 or F_4 , and the same three geometric axioms are imposed. The conjecture is deliberately stated in a way that can be attacked with tools from representation theory and invariant theory, and it reduces to the standard complex Born rule on associative subalgebras.

Several limitations remain. Our results are conditional on the geometric axioms, they are proved in finite dimension ($n \geq 3$), and the link to gravitational or torsionful dynamics is at present only kinematical. Nonetheless, the geometric formulation developed here provides a compact and flexible tool: it recasts the Born rule as a statement about measures on highly symmetric state spaces, and it

opens a precise pathway toward quaternionic and octonionic extensions relevant to quantum gravity and unification programmes.

In summary, the complex projective case developed here is fully rigorous: for $n \geq 3$ the Born rule is uniquely fixed by the geometry of $\mathbb{CP}^{n-1}$ under Assumptions (A1)–(A3). By contrast, the octonionic extension is at present only partially established, through the 7-state discrete model of Appendix A [Finite Toy Model from the Seven Imaginary Octonions](#), and otherwise remains conjectural.

Novelty and broader impact.

Beyond supplying an alternative proof of the Born rule, the contribution of this work is three-fold. (i) *Geometry-first Gleason*. By replacing the usual Hilbert-space axioms with the bare Fubini–Study metric, full $PU(n)$ symmetry and frame additivity, we isolate a *purely geometric* core that already forces the quadratic rule. In doing so, inner products and density operators emerge *a posteriori*, clarifying which part of the Born postulate is of geometric origin and which part is algebraic. This shift of perspective is valuable wherever linear Hilbert structure may itself be emergent, e.g. in certain quantum-gravity or generalized-probabilistic frameworks. (ii) *Local spinor application*. We show that any relativistic theory with Dirac, Weyl or Majorana fields inherits the Fubini–Study geometry fibre-wise, hence the Born measure follows *locally*, independent of the global structure of the solution space or of curvature/torsion effects. This establishes a crisp modular separation between (a) a kinematic probability geometry that is immutable, and (b) dynamical evolution equations that may vary from the free Dirac case up to Einstein–Cartan or beyond. Such locality has not been spelled out in prior treatments that focus on global Hilbert spaces. (iii) *Exceptional-algebra outlook*. By recasting the uniqueness problem on octonionic projective manifolds we connect the probability issue to the symmetry of the exceptional groups G_2 , F_4 and E_8 , furnishing an explicit conjecture testable with tools from non-associative geometry. The seven-state Fano toy model provided in the appendix is, to our knowledge, the first finite quantum system whose pure states are literally the imaginary units of $\mathbb{O}$ and that nonetheless obeys a quadratic Born-type rule. Together, these results offer a compact, geometry-centric template that can be transplanted to spin-foam, twistor or GPT settings, and they sharpen foundational debates by pinpointing exactly which physical desiderata—locality, symmetry, additivity—must be relaxed if one aims to modify or go beyond the Born rule. That being said, the octonionic case is only partially treated here: we give a discrete 7-state model and a uniqueness result for G_2 -invariant measures on S^6 , while a full Gleason-type theorem in the octonionic setting will be addressed in future work.

Appendix A. A Finite Toy Model from the Seven Imaginary Octonions

The seven imaginary units $\{e_1, \dots, e_7\}$ of the octonions multiply according to the Fano plane. The group $\Gamma \cong 2^3 \rtimes GL(3, 2)$ (order $8 \times 168 = 1344$) is the automorphism group of the Fano plane and acts transitively on the set

$$\mathcal{S}_7 = \{\pm e_1, \pm e_2, \dots, \pm e_7\} / \{\text{global sign}\},$$

which consists of seven pure states (one per imaginary direction, up to overall sign).

We now impose discrete analogues of Axioms (A1)–(A3): locality (the probability depends only on whether $i = j$ or $i \neq j$), Γ -invariance, and additivity over the complete set of seven states. A simple counting argument then shows that the only compatible transition probability on $\mathcal{S}_7$ is the sharp rule

$$P(s_i \rightarrow s_j) = \delta_{ij}. \quad (\text{A1})$$

When complex amplitudes $\{a_i\}$ with $\sum_{i=1}^7 |a_i|^2 = 1$ are assigned to these seven directions, the unique Γ -invariant extension of the rule above is

$$P(a \rightarrow s_j) = |a_j|^2, \quad j = 1, \dots, 7, \quad (\text{A2})$$

which is precisely the quadratic Born rule on the complex amplitudes a_i .

This toy model illustrates how a finite geometry derived purely from the octonions can admit a high degree of symmetry and, upon introducing ordinary complex amplitudes on its states, reproduce the standard quadratic probability rule. It is, to our knowledge, the first quantum system whose pure states are literally the imaginary units of the octonions and that rigorously satisfies the Born rule.

Appendix B. A Partial Result for the Unit Imaginary Octonions

Let

$$S^6 = \{v \in \text{Im}(\mathbb{O}) : \|v\| = 1\} \subset \mathbb{R}^7,$$

where $\|v\|^2$ is the standard Euclidean norm induced by the octonionic quadratic form. The exceptional Lie group G_2 acts transitively on S^6 by octonion automorphisms, with stabilizer isomorphic to $\text{SU}(3)$.

Definition A1 (G_2 -invariant measures on S^6). *A Borel probability measure μ on S^6 is called G_2 -invariant if $\mu(gE) = \mu(E)$ for all $g \in G_2$ and all Borel sets $E \subset S^6$.*

Theorem A1 (Uniqueness of the G_2 -invariant measure on S^6). *There exists a unique G_2 -invariant Borel probability measure μ_{G_2} on S^6 . Moreover, μ_{G_2} coincides with the normalized Riemannian volume measure induced by the round metric on S^6 , which in turn is determined by the G_2 -invariant quadratic form $\|v\|^2$ on $\text{Im}(\mathbb{O})$.*

Proof. The group G_2 is compact and admits a unique (up to scale) bi-invariant Haar measure. The orbit map $G_2 \rightarrow S^6, g \mapsto g \cdot e_1$ (for a fixed unit imaginary octonion e_1) identifies S^6 with the compact homogeneous space $G_2/\text{SU}(3)$. General results on compact homogeneous spaces guarantee the existence of a unique G_2 -invariant probability measure on S^6 , obtained as the pushforward of Haar measure and normalized so that $\mu_{G_2}(S^6) = 1$.

The representation of G_2 on $\text{Im}(\mathbb{O}) \cong \mathbb{R}^7$ is irreducible and admits a unique (up to scale) G_2 -invariant inner product, which coincides with the Euclidean metric associated with the octonionic quadratic norm $\|v\|^2$. This inner product induces the standard round metric on S^6 , whose volume form is G_2 -invariant and hence agrees with μ_{G_2} up to normalization. $\square$

Remark A1. *Theorem A1 establishes the analogue of the first step in the complex case: once G_2 -invariance is imposed, the underlying metric and volume measure on the state space are uniquely determined by a single quadratic invariant (the octonionic norm). In this sense, any “locality + symmetry” axiom on S^6 necessarily sees a quadratic geometry, just as in the complex projective setting where $\text{PU}(n)$ -invariance singles out the Fubini–Study metric.*

A full Gleason-type result would require classifying G_2 -invariant, additive probability assignments on suitable families of “octonionic subspaces”. This is obstructed by non-associativity and remains open. However, when one restricts to an associative subalgebra (e.g. a complex slice $\mathbb{C} \subset \mathbb{O}$), the geometry reduces to the standard complex projective case, and the usual Born rule is recovered from Theorem 1.

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